Note!

$$X(t) = \{ X(s,t) : s \in S,\ t \in T \}$$

## Statistics of RP/SP:

1) **Equality of RP:** Two RP, $x(t)$ and $y(t)$ are said to be equal if $X(s,t) = Y(s,t)$ $\forall s.$ $(\forall t)$

2) **CDF of RP:** Let us assume a RP $X(t)$ at a given time $t$. Then the CDF of $X(t)$ at $t$

Parameter

is given by

$$F(x,t) = P(X(t) \leq x) \quad \text{I}^{st} \text{ order CDF}.$$

→ Let us consider a RP $X(t)$ at two instances $t_1$ and $t_2$. Then the CDF of $X(t)$ at $t_1$ & $t_2$ is given by

$$F(x_1, x_2, t_1, t_2) = P(X(t_1) \leq x_1, X(t_2) \leq x_2)$$

$\text{II}^{nd}$ order CDF

→ $(x_1, t_1), (x_2, t_2) - , (x_n, t_n)$

$$F(x_1, x_2, - x_n, t_1, t_2, - t_n) = P(X(t_1) \leq x_1, X(t_2) \leq x_2, \ldots, X(t_n) \leq x_n).$$

⇒ $n^{th}$ order CDF.

3) **PDF of RP:** $\text{I}^{st}$ order CDF

$$f(x,t) = \text{PDF} = \frac{\partial}{\partial x} F(x,t) \equiv \frac{d}{dx} F(x,t)$$

$$\text{II}^{nd} \text{ order PDF} = \frac{\partial^2}{\partial x_1 \partial x_2} F(x_1, x_2, t_1, t_2) = f(x_1, x_2, t_1, t_2).$$

$n^{th}$ order PDF:

$$= \frac{\partial^n}{\partial x_1, \partial x_2 - \partial x_n} F(x_1, x_2, - x_n, t_1, t_2, - t_n)$$

$$\equiv f(x, x_1, - x_n, t_1, t_2, t_n)$$

4) <u>Mean of a RP</u>: Let $X(t)$ be a RP at instance $t$. Then the mean/expected value of $X(t)$ is given by

$$\eta(t) = E(X(t)) = \int_{-\infty}^{\infty} x\, f(x,t)\, dt$$

5) <u>Auto correlation of a RP</u>: Let $X(t)$ be a RP and $t_1$ and $t_2$ be two instances. Then, the auto-correlation can be defined as

$$R(t_1, t_2) = E(X(t_1)\, X(t_2))$$

$$= \int_{R_{X(t_1)}} \int_{R_{X(t_2)}} x_1 x_2\, f(x_1, x_2, t_1, t_2)\, dx_1\, dx_2$$

→ if $t_1 = t_2 = t$, then

$$R(t,t) = E(X^2(t)) \equiv \text{Average power of } X(t) \text{ at } t.$$

6) <u>Covariance of a RP</u>: Let $X(t)$ be RP and $t_1$ and $t_2$ be two instances. Then covariance can be defined as

$$C(t_1, t_2) = R(t_1, t_2) - \eta(t_1)\, \eta(t_2)$$

→ if $t_1 = t_2 = t$, then

$$C(t,t) = R(t,t) - \eta(t) \cdot \eta(t) = E(X^2(t)) - [E(X(t))]^2$$

⇒ Variance of $X(t)$ at $t$.

* **Uncorrelated RP:** A RP $X(t)$ is said to be Uncorrelated RP if the correlation coefficient is zero $\forall\ t_1, t_2$.

$\Rightarrow r(t_1, t_2) = 0 \quad \forall\ t_1, t_2$

$\Rightarrow \dfrac{C(t_1, t_2)}{\sqrt{C(t_1,t_1)\, C(t_2,t_2)}} = 0 \quad \forall (t_1, t_2)$

$\Rightarrow C(t_1, t_2) = 0 \quad \forall (t_1, t_2)$

* **Normal Random Process!** A RP. $X(t)$ is said to be normal if the joint distribution with respect to instances $t_1, t_2, -, t_n$ is normal.

$$\underbrace{X(t) \quad t_1, t_2, \text{---}, t_n, \text{---}}$$

$f(x_1, x_2, -x_n, t_1, t_2, -, t_n)$ is normal.

eg Let $X(t)$ be normal RP with $\eta(t) = 3$ and $C(t_1, t_2) = 4e^{-0.2|t_1 - t_2|}$. Find $P(X(5) \leq 2)$.

$X(t)$ is a ND RP $\Rightarrow X(t_i)$ is ND $\forall i$

$X(5) \sim N(3, 4)$ $\quad$ $Var(X(5)) = C(5,5)$
$= 4$ (HW)

$P(X(5) \leq 2) = P\left(\dfrac{X(5) - 3}{2} \leq \dfrac{2-3}{2}\right)$

$= P(Z \leq -0.5) = 0.691$ (HW)

7) Correlation Coefficient of a RP:

Let $X(t)$ be a RP and $t_1$ & $t_2$ be two instances. Then, the CC of $X(t)$ is given by

$$r(t_1, t_2) = \frac{C(t_1, t_2)}{\sqrt{C(t_1, t_1)\, C(t_2, t_2)}}$$

eg Let $X(t)$ be a RP with $\eta(t) = 3$ and $R(t_1, t_2) = 9 + 4e^{-0.2|t_1 - t_2|}$. Then find the mean, variance and covariance of $X(5)$ and $X(8)$.

$\Rightarrow$ $X(t)$ at $t$ mean $\equiv$ Expected value at $t$ $(E(X(t)))$
$\equiv \eta(t)$

$\eta(5) = E(X(5)) = 3$

$\eta(8) = E(X(8)) = 3$

$Var(X(5)) = C(5,5) = E(X^2(5)) - [E(X(5))]^2$ ——(1)

$E(X^2(5)) = R(5,5) = 9 + 4e^{-0.2 \times 0} = 13$

From eqn (1) $Var(X(5)) = 13 - 9 = 4$

$Var(X(8)) = 4$ (HW)

$Cov(X(5), X(8)) = C(5,8) = R(5,8) - \eta(5)\,\eta(8)$

$= \cancel{9} + 4e^{-0.2|-3|} - \cancel{9}$

$= 4e^{-0.6} = 2.195.$

2

* **Independent RP:** Two RP $X(t)$ and $Y(t)$ are said to be independent if $X(t_i)$ and $Y(t_i)$ are mutually independent $\forall i$.

**Markov Process:** A RP $X(t)$ is called a markov process if for any

$$t_0 < t_1 < t_2 < \cdots < t_{n-1} < t_n < t$$

the conditional distribution of $X(t)$ for given $X(t_0), X(t_1), X(t_2), \cdots, X(t_n)$ depends only on $X(t_n)$.

$$P(X(t) \leq x \mid X(t_n) = x_n, X(t_{n-1}) = x_{n-1}, \cdots, X(t_0) = x_0)$$
$$= P(X(t) \leq x \mid X(t_n) = x_n)$$

→ Memory less property

**Markov Chain:** A MP with discrete set of states is called markov chain.

- Discrete Time MC (DTMC)
- Continuous Time MC (CTMC)


**DTMC**

A DTMC is a MC having a countable no of states $(a_i)$. It is specified in terms of its

(1) **State Probability:** $p_i(n) = P(X_n = a_i)$ $i = 1, 2, 3, \ldots$

(2) **Transition** " : $p_{ij}(m, n) = P(X_n = a_j \mid X_m = a_i)$

$m$ → time instant, $a_i$ → state

# 1-Step Transition:


$$p_{ij} \equiv p_{ij}\underbrace{(n-1, n)} = P(X_n = a_j \mid X_{n-1} = a_i)$$

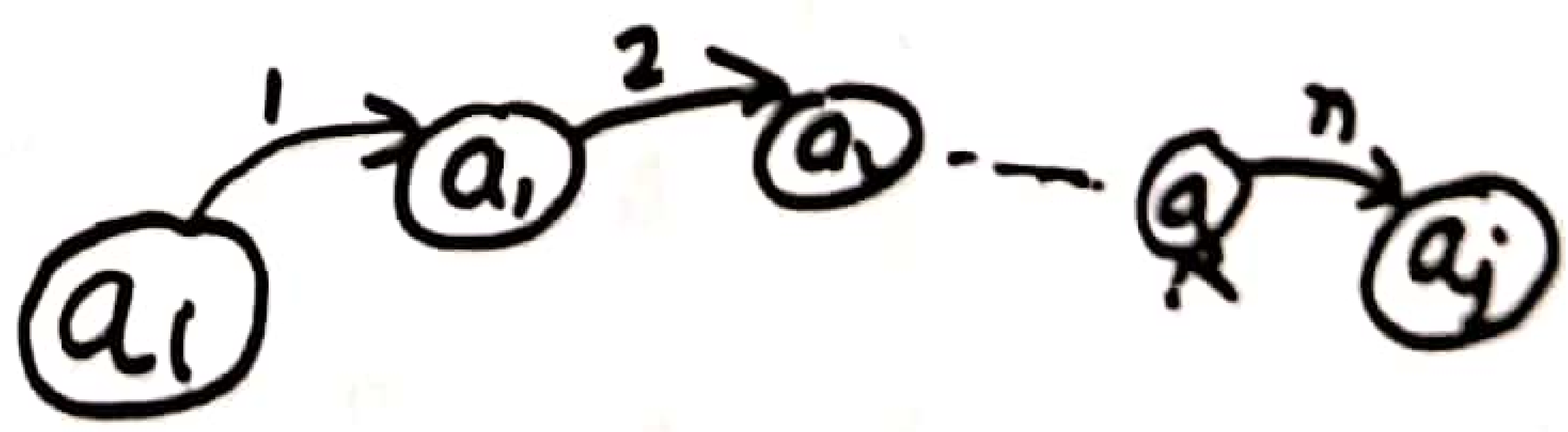

$(p_{12}$ or $p_{a_1 a_2})$

Consider all possible states $a_1, a_2, — a_m$

$$P = [p_{ij}] \qquad \begin{array}{c|c} & \begin{array}{cccc} a_1 & a_2 & — & a_m \end{array} \\ \begin{array}{c} a_1 \\ a_2 \\ \vdots \\ a_m \end{array} & \begin{bmatrix} p_{11} & p_{12} & — & p_{1m} \\ p_{21} & p_{22} & — & p_{2m} \\ \vdots & & — & \\ p_{m1} & — & — & p_{mm} \end{bmatrix} \end{array}$$

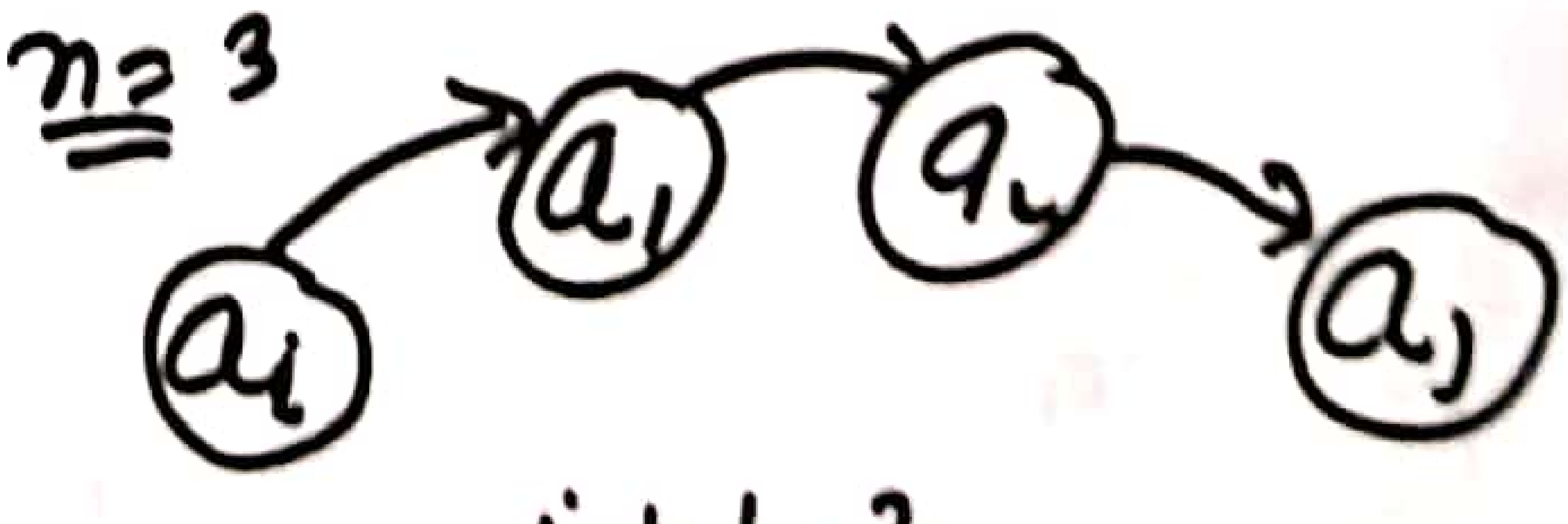
Transition Probabilities Matrix

**n-Step transition:** The n-step TP from $i$ to $j$ is the probability of being at $a_j$ exactly $n$ step after being at $a_i$.




$$p_{ij}^{(n)} = P(X_n = a_j \mid X_0 = a_i)$$

$n = 3$

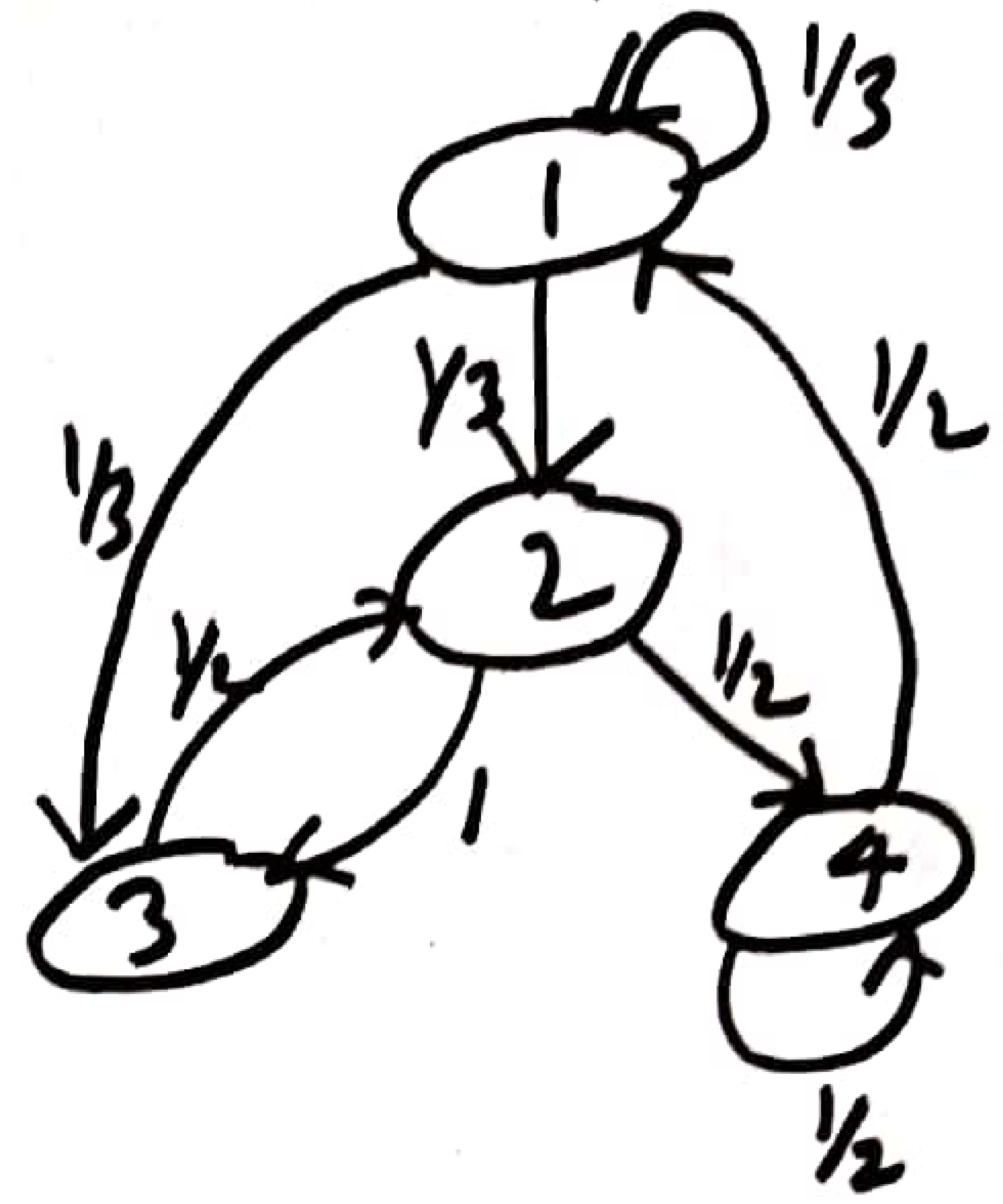



$i \neq 1, 2$
$j \neq 1, 2$

**Chapman - Kolmogrov theorem:**

Let $P$ be the TPM of a MC, Then n-step TPM ($P^{(n)}$) is given by

$$P^{(n)} = P^n$$




eg Consider the MC with 4 states given in figure

(i) Find the TPM

(ii) Compute the probability that the chain is in state 3 after 5 steps starting at state 1.

Soln (i) TPM

$$P = \begin{array}{c|cccc} & 1 & 2 & 3 & 4 \\ \hline 1 & 1/3 & 1/3 & 1/3 & 0 \\ 2 & 0 & 0 & 1/2 & 1/2 \\ 3 & 0 & 1 & 0 & 0 \\ 4 & 1/2 & 0 & 0 & 1/2 \end{array} \begin{array}{l} \rightarrow 1 \\ \rightarrow 1 \\ \rightarrow 1 \\ \rightarrow 1 \end{array} \Bigg\} \text{ Stochastic Matrix}$$

(ii)

$$P^{(5)} = P^5 = \begin{array}{c|cccc} & 1 & 2 & 3 & 4 \\ \hline 1 & 853/3888 & 509/1944 & 52/243 & 395/5296 \\ 2 & 173/864 & 85/432 & 31/108 & 91/288 \\ 3 & 37/144 & 29/72 & 1/9 & 11/48 \\ 4 & 499/2592 & 395/1296 & 71/324 & 245/864 \end{array}$$

$$p_{13}^{(5)} = \frac{52}{243}$$ m

$$p_{23}^{(5)} = 29/72$$

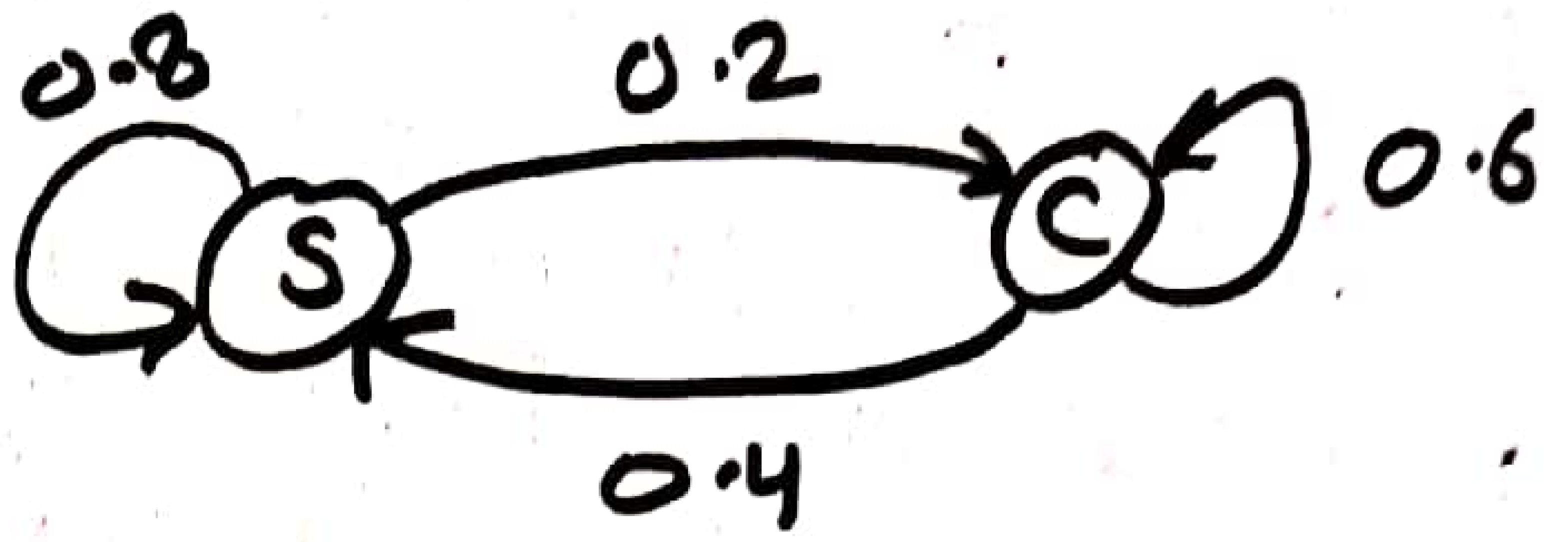
$\downarrow$ Current state

$\rightarrow$ Future state

Q In Jodhpur, if today is sunny, tomorrow will be sunny 80% of time. If today is cloudy tomorrow will be cloudy 60% of the time. Suppose today is sunny, what is the probability it will be cloudy day after tomorrow?

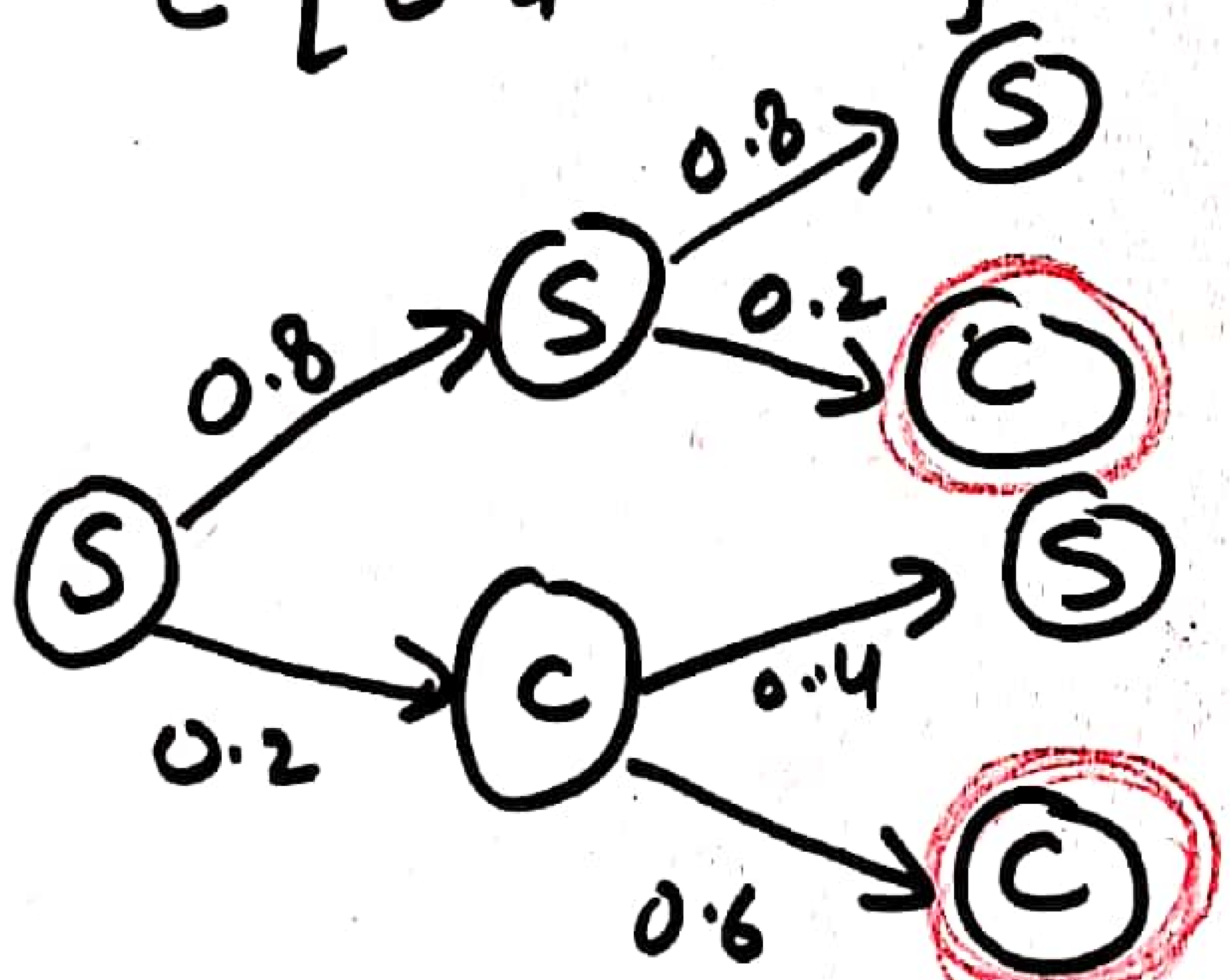
Sol<sup></sup>

Sol. States: $\{S, C\}$




TPM

$$P = \begin{array}{c c} & \begin{array}{cc} S & C \end{array} \\ \begin{array}{c} S \\ C \end{array} & \begin{bmatrix} 0.8 & 0.2 \\ 0.4 & 0.6 \end{bmatrix} \end{array}$$




Req. Probability: $0.8 \times 0.2 + 0.2 \times 0.6 = 0.28$

Method 2 Today is sunny $X_0 = \begin{array}{cc} S & C \\ [\,1 & 0\,] \end{array}$

$X_2 = X_0 \cdot P^2$

$= \underbrace{(X_0 P)}_{X_1} P$

$X_2 = X_1 P$

1) $X_0$

2) $X_1 = X_0 P$

3) $X_2 = X_1 P$

4) $X_3 = X_2 P$

⋮

(n) $X_n = X_{n-1} P$

$X_0 = [1 \quad 0]$

$$X_1 = X_0 \cdot P = [1 \quad 0] \begin{bmatrix} 0.8 & 0.2 \\ 0.4 & 0.6 \end{bmatrix} = [0.8 \quad 0.2]$$

$$X_2 = X_1 P = [0.8 \quad 0.2] \begin{bmatrix} 0.8 & 0.2 \\ 0.4 & 0.6 \end{bmatrix} = [0.28 \quad 0.72]$$

* Let $X_0$ be the initial probability vector. Then the probability vector after n-step transition will be

$$\underbrace{X_n = X_0 P^n} \qquad P \to TPM$$

$\Rightarrow$

$$X_1 = X_0 P$$
$$X_2 = X_1 P$$
$$\vdots$$

$$X_n = \underbrace{(X_0 P)}_{X_1} P^{n-1}$$
$$= (X_1 P) . P^{n-2}$$
$$= X_2 P^{n-2}$$
$$\vdots$$

## Steady State Distribution:

Let V be the PV ~~of~~ after some transitions. Then the distribution is called steady state if

$$V P = V.$$

$P \to TPM$

eg $X_0$ - Initial P. V

1st $X_1 = X_0 P$

$X_2 = X_1 P$

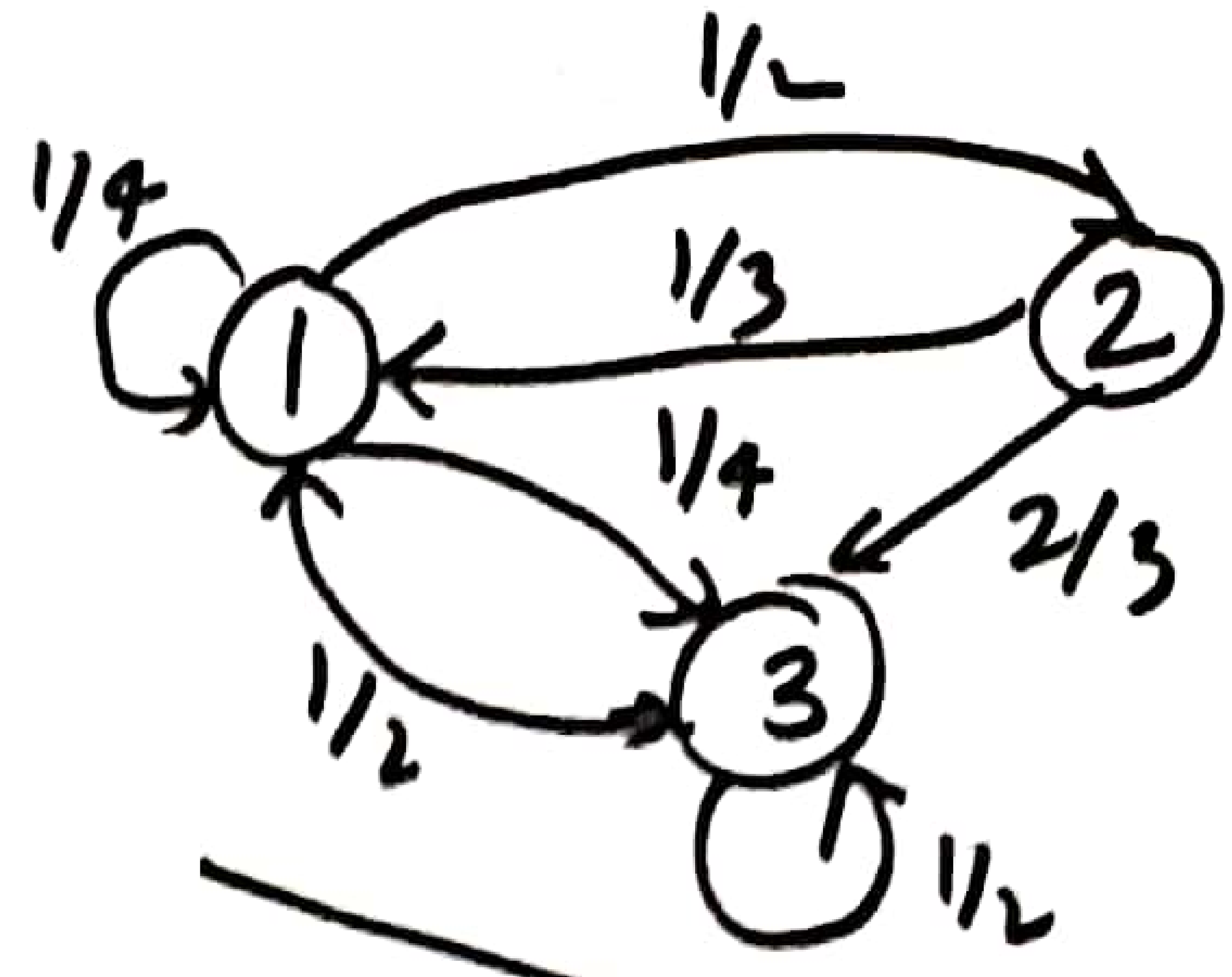
$\vdots$

$X_n = X_{n-1} P \Rightarrow \boxed{X_n = X_{n-1}}$

$X_{n+1} = X_n P \Rightarrow \boxed{X_{n+1} = X_n}$

Q (1) If $P(X_0=1) = 1/3$, find $P(X_0=1, X_1=2)$

(2) If $X_0 = (1/2 \quad 1/2 \quad 0)$, find $P(X_2=3)$




$$P = \begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array}\begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{bmatrix} 1/4 & 1/2 & 1/4 \\ 1/3 & 0 & 2/3 \\ 1/2 & 0 & 1/2 \end{bmatrix} \end{array}$$

Sol$^n$ (1) $P(X_0=1) = 1/3$ $\quad P(X_0=1, X_1=2)$

$$P(X_0=1, X_1=2)$$
$$= P(X_1=2 \mid X_0=1) \cdot P(X_0=1)$$
$$= \frac{1}{2} \cdot \frac{1}{3} = \frac{1}{6}$$

* $P(X_0=a_0, X_1=a_1, X_2=a_2, X_3=a_3) =$
$P(X_3=a_3 \mid X_2=a_2)$
$P(X_2=a_2 \mid X_1=a_1)$
$P(X_1=a_1 \mid X_0=a_0) \cdot P(X_0=a_0)$

(ii) $X_0 = [1/2, 1/2, 0]$

$$P(\underbrace{X_2}_{\text{Step}} = \underbrace{3}_{\text{State}})$$

$P(X_2=3)$ $\quad p^{(2)}$ at state 3

$$p^{(1)} = X_0 \cdot P = [1/2, 1/2, 0] \begin{bmatrix} 1/4 & 1/2 & 1/4 \\ 1/3 & 0 & 2/3 \\ 1/2 & 0 & 1/2 \end{bmatrix} = [\quad\quad]$$

$$p^{(2)} = p^{(1)} \cdot P = [\quad\quad] \begin{bmatrix} & & \\ & & \\ & & \end{bmatrix} = [\quad\quad]$$

$P(X_2=3) = ?$

(HW)

$=$