OPTIMIZATION TECHNIQUES-WEEK 2

Duality

- Provides the formulation of the same problem with an alternate perspective.
- Relates the solution for the LPP with the solution for the dual (and the relation between the two problems, in general).
- Asserts that a (finite) optimality can be achieved if it can be achieved from each perspective

Dual of an LPP

- If the original LPP (called the primal) is of the form

$$\left.\begin{array}{c} \max c^T X \\ AX \leq b \\ X \geq 0 \end{array}\right\} \qquad \text{(P1)}$$

- Then the dual of the LPP is given by :

$$\left.\begin{array}{c} min\ b^T Y \\ A^T Y \geq c \\ Y \geq 0 \end{array}\right\} \qquad \text{(P2)}$$

Dual of an LPP

- In general, the dual of an LPP can be obtained using the following table :

Primal	Dual
Is of maximization type	Is of minimization type
Has i-th variable ≥ 0	Has i-th equation of $\geq$ type
Has i-th variable ≤ 0	Has i-th equation of $\leq$ type
Has an unrestricted i-th variable	Has i-th equation of $=$ type
Has i-th equation of $\geq$ type	Has i-th variable ≤ 0
Has i-th equation of $\leq$ type	Has i-th variable ≥ 0
Has i-th equation of $=$ type	Has an unrestricted i-th variable
Has coefficient matrix A	Has coefficient matrix A^T

Example

- If the original LPP (called the primal) is of the form

$$\max x_1 - 2x_2 + 5x_3 \;\; s.t.$$

$$-2x_1 + x_2 + 3x_3 \leq 7$$
$$2x_1 + 3x_2 - x_3 = 10$$
$$x_1, x_2 \geq 0, x_3 \leq 0$$

- Then the dual of the LPP is given by :

$$min\, 7y_1 + 10y_2$$

$$-2y_1 + 2y_2 \geq 1$$

$$y_1 + 3y_2 \geq -2$$

$$3y_1 - y_2 \leq 5$$
$$y_1 \geq 0, y_2\ unrestricted$$

Example

- If the original LPP (called the primal) is of the form

$$\max 3x_1 + x_2 - 2x_3 \;\; s.t.$$

$$\begin{gathered} \mathrm{x}_1 + \mathrm{x}_2 - 3\mathrm{x}_3 \geq 3 \\ 5x_1 + x_2 + 2x_3 = 6 \\ 5x_1 - 2x_2 + x_3 = 12 \\ x_1 \;\geq 0, x_3 \leq 0, x_2 \text{ unrestricted} \end{gathered}$$

- Then the dual of the LPP is given by :

$$min\, 3y_1 + 6y_2 + 12y_3$$

$$\begin{gathered} y_1 + 5y_2 + 5y_3 \geq 3 \\ y_1 + y_2 - 2y_3 = 1 \\ -3y_1 + 2y_2 + y_3 \leq -2 \\ y_1 \leq 0, \;\; y_2, y_3 \; unrestricted \end{gathered}$$

Some Interesting Observations

- Relation of duality is symmetric (Dual of the Dual is Primal).
- If both primal and dual are feasible, then both have a finite optimal solution.
- Primal has a (finite) optimal solution if and only if dual as a (finite) optimal solution. Further, the optimal values for both the problems coincide.
- Primal has unbounded solution if and only if dual is infeasible.
- If x_0 and w_0 are feasible for primal (maximization) and dual (minimization) and $c^T x_0 = b^T w_0$ then x_0 and w_0 are optimal for their respective problems.
- If x_0 and w_0 are feasible for primal (maximization) and dual (minimization), then x_0 and w_0 are optimal for their respective problems if and only if $w_0^T(Ax_0 - b) = x_0^T(A^T w_0 - c) = 0$.

Example

Example :

$$\min 2x_1 + x_2 \quad \text{s.t.}$$

$$3x_1 + x_2 = 3$$

$$4x_1 + 3x_2 \geq 6$$
$$x_1 + 2x_2 \leq 3$$
$$x_1, x_2 \geq 0$$

Then the dual of the above problem can be written as :

$$\max 3w_1 + 6w_2 + 3w_3 \quad \text{s.t.}$$
$$3w_1 + 4w_2 + w_3 \leq 2$$

$$w_1 + 3w_2 + 2w_3 \leq 1$$
$$w_1 \text{unrestricted}, w_2 \geq 0, w_3 \leq 0$$

Example :

After introducing artificial variables, the modified Primal problem is :

$$\max -2x_1 - x_2 - Mx_5 - Mx_6 \text{ s.t.}$$
$$3x_1 + x_2 + x_5 = 3$$
$$4x_1 + 3x_2 - x_3 + x_6 = 6$$
$$x_1 + 2x_2 + x_4 = 3$$
$$x_i \geq 0 \; \forall i$$

First Simplex Table :

		c_j :	-2	-1	0	0	$-M$	$-M$
C_B	B	b	a_1	a_2	a_3	a_4	a_5	a_6
$-M$	x_5	3	3	1	0	0	1	0
$-M$	x_6	6	4	3	-1	0	0	1
0	x_4	3	1	2	0	1	0	0
		$z_j - c_j$:	$-7M + 2$	$-4M + 1$	M	0	0	0

Example :

As $z_1 - c_1$ is most negative, x_1 enters the basis. Further as $\frac{b_1}{a_{21}}$ is least among the positive ratios, x_5 leaves the basis. Thus the updated simplex table is :

		c_j :	-2	-1	0	0	$-M$	$-M$
C_B	B	b	a_1	a_2	a_3	a_4	a_5	a_6
-2	x_1	1	1	$\frac{1}{3}$	0	0	$\frac{1}{3}$	0
$-M$	x_6	2	0	$\frac{5}{3}$	-1	0	$-\frac{4}{3}$	1
0	x_4	2	0	$\frac{5}{3}$	0	1	$-\frac{1}{3}$	0
		$z_j - c_j$:	0	$-\frac{5M}{3} + \frac{1}{3}$	M	0	$\frac{7M}{3} - \frac{2}{3}$	0

Example :

Again, as $z_2 - c_2$ is most negative, x_2 enters the basis. Further as $\frac{b_2}{a_{22}}$ is least among the positive ratios, x_6 leaves the basis. Thus the updated simplex table is :

		c_j :	-2	-1	0	0	$-M$	$-M$
C_B	B	b	a_1	a_2	a_3	a_4	a_5	a_6
-2	x_1	$\frac{3}{5}$	1	0	$\frac{1}{5}$	0	$\frac{3}{5}$	$-\frac{1}{5}$
-1	x_2	$\frac{6}{5}$	0	1	$-\frac{3}{5}$	0	$-\frac{4}{5}$	$\frac{3}{5}$
0	x_4	0	0	0	1	1	1	-1
		$z_j - c_j$:	0	0	$\frac{1}{5}$	0	$M - \frac{2}{5}$	$M - \frac{1}{5}$

As $z_j - c_j \geq 0 \; \forall \; j$, (and all artificial variables are zero), the current solution is optimal for the given problem. Thus optimal solution is $x_1 = \frac{3}{5}, x_2 = \frac{6}{5}$ and optimal value is $\frac{12}{5}$.

Example :

Further, as z_j (in the optimal table) of the basic variables in the first simplex table provides the optimal solution to the dual, the optimal solution to the dual is $w_1 = \frac{2}{5}, w_2 = \frac{1}{5}, w_3 = 0$

Optimal Table for the Primal

		c_j :	-2	-1	0	0	$-M$	$-M$
C_B	B	b	a_1	a_2	a_3	a_4	a_5	a_6
-2	x_1	$\frac{3}{5}$	1	0	$\frac{1}{5}$	0	$\frac{3}{5}$	$-\frac{1}{5}$
-1	x_2	$\frac{6}{5}$	0	1	$-\frac{3}{5}$	0	$-\frac{4}{5}$	$\frac{3}{5}$
0	x_4	0	0	0	1	1	1	-1
		$z_j - c_j$:	0	0	$\frac{1}{5}$	0	$M - \frac{2}{5}$	$M - \frac{1}{5}$

and optimal value is $\frac{12}{5}$ (same as optimal value for the primal).

Example

Example :

$$\max 3x_1 + 2x_2 \text{ s.t.}$$

$$x_1 + x_2 \geq 1$$
$$x_1 + x_2 \leq 7$$
$$x_1 + 2x_2 \leq 10$$
$$x_2 \leq 3$$
$$x_1, x_2 \geq 0$$

Then the dual of the above problem can be written as :

$$min -w_1 + 7w_2 + 10w_3 + 3w_4 \text{ s.t.}$$
$$-w_1 + w_2 + w_3 \geq 3$$
$$-w_1 + w_2 + 2w_3 + w_4 \geq 2 ,$$
$$w_1, w_2, w_3, w_4 \geq 0$$

Example :

Introducing artificial variables, the modified problem is :

$$\max w_1 - 7w_2 - 10w_3 - 3w_4 - Mw_7 - Mw_8 \text{ s.t.}$$
$$-w_1 + w_2 + w_3 - w_5 + w_7 = 3$$
$$-\mathrm{w}_1 + \mathrm{w}_2 + 2\mathrm{w}_3 + \mathrm{w}_4 - \mathrm{w}_6 + \mathrm{w}_8 = 2$$
$$w_i \geq 0 \; \forall i$$

First Simplex Table :

		c_j :	1	-7	-10	-3	0	0	$-M$	$-M$
C_B	B	b	a_1	a_2	a_3	a_4	a_5	a_6	a_7	a_8
$-M$	w_7	3	-1	1	1	0	-1	0	1	0
$-M$	w_8	2	-1	1	2	1	0	-1	0	1
		$z_j - c_j$:	$2M - 1$	$-2M + 7$	$-3M + 10$	$-M + 3$	M	M	0	0

Example :

As $z_3 - c_3$ is most negative, w_3 enters the basis. Further as $\frac{b_2}{a_{32}}$ is least among the positive ratios, w_8 leaves the basis. Thus the updated simplex table is :

		c_j :	1	-7	-10	-3	0	0	$-M$	$-M$
C_B	B	b	a_1	a_2	a_3	a_4	a_5	a_6	a_7	a_8
$-M$	w_7	2	$-\frac{1}{2}$	$\frac{1}{2}$	0	$-\frac{1}{2}$	-1	$\frac{1}{2}$	1	$-\frac{1}{2}$
-10	w_3	1	$-\frac{1}{2}$	$\frac{1}{2}$	1	$\frac{1}{2}$	0	$-\frac{1}{2}$	0	$\frac{1}{2}$
		$z_j - c_j$:	$\frac{M}{2}+4$	$-\frac{M}{2}+2$	0	$\frac{M}{2}-2$	M	$-\frac{M}{2}+5$	0	$\frac{3M}{2}-5$

As $z_2 - c_2$ is most negative, w_2 enters the basis. Further as $\frac{b_2}{a_{22}}$ is least among the positive ratios, w_3 leaves the basis. Thus the updated simplex table is :

Example :

Next Simplex Table :

		c_j :	1	-7	-10	-3	0	0	$-M$	$-M$
C_B	B	b	a_1	a_2	a_3	a_4	a_5	a_6	a_7	a_8
$-M$	w_7	1	0	0	-1	-1	-1	1	1	-1
-7	w_2	2	-1	1	2	1	0	-1	0	1
		$z_j - c_j$:	6	0	$M-4$	$M-4$	M	$-M+7$	0	$2M-7$

As $z_6 - c_6$ is most negative, w_6 enters the basis. Further as $\frac{b_1}{a_{16}}$ is least among the positive ratios, w_7 leaves the basis. Thus the updated simplex table is :

Example :

Next Simplex Table :

		c_j :	1	-7	-10	-3	0	0	$-M$	$-M$
C_B	B	b	a_1	a_2	a_3	a_4	a_5	a_6	a_7	a_8
0	w_6	1	0	0	-1	-1	-1	1	1	-1
-7	w_2	3	-1	1	1	0	-1	0	1	0
		$z_j - c_j$:	6	0	3	3	7	0	$M-7$	M

As $z_j - c_j \geq 0 \ \forall j$, the above table corresponds to the optimal solution to problem solved. Further as the solution to its dual is the z_j (in the optimal table) of basic variables of the first simplex table, the solution to the originally considered problem is $x_1 = 7, x_2 = 0$ and the optimal value is 21 (same for both primal and dual).

Dual Simplex Method

- Provides an alternate method to solve a certain class of LPPs
- Can be used to solve LPPs for which $z_j - c_j$ is non-negative for all j in the first iteration (when the solution is allowed to go negative).
- Arrives at the optimal solution if both the column b and $z_j - c_j$ $(\forall\ j)$ are non-negative.
- Has applications to problems like integer programming problem, addition or deletion of constraints in a LPP.

Dual Simplex Method-Algorithm

- Represent the problem as a maximization problem where all constraints are of " $\leq$ " type (with all variables constrained to be non-negative).
- Compute the first simplex table. If some $z_j - c_j$ is negative, then dual simplex table is not applicable.
- If $z_j - c_j \geq 0 \ \forall j$, and column b is non-negative then table corresponds to an optimal solution to the given problem.
- If some entry in column b (say b_i) is negative, then choose the most negative variable to leave the simplex table (say b_i). Compute the ratios $\{ \frac{z_j - c_j}{a_{ij}} : a_{ij} < 0 \}$. The variable corresponding to largest ratio enters the simplex table.
- Update the simplex table and proceed towards the optimal solution (by repeating steps 2 to 5).

Example

Example :

$$\min 3x_1 + x_2 \text{ s.t.}$$

$$x_1 + x_2 \geq 1$$

$$2x_1 + 3x_2 \geq 2$$

$$x_1, x_2 \geq 0$$

Then the above problem can be written as :

$$\max -3x_1 - x_2 \text{ s.t.}$$

$$-x_1 - x_2 + x_3 = -1$$

$$-2x_1 - 3x_2 + x_4 = -2$$

$$x_1, x_2 \geq 0$$

Example :

First Simplex Table :

		c_j :	-3	-1	0	0
C_B	B	b	a_1	a_2	a_3	a_4
0	x_3	-1	-1	-1	1	0
0	x_4	-2	-2	-3	0	1
		$z_j - c_j$:	3	1	0	0

As $z_j - c_j \geq 0 \ \forall j$, dual simplex method is applicable. As b_2 is most negative, x_4 leaves. Further as $\frac{z_2 - c_2}{a_{22}}$ is largest (least modulus), x_2 enters the simplex table in the next iteration.

Example :

Next Simplex Table :

		c_j :	-3	-1	0	0
C_B	B	b	a_1	a_2	a_3	a_4
0	x_3	$-\frac{1}{3}$	$-\frac{1}{3}$	0	1	$-\frac{1}{3}$
-1	x_2	$\frac{2}{3}$	$\frac{2}{3}$	1	0	$\frac{-1}{3}$
		$z_j - c_j$:	$\frac{7}{3}$	0	0	$\frac{1}{3}$

As b_1 is most negative, x_3 leaves. Further as $\frac{z_4 - c_4}{a_{14}}$ is largest (least modulus), x_4 enters the simplex table in the next iteration.

Example :

Next Simplex Table :

		c_j :	-3	-1	0	0
C_B	B	b	a_1	a_2	a_3	a_4
0	x_4	1	1	0	-3	1
-1	x_2	1	1	1	-1	0
		$z_j - c_j$:	2	0	1	0

As all b_i are non-negative, the above table corresponds to the optimal solution for the given problem and the optimal solution is $x_1 = 0, x_2 = 1$.

Example

Example :

$$\min 6x_1 + 7x_2 + 3x_3 + 5x_4 \quad \text{s.t.}$$
$$5x_1 + 6x_2 - 3x_3 + 4x_4 \geq 12$$
$$x_2 + 5x_3 - 6x_4 \geq 10$$
$$2x_1 + 5x_2 + x_3 + x_4 \geq 8$$
$$x_1, x_2 \geq 0$$

Then the above problem can be written as :

$$\max -6x_1 - 7x_2 - 3x_3 - 5x_4 \quad \text{s.t.}$$
$$-5x_1 - 6x_2 + 3x_3 - 4x_4 + x_5 \leq -12$$
$$-x_2 - 5x_3 + 6x_4 + x_6 \leq -10$$
$$-2x_1 - 5x_2 - x_3 - x_4 + x_7 \leq -8$$
$$x_i \geq 0, \forall i$$

Example :

First Simplex table :

		c_j :	-6	-7	-3	-5	0	0	0
C_B	B	b	a_1	a_2	a_3	a_4	a_5	a_6	a_7
0	x_5	-12	-5	-6	3	-4	1	0	0
0	x_6	-10	0	-1	-5	6	0	1	0
0	x_7	-8	-2	-5	-1	-1	0	0	1
		$z_j - c_j$:	6	7	3	5	0	0	0

As $z_j - c_j \geq 0\ \ \forall j$, dual simplex method is applicable. As b_1 is most negative, x_5 leaves. Further as $\frac{z_2 - c_2}{a_{22}}$ is largest (least modulus), x_2 enters the simplex table in the next iteration.

Example :

Next Simplex table :

		c_j :	-6	-7	-3	-5	0	0	0
C_B	B	b	a_1	a_2	a_3	a_4	a_5	a_6	a_7
-7	x_2	2	$\frac{5}{6}$	1	$-\frac{1}{2}$	$\frac{2}{3}$	$-\frac{1}{6}$	0	0
0	x_6	-8	$\frac{5}{6}$	0	$-\frac{11}{2}$	$\frac{20}{3}$	$-\frac{1}{6}$	1	0
0	x_7	2	$\frac{13}{6}$	0	$-\frac{7}{2}$	$\frac{7}{3}$	$-\frac{5}{6}$	0	1
		$z_j - c_j$:	$\frac{1}{6}$	0	$\frac{13}{2}$	$\frac{1}{3}$	$\frac{7}{6}$	0	0

As b_2 is negative, x_6 leaves. Further as $\frac{z_3-c_3}{a_{23}}$ is largest (least modulus), x_3 enters the simplex table in the next iteration. Thus the updated simplex table is :

Example :

Next Simplex table :

		c_j :	-6	-7	-3	-5	0	0	0
C_B	B	b	a_1	a_2	a_3	a_4	a_5	a_6	a_7
-7	x_2	$\frac{30}{11}$	$\frac{25}{33}$	1	0	$\frac{2}{33}$	$-\frac{5}{33}$	$-\frac{1}{11}$	0
-3	x_3	$\frac{16}{11}$	$\frac{-5}{33}$	0	1	$\frac{-40}{33}$	$\frac{1}{33}$	$-\frac{2}{11}$	0
0	x_7	$\frac{78}{11}$	$\frac{18}{11}$	0	0	$-\frac{21}{11}$	$\frac{-8}{11}$	$-\frac{7}{11}$	1
		$z_j - c_j$:	$\frac{38}{33}$	0	0	$+ve$	$\frac{32}{33}$	$\frac{13}{11}$	0

As b is non-negative the table corresponds to the optimal solution for the given problem and the optimal solution is $x_1 = 0, x_2 = \frac{30}{11}, x_3 = \frac{16}{11}, x_4 = 0$ $(optimal\ value\ is\ \frac{258}{11}$.

Addition of Constraint

- Suppose we have a situation where we modelled a system and solved the associated LPP.
- Now with the change in situation, a new constraint has come up and the LPP needs to be solved with this additional constraint (added to the previous set of constraints).
- Do we have to solve the LPP again ? Or can we modify the optimal table for the previously solved problem such that it corresponds to a BFS for the updated problem (and we can solve further to obtain the optimal solution to the modified problem.
- If the optimal solution to the original LPP satisfies the new constraint then the optimal solution to the original problem is the optimal solution to the modified problem.
- If not, to address the changes brought in by the new constraint(s), express the constraint(s) as " $\leq$ " type and take it to the simplex table [note that $z_j - c_j$ do not change as the cost of the new basic variable (slack variable of the additional constraint) is zero].
- The simplex table corresponds to a BFS of the modified problem. Use dual simplex method to obtain the optimal solution to the modified problem.

Example

Example :

$$\max 6x_1 - 2x_2 \quad \text{s.t.}$$

$$2x_1 - x_2 \leq 2$$
$$x_1 \leq 4$$
$$x_1, x_2 \geq 0$$

Then above problem can be written as :

$$\max 6x_1 - 2x_2 \quad \text{s.t.}$$
$$2x_1 - x_2 + x_3 = 2$$
$$x_1 + x_4 = 4$$
$$x_i \geq 0 \quad \forall i$$

Example :

First Simplex Table :

		c_j :	6	-2	0	0
C_B	B	b	a_1	a_2	a_3	a_4
0	x_3	2	2	-1	1	0
0	x_4	4	1	0	0	1
		$z_j - c_j$:	-6	2	0	0

As $z_1 - c_1 < 0$, x_1 enters. Further as $\frac{b_1}{a_{11}}$ is the least ratio (among the positive ratios), x_3 leaves.
Thus the updated simplex table is :

Example :

Next Simplex Table :

		c_j :	6	-2	0	0
C_B	B	b	a_1	a_2	a_3	a_4
6	x_1	1	1	$-\frac{1}{2}$	$\frac{1}{2}$	0
0	x_4	3	0	$\frac{1}{2}$	$-\frac{1}{2}$	1
		$z_j - c_j$:	0	-1	3	0

As $z_2 - c_2 < 0$, x_2 enters. Further as $\frac{b_2}{a_{22}}$ is the least ratio (among the positive ratios), x_4 leaves.
Thus the updated simplex table is :

Example :

Next Simplex Table :

		c_j :	6	-2	0	0
C_B	B	b	a_1	a_2	a_3	a_4
6	x_1	4	1	0	0	1
-2	x_2	6	0	1	-1	2
		$z_j - c_j$:	0	0	2	2

As $z_j - c_j \geq 0 \ \forall j$, the current table corresponds to the optimal solution $x_1 = 4, x_2 = 6$ (with optimal cost 12)

Example :

Suppose we have to solve the problem with an additional constraint $2x_1 + 3x_2 \leq 6$

Writing the equation in ≤" form and taking it to simplex table gives :

		c_j :	6	-2	0	0	0
C_B	B	b	a_1	a_2	a_3	a_4	a_5
6	x_1	4	1	0	0	1	0
-2	x_2	6	0	1	-1	2	0
0	x_5	6	2	3	0	0	1
		$z_j - c_j$:	0	0	2	2	0

Note that although $x_1, x_2$and x_5 are basic variables, the corresponding not identity. Transforming the matrix for basic variables as identity matrix (using row operations, we obtain the simplex table to be :

Example :

Updated table :

		c_j :	6	-2	0	0	0
C_B	B	b	a_1	a_2	a_3	a_4	a_5
6	x_1	4	1	0	0	1	0
-2	x_2	6	0	1	-1	2	0
0	x_5	-20	0	0	3	-8	1
		$z_j - c_j$:	0	0	2	2	0

Applying dual simplex method, as b_3 is negative, x_5 leaves. Further, as $\frac{z_4-c_4}{a_{34}}$ is largest ratio (among negative entries, x_4 enters. Thus the updated simplex table is :

Example :

Updated table :

		c_j :	6	-2	0	0	0
C_B	B	b	a_1	a_2	a_3	a_4	a_5
6	x_1	$\frac{3}{2}$	1	0	$\frac{3}{8}$	0	$\frac{1}{8}$
-2	x_2	1	0	1	$-\frac{1}{4}$	0	$\frac{1}{4}$
0	x_5	$\frac{5}{2}$	0	0	$-\frac{3}{8}$	1	$-\frac{1}{8}$
		$z_j - c_j$:	0	0	$\frac{11}{4}$	0	$\frac{1}{4}$

As $z_j - c_j \geq 0\ \forall\, j$, the current table corresponds to the optimal solution and the optimal solution to the modified problem is $x_1 = \frac{3}{2}, x_2 = 1$ and the optimal value is 7.