$u_1 \quad u_2 \quad u_3 \quad u_4$

L $\quad \{0, 1, 3, 5\}$

$\min \; v^T L v = \lambda$ | $v^* = u_1$

s.t.

$v^T v = 1$

$\min \; v^T L v$ s.t. $\underline{v^T v = 1 \text{ and } v^T u_1 = 0}$.

$\rightarrow \; Lv = \lambda v$

$\downarrow$

$\min \; \lambda$ s.t. $Lv = \lambda v$ and $v^T u_1 = 0$

$\underline{v^* = u_2}$.

$L = D - W$

$L^T = D^T - W^T$

$= D - W$

$L^T = L$.

$\langle u_i, u_j \rangle = 0$ if $i \neq j$.

$$\min \; v^T L v \quad \text{s.t.} \quad v^T v = 1$$
$$\text{and} \quad v^T u_1 = 0, \; v^T u_2 = 0, \; \ldots, \; v^T u_{k-1} = 0$$

$$v^* = u_k$$

$$\min_{v_1, v_2, \ldots v_k} \; v_1^T L v_1 + v_2^T L v_2 + \cdots + v_k^T L v_k$$

Such that $\langle v_i, v_j \rangle = 0$ if $i \neq j$
$= 1$ if $i = j$

$$f(v_1, v_2, \ldots, v_k) = \sum_{j=1}^{k} v_j^T L v_j + \sum_{j=1}^{k} \lambda_j (1 - v_j^T v_j)$$

$$\nabla_{v_i} f = 2 L v_i - \lambda_i v_i = \vec{0}$$

$$L v_i = \lambda_i v_i \qquad \forall i = 1, \ldots k$$

$$v_i^T L v_i = \lambda_i v_i^T v_i$$

$$v_i^T L v_i = \lambda_i \qquad \forall i = 1, \ldots k$$

$\min \quad \lambda_1 + \lambda_2 + \lambda_3 + \dots + \lambda_k$

such that $\quad L v_i = \lambda_i v_i \quad \forall i = 1, \dots, k$

and $\quad \underline{\langle v_i, v_j \rangle = 0} \quad i \neq j$

$u_1 \quad u_2 \quad u_3 \quad u_4 \quad u_5$

$\lambda_1, \quad \lambda_2, \quad \lambda_3, \lambda_4, \lambda_5$

$0 \quad 2 \quad 5 \quad 7 \quad 9$

$K = 3$

case 1

$\lambda_1 + \lambda_4 + \lambda_5$

$0 + 7 + 9$

16

case 2

$\lambda_1 + \lambda_2 + \lambda_3$

$0 + 2 + 5$

7

Solution $\quad v_1^* = u_1, \quad v_2^* = u_2, \quad v_3^* = u_3.$

$$\min_{v_1, v_2 \ldots v_k} \sum_{i=1}^{k} v_i^T L v_i \qquad \text{s.t. } \langle v_i, v_j \rangle = 1 \text{ if } i = j$$
$$= 0 \text{ if } i \neq j$$

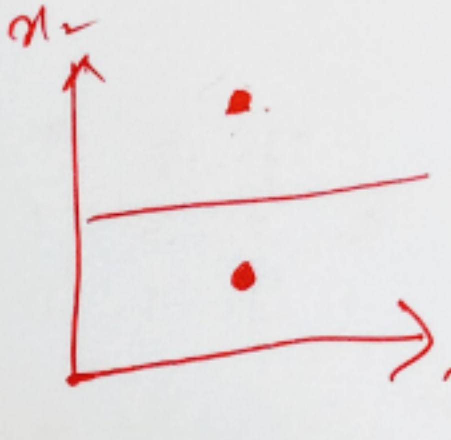
$$H = [v_1 \; v_2, \; \ldots v_k] \in \mathbb{R}^{n \times k}$$

$$\sum_{i=1}^{k} v_i^T L v_i = \text{Trace}(H^T L H)$$

$$H^T H = I$$

$$\min_{\substack{H \in \mathbb{R}^{n \times k} \\ H^T H = I}} \text{Trace}(H^T L H)$$

$$H = \begin{array}{cc} x_1 & x_2 \\ \end{array}$$
$$H = \begin{bmatrix} 1 & 0.5 \\ 1 & 0.5 \\ 1 & 0.5 \\ 1 & -0.5 \\ 1 & -0.5 \\ 1 & -0.5 \end{bmatrix}$$



Spectral Embedding

x_1 x_2

$$e^{-\|x_1 - x_2\|_2^2/\sigma^2}$$

$$H = \begin{bmatrix} r_1^T \\ r_2^T \\ \vdots \\ r_n^T \end{bmatrix}$$

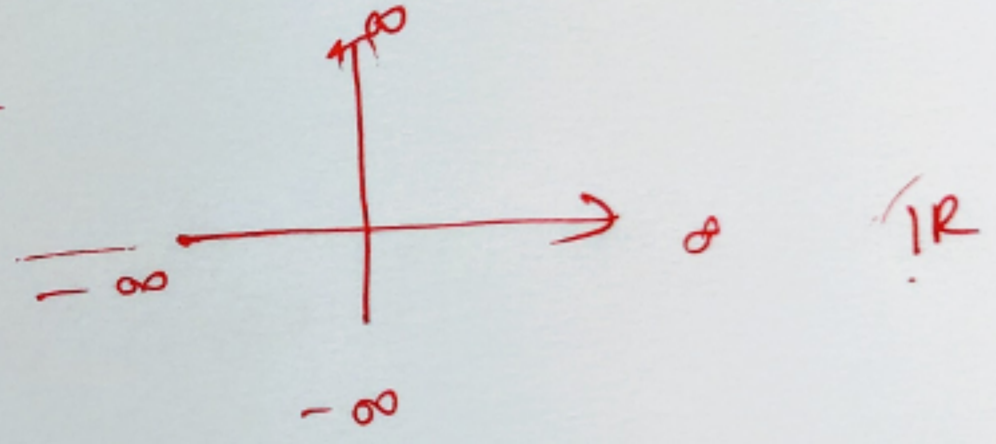
$$\bar{X} = [r_1 \cdots r_n]$$

R   V.

$S \subset V$

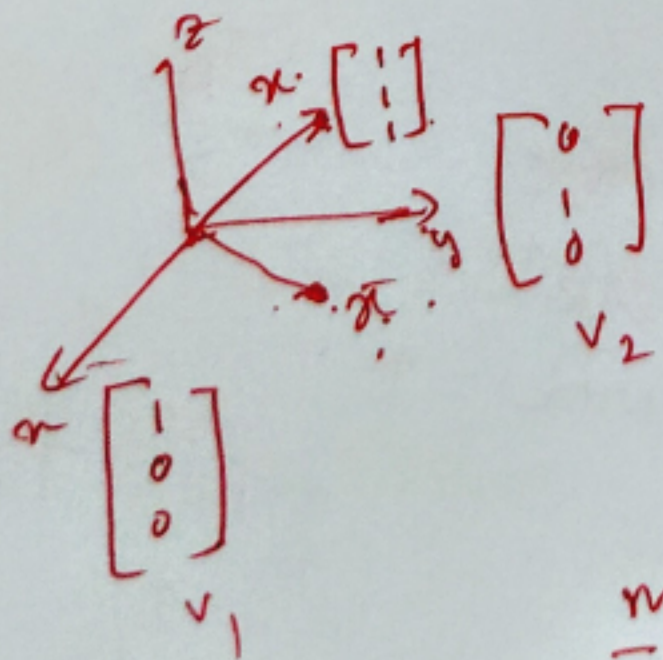
if S is again a vector space
then S is called a subspace
of the vector space V.

$V = \mathbb{R}^2$



$S \subseteq \mathbb{R}^2$.

$S = \left\{ \begin{bmatrix} x \\ 0 \end{bmatrix} : x \in \mathbb{R} \right\}$



$\bar{x} \in$ x-y plane R

$\bar{x} = y_1 \vec{v_1} + y_2 \vec{v_2}$ $V = [v_1 \;\; v_2]$

$\bar{x} = Vy$

$\min_{\bar{x} \in R} \| x - \bar{x} \|_2^2 \Leftrightarrow \min_{y \in \mathbb{R}^2} \| x - Vy \|_2^2$

$$\min_{U \in \mathbb{R}^{d \times k},\ U^T U = I} \sum_{i=1}^{n} \| x_i - U U^T x_i \|_2^2 \qquad \|x\|_2^2 = x^T x$$

$$\begin{aligned}
\| x_i - U U^T x_i \|_2^2 &= (x_i - U U^T x_i)^T (x_i - U U^T x_i) \\
&= (x_i^T - x_i^T U U^T)(x_i - U U^T x_i) \\
&= x_i^T x_i - \underline{x_i^T U U^T x_i} + x_i^T U \boxed{U^T U} U^T x_i \\
&\quad - \underline{x_i^T U U^T x_i} \\
&= x_i^T x_i - 2\, x_i^T U U^T x_i + x_i^T U U^T x_i \\
&= x_i^T x_i - x_i^T U U^T x_i
\end{aligned}$$

$$\min_{U \in \mathbb{R}^{d \times k},\ U^T U = I} \sum_{i=1}^{k} \cancel{x_i^T x_i} - x_i^T U U^T x_i$$

$$\max_{U \in \mathbb{R}^{d \times k},\ U^T U = I} \sum_{i=1}^{k} x_i^T U U^T x_i \qquad \| X = [X_1 \ldots X_n]$$

$$\Downarrow$$

$$\min_{U^T U = I,\ U \in \mathbb{R}^{d \times k}} \operatorname{Trace}(X^T U U^T X)$$

$\max_{U^T U = I,\ U \in \mathbb{R}^{d \times k}}$ $\text{Trace}(X^T U U^T X)$

$\text{Trace}(U^T X X^T U)$

$A \in \mathbb{R}^{m \times n}$

$B \in \mathbb{R}^{n \times m}$

$\text{Trace}(A B)$

$= \text{Trace}(A^T B^T)$

$$\max_{\substack{U \in \mathbb{R}^{d \times k} \\ U^T U = I}} \text{Tarce}(U^T A U)$$

$A = X X^T$.

$\mathcal{X} = \{x_1 \ldots x_n\}$ $x_i \in \mathbb{R}^d$

$X = [x_1 \ldots x_n] \in \mathbb{R}^{d \times n}$

$\underline{y_i = W x_i}$

$\bar{x} = U W x$

$U \ni U^T U = I$ and

$\underline{W = U^T}$