Probabilities with MC: $X(t)$ $a_0, a_1, a_2, — a_n$

(I) $P(X_0 = a_0, X_1 = a_1, X_2 = a_2, ---, X_n = a_n)$

$= P(X_n = a_n \mid X_{n-1} = a_{n-1}) \cdot P(X_{n-1} = a_{n-1} \mid X_{n-2} = a_{n-2})$

$\ldots\ldots P(X_1 = a_1 \mid X_0 = a_0)\ P(X_0 = a_0)$

(II) $P(X_n = a_j) \equiv j^{th}$ state in $V_0 P^n$

Step State

Initial Probability Vector

Given $V_0 \Rightarrow V_1 = V_0 P,\ V_2 = V_1 P,$ —.

$\ldots V_{n+1} = V_n \cdot P$

$\equiv V_{n+1} = [- - - \underset{j^{th}}{O} - -]_{n+1}$

Classification of States:

The state of a M.C can be classified as recurrent or transient based on whether they are visited again on that state.

1) Recurrent: State a_i of a MC is said to be recurrent if starting from a_i, the return to state a_i is certain.

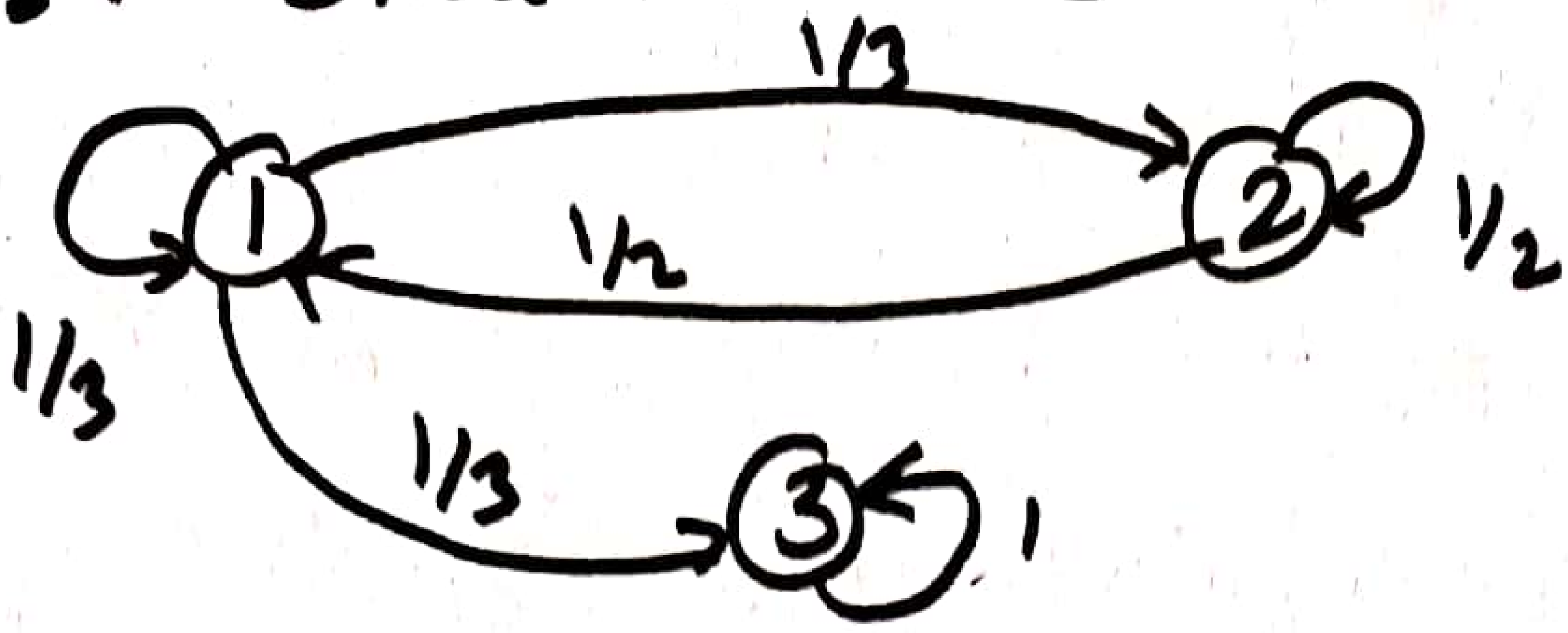
2) Transient: State a_i of a MC is said to be transient if starting from a_i, the return to state a_i is uncertain.

→ (1) Find $F_{ii} \equiv$ Probability of returning to state a_i

$= f_{ii}^{(1)} + f_{ii}^{(2)} + \cdots = \sum_{n=1}^{\infty} f_{ii}^{(n)}$

(2) if $F_{ii} = 1$, then recurrent / $F_{ii} < 1$, then transient

Q1 Categorized state 1 and 3 in the follwg MC



(HW)

State 1 (1)

No of step	Path	$f_{ii}^{(n)}$
1	$1 \to 1$ (loop)	$1/3 = f_{11}^{(1)}$
2	$1 \to 2 \to 1$	$1/3 \cdot 1/2 = f_{11}^{(2)}$
3	$1 \to 2 \to 2 \to 1$	$1/3 \cdot (1/2)^2 = f_{11}^{(3)}$
4	$1 \to 2 \to 2 \to 2 \to 1$	$1/3 (1/2)^3 = f_{11}^{(4)}$
$\vdots$	$\vdots$	$\vdots$

$$F_{11} = \sum_{n=1}^{\infty} f_{11}^{(n)} = \frac{1}{3} + \frac{1}{3}\left(\frac{1}{2}\right) + \frac{1}{3}\left(\frac{1}{2}\right)^2 + \frac{1}{3}\left(\frac{1}{2}\right)^3 + \text{---}$$

$$= \frac{1}{3}\underbrace{\left(1 + \frac{1}{2} + \left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^3 + \text{---}\right)}$$

$$= \frac{1}{3} \cdot \frac{1}{1-\frac{1}{2}} = \frac{2}{3} < 1 \left[\because 1 + x + x^2 + \text{---} = \frac{1}{1-x}\right.$$

$\Downarrow$

state 1 is transient.

State 3 (3)

No. of step	Path	$f_{ii}^{(n)}$
1	$3 \to 3$ (loop)	$1 \equiv f_{33}^{(1)}$
2	No path exist	$0 \equiv f_{33}^{(2)}$
$\vdots$	$\vdots$	$\vdots$

$$F_{33} = \sum_{n=1}^{\infty} f_{33}^{(n)}$$

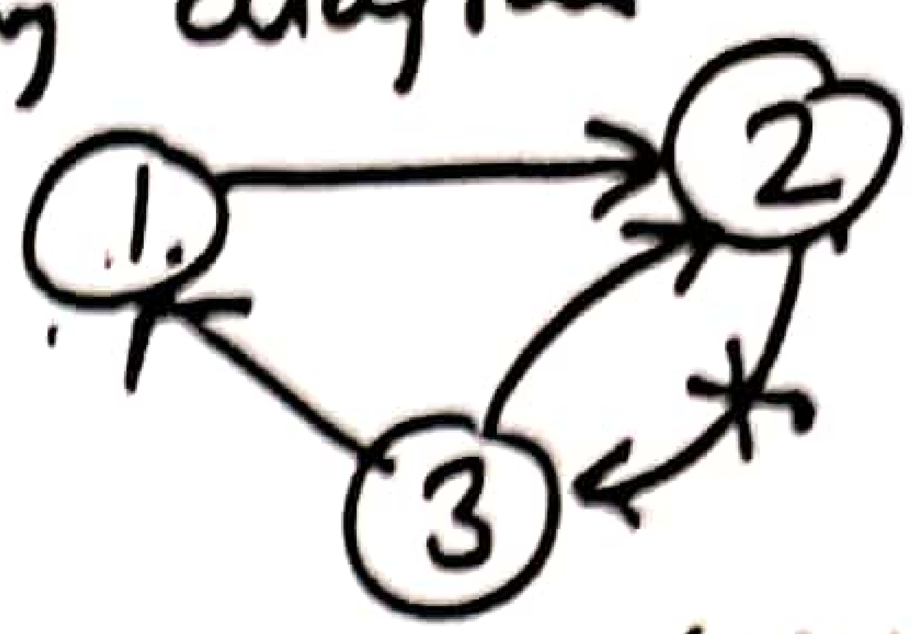
$= 1 + 0 + 0 + \text{---}$

$= 1$

$\Rightarrow$ 3 is recurrent

* **Irreducible MC**: If all states are reachable from all other states, then MC is called irreducible.

eg Consider the MC given in following diagram. Prove that it is irreducible.



(HW)

(1) 2: $2 \to 3 \to 1$

3: $3 \to 1$

(2) 1: $2 \to 1$

3: $3 \to 2$

Not loop $(3 \to 3)$

HW: Classify the States

(3) 1: $1 \to 2 \to 3$

2: $2 \to 3$

$\Rightarrow$ All states are reachable from all other states of MC

$\Rightarrow$ Given MC is irreducible.

Poisson Process:

D.RP

If $X(t)$ be the no of occurance of a certain event in $(0, t)$, then the underlying process is called Poisson Process if it satisfy the following postulates:

(i) $P(1 \text{ occurance in } (t, t+\Delta t)) = \lambda \Delta t + o(\Delta t)$

parameter

(ii) $P(2 \text{ or more occurance in } (t, t+\Delta t)) = o(\Delta t) \to 0$

(iii) $X(t)$ is independent of occurences of the event in any interval prior or after $(0, t)$.

(iv) Probability that the event occurs a specified no of time in $(t_0, t_0 + t)$ depends only on t but not on t_0.

Probability Law for PP:

1st order PMF: $P_n(t) = P(X(t) = n) = \dfrac{e^{-\lambda t} (\lambda t)^n}{\lfloor n} = \phi(\lambda t)$

$n = 0, 1, 2, \ldots$

PP $\to$ λt

2nd order PMF $P_{m,n}(t) = P[X(t_1) = m, X(t_2) = n]$

$$= \begin{cases} \dfrac{e^{-\lambda t_2} (\lambda t_1)^m (t_2 - t_1)^{n-m}}{\lfloor m \ \lfloor n-m} & ; n \geq m \\ 0 & \text{o.w} \end{cases}$$

Statistics of PP: (1) $E(X(t)) = \lambda t$ (2) $Var(X(t)) = \lambda t$

(2) $R(t_1, t_2) = \lambda^2 t_1 t_2 + \lambda \min(t_1, t_2)$ | $t_1 \leq t_2$, $R = \lambda^2 t_1 t_2 + \lambda t_1$

(4) $C(t_1, t_2) = \lambda \min(t_1, t_2) = \begin{cases} \lambda t_1 & t_1 \le t_2 \\ \lambda t_2 & t_2 \le t_1 \end{cases}$

(5) $r(t_1, t_2) = \dfrac{C(t_1, t_2)}{\sqrt{Var(X(t_1))}\sqrt{Var(X(t_2))}}$

$= \dfrac{\lambda \min(t_1, t_2)}{\sqrt{\lambda t_1}\sqrt{\lambda t_2}} = \begin{cases} \sqrt{t_1/t_2} & t_1 \le t_2 \\ \sqrt{t_2/t_1} & t_2 \le t_1 \end{cases}$

Properties:

(1) PP is a MP. $\Rightarrow$ Memory less

(2) Sum of two independent PP is a PP.

(3) Diff " " " " is not a PP.

(4) The interarrival time of a PP (interval between two successive occurance) with parameter λ has an exponential distribution.

* The two consecutive occurance of events be E_i and E_{i+1} with T be the interval b/w occurance of E_i and E_{i+1}. Then, T be a RV and follow exponential distribution.

PDF of T: $f(t) = \lambda e^{-\lambda t}$ $(t \ge 0)$

(5) If the # of occurance of an event E is an interval of length of t is a PP with λ and if each occurance of E has a constant probabity p of being recorded. If every recording is independent, then number N(t) of recorded occurance in t is also a PP with (λp).

$$P(N(t)=n) = \frac{e^{-\lambda p t}(\lambda p t)^n}{\lfloor n}$$

$\lambda \quad t = \lambda t$

$\downarrow$

$\lambda p \quad t \equiv (\underline{\lambda p} t)$

Q) For a PP with arrival rate 30 car/hour. Find the following:

(1) Expected No of arrival in first 10 min of an hour.

(2) Prob of exactly 4 arrival in first 10 min of an hour

(3) Prob of 35 or more arrival in an hour given 8 arrival in first 10 min.

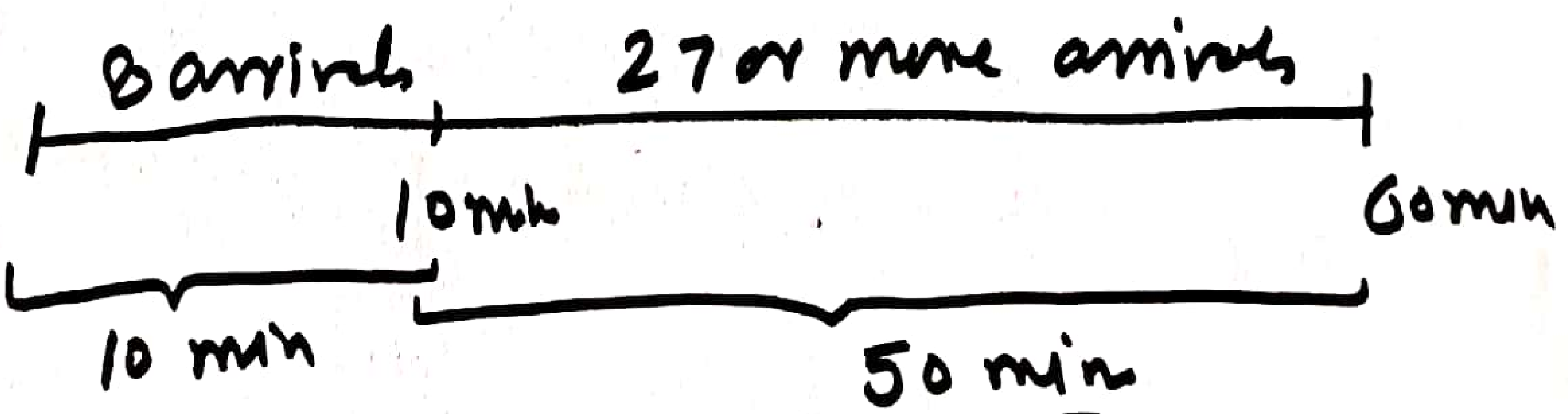
Sol^n (1) $\lambda = 30 \quad t = 10$ min (in an hour) $= 1/6$ hour

(1) $E(X(t)) = \lambda t = 30 \times \frac{1}{6} = 5$ m

(II) $\lambda = 30 \quad t = 10$ min in an hour $= 1/6$ hour

$$P(X(t)=4) = \frac{e^{-\lambda t}(\lambda t)^4}{\lfloor 4} = \frac{e^{-5}(5)^4}{\lfloor 4} = 0.1755$$

(III)



$\lambda t = 30 \cdot \frac{1}{6}$ $\qquad \lambda t = 30 \cdot \frac{5}{6} = 25$

$= 5$

$X_1(t)$ $\qquad X_2(t) \equiv P(X_2(t) \geq 27)$

$$P(X_1(t)=8) = \frac{e^{-5}(5)^8}{\lfloor 8} =$$ (HW)

$$P(X_2(t) \geq 27) = P(X_2(t) = 27) + P(X_1(t) = 28) + \cdots$$

$$\Downarrow$$

$$1 - P(X_2(t) \leq 26) = \sum_{i=27}^{\infty} P(X_2(t) = i).$$

$$= 1 - \sum_{i=0}^{26} P(X_2(t) = i)$$

$$= 0.3706$$

Birth and Death Process:

$X(t) \rightarrow$ population at time t

$P_n(t) \rightarrow$ Prob. n ~~individual~~ birth/death at time t

Birth $\rightarrow$ increases the population

Death $\rightarrow$ decreases the population

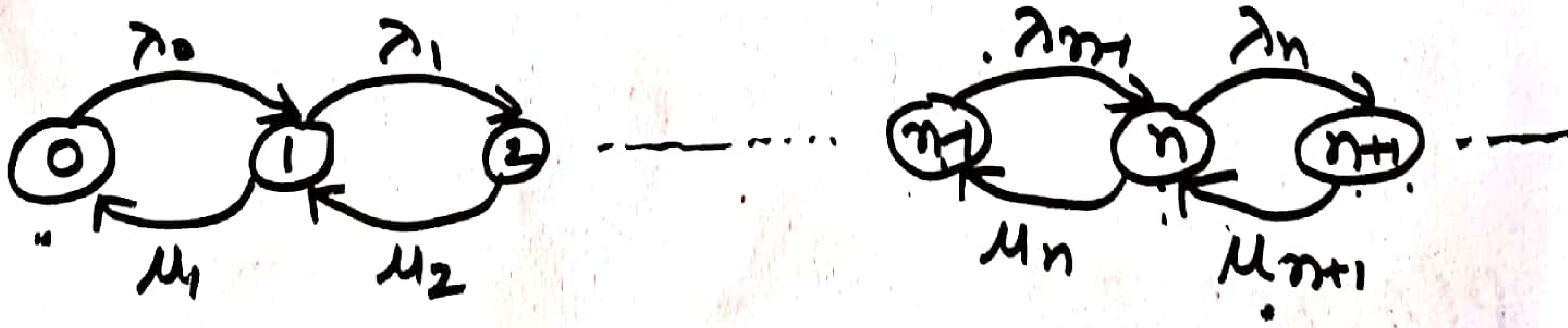
T_B : time until next birth $\equiv$ Birth process follow the PP

$T_B \equiv$ Exponential distributed RV with parameter (λ_n)

T_D : time until next death / unit time

$\equiv$ Exponential distributed RV with parameter (μ_n)

Population Transition Diagram:



Rate of change in the population: (n state)

$$P_n'(t) = \underbrace{P_{n-1}(t)\,\lambda_{n-1} + P_{n+1}(t)\,\mu_{n+1}}_{\text{inflow at } n} - \underbrace{P_n(t)\,(\lambda_n + \mu_n)}_{\text{outflow}} \quad \text{—(1)}$$

$$P_0'(t) = P_1(t)\,\mu_1 - P_0(t)\,\lambda_0 \quad \text{—(2)}$$

eq^n(1) & eq^n(2) togeather is called Birth and Death process model.

Steady state of BDP: No rate of change.

$$P_n'(t) = 0 \quad \forall n$$

$$\Rightarrow P_0'(t) = 0$$

from eq^n (1) & (2)

$$\left.\begin{aligned} &P_{n-1}(t)\,\lambda_{n-1} + P_{n+1}(t)\,\mu_{n+1} - P_n(t)\,(\lambda_n + \mu_n) = 0 \\ &P_1(t)\,\mu_1 - P_0(t)\,\lambda_0 = 0 \end{aligned}\right\} \text{—(3)}\ \text{—(4)}$$

ES in the BDP.

Eq^n (3) & (4) $\quad \boxed{AX = 0}$

* Birth → Arrival of a costomer

Death → Departure of a served costomer.

Queuing theory:

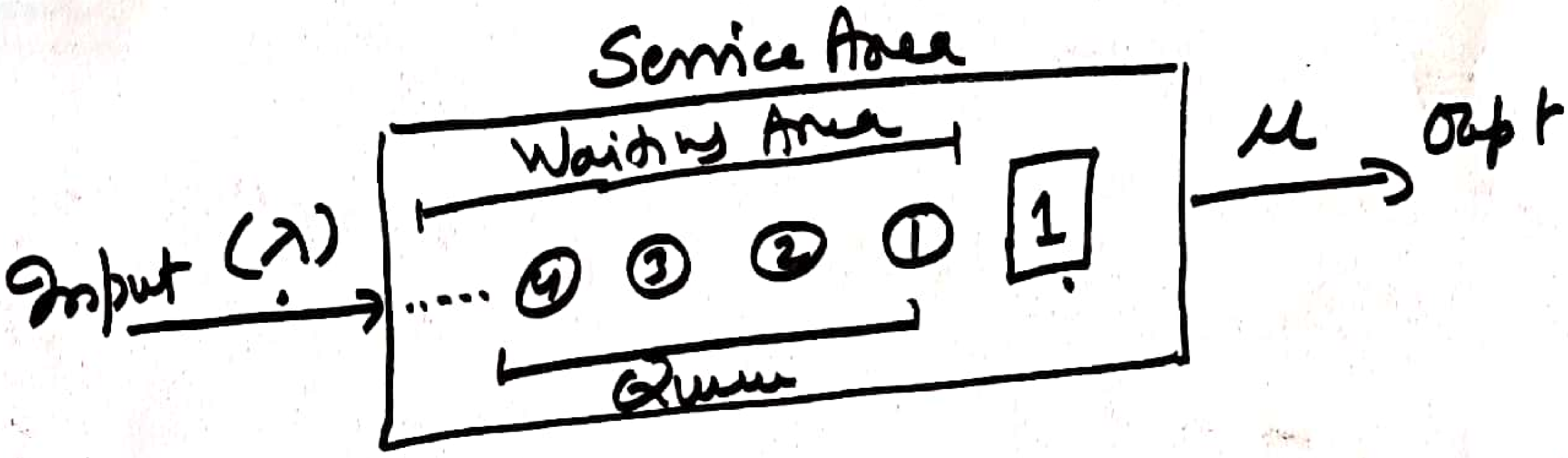
QT tries to provide a balance b/w cost associated and the waiting time of Customers.

OR

balance b/w current demand of a particular service and the capacity to provide the service.

Assume: Birth ≡ Arrival of a customer
Death ≡ Departure of a served customer.

Single Service Desk Case:



M/M/1 Queuing Model

→ M stand for 'Markov' indicating the no of arrival at t and completed service in t follow the PP/MP/MC.

M/M/1 model — ∞/FIFO, k/FIFO

Rate of change in the population: (n state)

$$P_n'(t) = \underbrace{P_{n-1}(t)\,\lambda_{n-1} + P_{n+1}(t)\,\mu_{n+1}}_{\text{inflow at } n} - \underbrace{P_n(t)\,(\lambda_n+\mu_n)}_{\text{outflow}} \quad \text{—(1)}$$

$$P_0'(t) = P_1(t)\,\mu_1 - P_0(t)\,\lambda_0 \quad \text{—(2)}$$

eq^n(1) & eq^n(2) togeather is called Birth and Death process model.

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$$\Rightarrow P_0'(t) = 0$$

from eq^n (1) & (2)

$$P_{n-1}(t)\,\lambda_{n-1} + P_{n+1}(t)\,\mu_{n+1} - P_n(t)\,(\lambda_n+\mu_n) = 0 \quad \text{—(3)}$$

$$P_1(t)\,\mu_1 - P_0(t)\,\lambda_0 = 0 \quad \text{—(4)}$$

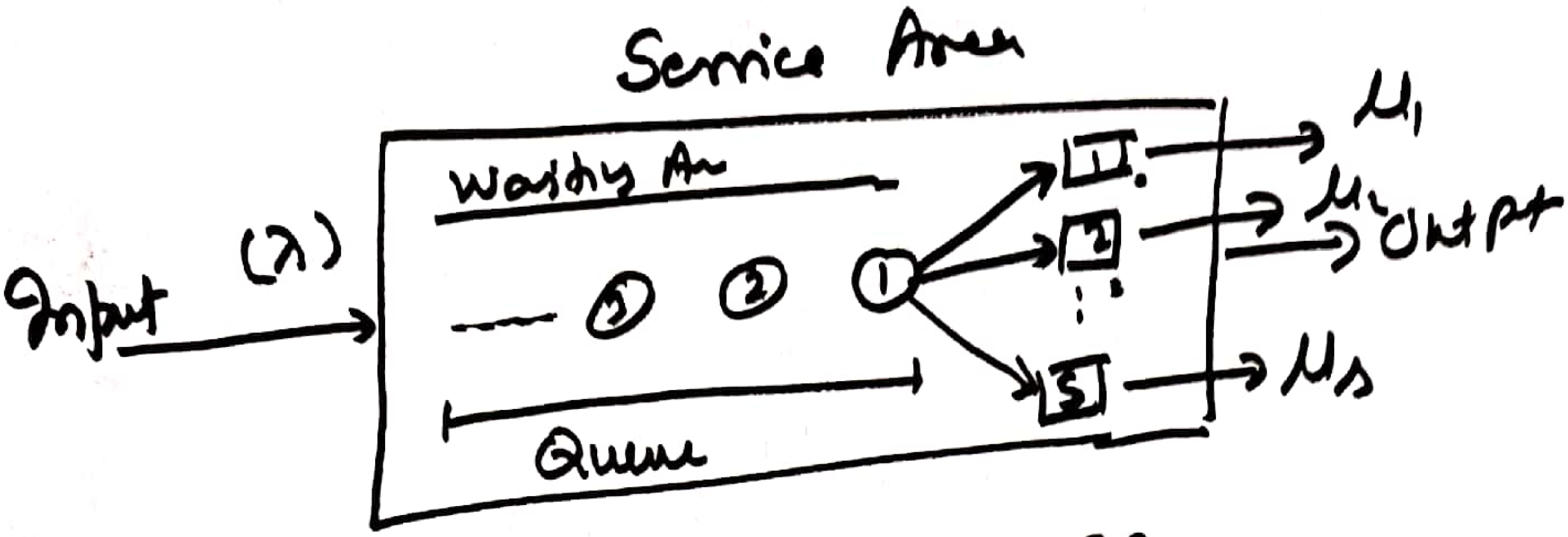
ES in the BDP.

Eq^n (3) & (4) $\boxed{AX = 0}$

* Birth → Arrival of a customer

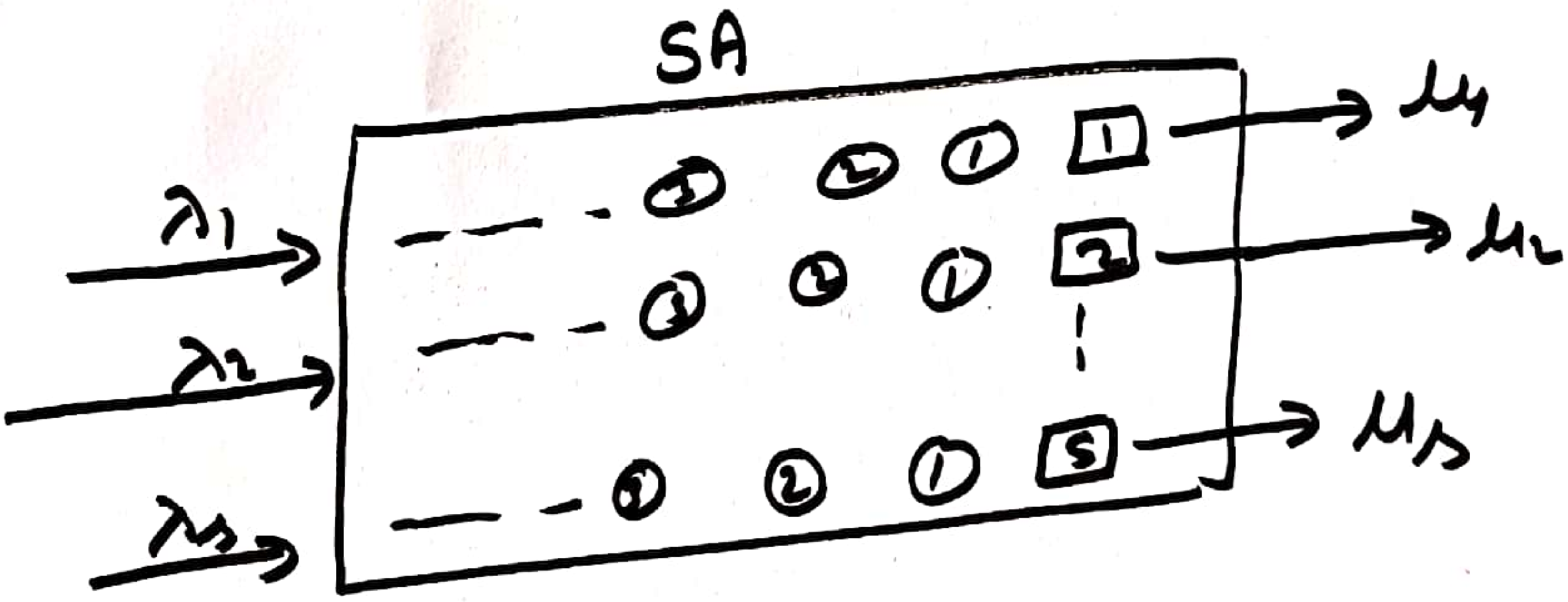
Death → Departure of a served customer.

Multiple Service Desk (in Parallel)



M/M/s Q Model — ∞/FIFO
— K/FIFO

Multiple Service Desk (in Series)



M/M/s Model (with Series)

∞/FIFO K/FIFO