Optimization

MAL 7023

The course is mainly a course on Linear Programming with an introduction to non-linear programming.

Linear Programming/Linear Optimization

An optimization problem in which the function to be optimized (maximized or minimized) and the constraints are linear functions is called a Linear programming problem

example. Max $2x_1 + 3x_2 - x_3 + x_4$

s.t $2x_1 + 5x_2 + 3x_3 - x_4 = 6$

$x_1 + 2x_2 + 2x_3 + x_4 = 10$

If the constraints in the problem are of "$\leq$" type (or "$\geq$" type) then we shall add (or subtract) suitable quantity to make the equations of equality type

example

$$\begin{aligned} \max\ & 3x_1 + 2x_2 \\ \text{s.t } & x_1 + 2x_2 \leq 4 \\ & 3x_1 + 4x_2 \leq 12 \\ & x_1, x_2 \geq 0 \end{aligned}$$

[can be written as] $\Rightarrow$

$$\begin{aligned} \max\ & 3x_1 + 2x_2 \\ \text{s.t } & x_1 + 2x_2 + x_3 = 4 \\ & 3x_1 + 4x_2 + x_4 = 12 \\ & x_1, x_2, x_3, x_4 \geq 0 \end{aligned}$$

where "x_3" is added to convert $x_1 + 2x_2 \leq 4$ into equality and "x_4" is added to convert $3x_1 + 4x_2 \leq 12$ into equality.

Thus standard form of LPP in this course is

$$\left.\begin{aligned} \max\ & c_1x_1+c_2x_2+\dots+c_nx_n \\ \text{s.t.}\ & AX=b \\ & X\geq 0 \end{aligned}\right\} \quad \text{— (A)}$$

where A is a $m\times n$ matrix, $X=\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$ is coloum of variables & b is $m\times 1$ matrix. Note that $X\geq 0$ means $x_i\geq 0$ $\forall i=1,2,\dots n$ and $\text{Rank}(A)=\min(m,n)$.

Solution to a given LPP

Note that if $\{x\in \mathbb{R}^n : AX=b, X\geq 0\}$ is the feasible region, the feasible region is described by m equations in n variables (with $X\geq 0$).

To obtain possible candidates for maximization or minimization, substitute $(n-m)$ variables as zero and solve for remaining m variables. If solving for these m variables gives a unique solution (for these m variables) the the solution is called the basic feasible solution

example : Find the basic feasible solutions of region described by the equations

$$x_1+2x_2+x_3=4$$
$$2x_1+x_2+2x_3=5$$

2 equations, 3 variables, put any one equal to zero & solve for other two.

$x_1=0 \Rightarrow x_2=1,\ x_3=2$

$x_2=0 \Rightarrow$ no solution

$x_3=0 \Rightarrow x_1=2,\ x_2=1$

$\Rightarrow (0,1,2)$ and $(2,1,0)$ are basic feasible solutions to the given feasible region.

example :- Find the basic feasible solutions for the feasible region described by the equations

$$x_1 + x_2 + x_3 = 4$$
$$2x_1 + 2x_2 - x_3 = 8$$

Here $n = 3$, $m = 2$ and the Basic feasible solutions will be obtained by setting $n-m$ variables to zero and solving for the remaining.

$x_1 = 0 \Rightarrow x_2 = 4,\ x_3 = 0$ — ①

$x_2 = 0 \Rightarrow x_1 = 4,\ x_3 = 0$ — ②

$x_3 = 0$ yields $x_1 + x_2 = 4$ and $2x_1 + 2x_2 = 8$. As these cannot be solved uniquely in (x_1, x_2), there is no basic feasible solution corresponding to $x_3 = 0$.

As a terminology, the $n-m$ variables set to zero and remaining are solved to obtain a solution x^*, then the $n-m$ variables set to zero are non-basic variables and the m variables (solved) are the basic variables for the solution x^*.

In the above example, For the solution $(0,4,0)$ described by ①
x_1 is non-basic variable and (x_2, x_3) are basic variables [for $(0,4,0)$].

Similarly for the solution $(4,0,0)$ described by ②, x_2 is non-basic variable and (x_1, x_3) are basic variables [for the solution $(4,0,0)$].

Result :- For a given LPP, if the LPP has an optimal solution, then atleast one of the basic feasible solutions is optimal.

Thus to find an optimal solution to the given problem, one approach is to find all basic feasible solutions, evaluate function value at each of these points and find the optimal solution by sheer comparison.

example :
$$\left.\begin{array}{l} \max\ x_1 + 2x_2 \\ \text{s.t.}\ x_1 + x_2 \leq 4 \\ 2x_1 + x_2 \leq 6 \\ x_1, x_2 \geq 0 \end{array}\right\} - (1)$$

The problem can also be writen as
$$\begin{array}{l} \max\ x_1 + 2x_2 + 0x_3 + 0 \cdot x_4 \\ \text{s.t.}\ x_1 + x_2 + x_3 = 4 \\ 2x_1 + x_2 + x_4 = 6 \\ x_1, x_2, x_3, x_4 \geq 0 \end{array}$$

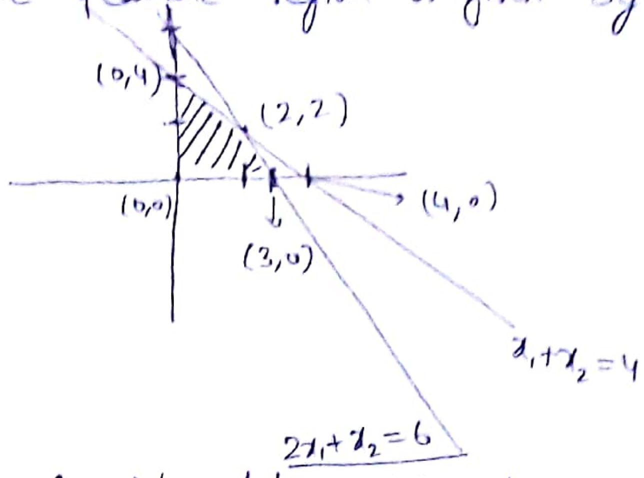
Finding all basic feasible solutions

1. $x_1 = 0,\ x_2 = 0 \Rightarrow x_3 = 4,\ x_4 = 6$ pt is $(0, 0, 4, 6)$
2. $x_1 = 0,\ x_3 = 0 \Rightarrow x_2 = 4,\ x_4 = 2$ pt is $(0, 4, 0, 2)$
3. $x_1 = 0,\ x_4 = 0 \Rightarrow x_2 = 6,\ x_3 = -2$ (not feasible as $x_3 \geq 0$ is a condition)
4. $x_2 = 0,\ x_3 = 0 \Rightarrow x_1 = 4,\ x_4 = -2$ (not feasible as $x_4 \geq 0$ is a condition)
5. $x_2 = 0,\ x_4 = 0 \Rightarrow x_1 = 3,\ x_3 = 1$ pt is $(3, 0, 1, 0)$
6. $x_3 = 0,\ x_4 = 0 \Rightarrow x_1 = 2,\ x_2 = 2$ pt is $(2, 2, 0, 0)$

finding value at each of the obtained points, optimal solution lies at $(0, 4)$ and optimal value is 8.

Geometrically, the basic feasible solutions are corner points of the feasible region (as described below).

For the problem ①, the feasible region is given by



Note that the four basic feasible solutions obtained in the Example ① are corner points of the feasible region as indicated in the diagram above.

Thus as described before, one may find all corner points of the feasible region (or all basic feasible solutions) and compare the function values at these points to obtain the optimal solution.

One may verify that if the feasible region is described by the equations

$$x_1+x_2 \geq 4$$
$$2x_1+x_2 \leq 6$$
$$x_1, x_2 \geq 0$$

then feasible region is the triangle formed by points (2,2), (3,0) & (4,0) and the basic feasible solutions for this region are vertices of the triangle i.e. (2,2), (3,0) and (4,0).

Simplex method for solving LPP

Let the original LPP be of the form

$$\max \; c^T X \quad (= c_1 x_1 + c_2 x_2 + \dots c_n x_n)$$

$$\text{s.t.} \quad AX \leq b$$

$$X \geqslant 0$$

which is transformed into the following problem (after converting "$\leq$" eqns into equality)

$$\left.\begin{aligned} &\max \; c^T X \\ &\text{s.t} \;\; AX + I X_s = b \\ &\quad X \geqslant 0 \end{aligned}\right\} \quad - \; (*)$$

If A is a $m \times n$ matrix (of full Rank) then any basic feasible solution has m basic variables and $n-m$ non-basic variables. Let us write the information present in the LPP in tabular form (as described below):

			c_1	c_2	$c_3 \dots$	c_n	0	$0 \dots$	0
C_B	B	b	a_1	a_2	$a_3 \dots$	a_n	a_{n+1}	$a_{n+2} \dots$	a_{n+m}
0	x_{n+1}	b_1	a_{11}	a_{12}	$a_{13} \dots$	a_{1n}	1	$0 \; 0 \dots$	0
0	x_{n+2}	b_2	a_{21}	a_{22}	$a_{23} \dots$	a_{2n}	0	$1 \; 0 \dots$	0
$\vdots$	$\vdots$	$\vdots$							
0	x_{n+m}	b_m	a_{m1}	a_{m2}	$a_{m3} \dots$	a_{mn}	0	$0 \; 0 \dots$	1

Note that the table is obtained by writing each of the equations $b = AX$ (row-wise). As x_{n+i} is added in i-th equation to convert "$\leq$" type equation into equality, the coloums of $a_{n+1}, a_{n+2}, \ldots a_m$ together form an identity matrix.

Note that if we substitute $x_1 = x_2 = \ldots x_n = 0$ we obtain $x_{n+1} = b_1, x_{n+2} = b_2, \ldots x_{n+m} = b_m$ and hence the solution $(0, 0, \ldots 0, b_1, b_2, \ldots b_m)$ is a basic feasible solution. Further, it may be noted that while original variables $x_1, x_2, \ldots x_n$ are non-basic variables for this solution, the variables $x_{n+1}, x_{n+2}, \ldots x_{n+m}$ are basic variables for this solution. The information is captured in ~~tab~~ coloum B of the constructed table. Finally the cost of the basic variables is metioned alongside the basic variables (coloum C_B) and cost of each variable is mentioned above coloum of the respective variable.

Note that comparing the coloums B and b yields the solution obtained in previous para ($x_{n+1} = b_1, x_{n+2} = b_2, \ldots x_{n+m} = b_m$) and thus the simplex table corresponds to the basic feasible solution obtained in the previous para.

In the process of obtaining the optimal solution to the given problem, we shall update the table iteratively moving closer and closer to optimal solution and finally reach optimality in finite iterations

Let us introduce two more notions to make our journey easier

For each variable x_j we ~~not denote~~ define.

$$z_j = \sum_{j=1}^{m} c_{B_i}\, a_{ij} \quad \left[\begin{array}{l}\text{multiplication of coloum } C_B \\ \text{with coloum of } x_j \text{ (i.e. } a_j)\end{array}\right]$$

and finally define net evaluation at x_j as $z_j - c_j$ (where c_j is the cost of j-th variable). Let us append the net evaluations at each of the variables below their coloums (in the table). The modified table now looks like :

			c_1	c_2	c_3	c_n	0	0	0
C_B	B	b	a_1	a_2	a_3	a_n	a_{n+1}	a_{n+2} . . .	a_{n+m}
0	x_{n+1}	b_1	a_{11}	a_{12}	a_{13}	a_{1n}	1	0	0
0	x_{n+2}	b_2	a_{21}	a_{22}	a_{23}	a_{2n}	0	1	0
$\vdots$	$\vdots$	$\vdots$							1
						a_{mn}	0	0	
0	x_{n+m}	b_m	a_{m1}	a_{m2}	a_{m3}		0	0	0
			z_1-c_1	z_2-c_2	z_3-c_3	z_n-c_n			

			c_1	c_2	c_3	c_n	0	0	0
C_B	B	b	a_1	a_2	a_3	a_n	a_{n+1}	a_{n+2}	a_{n+m}
0	x_{n+1}	b_1	a_{11}	a_{12}	a_{13}	a_{1n}	1	0	0
0	x_{n+2}	b_2	a_{12}	a_{22}	a_{23}	a_{2n}	0	1	0
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$		
0	x_{n+m}	b_m	a_{1m}	a_{2m}	a_{3m}	a_{mn}	0	0	1
			z_1-c_1	z_2-c_2	z_3-c_3	z_4-c_4	0	0	0

Some Results

① If $z_j - c_j \geq 0 \ \forall j = 1, 2, \ldots n+m$, then the simplex table corresponds to the optimal solution for the given problem

② If $z_j - c_j < 0$ for some variables, pick the variable (coloum) with most negative $z_j - c_j$ and ~~computing compute the~~ compute the ratios $\left\{ \frac{b_i}{a_{ij_0}} : a_{ij} > 0 \right\}$ (j_0 is coloum with most $-$ve $z_j - c_j$)

If $\frac{b_r}{a_{rj_0}}$ is the minimum ratio then replacing the r-th basic variable with x_{j_0} gives an Basic feasible solution with greater value of objective function (and hence is closer to the optimal solution).

③ If for some variable x_j, we have $z_j - c_j < 0$ and $a_{ij} < 0 \ \forall i$, i.e. $z_j - c_j$ is $-$ve all entries in that coloum are ≤ 0, then the given LPP has an unbounded solution.

Methodology :- While computing BFS (basic feasible solutions), we shall always mantain the matrix corresponding to basic variables is the identity matrix. Thus if r-th basic variable leaves the basis and x_j is required to enter, then we shall transform coloum of x_j (i.e. a_j) into coloum of r^{th} basic variable (which is leaving).

Example

$$\left[\begin{array}{l} \max\ 3x_1 + 2x_2 \\ \text{s.t. } x_1 + x_2 \le 4 \\ \quad 2x_1 + x_2 \le 6 \\ \quad x_1, x_2 \ge 0 \end{array}\right]$$

which is same as

$$\begin{array}{l} \max\ 3x_1 + 2x_2 \\ \text{s.t. } x_1 + x_2 + x_3 = 4 \\ \quad 2x_1 + x_2 + x_4 = 6 \\ \quad x_1, x_2, x_3, x_4 \ge 0 \end{array}$$

First Simplex table

				3	2	0	0
	C_B	B	b	a_1	a_2	a_3	a_4
	0	x_3	4	1	1	1	0
←	0	x_4	6	2	1	0	1
		$z_j - c_j$:		-3 ↑	-2	0	0

As $z_j - c_j$ is least for a_1, a_1 will enter the basis. Further as the ratio $\frac{b_2}{a_{12}}$ is least, x_4 will leave the basis, Thus new table is

				3	2	0	0
	C_B	B	b	a_1	a_2	a_3	a_4
←	0	x_3	1	0	$1/2$	1	$-1/2$
	3	x_1	3	1	$1/2$	0	$1/2$
		$z_j - c_j$:		0	$-1/2$ ↑	0	$3/2$

$z_2 - c_2$ is negative and $\frac{b_1}{a_{21}}$ is least. Thus x_2 enters the basis and x_3 leaves the basis. Thus the new table is

			3	2	0	0
C_B	B	b	a_1	a_2	a_3	a_4
2	x_2	2	0	1	2	-1
3	x_1	2	1	0	-1	1
	$z_j - c_j$:		0	0	1	1

$z_j - c_j \geq 0 \;\forall j = 1,2,3,4. \Rightarrow$ Current table corresponds to the optimal solution to the given problem.

Thus optimal solution is $(x_1^* = 2,\ x_2^* = 2)$ opt. value $= 10$.

Some Observations

① Note that the BFS corresponding to each table in the previous problem are $(0,0)$, $(3,0)$ and $(2,2)$ respectively. The function value to be optimized $(3x_1 + 2x_2)$ increases at each step and reaches optimality in the final step [at $(2,2)$].

② Simplex method actually compares the value of current BFS with neighbouring BFS and moves towards improvement (which need not be maximum improvement). It does this at each iteration and once it has no scope of improvent in neighbouring BFS, it is a point of local maxima (or minima) and hence is the optimal solution to the given problem (why??).

Excercise : $\max\ 4x_1 + 3x_2$

s.t. $x_1 + x_2 \le 8$

$2x_1 + x_2 \le 10$

$x_1, x_2 \geq 0$

Before we move further, let us solve one more problem.

$$\left\{\begin{array}{l} \max\ 6x_1 - 2x_2 \\ \text{s.t. } 2x_1 - x_2 \le 2 \\ \quad x_1 \le 4 \\ \quad x_1, x_2 \geq 0 \end{array}\right\}$$

[which can be written as]

$$\begin{array}{l} \max\ 6x_1 - 2x_2 \\ \text{s.t. } 2x_1 - x_2 + x_3 = 2 \\ \quad x_1 + x_4 = 4 \\ \quad x_1, x_2, x_3, x_4 \geq 0 \end{array}$$

First Simplex table

				6	-2	0	0
	C_B	B	b	a_1	a_2	a_3	a_4
←	0	x_3	2	2	-1	1	0
	0	x_4	4	1	0	0	1
		$z_j - c_j$:		-6 ↑	2	0	0

$z_1 - c_1$ is least (most negative) thus x_1 enters. Further in column a_1

$\frac{b_1}{a_{11}}$ is least ratio (among positive entries) and thus x_3 leaves

the basis. Thus the new table is

			6	-2	0	0
C_B	B	b	a_1	a_2	a_3	a_4
6	x_1	1	1	$-\frac{1}{2}$	$\frac{1}{2}$	0
0	x_4	3	0	$\frac{1}{2}$	$-\frac{1}{2}$	1
	$z_j - c_j$:		0	-1	3	0

$z_2 - c_2$ is negative $\Rightarrow x_2$ enters the basis. Further $\frac{b_2}{a_{22}}$ is least ratio (among positive entries) & thus x_4 leaves the basis. Thus the updated table is

			6	-2	0	0
C_B	B	b	a_1	a_2	a_3	a_4
6	x_1	4	1	0	0	1
-2	x_2	6	0	1	-1	2
			0	0	2	2

$z_j - c_j \geq 0 \ \forall j = 1,2,3,4 \Rightarrow$ above table corresponds to optimal solution to the given problem. Optimal soln is $(x_1^* = 4, x_2^* = 6)$. opt. value $= 12$.

One may verify that the problem

$$\max\ 2x_1 + 5x_2$$
$$\text{s.t}\quad x_1 - 3x_2 \le 5$$
$$2x_1 - x_2 \le 7$$
$$x_1, x_2 \geq 0$$

has an unbounded solution.

Charne's M method for Solving LPP

So far we have addressed the problem of solving LPP where all eqns are of "$\le$" type (with $b \geq 0$). In such situations, as we add new variables to convert constraints into equalities, identity matrix is available in the first iteration (coloums of additional "slack" variables) which ~~are~~ is subsequently preserved and used to solve the given LPP.

However, a general LPP may contain constraints of "$\geq$" type or "$=$" type and hence identity matrix in the first iteration is not readily available. To tackle the situation we introduce "artificial" variables (to obtain initial identity matrix) and solve the problem. If in the optimal solution, we have all artificial variables equal to zero, then the corresponding solution is optimal for the original problem. If some artificial variable is non-zero in the optimal solution, the original problem has no feasible (and hence optimal) solution.

As artificial variables are not originally a part of given problem, they are assigned a very large negative cost $(-M)$ in the linear programming problem (why??).

<u>Example</u>:

$$\left.\begin{aligned} &\text{Max } -2x_1 - x_2 \\ &\text{s.t. } 3x_1 + x_2 = 3 \\ &\quad 4x_1 + 3x_2 \geq 6 \\ &\quad x_1 + 2x_2 \leq 3 \\ &\quad x_1, x_2 \geq 0 \end{aligned}\right\} \xrightarrow{\text{same as}} \begin{aligned} &\max\ -2x_1 - x_2 \\ &\text{s.t. } 3x_1 + x_2 = 3 \\ &\quad 4x_1 + 3x_2 - x_3 = 6 \\ &\quad x_1 + 2x_2 + x_4 = 3 \\ &\quad x_1, x_2, x_3, x_4 \geq 0 \end{aligned}$$

As the problem on RHS does not have an identity matrix to start with, we introduce artificial variables (with cost $-M$) in the problem. The new problem obtained is

$$\begin{aligned} &\max\ -2x_1 - x_2 - Mx_5 - Mx_6 \\ &\text{s.t. } 3x_1 + x_2 + x_5 = 3 \\ &\quad 4x_1 + 3x_2 - x_3 + x_6 = 6 \\ &\quad x_1 + 2x_2 + x_4 = 3 \\ &\quad x_1, x_2, x_3, x_4, x_5, x_6 \geq 0 \end{aligned}$$

Now as identity matrix is available in the matrix (coloums of x_5, x_6 & x_4) we solve the problem using simplex table.

First table :

C_B	B	b	a_1 (−2)	a_2 (−1)	a_3 (0)	a_4 (0)	a_5 (−M)	a_6 (−M)
$-M$	x_5	3	3	1	0	0	1	0
$-M$	x_6	6	4	3	−1	0	0	1
0	x_4	3	1	2	0	1	0	0
	$z_j - c_j$:		$-7M-2$	$-4M+1$	M	0	0	0

$z_1 - c_1$ is most negative $\Rightarrow x_1$ enters the basis, $\frac{b_1}{a_{11}}$ is least ratio and thus x_5 leaves the basis. The updated simplex table is

C_B	B	b	a_1 (−2)	a_2 (−1)	a_3 (0)	a_4 (0)	a_5 (−M)	a_6 (−M)
−2	x_1	1	1	1/3	0	0	1/3	0
$-M$	x_6	2	0	5/3	−1	0	−4/3	1
0	x_4	2	0	5/3	0	1	−1/3	0
	$z_j - c_j$:		0	$-\frac{5M}{3}+\frac{1}{3}$	M	0	$\frac{7M}{3}-\frac{2}{3}$	0

$z_2 - c_2$ is −ve $\Rightarrow x_2$ enters $\frac{b_2}{a_{22}}$ is least $\Rightarrow x_6$ leaves. Thus new table is

C_B	B	b	a_1 (−2)	a_2 (−1)	a_3 (0)	a_4 (0)	a_5 (−M)	a_6 (−M)
−2	x_1	3/5	1	0	1/5	0	3/5	−1/5
−1	x_2	6/5	0	1	−3/5	0	−4/5	3/5
0	x_4	0	0	0	1	1	1	−1
			0	0	1/5	0	$M-\frac{2}{5}$	$M-\frac{1}{5}$

As all $z_j - c_j \geq 0$ and all artificial variables are zero, the above table corresponds to optimal solution to the original problem. Thus optimal solution is $\left(x_1^* = \frac{3}{5}, x_2^* = \frac{6}{5}\right)$ & optimal value is $\left(-\frac{12}{5}\right)$.

<u>Excercise</u> :- $\max\ 3x_1 + 2x_2 + 3x_3$

$$\text{s.t.}\quad 2x_1 + x_2 + x_3 \leq 2$$
$$3x_1 + 4x_2 + 2x_3 \geq 8$$
$$x_1, x_2, x_3 \geq 0$$

<u>Remarks</u> :-

① So far throughout we have addressed problems of maximization type, if the problem is of minimization, we may maximize $-f(X)$ instead of minimizing $f(X)$ $\left[\text{as } \min f(X) = -\max[-f(X)]\right]$.

② In Charne's M method, if an artificial variable leaves the basis it cannot enter again [as its cost is $-M$ and problem is of maximization type]. Thus one may not compute the coloum of artificial variable after it has left the basis. [However the coloum of artificial variable are useful for other reasons and will be seen later].

③ In all the problems solved so far, in each table, $z_j - c_j$ is zero for basic variables. Is it a coincidence or can it be generalised ??

(Think !!!)