Tutorial - I

① (a)

$$f(x) = 6x(1-x), \quad x \in [0,1]$$

NOW, $\int_0^1 f(x)\,dx$

$$= 6\int_0^1 x(1-x)\,dx = 6\left(\frac{1}{2}-\frac{1}{3}\right) = 1.$$

$\therefore$ $f(x)$ is a pdf.

(b)

$$P(X<b) = P(X>b)$$

$$\Rightarrow \int_0^b f(x)\,dx = \int_b^1 f(x)\,dx$$

$$\Rightarrow 2\left(\frac{b^2}{2}-\frac{b^3}{3}\right) = \frac{1}{6}$$

$$\Rightarrow 4b^3 - 6b^2 + 1 = 0$$

$$\Rightarrow \left(b-\frac{1}{2}\right)\left(4b^2-4b-2\right) = 0$$

$\therefore$ $b = \frac{1}{2}$ is solution of this equation

$\therefore$ $b = \frac{1}{2}$.

(c)

$$E(X) = \int_0^1 6x^2(1-x)\,dx$$

$$= 6\left[\frac{1}{12}\right] = \frac{1}{2}$$

$$E(X^2) = 6\int_0^1 x^3(1-x)\,dx$$

$$= 6 \cdot \frac{1}{20} =$$

$\therefore$ Mean $= \frac{1}{2}$

variance $= \frac{6}{20} - \left(\frac{1}{2}\right)^2 = \frac{6}{20} - \frac{1}{4} = \frac{6-5}{20} = \frac{1}{20}$

median $= b = \frac{1}{2}$

~~...~~ NOW $f'(x) = 6(1-2x) = 0 \Rightarrow x = 1/2$

$$f''(x)\big|_{x=1/2} < 0$$

$\therefore$ mode $= \frac{1}{2}$

(2.)

$$\int_0^3 f(x)\,dx = 1$$

$$\Rightarrow \int_0^1 ax\,dx + \int_1^2 a\,dx + \int_2^3 (-ax+3a)\,dx = 1$$

$$\Rightarrow a\cdot\tfrac{1}{2} + a\cdot 1 + \left\{-\tfrac{a}{2}(9-4) + 3a(3-2)\right\} = 1$$

$$\Rightarrow a\left\{\tfrac{1}{2} + 1 - \tfrac{5}{2}\right\} = 1$$

$$\Rightarrow a = \tfrac{1}{2}$$

$P(X \leq 1.5)$

$$= \int_0^{1.5} f(x)\,dx$$

$$= \int_0^1 f(x)\,dx + \int_1^{1.5} f(x)\,dx$$

$$= \tfrac{1}{2}\cdot\tfrac{1}{2} + \tfrac{1}{2}(1.5-1)$$

$$= \tfrac{1}{4} + \tfrac{1}{4} = \tfrac{1}{8}$$

NOW $\quad F(x) = \int_0^x f(x)\,dx$

$$\therefore F(x) = \begin{cases} 0 & x \leq 0 \\ \int_0^x ax\,dx & 0 < x \leq 1 \\ \int_0^1 ax\,dx + \int_1^x a\,dx & 1 < x \leq 2 \\ \int_0^1 ax\,dx + \int_1^2 a\,dx + \int_2^x (-ax+3a)\,dx & 2 < x \leq 3 \\ 1 & x > 3 \end{cases}$$

$$\Rightarrow F(x) = \begin{cases} 0 & x \leq 0 \\ \frac{x^2}{4} & 0 < x \leq 1 \\ \frac{1}{4} + \frac{1}{2}(x-1) & 1 < x \leq 2 \\ \frac{1}{4} + \frac{1}{2} + \frac{1}{2}\left[3(x-2) - \frac{1}{2}(x^2-4)\right] & 2 < x \leq 3 \\ 1 & x > 3. \end{cases}$$

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3)

$$f(x) = \frac{2a}{\pi}\left(\frac{1}{a^2+x^2}\right)$$

$$E(X) = \frac{2a}{\pi}\int_{-a}^{a}\frac{x}{a^2+x^2}\,dx = 0 \quad [\because \text{odd}]$$

$$E(X^2) = \frac{2a}{\pi}\int_{-a}^{a}\frac{x^2}{a^2+x^2}\,dx$$

$$= \frac{4a}{\pi}\int_{0}^{a}\left(1-\frac{a^2}{a^2+x^2}\right)dx$$

$$= \frac{4a}{\pi}\left[a - \int_{0}^{a}\frac{a^2}{a^2+x^2}\,dx\right]$$

$$= \frac{4a}{\pi}\left[a - \frac{a^2}{a}\cdot\left[\tan^{-1}\frac{x}{a}\right]_0^a\right]$$

$$= \frac{4a}{\pi}\left[a - a\left(\frac{\pi}{4}-0\right)\right]$$

$$= \frac{4a^2}{\pi}\left(1-\frac{\pi}{4}\right)$$

$$= \frac{4a^2}{\pi} - 1$$

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(5)

$F(-\infty) = 0$

$F(\infty) = 1$

$\lim_{x \to a - a^-} F(x) = 0 = f(a^-)$

$\lim_{x \to a^-} f(x) = \frac{1}{2}(1+1) = 1 = f(a)$

$\therefore$ f is right Continuous at each point.

$f(x)$ is $\uparrow$ in $[-a, a]$

$\therefore$ ~~the~~ $f(x)$ is $\uparrow$ in whole domain

(4)(a) $Md_A = \int_{R_x} |x-A| f(x)\, dx$

mean deviation is least when A is median.

$\Rightarrow g(A) = Md_A = \int_{-\infty}^{\infty} |x-A| f(x)\, dx$

$= \int_{-\infty}^{A} (A-x) f(x)\, dx + \int_{A}^{\infty} (x-A) f(x)\, dx$

$\Rightarrow g'(A) = \int_{-\infty}^{A} f(x)\, dx + \int_{A}^{\infty} -f(x)\, dx$

$\therefore g'(A) = 0 \Rightarrow \int_{-\infty}^{A} f(x)\, dx = \int_{A}^{\infty} f(x)\, dx.$

i.e if $A = A_0$ is median of X then $g'(A_0) = 0$.

NOW $g''(A) = f(A) + (f(A)) = 2f(A)$

$\therefore$ at $A = A_0$

$g''(A_0) = 2f(A_0) \geq 0$

If $f(A) > 0$

$g(A)$ is minimum at $A = A_0$

~~and~~ If $f(A_0) \geq 0$ then by Sign Scheme, we can say that median $(A = A_0)$ is the minimum point.

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