OPTIMIZATION TECHNIQUES-WEEK 3

Integer Programming Problem

- Refers to an LPP where some (or all) of the variables are constrained to be integer.

- If all the variables are constrained to be integer, the LPP is referred as a All Integer Programming Problem (AIPP).

- If some (not all) of the variables are constrained to be integer, the LPP is referred as a Mixed Integer Programming Problem (MIPP).

- Note that rounding off of the optimal solution of the LPP (ignoring the integer constraints) does not yield the optimal solution of the associated IPP.

Gomory Cut Constraint Method for AIPP

- Solve the given LPP ignoring the integer constraints. If the optimal solution to the LPP is feasible to the IPP, then the solution obtained is optimal for the IPP.

- If some of the basic variables are not integers, then, pick the basic variable with greatest fractional part (say x_{b_i}). Introduce the constraint

$$-\sum_{i=1}^{n} f_{ij}\, x_j \leq -f_{b_i}$$

in the problem, where f_{ij} is the fractional part of a_{ij} and f_{b_i} is fractional part of b_i.

- Solve the modified problem using dual simplex method. If the optimal solution satisfies the integer constraints, the solution obtained is optimal for the given IPP. If not, go to step 2 and repeat the process.

Example

$$\max 3x_1 + x_2 \quad s.t.$$

$$14\mathrm{x}_1 + 6\mathrm{x}_2 \leq 21$$
$$x_1, x_2 \geq 0$$

- Then above LPP can be written as :

$$max\ 3x_1 + x_2$$

$$14x_1 + 6x_2 + x_3 = 21$$

$$x_1, x_2, x_3 \geq 0,$$

Example

- First Simplex Table :

			3	1	0
$\boldsymbol{C_B}$	$\boldsymbol{B}$	$\boldsymbol{b}$	$\boldsymbol{a_1}$	$\boldsymbol{a_2}$	$\boldsymbol{a_3}$
0	x_3	21	14	6	1
		$z_j - c_j$:	-3	-1	0

x_1 enters, x_3 leaves

Example

- Next Simplex Table :

			3	1	0
C_B	B	b	a_1	a_2	a_3
3	x_1	$\frac{3}{2}$	1	$\frac{3}{7}$	$\frac{1}{14}$
		$z_j - c_j$:	0	$\frac{2}{7}$	$\frac{3}{14}$

$z_j - c_j \geq 0\ \forall j$, thus above table corresponds to optimal solution which is non integer. Introduce the constraint :

$$-\frac{3}{7}x_2 - \frac{1}{14}x_3 \leq -\frac{1}{2}$$ which is same as

$$-\frac{3}{7}x_2 - \frac{1}{14}x_3 + x_4 = -\frac{1}{2}$$

Example

- Next Simplex Table :

			3	1	0	0
C_B	B	b	a_1	a_2	a_3	a_4
3	x_1	$\frac{3}{2}$	1	$\frac{3}{7}$	$\frac{1}{14}$	0
0	x_4	$-\frac{1}{2}$	0	$-\frac{3}{7}$	$-\frac{1}{14}$	1
		$z_j - c_j$:	0	$\frac{2}{7}$	$\frac{3}{14}$	0

Apply dual simplex method, x_4 leaves, x_2 enters.

Example

- Next Simplex Table :

			3	1	0	0
C_B	B	b	a_1	a_2	a_3	a_4
3	x_1	1	1	0	0	1
1	x_2	$\frac{7}{6}$	0	1	$\frac{1}{6}$	$-\frac{7}{3}$
		$z_j - c_j$:	0	0	$\frac{1}{6}$	$\frac{2}{3}$

Solution is non integer. Thus, introduce the constraint :

$-\frac{1}{6}x_3 - \frac{2}{3}x_4 \leq -\frac{1}{6}$ which is same as

$$--\frac{1}{6}x_3 - \frac{2}{3}x_4 + x_5 = -\frac{1}{6}$$

Example

- Next Simplex Table :

			3	1	0	0	0
$\boldsymbol{C_B}$	$\boldsymbol{B}$	$\boldsymbol{b}$	$\boldsymbol{a_1}$	$\boldsymbol{a_2}$	$\boldsymbol{a_3}$	$\boldsymbol{a_4}$	$\boldsymbol{a_5}$
3	x_1	1	1	0	0	1	0
1	x_2	$\frac{7}{6}$	0	1	$\frac{1}{6}$	$-\frac{7}{3}$	0
0	x_5	$-\frac{1}{6}$	0	0	$-\frac{1}{6}$	$-\frac{2}{3}$	1
		$z_j - c_j$:	0	0	$\frac{1}{6}$	$\frac{2}{3}$	0

Apply Dual Simplex method. x_5 leaves, x_3 enters.

Example

- Next Simplex Table :

			3	1	0	0	0
$\boldsymbol{C_B}$	$\boldsymbol{B}$	$\boldsymbol{b}$	$\boldsymbol{a_1}$	$\boldsymbol{a_2}$	$\boldsymbol{a_3}$	$\boldsymbol{a_4}$	$\boldsymbol{a_5}$
3	x_1	1	1	0	0	1	0
1	x_2	1	0	1	0	-3	1
0	x_3	1	0	0	1	4	-6
		$z_j - c_j$:	0	0	0	0	1

Solution obtained is an integer solution and hence is optimal to the given IPP.
Optimal solution : (1,1) and optimal value is 4.

Example

$$\max x_1 + 2x_2 \;\; s.t.$$

$$2x_2 \leq 7$$

$$x_1 + x_2 \leq 7$$
$$2x_1 \leq 11$$
$$x_1, x_2 \geq 0, integers$$

- Then above LPP can be written as :

$$max\, x_1 + 2x_2\, s.t.$$
$$2x_2 + x_3 \;= 7$$

$$x_1 + x_2 + x_4 = 7$$
$$2x_1 + x_5 \quad = 11$$
$$x_1, x_2 \geq 0, integers$$

Example

- First Simplex Table :

			1	**2**	**0**	**0**	**0**
$\boldsymbol{C_B}$	$\boldsymbol{B}$	$\boldsymbol{b}$	$\boldsymbol{a_1}$	$\boldsymbol{a_2}$	$\boldsymbol{a_3}$	$\boldsymbol{a_4}$	$\boldsymbol{a_5}$
0	x_3	7	0	2	1	0	0
0	x_4	7	1	1	0	1	0
0	x_5	11	2	0	0	0	1
		$z_j - c_j$:	-1	-2	0	0	0

Apply Simplex method. x_2 enters, x_3 leaves.

Example

- Next Simplex Table :

			1	2	0	0	0
$\boldsymbol{C_B}$	$\boldsymbol{B}$	$\boldsymbol{b}$	$\boldsymbol{a_1}$	$\boldsymbol{a_2}$	$\boldsymbol{a_3}$	$\boldsymbol{a_4}$	$\boldsymbol{a_5}$
2	x_2	$\frac{7}{2}$	0	1	$\frac{1}{2}$	0	0
0	x_4	$\frac{7}{2}$	1	0	$-\frac{1}{2}$	1	0
0	x_5	11	2	0	0	0	1
		$z_j - c_j$:	-1	0	1	0	0

x_1 enters, x_4 leaves.

Example

- Next Simplex Table :

			1	2	0	0	0
C_B	B	b	a_1	a_2	a_3	a_4	a_5
2	x_2	$\frac{7}{2}$	0	1	$\frac{1}{2}$	0	0
1	x_1	$\frac{7}{2}$	1	0	$-\frac{1}{2}$	1	0
0	x_5	4	0	0	1	-2	1
		$z_j - c_j$:	0	0	$\frac{1}{2}$	1	0

$z_j - c_j \geq 0 \ \forall j$, thus above table corresponds to optimal solution which is non integer. Introduce the constraint :

$-\frac{1}{2}x_3 \leq -\frac{1}{2}$ which is same as

$$-\frac{1}{2}x_3 + x_6 = -\frac{1}{2}$$

Example

- Next Simplex Table :

			1	2	0	0	0	0
$\boldsymbol{C_B}$	$\boldsymbol{B}$	$\boldsymbol{b}$	$\boldsymbol{a_1}$	$\boldsymbol{a_2}$	$\boldsymbol{a_3}$	$\boldsymbol{a_4}$	$\boldsymbol{a_5}$	$\boldsymbol{a_6}$
2	x_2	$\frac{7}{2}$	0	1	$\frac{1}{2}$	0	0	0
1	x_1	$\frac{7}{2}$	1	0	$-\frac{1}{2}$	1	0	0
0	x_5	4	0	0	1	-2	1	0
0	x_6	$-\frac{1}{2}$	0	0	$-\frac{1}{2}$	0	0	1
		$z_j - c_j$:	0	0	$\frac{1}{2}$	1	0	0

Apply Dual Simplex Method. x_6 leaves, x_3 enters.

Example

- Next Simplex Table :

			1	2	0	0	0	0
C_B	B	b	a_1	a_2	a_3	a_4	a_5	a_6
2	x_2	3	0	1	0	0	0	1
1	x_1	4	1	0	0	1	0	-1
0	x_5	3	0	0	0	-2	1	2
0	x_6	1	0	0	1	0	0	-2
		$z_j - c_j$:	0	0	0	1	0	1

$z_j - c_j \geq 0\ \forall j$ and solution satisfies the integer constraints. Thus solution is optimal for given IPP. Optimal solution is $x_1 = 4, x_2 = 3$ and optimal value is 10.

Branch and Bound Method

- Solve the given LPP ignoring the integer constraints. If the optimal solution to the LPP is feasible to the IPP, then the solution obtained is optimal for the IPP.

- If some of the basic variables are not integers, then, pick the basic variable with greatest fractional part (say $x_{b_i} = r$). Branch the problem out in two parts by including the constraints $x_{b_i} \leq [r]$ and $x_{b_i} \geq [r] + 1$ (respectively along each branch) and solve the individual branches obtained.

- Continue till each of the branch terminates (either in integer solution or infeasible problem).

- Compare the integer solutions obtained along different branches to obtain the optimal solution to the given IPP.

Example

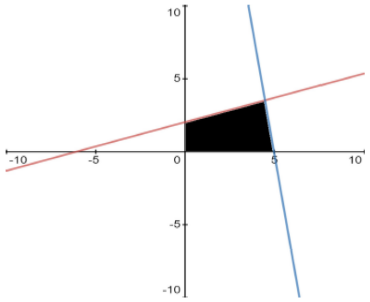
$$\max 7x_1 + 9x_2 \ s.t.$$

$$-x_1 + 3x_2 \leq 6$$

$$7x_1 + x_2 \leq 35$$

$$0 \leq x_1, x_2 \leq 7, integers$$

Firstly solving the LPP (graphically) ignoring the integer constraints we get



Graphical Plot for original LPP :

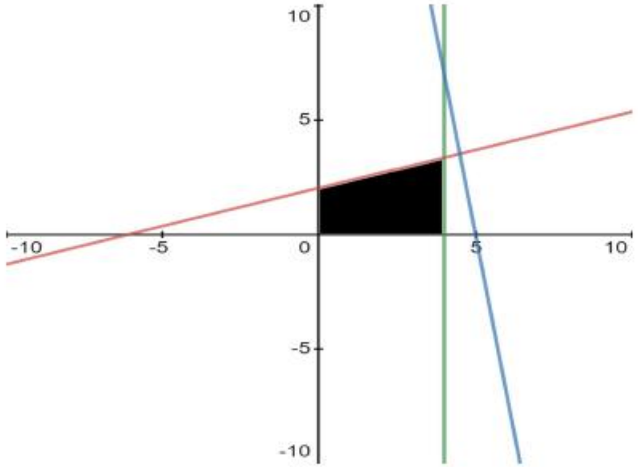
BFS : (0,0), (0,2), (5,0) and (9/2, 7/2)

Optimal Solution : (9/2, 7/2)

optimal solution : $\left(x_1 = \frac{9}{2}, x_2 = \frac{7}{2}\right)$

Branching the Problem Out :

Branch 1 : introduce constraint $x_1 \leq 4$



Graphical Solution for Branch 1 of Level 1 :

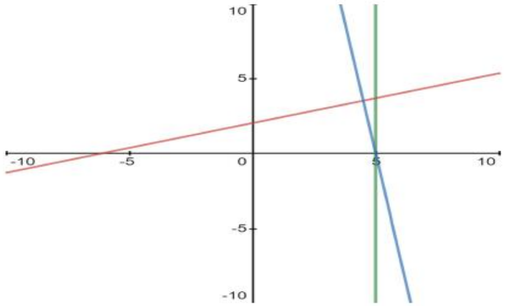
BFS: (0,0), (0,2), (4,10/3) and (4,0)

Optimal Solution : (4, 10/3)

optimal solution : $\left(x_1 = 4,\ x_2 = \frac{10}{3}\right)$

Branching the Problem Out :

Branch 2 : introduce the constraint $x_1 \geq 5$



Graphical Solution to Brach 2 of Level 1:

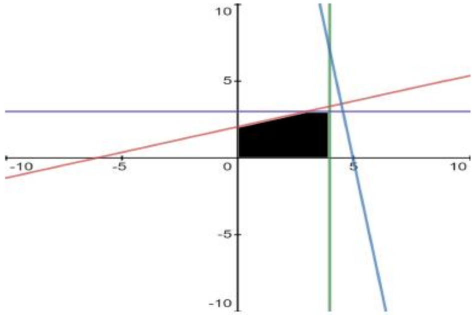
BFS : (5,0) [feasible region is a singleton]

Optimal Solution : (5,0)

Optimal Solution $(x_1 = 5,\ x_2 = 0)$ Branch ends.

Branching the Problem Out (Level 2) :

<u>Branch 1</u> : Introduce the constraint $x_2 \leq 3$



Graphical Solution to Branch 1 of Level 2 :

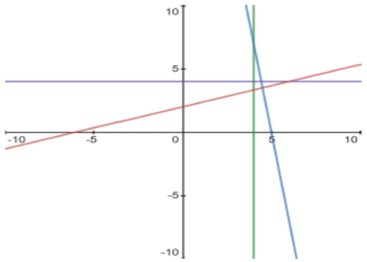
BFS : (0,0), (0,2), (3,3), (4,3) and (4,0)

Optimal Solution : (4,3)

optimal Solution : $(x_1 = 4 \; , \; x_2 = 3)$

Branching the Problem Out (Level 2) :

Branch 2 : introduce the constraint $x_2 \geq 4$



Graphical Solution to Branch 2 of Level 2 :

Feasible region is empty.

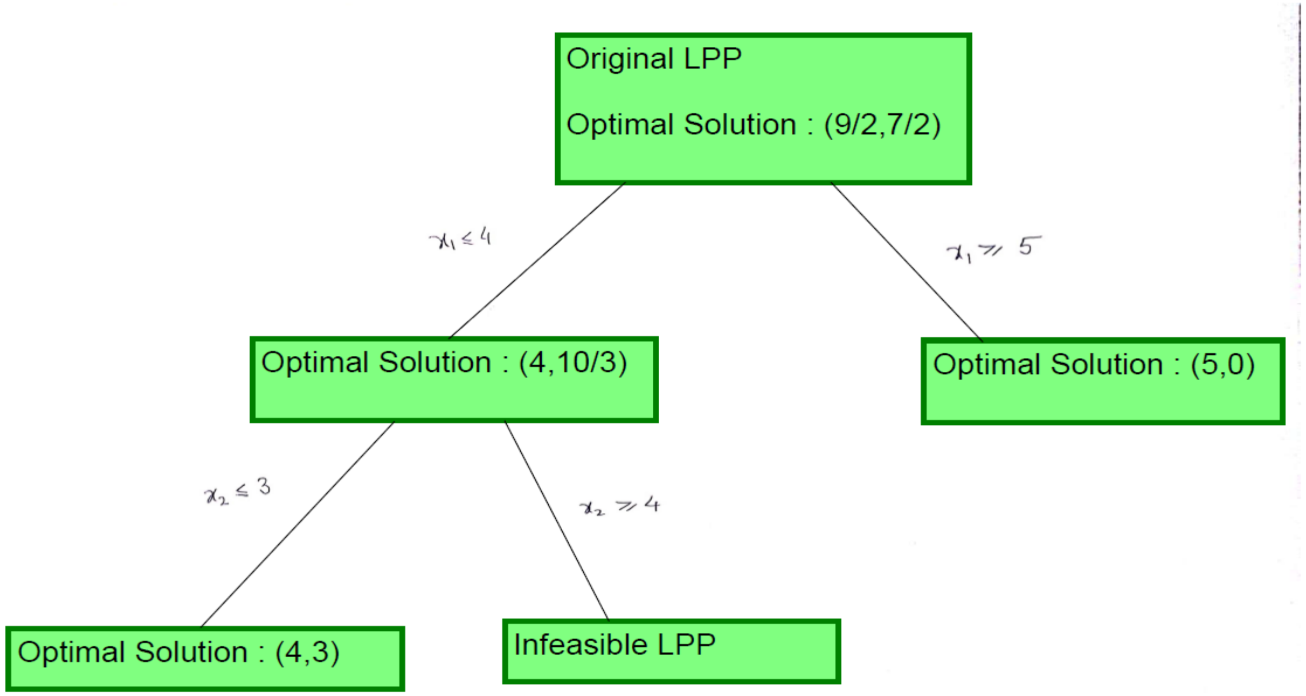
LPP is infeasible.

Infeasible solution

Optimal Solution :

- Comparing the integer solution along different branches, the optimal solution to the given IPP is $x_1 = 4, x_2 = 3$ and optimal value is 55.

Graphical Representation :



Some Important Observations

- Gomory's Cut Constraint method introduces a constraint that cuts off a part of the feasible region containing the optimal solution (if the LPP has non-integer optimal solution while ignoring the integer constraints).

- However, the constraint introduced does not cut off any integer points and hence preserves all the potential candidates for the optimal solution.

- Gomory's method does not distinguish between the different variables and hence cannot be applied to MIPP (although variants of this method are available for MIPP).

- Branch and Bound method is applicable to a wider class of problems (to all IPPs).

- Has applications for finding solutions to integer lattice problems in areas like Telecommunication networks, cellular networks, unmanned automated vehicles and others.