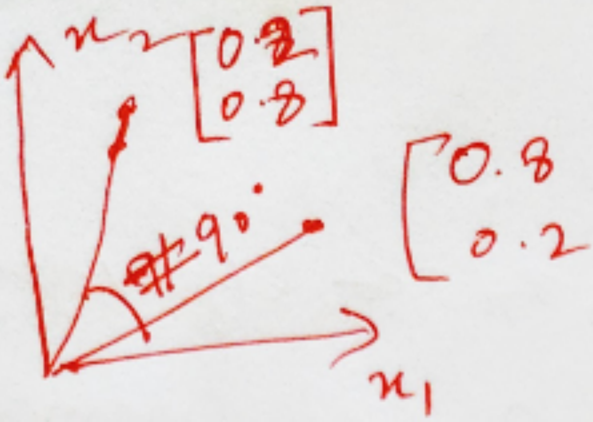


$$u = \begin{bmatrix}1\\0\end{bmatrix}$$

$$v = \begin{bmatrix}0\\1\end{bmatrix}$$

$$\langle u, v\rangle = u^T v = 1\cdot 0 + 0\cdot 1 = 0.$$

$$A = \begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix} = \begin{bmatrix}a_1 & a_2 & a_3\end{bmatrix}$$

$$\langle a_i, a_j\rangle = \begin{cases}1 & \text{if } i=j\\0 & \text{if } i\neq j\end{cases}$$

$$\langle a_1, a_2\rangle = 1\cdot 0 + 0\cdot 1 + 0\cdot 0 = 0$$

$$\langle a_2, a_3\rangle = 0\cdot 0 + 1\cdot 0 + 0\cdot 1 = 0$$

$$\langle a_3, a_1\rangle = 1\cdot 0 + 0\cdot 0 + 0\cdot 1 = 0$$

$$\langle a_1, a_1\rangle = 1\cdot 1 + 0\cdot 0 + 0\cdot 0 = 1$$

$$\langle a_2, a_2\rangle = 1$$

$$\langle a_3, a_3\rangle = 1$$

$A = A^T$, $A \in \mathbb{R}^{n \times n}$

$A x_i = \lambda_i x_i$

$x_1, x_2, \dots, x_n \leftarrow$ eigenvectors of A.

$\lambda_1, \lambda_2, \dots, \lambda_n \leftarrow$ eigenvalues of A

Spectral Theorem:

$\langle x_i, x_j \rangle = 0$ if $i \neq j$ $\forall i, j$

$\lambda_i \neq \lambda_j$

$x_i \quad x_j$

$A x_i = \lambda_i x_i$ — ①

$A x_j = \lambda_j x_j$ — ②

$\langle x_j, A x_i \rangle = \langle x_j, \lambda_i x_i \rangle$

$x_j^T A x_i = x_j^T \lambda_i x_i$

$x_j^T A x_i = \lambda_i x_j^T x_i$ — ③

$1 \times n \quad n \times n \quad n \times 1$

$(x_j^T A x_i)^T = x_j^T A x_i$

$x_i^T A^T x_j = x_j^T A x_i$ — ④

$x_i^T \underbrace{A^T}_{} x_j = \cancel{x_j^T A x_i} \lambda_i x_j^T x_i$

$x_i^T A x_j = \lambda_i x_j^T x_i$

$x_i^T \lambda_j x_j = \lambda_i x_j^T x_i$

$$\lambda_j x_i^T x_j = \lambda_i x_j^T x_i$$

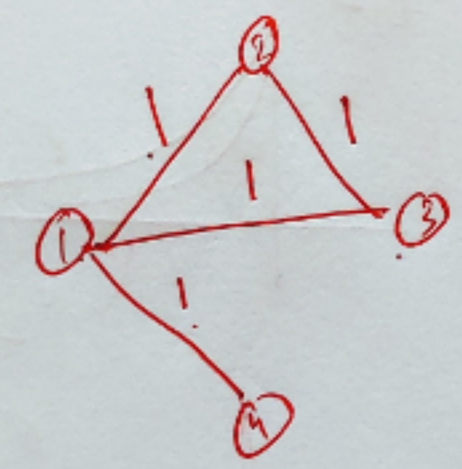
$$\underbrace{(\lambda_i - \lambda_j)}_{\neq 0} x_i^T x_j = 0$$

$$x_i^T x_j = 0$$

$$\langle x_i, x_j \rangle = 0$$

$A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times n}$ $B = [b_1 \; b_2 \; \dots \; b_n]$

$$\text{Trace}(B^T A B) = \sum_{i=1}^{n} b_i^T A b_i$$



$$W = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}_{4\times 4}$$

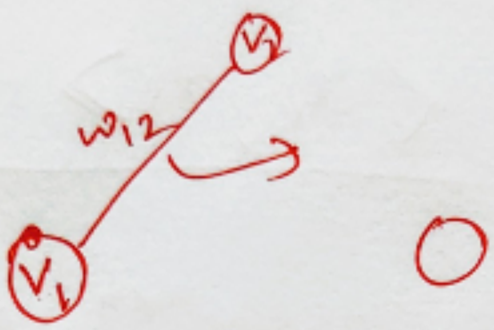
$$D = \begin{bmatrix} 3 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$L = D - W$$

$$= \begin{bmatrix} 3 & -1 & -1 & -1 \\ -1 & 2 & -1 & 0 \\ -1 & -1 & +2 & 0 \\ -1 & 0 & 0 & 1 \end{bmatrix}$$

$\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$

$\begin{bmatrix} 0 \\ 0 \end{bmatrix}$ $\begin{bmatrix} 2 \\ 0 \end{bmatrix}$

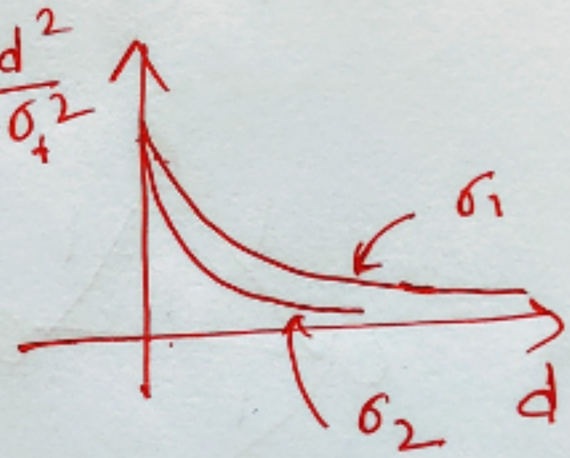


$$w_{12} = e^{-\frac{\|x_1 - x_2\|_2^2}{\sigma^2}}$$

$$= e^{-\frac{1}{\sigma^2}}$$

$$w_{13} = e^{-\frac{4}{\sigma^2}}$$

$e^{-\frac{d^2}{\sigma_1^2}}$



$\sigma_1 < \sigma_2$

$$\text{Ratio Cut}(C_1, C_2, \dots C_k) = \sum_{i=1}^{k} \frac{1}{|C_i|} \sum_{r \in C_i} \sum_{s \notin C_i} W_{r,s}$$

$$= \text{trace}(H^T L H)$$

$$\text{Trace}(H^T L H) = \sum_{i=1}^{K} \underbrace{h_i^T L h_i}$$

$$a^T B a = \sum_{i=1}^{n} \sum_{j=1}^{n} a_i a_j B_{ij}$$

$$v^T L v = v^T (D - W) v$$

$$= v^T D v - v^T W v$$

$$= \sum_{r=1}^{n} v_r^2 d_r - \sum_{r=1}^{n} \sum_{s=1}^{n} v_r v_s W_{r,s}$$

$$= \frac{1}{2} \sum_{r=1}^{n} v_r^2 d_r + \frac{1}{2} \sum_{s=1}^{n} v_s^2 d_s - \sum_{r=1}^{n} \sum_{s=1}^{n} v_r v_s W_{r,s}$$

$$= \frac{1}{2} \left[\sum_{r=1}^{n} v_r^2 d_r - 2 \sum_{r=1}^{n} \sum_{s=1}^{n} v_r v_s W_{r,s} + \sum_{s=1}^{n} v_s^2 d_s \right]$$

$$= \frac{1}{2} \left[\sum_{r=1}^{n} \sum_{s=1}^{n} W_{r,s} v_r^2 - 2 \sum_{r=1}^{n} \sum_{s=1}^{n} v_r v_s W_{r,s} + \sum_{s=1}^{n} \sum_{r=1}^{n} W_{r,s} v_s^2 \right]$$

$$= \frac{1}{2} \sum_{r=1}^{n} \sum_{s=1}^{n} W_{r,s} (v_r^2 - 2 v_r v_s + v_s^2)$$

$$= \frac{1}{2} \sum_{r=1}^{n} \sum_{s=1}^{n} W_{r,s} (v_r - v_s)^2$$

$$v^T L v = \frac{1}{2}\sum_{r=1}^{n}\sum_{s=1}^{n} w_{r,s}(v_r - v_s)^2$$

let $v = h_i$

$$h_i^T L h_i = \frac{1}{2}\sum_{r=1}^{n}\sum_{s=1}^{n} w_{r,s}\underbrace{(h_{i,r} - h_{i,s})^2}$$

if v_r and v_s belong to the same group

$h_{i,r} = \frac{1}{\sqrt{G_i}}$

$h_{i,s} = \frac{1}{\sqrt{G_i}}$

$\Rightarrow (h_{i,r} - h_{i,s})^2 = 0$

$$= \frac{1}{2}\sum_{r=1}^{n}\sum_{s=1}^{n}$$

$$h_i^T L h_i = \frac{1}{|G_i|}\sum_{r\in G_i}\sum_{s\notin G_i} w_{r,s}$$

$$\sum_{i=1}^{k} h_i^T L h_i = \sum_{i=1}^{k}\frac{1}{|G_i|}\sum_{r\in G_i,}\sum_{s\notin G_i} w_{r,s}$$

$$\text{Trace}(H^T L H) = \text{RatioCut}(G_1, G_2, \ldots, G_k)$$

$$\mathrm{trac}(H^T L H)$$

$$= \sum_{i=1}^{n} h_i^T L h_i$$

$$\min_{v \in \mathbb{R}^n,\, v^T v = 1} v^T L v \qquad v^T v = 1$$

$$f(v) = v^T L v + \lambda(1 - v^T v)$$

$$\nabla_v f(v) = 2Lv - 2\lambda v = 0$$

$$\Rightarrow Lv = \lambda v$$

$$v^T L v = \lambda$$

$$\min \lambda \quad \text{s.t.} \quad Lv = \lambda v$$

$v \to$ eigenvector corresponding to the ~~lowest~~ smallest eigenvalue of this matrix L.