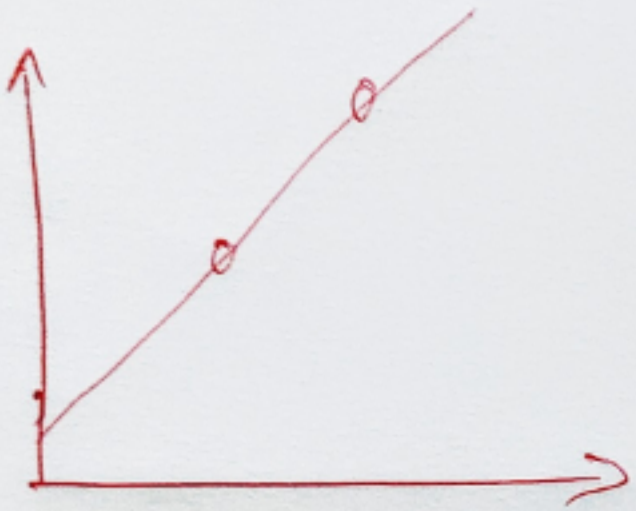
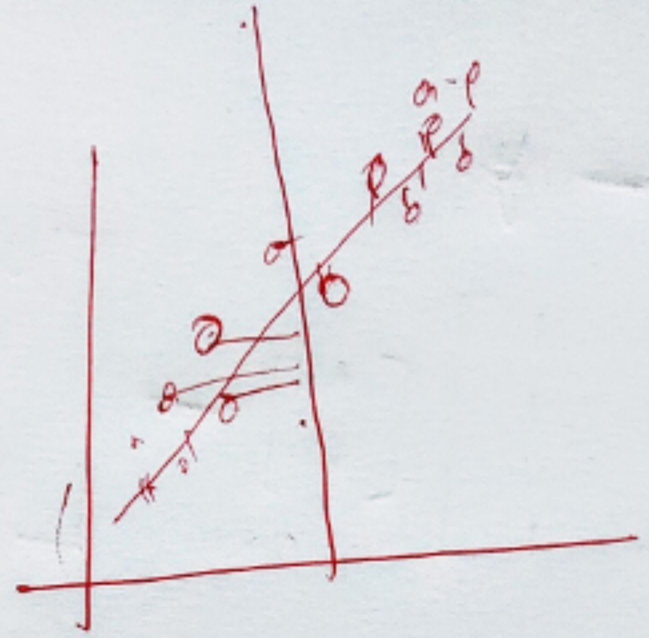
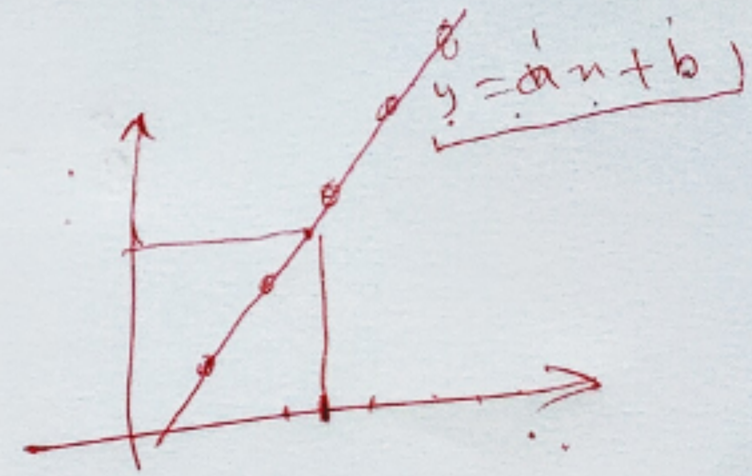




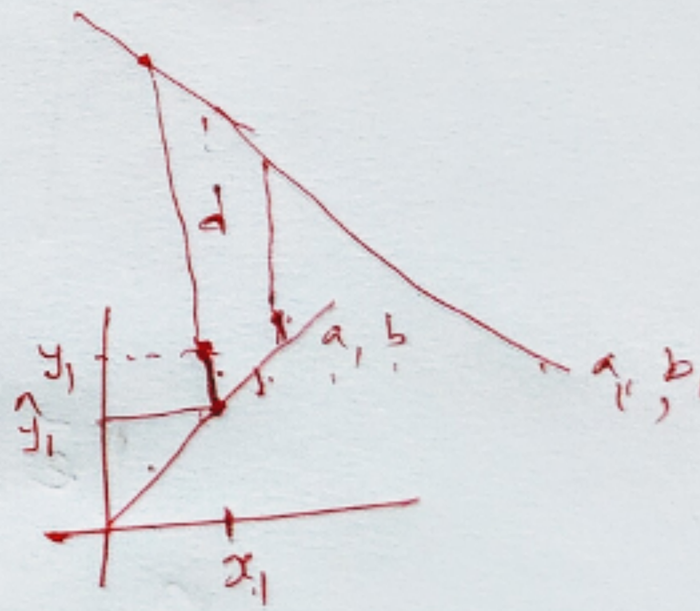


$$\{(x_1, y_1), (x_2, y_2), \ldots, (x_m, y_m)\}$$

$$y = ax + b$$

$$ax_1 + b \neq y_1$$

$$(\hat{y}_1 - y_1)^2 \quad \forall\ i \in [m]$$



$$\sum_{i=1}^{m} (\hat{y}_i - y_i)^2 = \sum_{i=1}^{m} (a x_i + b - y_i)^2$$

$$A = \begin{bmatrix} x_1 & 1 \\ x_2 & 1 \\ \vdots & \vdots \\ x_m & 1 \end{bmatrix}_{m\times 2} \qquad \mathbf{x} = \begin{bmatrix} a \\ b \end{bmatrix}_{2\times 1}, \qquad \mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_m \end{bmatrix}$$

$$\vec{e} = Ax - y$$

$$\underbrace{\begin{bmatrix} x_1 & 1 \\ x_2 & 1 \\ \vdots & \vdots \\ x_m & 1 \end{bmatrix}}_{m\times 2} \underbrace{\begin{bmatrix} a \\ b \end{bmatrix}}_{2\times 1} = \begin{bmatrix} y_1 \\ y_2 \\ \\ y_m \end{bmatrix}$$

$$\vec{e} = \underbrace{\begin{bmatrix} ax_1 + b \\ ax_2 + b \\ \vdots \\ ax_m + b \end{bmatrix}}_{m\times 1} - \underbrace{\begin{bmatrix} y_1 \\ y_2 \\ \\ y_m \end{bmatrix}}_{m\times 1} = \begin{bmatrix} ax_1 + b - y_1 \\ ax_2 + b - y_2 \\ \vdots \\ ax_m + b - y_m \end{bmatrix}$$

$$\|\vec{e}\|_2^2 = \sum_{i=1}^{m} (ax_i + b - y_i)^2$$

$$\vec{v} = [v_1 \ v_2 \ \cdots \ v_m]^T$$

$$\|v\|_2^2 = v_1^2 + v_2^2 + \cdots + v_m^2 = \sum_{i=1}^{m} v_i^2$$

$$\sum_{i=1}^{m} (ax_i + b - y_i)^2 = \|\vec{e}\|_2^2 = \|Ax - y\|_2^2$$

$$\min_{a,b} \sum_{i=1}^{m} (ax_i + b - y_i)^2 \iff \min_{x} \|Ax - y\|_2^2$$

$$f(x) = \|Ax - y\|_2^2 = (Ax-y)^T(Ax-y) \qquad \|v\|_2^2 = v^T v$$

$$\cancel{\nabla} f = (x^T A^T - y^T)(Ax - y)$$

$$= x^T A^T A x - x^T A^T y - y^T A x + y^T y$$

$$f(x) = x^T A^T A x - 2x^T A^T y + y^T y$$

$$\nabla_x f = 2A^T A x - 2A^T y = \vec{0}$$

$$A^T A x = A^T y$$

$$\boxed{x^* = (A^T A)^{-1} A^T y} = \begin{bmatrix} \cdot \\ \cdot \end{bmatrix}$$

Normal Equation.

$$\begin{bmatrix} x_1 \\ x_2 \\ \\ \\ x_d \end{bmatrix} \quad \underbrace{\begin{bmatrix} x_1 \\ x_3 \\ \vdots \\ x_{d-3} \end{bmatrix}_{k\times 1}} \quad \begin{bmatrix} x_2 \\ x_4 \\ \vdots \\ x_{d.u} \end{bmatrix}_{k\times 1}$$

$$\| \quad \| \qquad \| \quad \| \qquad =$$

$A \quad \{ x_1^{(1)}, x_2^{(1)}, y$

$$\{ (x_1^{(1)}, x_2^{(1)}, y_1) \, \ldots, (x_1^{(m)}, x_2^{(m)}, \ldots y_m) \}$$

$$A = \begin{bmatrix} x_1^{(1)} & x_2^{(1)} & 1 \\ x_1^{(2)} & x_2^{(2)} & 1 \\ & & 1 \\ x_1^{(m)} & x_2^{(m)} & 1 \end{bmatrix} \quad x = \begin{bmatrix} a \\ b \\ c \end{bmatrix}, \quad y = \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix}$$

$$\underline{x^* = (A^T A)^{-1} A^T y}$$

$$X = \begin{bmatrix} x_1 & x_2 & \cdots & x_m \end{bmatrix} \in \mathbb{R}^{d \times m}$$

$$X = \begin{bmatrix} x_{1,1} & x_{2,1} & \cdots & x_{m,1} \\ x_{1,2} & x_{2,2} & \cdots & x_{m,2} \\ \vdots & & & \vdots \\ x_{1,d} & x_{2,d} & \cdots & x_{m,d} \end{bmatrix}$$

2nd, jth

$$v = \begin{bmatrix} x_{1,j} & x_{2,j} & \cdots & x_{m,j} \end{bmatrix}^T \qquad y = \begin{bmatrix} y_1 & y_2 & \cdots & y_m \end{bmatrix}^T$$

$$a x_{i,j} + b = y_i$$

$$\sum_{i=1}^{m} (a x_{i,j} + b - y_i)^2 = \|av + b\mathbf{1} - y\|_2^2$$

$$= \left\| \begin{bmatrix} a x_{1,j} \\ a x_{2,j} \\ \vdots \\ a x_{m,j} \end{bmatrix} + \begin{bmatrix} b \\ b \\ \vdots \\ b \end{bmatrix} - \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_m \end{bmatrix} \right\|_2^2$$

$$\|av + b\mathbf{1} - y\|_2^2$$

$$f(a,b) = \|av + b\mathbb{1} - y\|_2^2 \qquad \mathbb{1} = \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}_{m \times 1}$$

$$= (av + b\mathbb{1} - y)^T (av + b\mathbb{1} - y)$$

$$= (av^T + b\mathbb{1}^T - y^T)(av + b\mathbb{1} - y)$$

$$= a^2 v^T v + bav^T\mathbb{1} - av^T y + ba\mathbb{1}^T v + b^2\mathbb{1}^T\mathbb{1} - b\mathbb{1}^T y$$
$$- ay^T v - by^T\mathbb{1} + y^T y$$

$$= a^2 v^T v + 2ba \underbrace{v^T\mathbb{1}}_{=0} - 2ay^T v + b^2\mathbb{1}^T\mathbb{1} - 2b\underbrace{y^T\mathbb{1}}_{=0} + y^T y$$

$$f(a,b) = a^2 v^T v - 2ay^T v + b^2\mathbb{1}^T\mathbb{1} + y^T y$$

$v \;,\; y$

$\text{mean}(v) = 0, \qquad \text{mean}(y) = 0$

$\bar{v} = \text{mean}(v) \qquad \bar{y} = \text{mean}(y)$

$v_i \leftarrow v_i - \bar{v} \qquad y_i \leftarrow y_i - \bar{y}$

$$\bar{v} = \frac{1}{m}\sum_{i=1}^{m} v_i = \frac{1}{m}\mathbb{1}^T v = \frac{1}{m}[1\ 1 \cdots 1]\begin{bmatrix} v_1 \\ \vdots \\ v_m \end{bmatrix}$$

$$= \frac{1}{m}(v_1 + v_2 + \cdots v_m)$$

$$= \frac{1}{m}\sum_{i=1}^{m} v_i$$

$\text{mean}(v) = 0$

$\Rightarrow \quad \mathbb{1}^T v = 0$

$\mathbb{1}^T y = 0$

$$\frac{\partial f(a,b)}{\partial b} = 2b\,1^T 1 = 2bm = 0 \Rightarrow \boxed{b=0}$$

$$\frac{\partial f(a,b)}{\partial a} = 2a\,v^T v - 2\,y^T v = 0$$

$$\boxed{a = \frac{y^T v}{v^T v}}$$

$$\cancel{\| a v}$$

$$f(a,b) = a^2 v^T v - 2a\,y^T v + y^T y$$

$$= \frac{(y^T v)^2}{(v^T v)^2}(v^T v) - 2\,\frac{(y^T v)}{(v^T v)}\,y^T v + y^T y$$

$$= y^T y - \frac{(y^T v)^2}{v^T v}$$

$$= y^T y\left[1 - \frac{(y^T v)\cdot(y^T v)}{(v^T v)\cdot(y^T y)}\right]$$

$$f(a,b) = y^T y \left(1 - \rho_{y,v}^2\right)$$

$$\rho_{y,v} = \frac{\frac{1}{m}(y^T v)}{\sqrt{\frac{v^T v}{m}}\sqrt{\frac{y^T y}{m}}}$$

$$\rho_{x,y} = \frac{E[(x-\bar{x})(y-\bar{y})]}{\sqrt{\sigma_x^2 \cdot \sigma_y^2}}, \quad -1 \le \rho_{x,y} \le 1$$

$$= \frac{E(xy)}{\sqrt{\sigma_x^2 \sigma_y^2}}$$

$\frac{1}{m}\sum x_i y_i$

$\sigma_x^2 \quad \frac{1}{m}\sum x_i^2$

$\sigma_y = \frac{1}{m}\sum y_i^2$

if $\rho_{y,v} = 1$

$\Rightarrow f(a,b) = 0$

1^{st} $\quad 2^{nd}$ $\quad d^{th}$

ρ_{y,v_1} $\quad \rho_{y,v_2}$ $\quad \rho_{y,v_d}$