Addition of a Constraint

Suppose that for a given problem, the corresponding LPP has been obtained and has been solved. Now, if due to some unexpected factors, some new constraints come up and the LPP needs to be solved with these new constraints included (along with the previous ones).

Question: Do we have to solve the modified LPP all over again? Can we use the optimal solution (table) of the previously solved problem to obtain the optimal solution to the modified problem?

Answer: The answer is yes. The optimal table of the original problem can be used to obtain a BFS (and its table) for the modified problem. The table can be updated (iteratively) using simplex method to obtain the optimal solution to the modified problem.

Methodology: Note that if the optimal solution to the original problem satisfies the additional constraints, then the optimal solution to the original problem is the optimal solution to the modified problem

If the optimal solution to the original problem does not satisfy the additional constraints then,

1. Express the additional constraint (s) in "$\leq$" form.
2. Take the constraints to the simplex table [after introducing the slack variable (s)].
3. Tranform the matrix corresponding to basic variables as identity matrix (using Row operations only).
4. Apply simplex table to the table obtained (in the previous step) to obtain the optimal solution to the modified problem

<u>Example</u> : Solve the LPP described by the equations given below :

$$\max\ 6x_1 - 2x_2$$

$$\text{s.t.}\quad 2x_1 - x_2 \leq 2$$

$$x_1 \leq 4$$

$$x_1, x_2 \geq 0$$

Using the optimal table for the above problem, find the optimal solution when the constraint "$2x_1 + 3x_2 \leq 6$" is also included in the above problem (with the previously existing set of constraints)?

The given problem can be written as

$$\max\ 6x_1 - 2x_2$$
$$\text{s.t. } 2x_1 - x_2 + x_3 = 2$$
$$x_1 + x_4 = 4$$
$$x_i \geq 0 \ \forall i = 1,2,3,4.$$

First Simplex table

			6	−2	0	0
C_B	B	b	a_1	a_2	a_3	a_4
← 0	x_3	2	2	−1	1	0
0	x_4	4	1	0	0	1
	$z_j - c_j$:		−6 ↑	2	0	0

x_1 enters, x_3 leaves

Second Simplex table

			6	−2	0	0
C_B	B	b	a_1	a_2	a_3	a_4
6	x_1	1	1	$-\frac{1}{2}$	$\frac{1}{2}$	0
0	x_4	3	0	$\frac{1}{2}$	$-\frac{1}{2}$	1
	$z_j - c_j$:		0	−1	3	0

x_2 enters, x_4 leaves.

Third Simplex table

			6	-2	0	0
C_B	B	b	a_1	a_2	a_3	a_4
6	x_1	4	1	0	0	1
-2	x_2	6	0	1	-1	2
	$z_j - c_j$:		0	0	2	2

As $z_j - c_j \geq 0 \ \forall j$, the above table corresponds to the optimal solution of the given problem. Optimal solution is $x_1 = 4$, $x_2 = 6$, optimal value is 12.

The new constraint to be introduced is $2x_1 + 3x_2 \leq 6$.

(4, 6) does not satisfy the additional constraint.

Representing the additional constraint as $2x_1 + 3x_2 + x_5 = 6$ and taking it to the simplex table, the update table is

			6	-2	0	0	0
C_B	B	b	a_1	a_2	a_3	a_4	a_5
6	x_1	4	1	0	0	1	0
-2	x_2	6	0	1	-1	2	0
0	x_5	6	2	3	0	0	1
			0	0	2	2	0

As coloums of x_1, x_2 and x_5 do not form identity matrix, reducing the matrix corresponding to basic variables as identity matrix (by the row operation $[R_3 \rightarrow R_3 - 2R_1 - 3R_2]$, we get,

First table for modified problem

C_B	B	b	a_1	a_2	a_3	a_4	a_5
6	x_1	4	1	0	0	1	0
-2	x_2	6	0	1	-1	2	0
0	x_5	-20	0	0	3	-8	1
			0	0	2	2	0

$z_j - c_j \geq 0 \; \forall j \Rightarrow$ Dual simplex method can be applied.

x_5 leaves, x_4 enters.

Second table for modified problem

			6	-2	0	0	0
C_B	B	b	a_1	a_2	a_3	a_4	a_5
6	x_1	$\frac{3}{2}$	1	0	$3/8$	0	$1/8$
-2	x_2	1	0	1	$-1/4$	0	$1/4$
0	x_4	$\frac{5}{2}$	0	0	$-\frac{3}{8}$	1	$-1/8$
			0	0	$\frac{11}{4}$	0	$1/4$

~~sad~~ as coloum b & $z_j - c_j$ are non-negative($\forall j$), the above table corresponds to optimal solution to the modified problem.

Optimal Solution to modified problem: $x_1 = \frac{3}{2}$, $x_2 = 1$, optimal value is 7.

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Integer Programming Problem

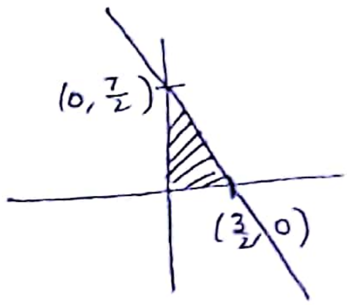
Suppose we are given an LPP where the variables are constrained to be integers. Then, solving the LPP using simplex method does not guarantee an integer solution and hence some alternate criteria needs to be implemented.

Note that rounding off the optimal solution of the problem without integer constraints does not give the optimal solution to the integer programming problem (IPP) as shown in the example below:

$$\max\ 3x_1 + x_2$$

$$\text{s.t } 14x_1 + 6x_2 \leq 21$$

$$x_1, x_2 \geq 0, \text{integers}$$



For the above problem, the feasible region is a triangle with vertices $(0,0)$, $(\frac{3}{2},0)$ and $(0,\frac{7}{2})$ and optimal solution is $(\frac{3}{2},0)$.

Rounding off of the solution yields the point $(1,0)$ or $(2,0)$.

However $(2,0)$ is not inside the feasible region and $(1,0)$ is not optimal as $(1,1)$ lies inside the feasible region and yields a greater value

Gomory's Cut Constraint Method (for All IPP)

Suppose we are given an all integer programming problem (where all variables are constrained to be integers). Gomory's Cut constraint method provides an algorithm to determine integer solution to an All IPP (or AIPP) using dual simplex algorithm

Note that as rows of the first simplex table are the constraints for the given problem, and we use only row operations to update the simplex table [which is essentially equivalent to solving the given equations], the rows of any simplex table are constraints for the given problem (are they equivalent to original set of constraints!!!).

Suppose we first solve the given LPP using simplex algorithm [ignoring the fact that x_i's are constrained to be integer]. If all x_i's turn out to be integer then the table corresponds to the optimal solution to the given AIPP.

If not, then Gomory Cut constraint method introduces another constraint into the problem which cuts out the present optimal solution from the feasible region but does not cut any potential candidate for maxima or minima, i.e. does not cut any point with all integer co-ordinates.

Suppose our LPP was a problem in two variables and the optimal table obtained (using simplex table) was

			c_1	c_2	c_3	c_4
C_B	B	b	a_1	a_2	a_3	a_4
c_1	x_1	b_1	a_{11}	a_{12}	a_{13}	a_{14}
c_2	x_2	b_2	a_{21}	a_{22}	a_{23}	a_{24}
	$z_j - c_j$:		0	0	$+ve$	$+ve$

Thus the optimal solution was $(x_1 = b_1,\ x_2 = b_2)$. Suppose one of x_1 or x_2 is non integer. Say x_1 is non-integer. Then the first row is equivalent to the equation

$$b_1 = a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + a_{14}x_4$$

Writing $a_{ij} = I_{ij} + f_{ij}$ $\left[\begin{array}{l} I_{ij} \text{ is integer part of } a_{ij} \\ f_{ij} \text{ is fractional part of } a_{ij} \end{array}\right]$

and $b_1 = I_{b_1} + f_{b_1}$ [I_{b_1} and f_{b_1} are integer & fractional part of b_1 resp.]

we get,

$$I_{b_1} + f_{b_1} = (I_{11} + f_{11})x_1 + (I_{12} + f_{12})x_2 + (I_{13} + f_{13})x_3 + (I_{14} + f_{14})x_4$$

Thus, $f_{b_i} - \sum_{i=1}^{4} f_{ii} x_i = \sum_{i=1}^{4} I_{1i} x_i - I_{b_1}$

Note that if all x_i's are integers then RHS is an integer.

$\therefore$ Further, as $f_{b_i} = \text{RHS} + \sum_{i=1}^{4} f_{ii} x_i$ we have

$$(*) \text{ --- } \quad f_{b_i} - \sum_{i=1}^{4} f_{ii} x_i \leq 0 \quad \left[\text{as } \sum_{i=1}^{4} f_{ii} x_i \text{ is non-negative}\right]$$

Thus introduce the above constraint $(*)$ in the optimal table and solve it using dual simplex algorithm. If all variables turn out to be integers then the optimal solution to AIPP is achieved. If not, repeat the process, [i.e. introduce another constraint] and move closer to the optimal solution for the given AIPP.

We now give the detailed algorithm for Gomory Cut Constraint method for AIPP.

Gomory Cut Constraint method for AIPP

1. Solve the LPP ignoring the integer constraints. If the solution obtained is an integer solution, it is optimal for the given problem.

2. If the solution obtained is non-integral, pick the basic variable with greatest fractional part (say it is in the i^{th} row). and introduce the constraint

$$-\sum_{j=1}^{n} f_{ij} x_j \leq -b_i$$

③ Solve the modified problem with dual simplex method and obtain the optimal solution

④ If the optimal solution is integer solution, it is optimal solution for the given AIPP.

⑤ If not, goto step 2.

It may be noted that the constraint introduced in step 2 cuts a region from the feasible (containing the optimal solution) region not containing any integer solution. Thus the optimal solution for the modified problem neccessarily changes. We keep on cutting regions from the feasible region till we reach an integer solution. As no integer solution has been deleted at any step, the integer solution obtained is optimal for the given problem (AIPP).

<u>Example</u>

$$\max\ 3x_1 + x_2$$
$$\text{s.t}\quad 14x_1 + 6x_2 \leq 21$$
$$x_1, x_2 \geq 0, \text{ integers}$$

The problem can be written as

$$\max\ 3x_1 + x_2$$
$$\text{s.t}\quad 14x_1 + 6x_2 + x_3 = 21$$
$$x_1, x_2 \geq 0, \text{ integers}$$

First Simplex table

			3	1	0
C_B	B	b	a_1	a_2	a_3
0	x_3	21	14	6	1
	$z_j - c_j$:		-3	-1	0

x_1 enters, x_3 leaves.

Second Simplex table

			3	1	0
C_B	B	b	a_1	a_2	a_3
3	x_1	$\frac{3}{2}$	1	$\frac{3}{7}$	$\frac{1}{14}$
	$z_j - c_j$:		0	$\frac{2}{7}$	$\frac{3}{14}$

$z_j - c_j \geq 0 \Rightarrow$ above solution is optimal (but non-integer).

Introduce the constraint

$$-0 \cdot x_1 - \frac{3}{7} x_2 - \frac{1}{14} x_3 \leq -\frac{1}{2}$$

which is same as

$$-\frac{3}{7} x_2 - \frac{1}{14} x_3 + x_4 = -\frac{1}{2}$$

Next Simplex table

			3	1	0	0
C_B	B	b	a_1	a_2	a_3	a_4
3	x_1	$\frac{3}{2}$	1	$\frac{3}{7}$	$1/14$	0
0	x_4	$-\frac{1}{2}$	0	$-\frac{3}{7}$	$-\frac{1}{14}$	1
			0	$\frac{2}{7}$	$\frac{3}{14}$	0

Apply dual Simplex method, x_4 leaves, x_2 enters

Next Simplex table

			3	1	0	0
C_B	B	b	a_1	a_2	a_3	a_4
3	x_1	1	1	0	0	1
1	x_2	$\frac{7}{6}$	0	1	$+1/6$	$-7/3$
			0	0	$1/6$	$2/3$

$z_j - c_j \geq 0 \;\forall j$ but soln is non-integer.

Introduce the constraint

$$-\frac{1}{6}x_3 - \frac{2}{3}x_4 \leq -\frac{1}{6}$$

which is same as

$$-\frac{1}{6}x_3 - \frac{2}{3}x_4 + x_5 = -\frac{1}{6}$$

Next Simplex table

			3	1	0	0	0
C_B	B	b	a_1	a_2	a_3	a_4	a_5
3	x_1	1	1	0	0	1	0
1	x_2	$\frac{7}{6}$	0	1	$\frac{1}{6}$	$-\frac{7}{3}$	0
0	x_5	$-\frac{1}{6}$	0	0	$-\frac{1}{6}$	$-\frac{2}{3}$	1
			0	0	$1/6$	$\frac{2}{3}$	0

Apply dual Simplex method, x_5 leaves, x_3 enters.

Next Simplex table

			3	1	0	0	0
C_B	B	b	a_1	a_2	a_3	a_4	a_5
3	x_1	1	1	0	0	1	0
1	x_2	1	0	1	0	1	1
0	x_3	1	0	0	1	4	-6
			0	0	0	4	1

$Z_j - C_j \geq 0 \; \forall j$ & solution is integer solution and thus is optimal for the given AIPP.

Thus optimal solution is $(x_1 = 1, x_2 = 1)$ and optimal value is 4.

Example :

$$\max \; x_1 + 2x_2$$
$$\text{s.t.} \quad 2x_2 \leq 7$$
$$x_1 + x_2 \leq 7$$
$$2x_1 \leq 11$$
$$x_1, x_2 \geq 0, \text{ integers}$$

The above problem can be written as

$$\max \; x_1 + 2x_2$$
$$\text{s.t.} \quad 2x_2 + x_3 = 7$$
$$x_1 + x_2 + x_4 = 7$$
$$2x_1 + x_5 = 11$$
$$x_1, x_2, x_3, x_4, x_5 \geq 0, \text{ integers}.$$

First Simplex table

			1	2	0	0	0
C_B	B	b	a_1	a_2	a_3	a_4	a_5
0	x_3	7	0	2	1	0	0
0	x_4	7	1	1	0	1	0
0	x_5	11	2	0	0	0	1
	$z_j - c_j$:		-1	-2	0	0	0

x_2 enters , x_3 leaves

Next Simplex table

		c_j	1	2	0	0	0
C_B	B	b	a_1	a_2	a_3	a_4	a_5
2	x_2	$\frac{7}{2}$	0	1	$\frac{1}{2}$	0	0
0	x_4	$\frac{7}{2}$	1	0	$-\frac{1}{2}$	1	0
0	x_5	11	2	0	0	0	1
			-1	0	1	0	0

x_1 enters, x_4 leaves

Next Simplex table

		c_j	1	2	0	0	0
C_B	B	b	a_1	a_2	a_3	a_4	a_5
2	x_2	$\frac{7}{2}$	0	1	$\frac{1}{2}$	0	0
1	x_1	$\frac{7}{2}$	1	0	$-\frac{1}{2}$	1	0
0	x_5	4	0	0	1	-2	1
			0	0	$\frac{1}{2}$	1	0

$z_j - c_j \geq 0 \ \forall j$ but solution is non-integer.

Introduce the constraint

$$-\frac{1}{2}x_3 \leq -\frac{1}{2}$$

which is same as $-\frac{1}{2}x_3 + x_6 = -\frac{1}{2}$

Next Simplex table

		c_j	1	2	0	0	0	0
C_B	B	b	a_1	a_2	a_3	a_4	a_5	a_6
2	x_2	$\frac{7}{2}$	0	1	$\frac{1}{2}$	0	0	0
1	x_1	$\frac{7}{2}$	1	0	$-\frac{1}{2}$	1	0	0
0	x_5	4	0	0	1	-2	1	0
0	x_6	$-\frac{1}{2}$	0	0	$-\frac{1}{2}$	0	0	1
			0	0	$\frac{1}{2}$	1	0	0

Apply dual simplex method, x_6 leaves, x_3 enters.

Next Simplex table

		c_j	1	2	0	0	0	0
C_B	B	b	a_1	a_2	a_3	a_4	a_5	a_6
2	x_2	3	0	1	0	0	0	1
1	x_1	4	1	0	0	1	0	-1
0	x_5	3	0	0	0	-2	1	2
0	x_3	1	0	0	1	0	0	-2
		$z_j - c_j$:	0	0	0	1	0	1

$z_j - c_j \geq 0 \; \forall j$ and solution is an integer solution & thus is optimal for the given AIPP.

optimal soln : $x_1 = 4, x_2 = 3$, optimal value is 10.

Branch and Bound Method

To find the optimal solution to the given IPP, Branch and Bound method branches out the problem by adding new constraints [if the solution obtained by ignoring integer constraints is not of desired form].

Methodology

① Solve the given problem ignoring the integer constraints. If the solution obtained satisfies the integer conditions, the solution is the optimal solution to the given IPP.

② In the optimal solution obtained in step 1, ~~become~~ choose variable x_i (which was constrained to be integer) which has greatest fractional part: (say $x_k = r$).

③ Branch the problem out in two parts by including the constraints (one constraint in each branch) and $x_k \leq [r]$ and $x_k \geq [r]+1$
solve the modified LPP branches.

④ Continue till each of the branches terminate (Note that any branch will either terminate in integer solution (as desired) or an infeasible LPP.

⑤ Compare the integer solution across different bronces to obtain the optimal solution to the given IPP.

Example

$$\max\ 7x_1 + 9x_2$$

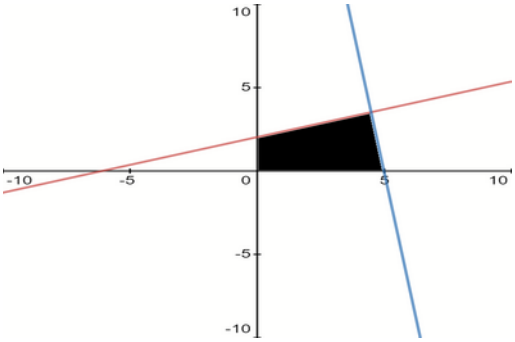
$$\text{s.t.}\quad -x_1 + 3x_2 \le 6$$

$$7x_1 + x_2 \le 35$$

$$0 \le x_1, x_2 \le 7$$

$$x_1, x_2 \text{ integers.}$$

Firstly solving the LPP (graphically) ignoring the integer constraints we get



Graphical Plot for original LPP :

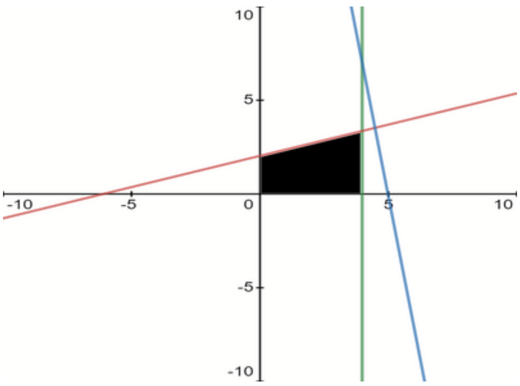
BFS : (0,0), (0,2), (5,0) and (9/2, 7/2)

Optimal Solution : (9/2, 7/2)

optimal solution : $\left(x_1 = \frac{9}{2},\ x_2 = \frac{7}{2}\right)$

Level 1

Branch 1 : introduce constraint $x_1 \le 4$



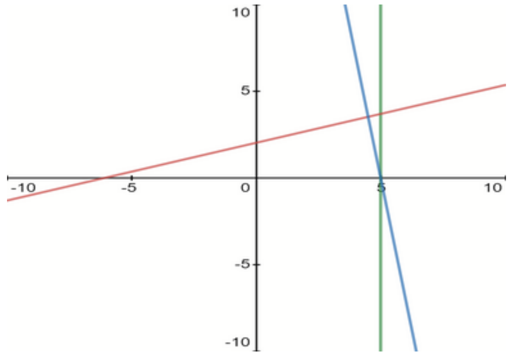
Graphical Solution for Branch 1 of Level 1 :

BFS: (0,0), (0,2), (4,10/3) and (4,0)

Optimal Solution : (4, 10/3)

optimal solution : $\left(x_1 = 4,\ x_2 = \frac{10}{3}\right)$

Branch 2 : introduce the constraint $x_1 \geq 5$



Graphical Solution to Brach 2 of Level 1:

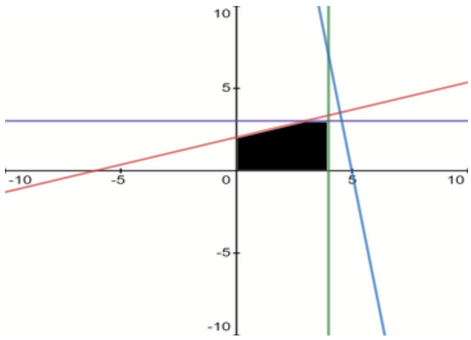
BFS : (5,0) [feasible region is a singleton]

Optimal Solution : (5,0)

optimal Solution $(x_1 = 5,\ x_2 = 0)$ Branch ends.

Level 2 As Branch 1 has non-integer solutions, branching it further, we get

Branch 1 : Introduce the constraint $x_2 \leq 3$



Graphical Solution to Branch 1 of Level 2 :

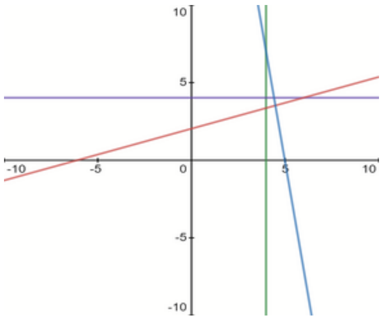
BFS : (0,0), (0,2), (3,3), (4,3) and (4,0)

Optimal Solution : (4,3)

optimal Solution : $(x_1 = 4\ ,\ x_2 = 3)$

~~Branch~~

Branch 2 : introduce the constraint $x_2 \geq 4$



Graphical Solution to Branch 2 of Level 2 :

Feasible region is empty.

LPP is infeasible.

Infeasible solution

Comparing the integer solution across different branches, the optimal soln to given IPP is $x_1 = 4$, $x_2 = 3$ optimal value is 55.

The branching process can be summarized by following daigram: