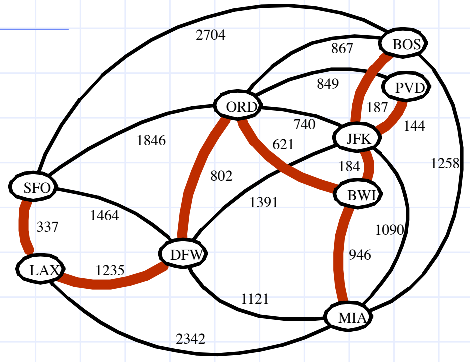
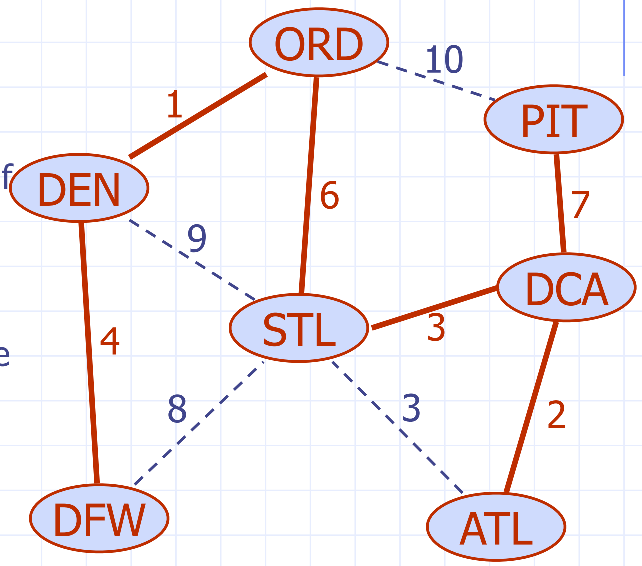
Minimum Spanning Trees



Minimum Spanning Trees

- Spanning subgraph
  - Subgraph of a graph G containing all the vertices of G
- Spanning tree
  - Spanning subgraph that is itself a (free) tree
- Minimum spanning tree (MST)
  - Spanning tree of a weighted graph with minimum total edge weight
- Applications
  - Communications networks
  - Transportation networks



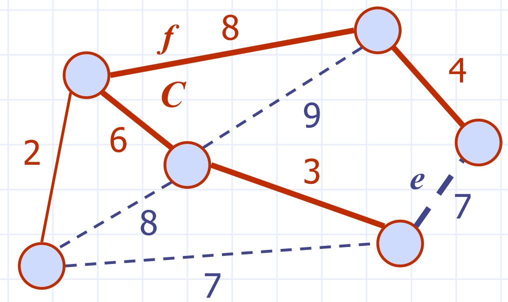
Cycle Property

Cycle Property:

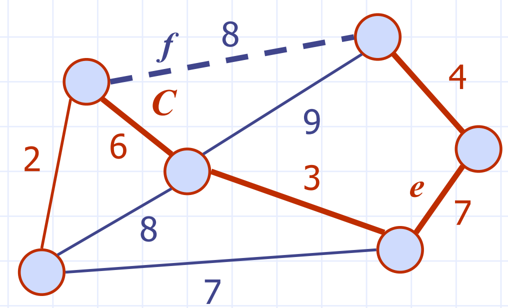
- Let T be a minimum spanning tree of a weighted graph G
- Let e be an edge of G that is not in T and C let be the cycle formed by e with T
- For every edge f of C, $weight(f) \le weight(e)$

Proof:

- By contradiction
- If $weight(f) > weight(e)$ we can get a spanning tree of smaller weight by replacing e with f



Replacing f with e yields a better spanning tree



Partition Property

Partition Property:

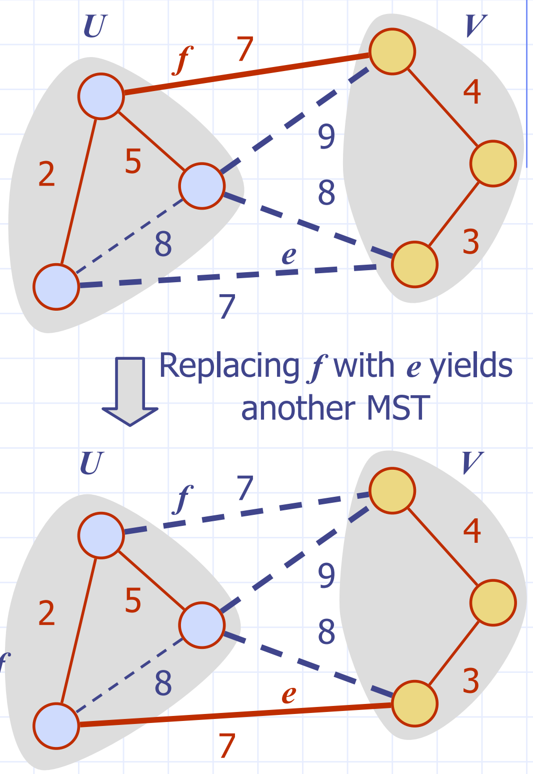
- Consider a partition of the vertices of G into subsets U and V
- Let e be an edge of minimum weight across the partition
- There is a minimum spanning tree of G containing edge e

Proof:

- Let T be an MST of G
- If T does not contain e, consider the cycle C formed by e with T and let f be an edge of C across the partition
- By the cycle property,

$$weight(f) \leq weight(e)$$

- Thus, $weight(f) = weight(e)$
- We obtain another MST by replacing f with e



Algorithm Characteristics

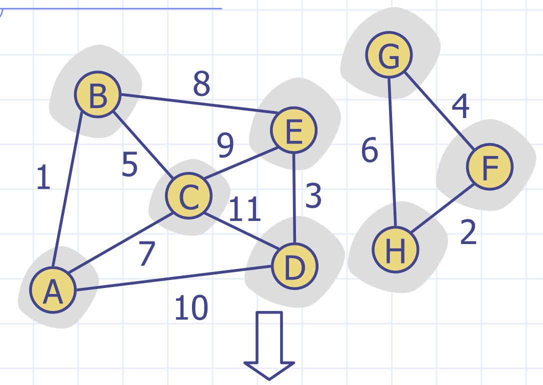
- Both Prim’s and Kruskal’s Algorithms work with undirected graphs
- Both work with weighted and unweighted graphs but are more interesting when edges are weighted
- Both are greedy algorithms that produce optimal solutions

Kruskal's Algorithm

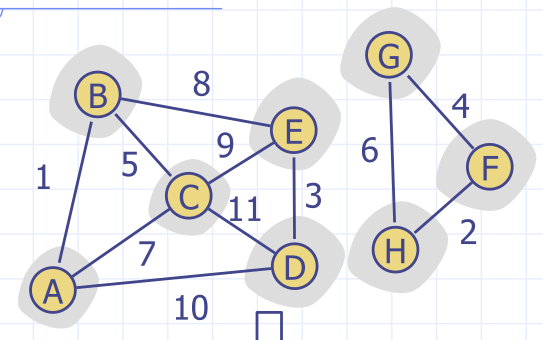
- Maintain a partition of the vertices into clusters
  - Initially, single-vertex clusters
  - Keep an MST for each cluster
  - Merge "closest" clusters and their MSTs
- A priority queue stores the edges outside clusters
  - Key: weight
  - Element: edge
- At the end of the algorithm
  - One cluster and one MST

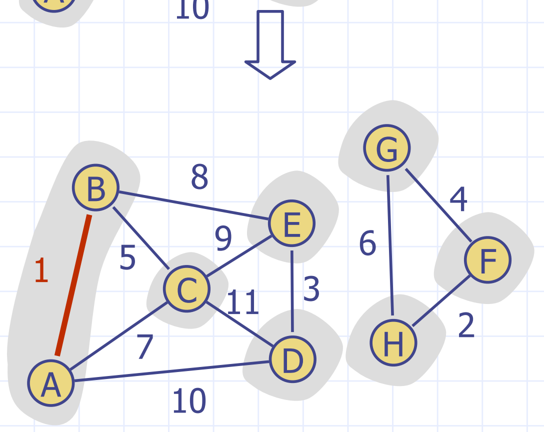
Algorithm *KruskalMST*(*G*)
  for each vertex *v* in *G* **do**
    Create a cluster consisting of *v*
  let *Q* be a priority queue.
  Insert all edges into *Q*
  $T \leftarrow \emptyset$
  {*T* is the union of the MSTs of the clusters}
  while *T* has fewer than $n - 1$ edges **do**
$e \leftarrow$ *Q.removeMin*().*getValue*()
    $[u, v] \leftarrow$ *G.endVertices*(*e*)
    $A \leftarrow$ *getCluster*(*u*)
    $B \leftarrow$ *getCluster*(*v*)
    if $A \neq B$ **then**
      Add edge *e* to *T*
      mergeClusters(*A*, *B*)
  return *T*

Example

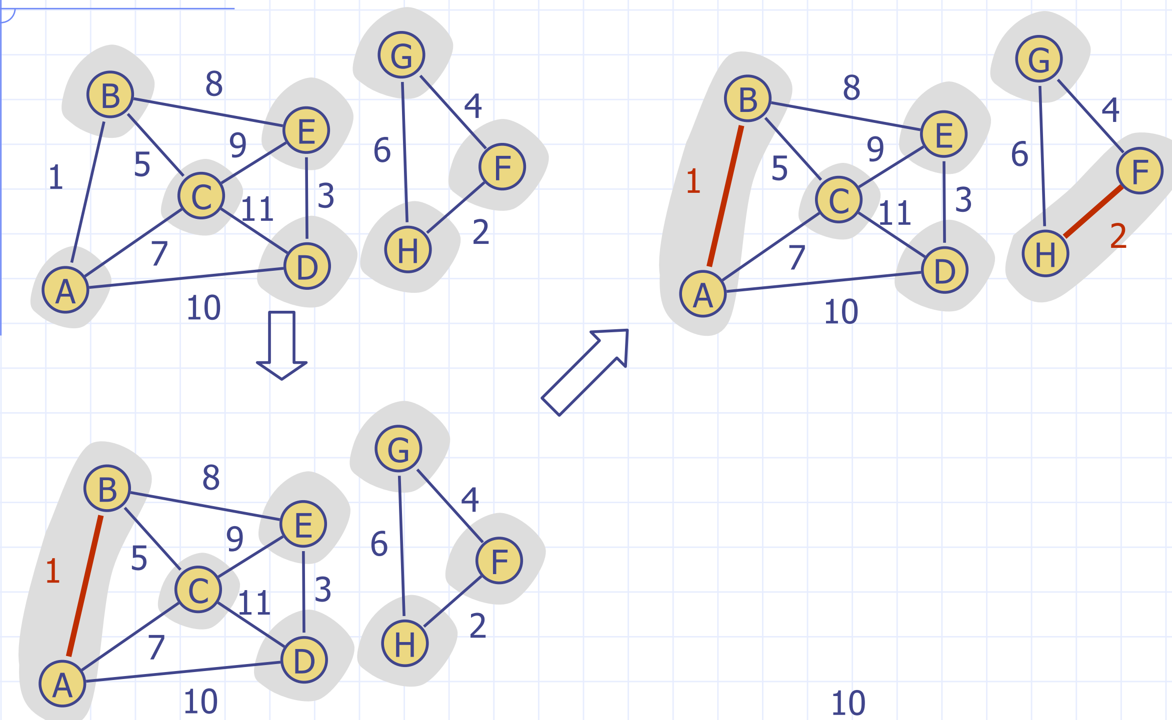


Example





Example



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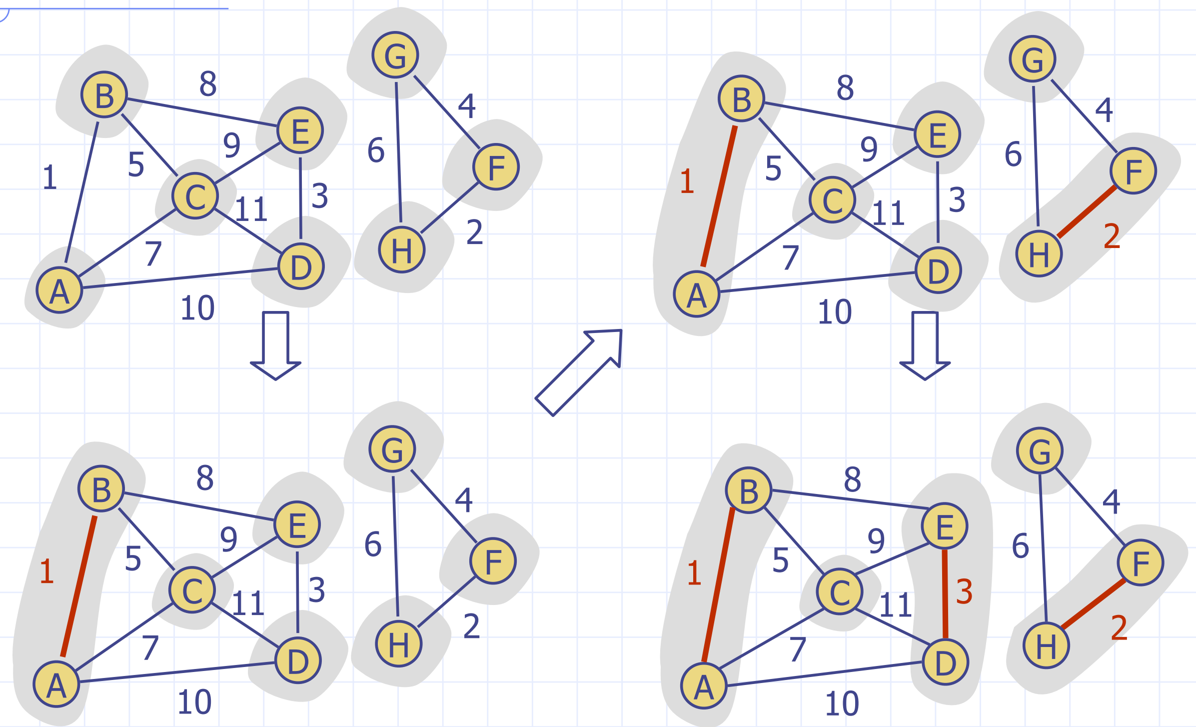
Example



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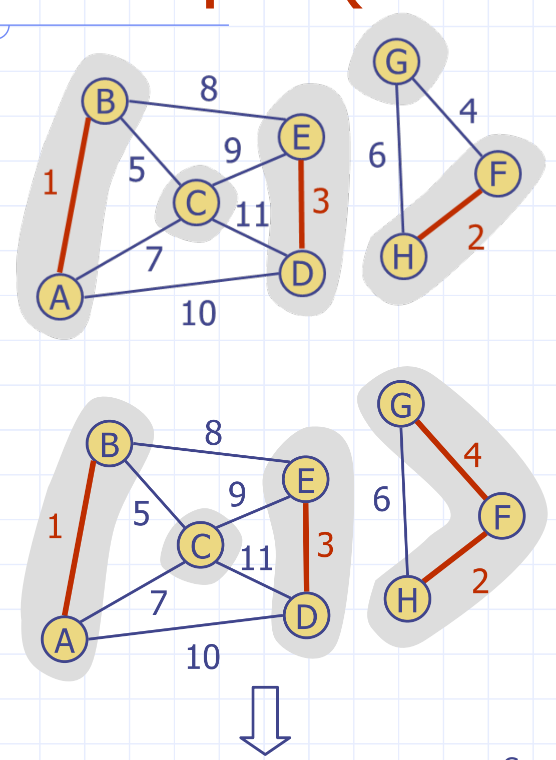
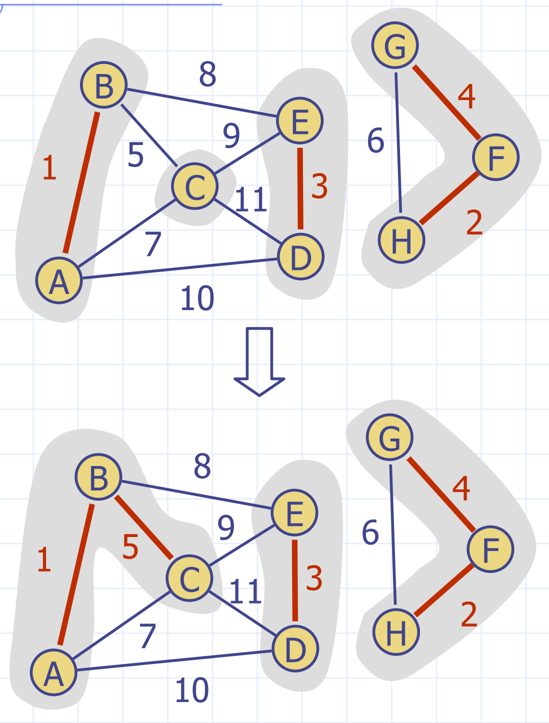
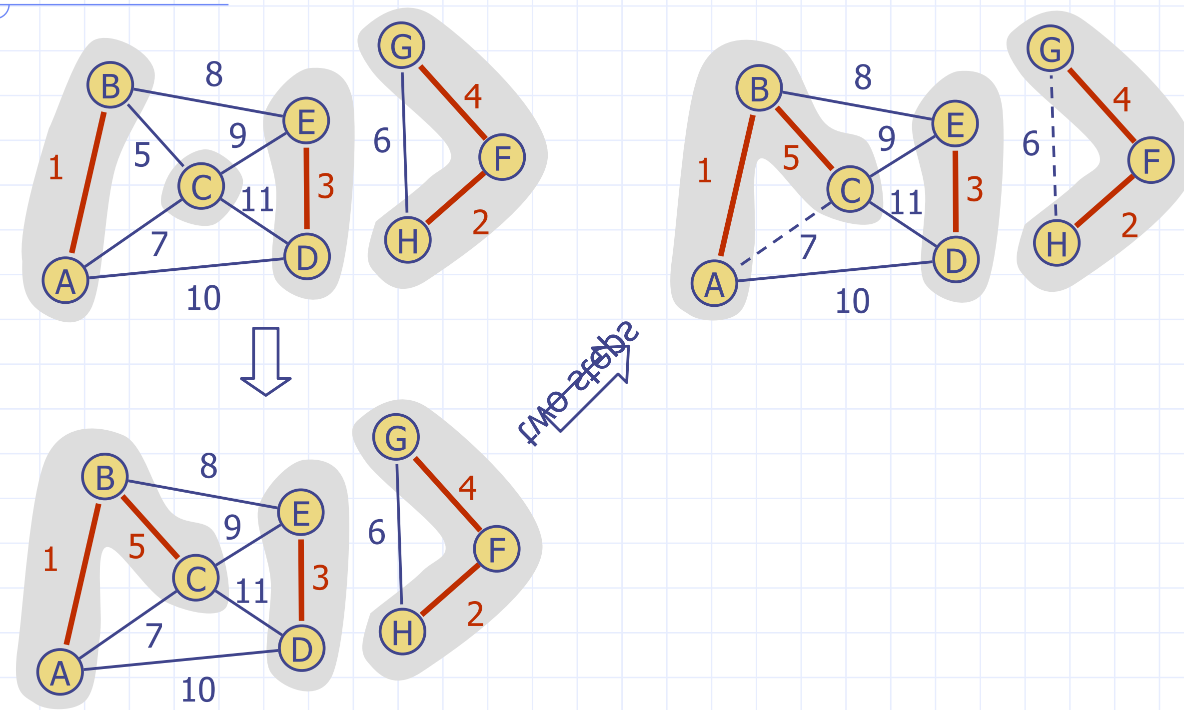


Example (contd.)



Example (contd.)



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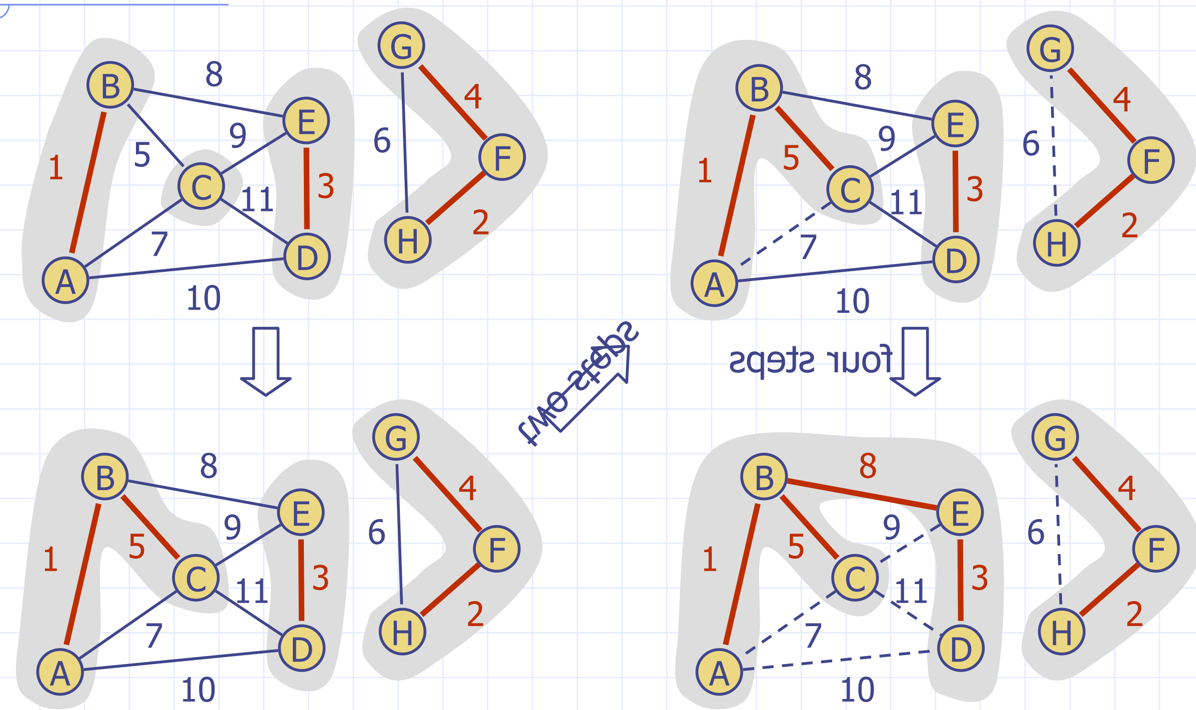

Data Structure for Kruskal's Algorithm

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- The algorithm maintains a forest of trees

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- A priority queue extracts the edges by increasing weight

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- An edge is accepted it if connects distinct trees

Data Structure for Kruskal's Algorithm

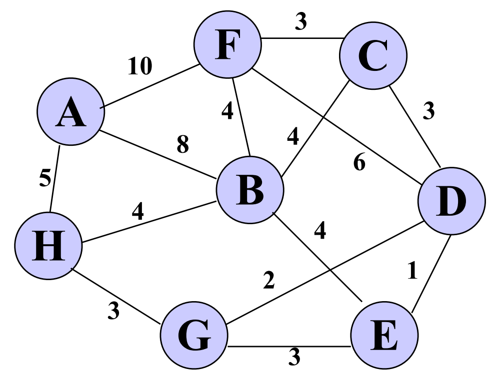
- The algorithm maintains a forest of trees
- A priority queue extracts the edges by increasing weight
- An edge is accepted it if connects distinct trees
- We need a data structure that maintains a partition, i.e., a collection of disjoint sets, with operations:
  - makeSet(u): create a set consisting of u
  - find(u): return the set storing u
  - union(A, B): replace sets A and B with their union

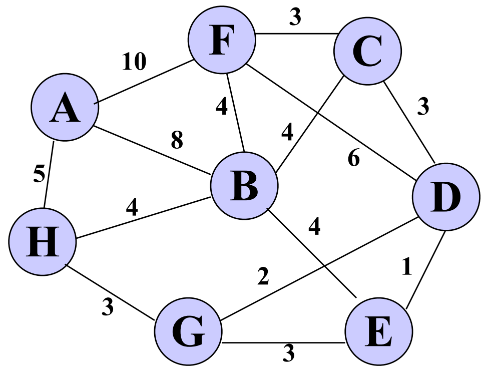
Kruskal’s Algorithm

- Two steps:
  - Sort edges by increasing edge weight
  - Select the first $|V| - 1$ edges that do not generate a cycle

Walk-Through

Consider an undirected, weight graph

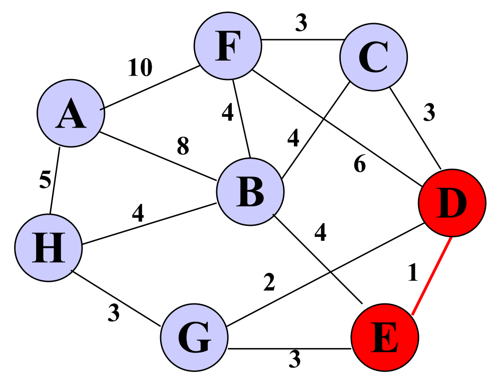




Sort the edges by increasing edge weight

edge	d_v	
(D,E)	1	
(D,G)	2	
(E,G)	3	
(C,D)	3	
(G,H)	3	
(C,F)	3	
(B,C)	4	

edge	d_v	
(B,E)	4	
(B,F)	4	
(B,H)	4	
(A,H)	5	
(D,F)	6	
(A,B)	8	
(A,F)	10	

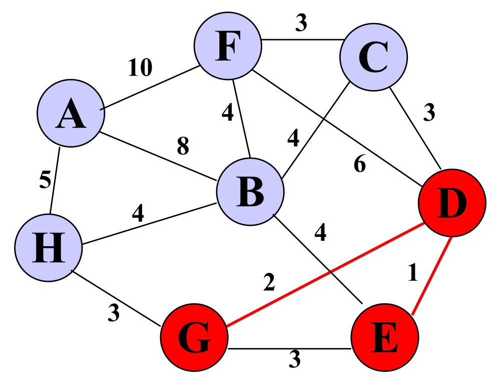


Select first |V|–1 edges which do not generate a cycle

edge	d_v	
(D,E)	1	√
(D,G)	2	
(E,G)	3	
(C,D)	3	
(G,H)	3	
(C,F)	3	
(B,C)	4	

edge	d_v	
(B,E)	4	
(B,F)	4	
(B,H)	4	
(A,H)	5	
(D,F)	6	
(A,B)	8	
(A,F)	10	

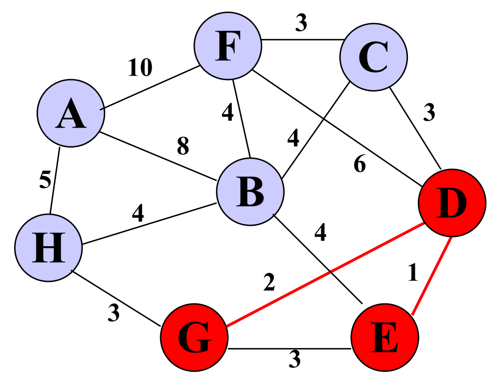
Select first |V|–1 edges which do not generate a cycle



edge	d_v	
(D,E)	1	√
(D,G)	2	√
(E,G)	3	
(C,D)	3	
(G,H)	3	
(C,F)	3	
(B,C)	4	

edge	d_v	
(B,E)	4	
(B,F)	4	
(B,H)	4	
(A,H)	5	
(D,F)	6	
(A,B)	8	
(A,F)	10	

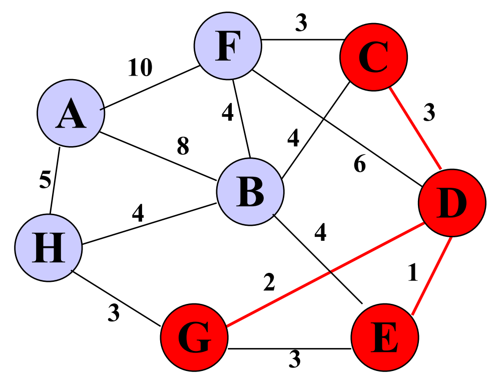
Select first |V|–1 edges which do not generate a cycle



edge	d_v	
(D,E)	1	√
(D,G)	2	√
(E,G)	3	χ
(C,D)	3	
(G,H)	3	
(C,F)	3	
(B,C)	4	

edge	d_v	
(B,E)	4	
(B,F)	4	
(B,H)	4	
(A,H)	5	
(D,F)	6	
(A,B)	8	
(A,F)	10	

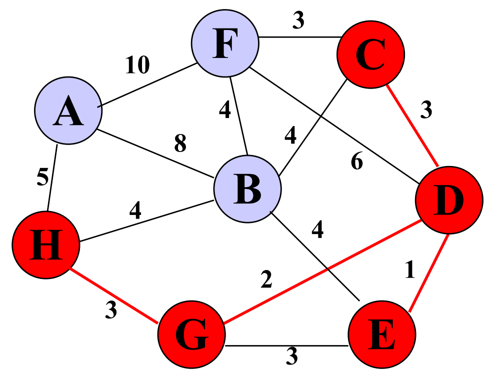
Accepting edge (E,G) would create a cycle



Select first |V|–1 edges which do not generate a cycle

edge	d_v	
(D,E)	1	√
(D,G)	2	√
(E,G)	3	χ
(C,D)	3	√
(G,H)	3	
(C,F)	3	
(B,C)	4	

edge	d_v	
(B,E)	4	
(B,F)	4	
(B,H)	4	
(A,H)	5	
(D,F)	6	
(A,B)	8	
(A,F)	10	

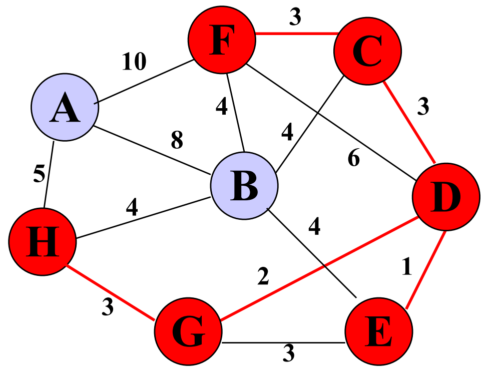


Select first |V|–1 edges which do not generate a cycle

edge	d_v	
(D,E)	1	√
(D,G)	2	√
(E,G)	3	χ
(C,D)	3	√
(G,H)	3	√
(C,F)	3	
(B,C)	4	

edge	d_v	
(B,E)	4	
(B,F)	4	
(B,H)	4	
(A,H)	5	
(D,F)	6	
(A,B)	8	
(A,F)	10	

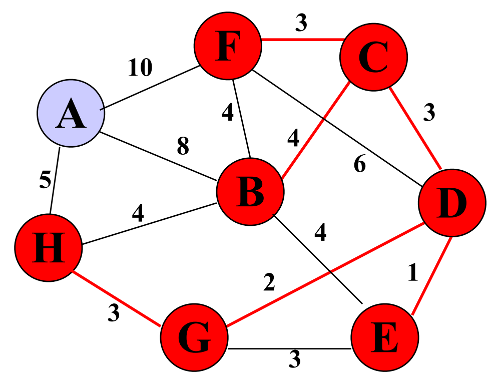
Select first |V|–1 edges which do not generate a cycle



edge	d_v	
(D,E)	1	√
(D,G)	2	√
(E,G)	3	χ
(C,D)	3	√
(G,H)	3	√
(C,F)	3	√
(B,C)	4	

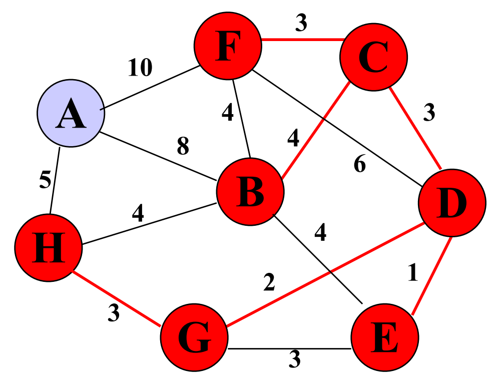
edge	d_v	
(B,E)	4	
(B,F)	4	
(B,H)	4	
(A,H)	5	
(D,F)	6	
(A,B)	8	
(A,F)	10	

Select first |V|–1 edges which do not generate a cycle



edge	d_v	
(D,E)	1	√
(D,G)	2	√
(E,G)	3	χ
(C,D)	3	√
(G,H)	3	√
(C,F)	3	√
(B,C)	4	√

edge	d_v	
(B,E)	4	
(B,F)	4	
(B,H)	4	
(A,H)	5	
(D,F)	6	
(A,B)	8	
(A,F)	10	



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(D,E)	1	√
(D,G)	2	√
(E,G)	3	χ
(C,D)	3	√
(G,H)	3	√
(C,F)	3	√
(B,C)	4	√

edge	d_v	
(B,E)	4	χ
(B,F)	4	
(B,H)	4	
(A,H)	5	
(D,F)	6	
(A,B)	8	
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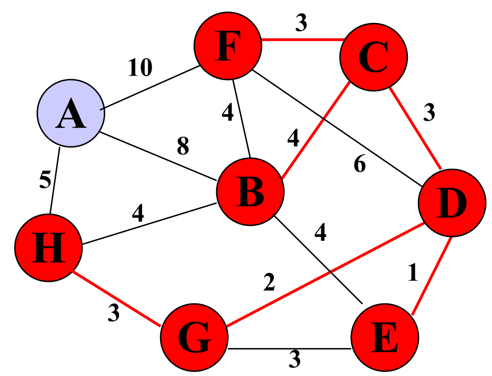
Select first |V|–1 edges which do not generate a cycle



edge	d_v	
(D,E)	1	√
(D,G)	2	√
(E,G)	3	χ
(C,D)	3	√
(G,H)	3	√
(C,F)	3	√
(B,C)	4	√

edge	d_v	
(B,E)	4	χ
(B,F)	4	χ
(B,H)	4	
(A,H)	5	
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(A,B)	8	
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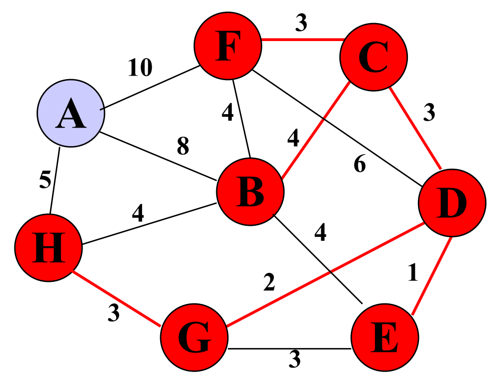
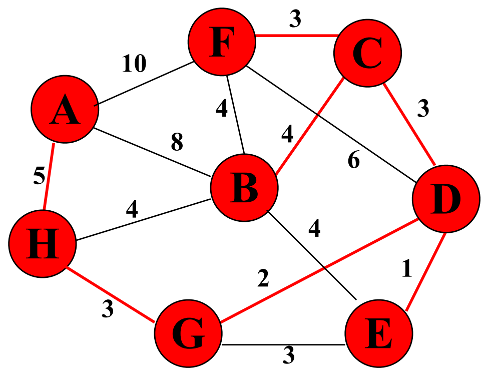
Select first |V|–1 edges which do not generate a cycle



edge	d_v	
(D,E)	1	√
(D,G)	2	√
(E,G)	3	χ
(C,D)	3	√
(G,H)	3	√
(C,F)	3	√
(B,C)	4	√

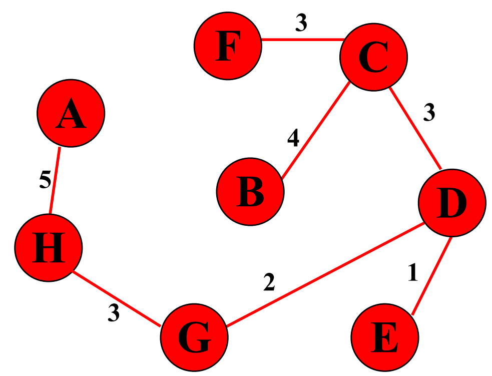
edge	d_v	
(B,E)	4	χ
(B,F)	4	χ
(B,H)	4	χ
(A,H)	5	
(D,F)	6	
(A,B)	8	
(A,F)	10	

Select first |V|–1 edges which do not generate a cycle



edge	d_v	
(D,E)	1	√
(D,G)	2	√
(E,G)	3	χ
(C,D)	3	√
(G,H)	3	√
(C,F)	3	√
(B,C)	4	√

edge	d_v	
(B,E)	4	χ
(B,F)	4	χ
(B,H)	4	χ
(A,H)	5	√
(D,F)	6	
(A,B)	8	
(A,F)	10	



Select first |V|–1 edges which do not generate a cycle

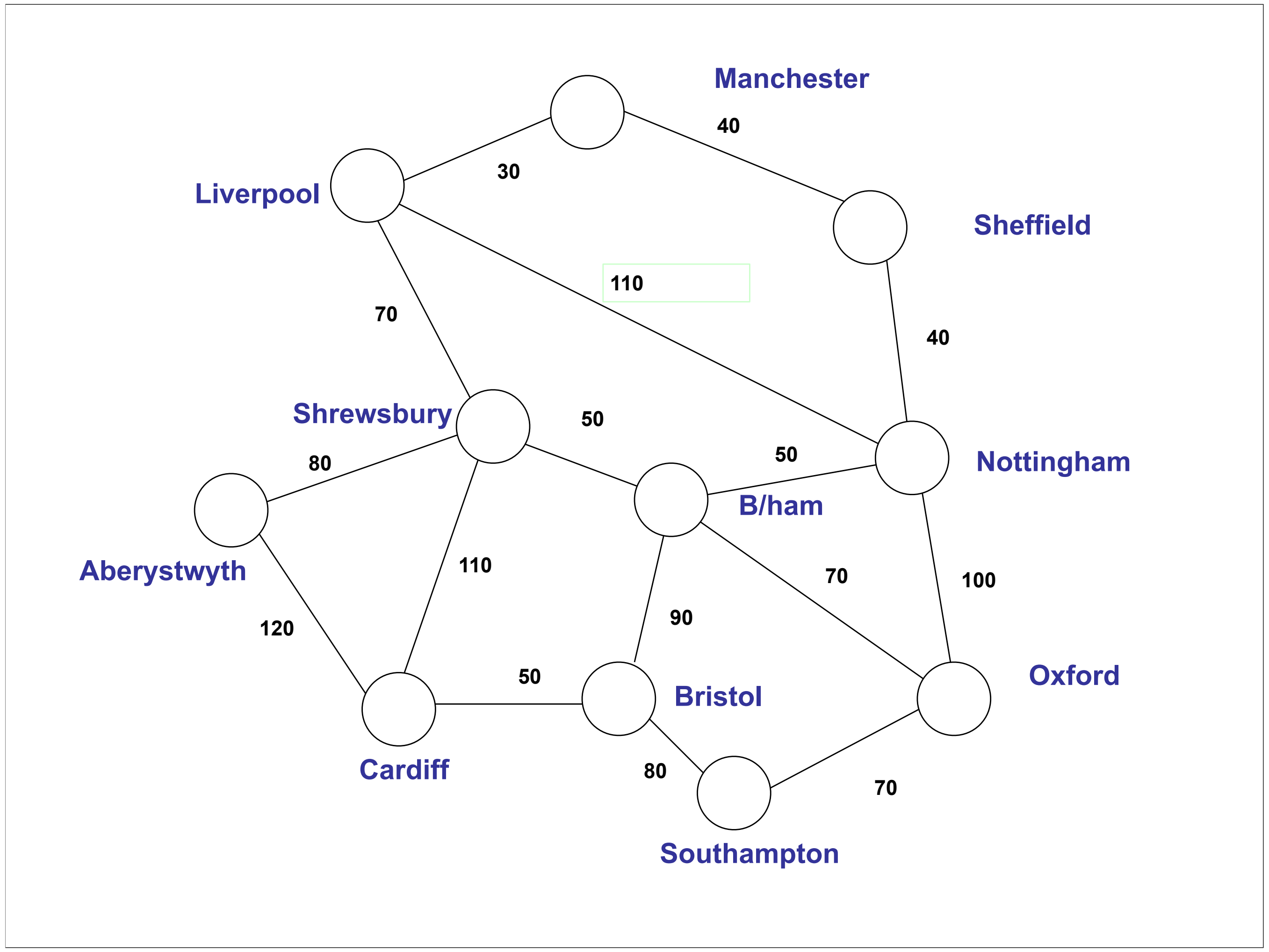
edge	d_v	
(D,E)	1	√
(D,G)	2	√
(E,G)	3	χ
(C,D)	3	√
(G,H)	3	√
(C,F)	3	√
(B,C)	4	√

edge	d_v	
(B,E)	4	χ
(B,F)	4	χ
(B,H)	4	χ
(A,H)	5	√
(D,F)	6	
(A,B)	8	
(A,F)	10	

$\}$ not considered

Done

Total Cost = $\Sigma\, d_v = 21$



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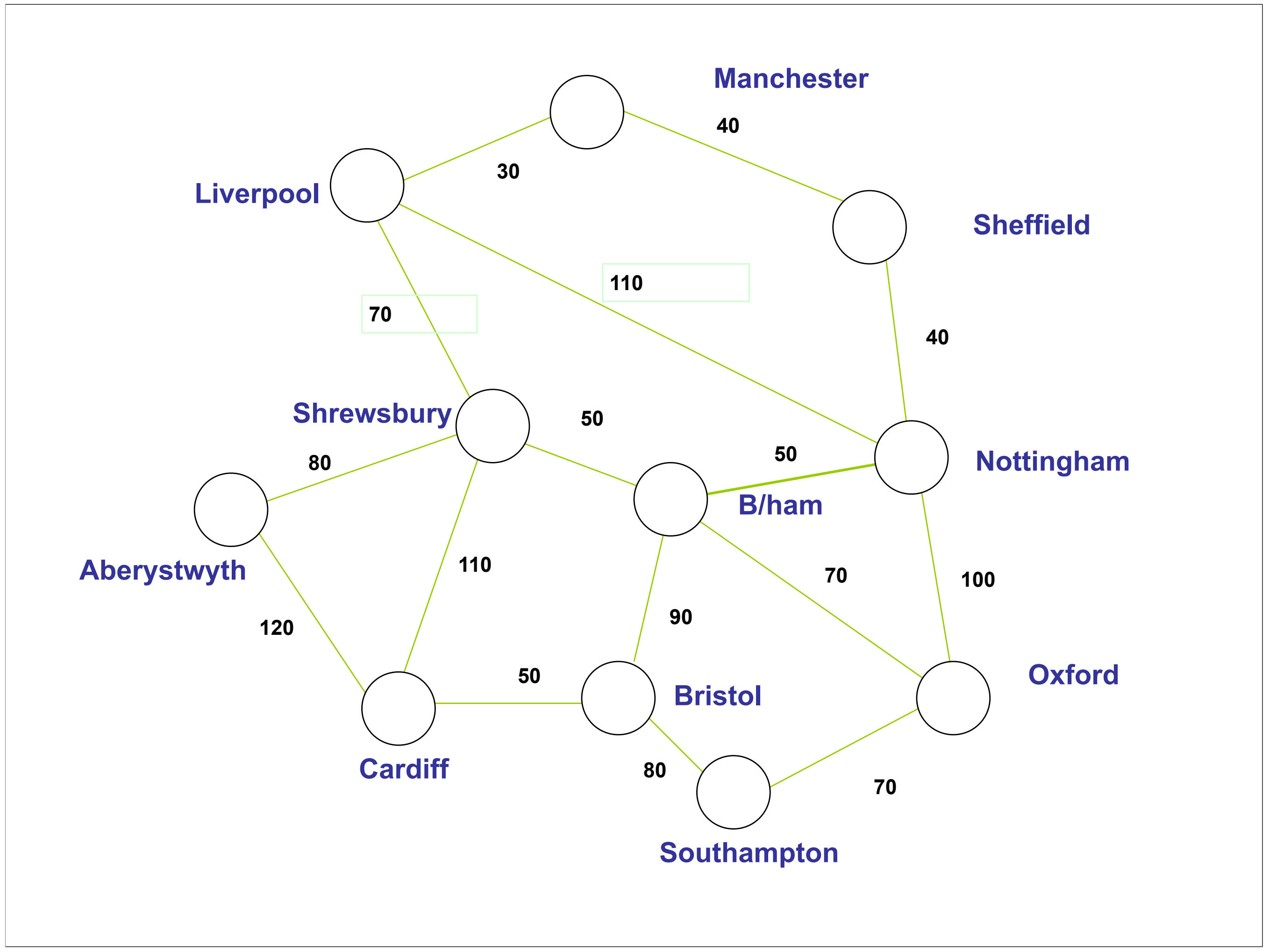
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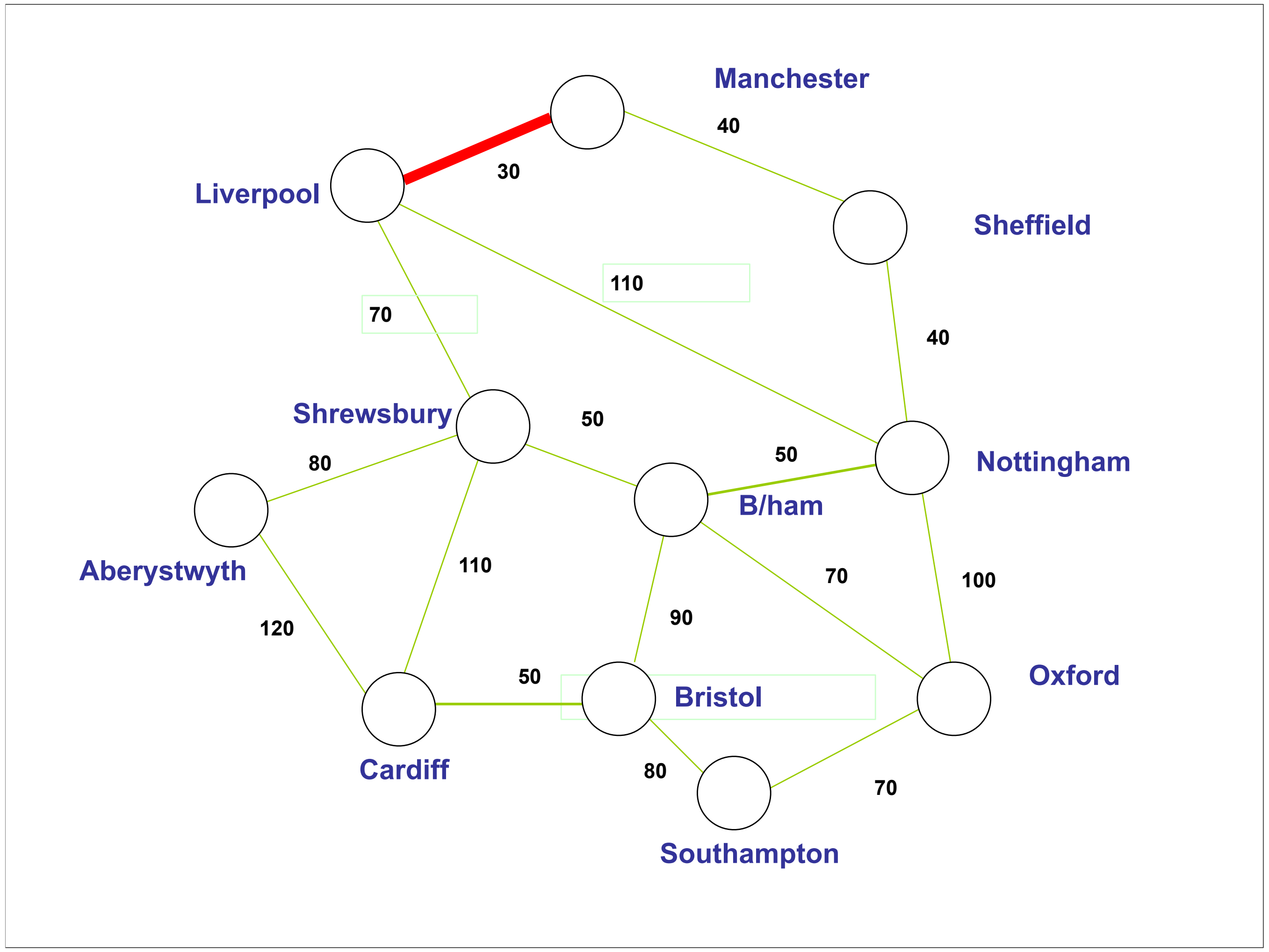

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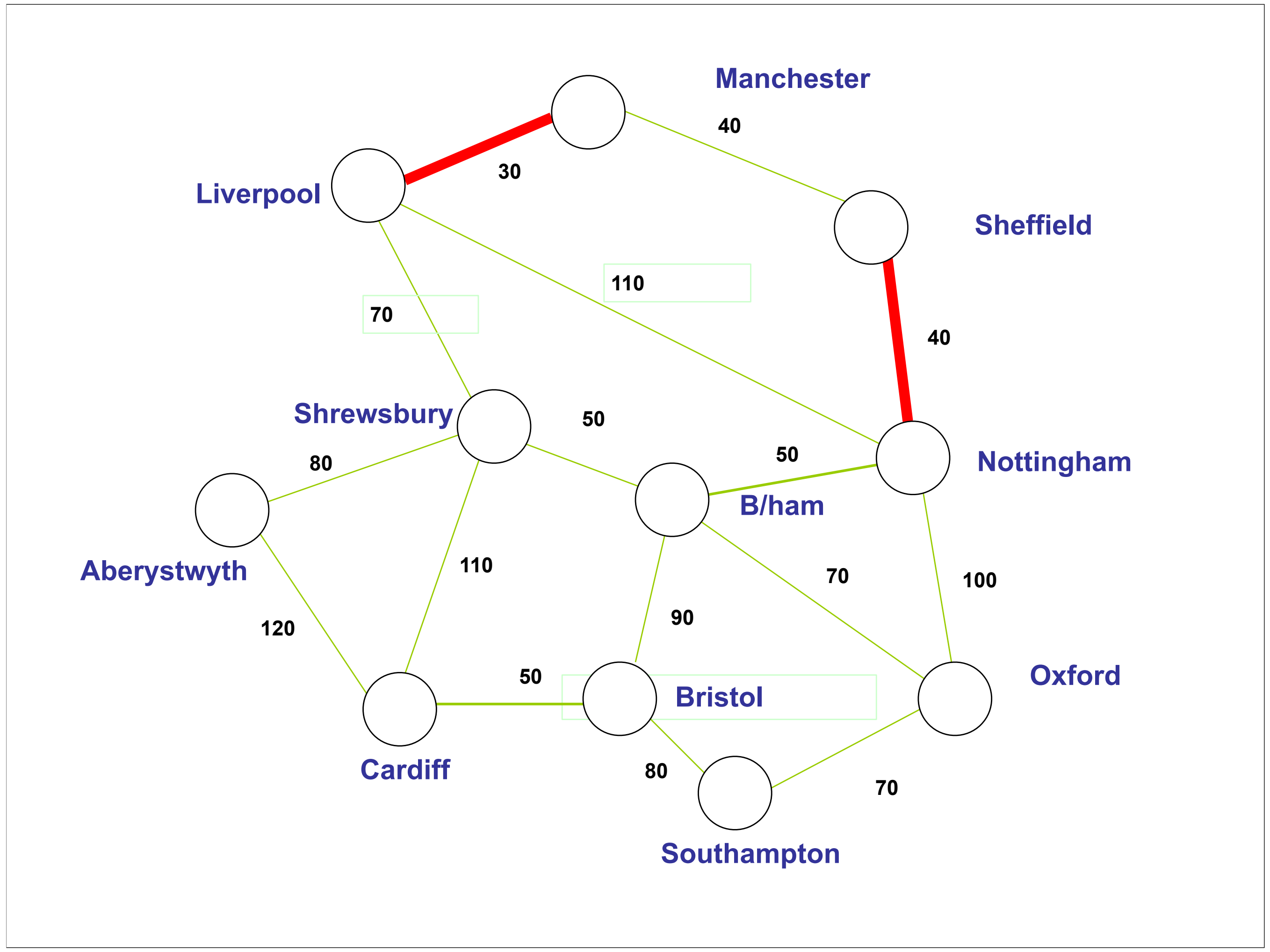
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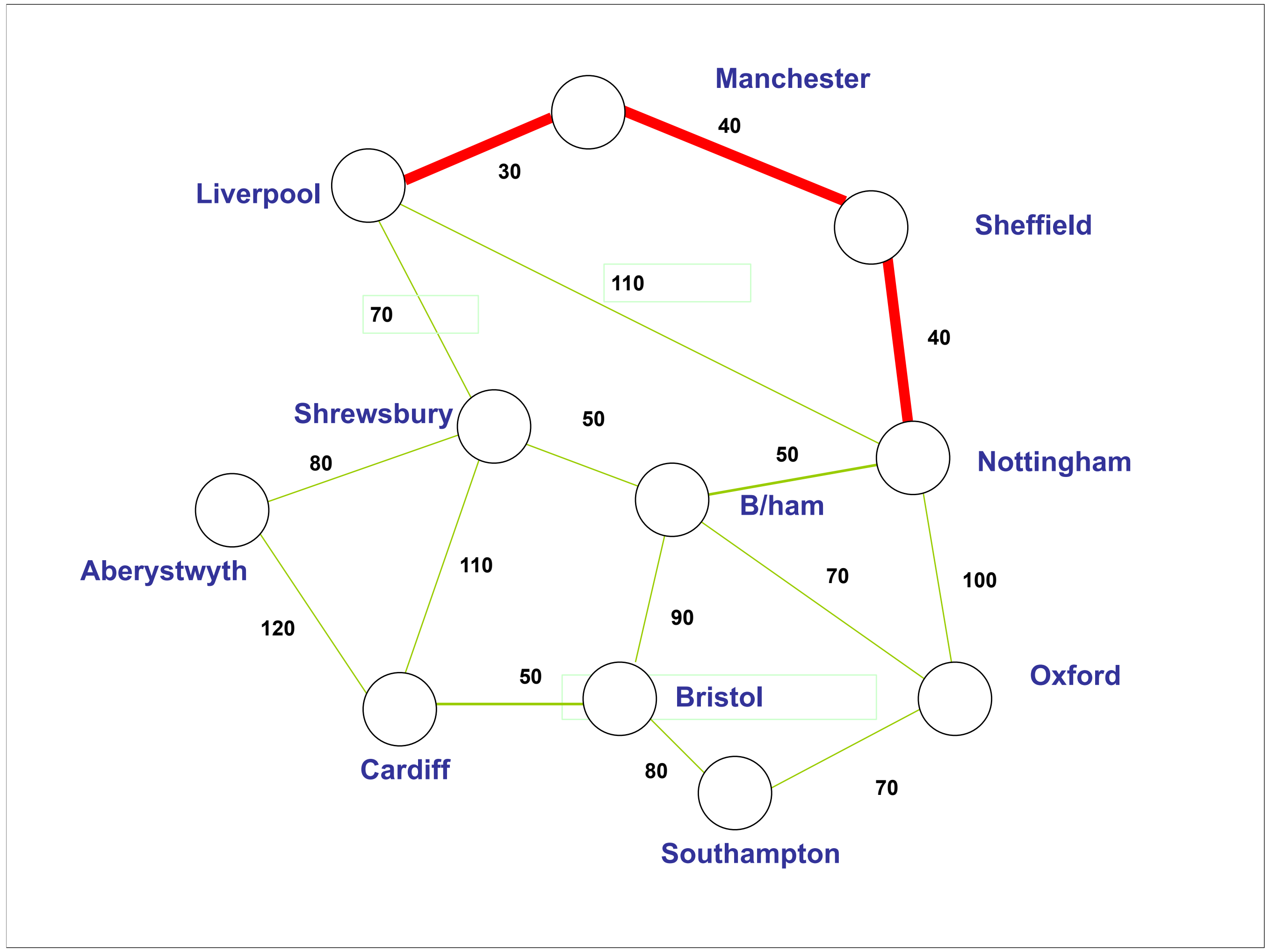
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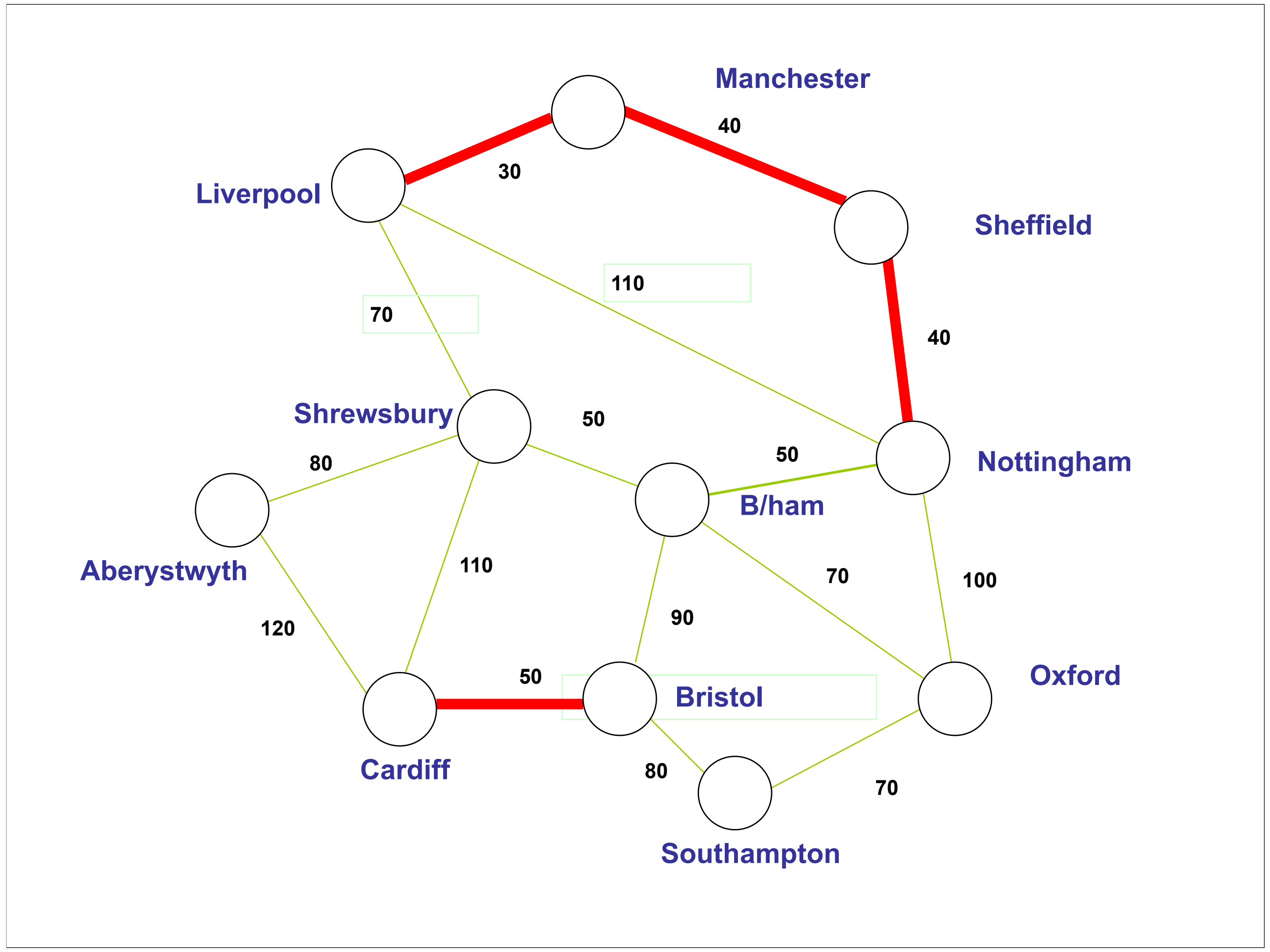
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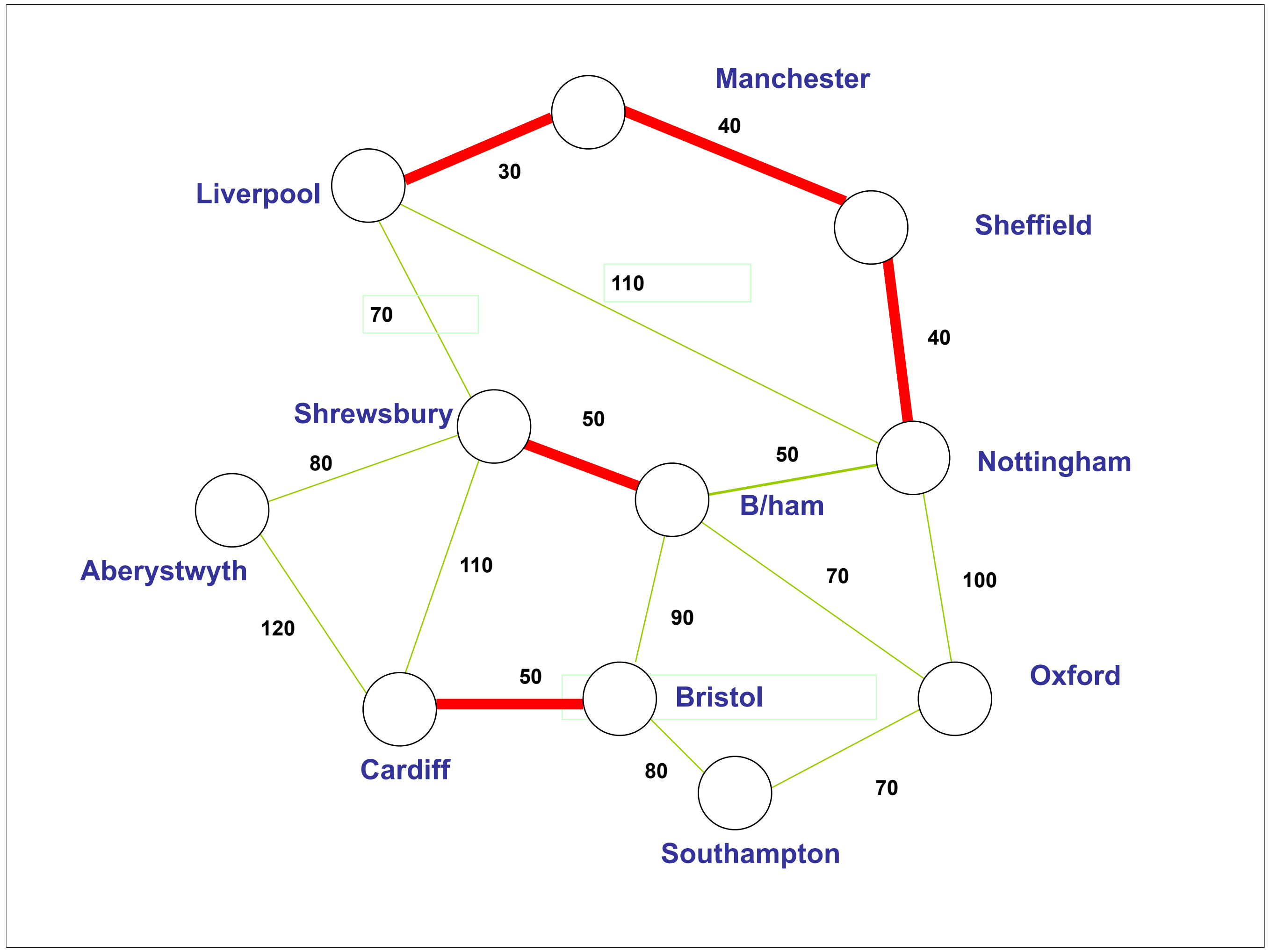
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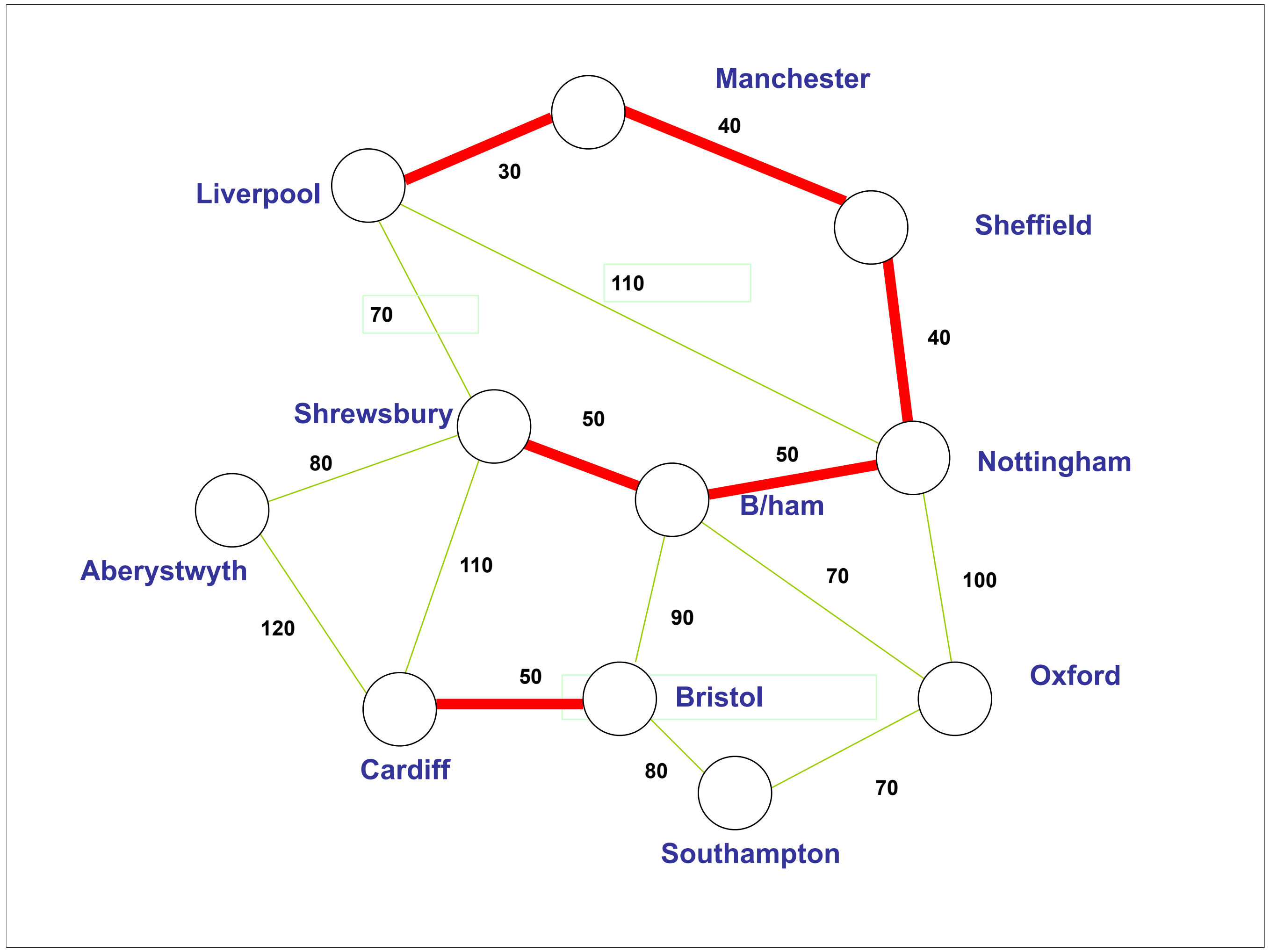

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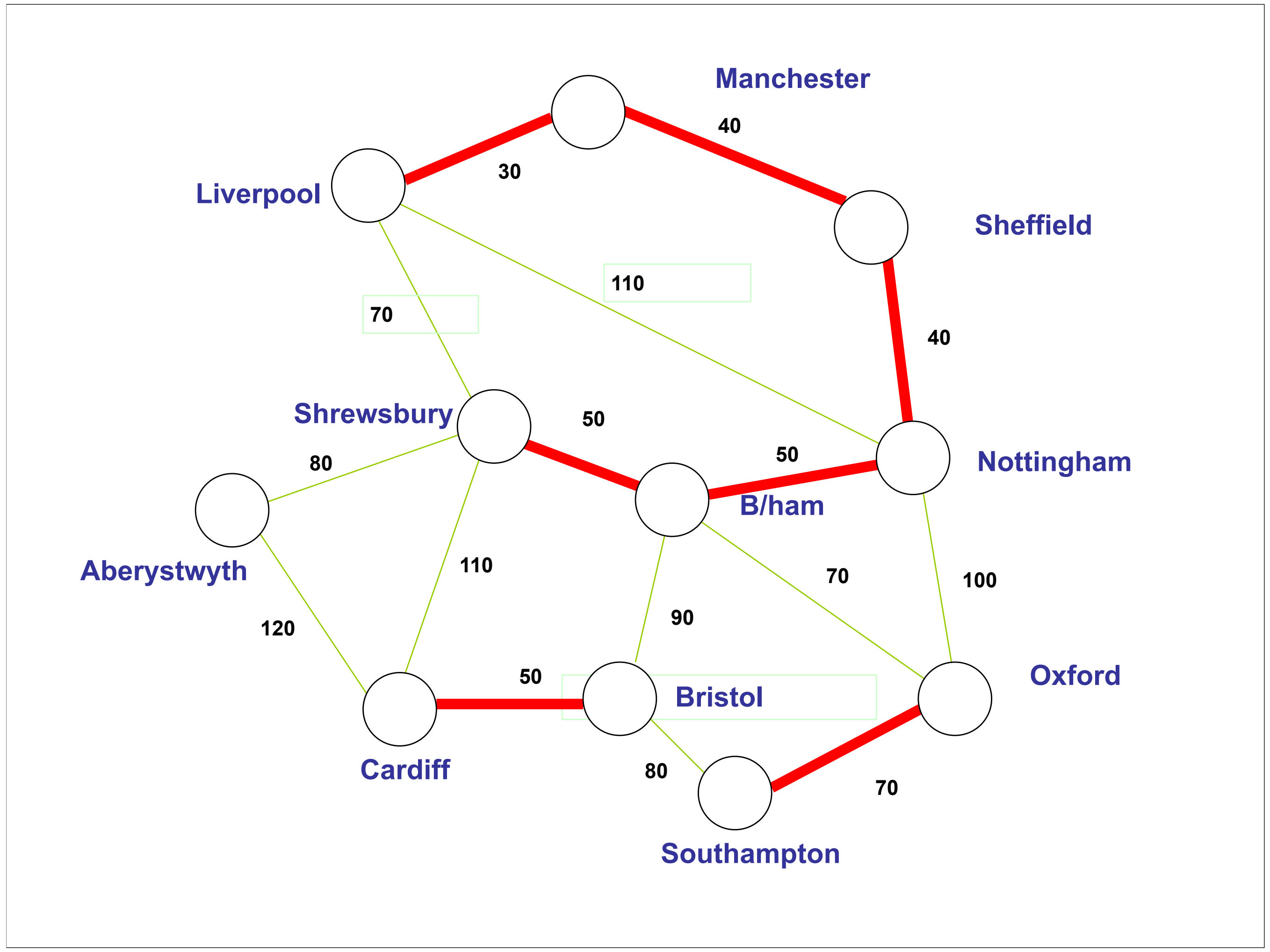
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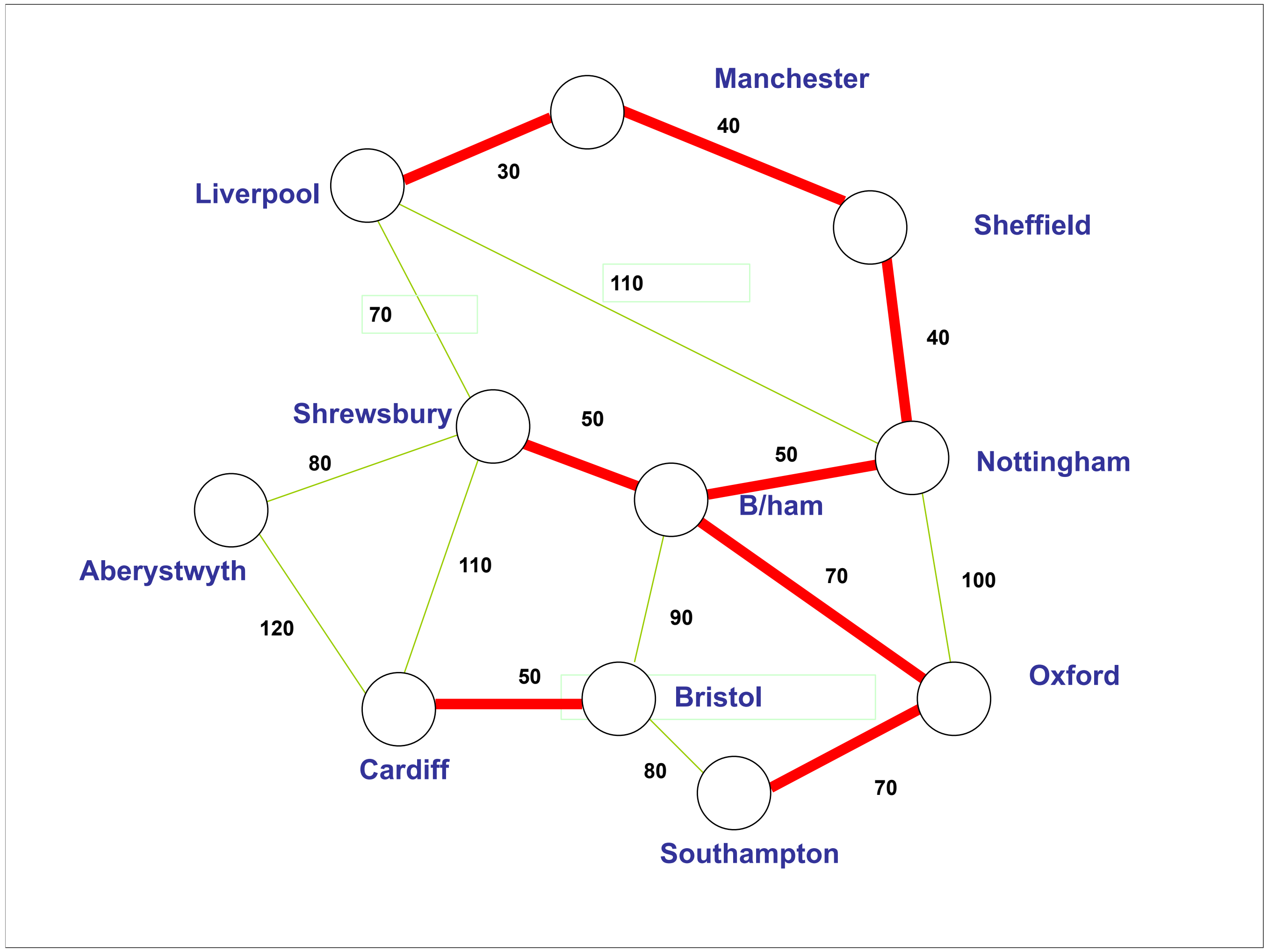

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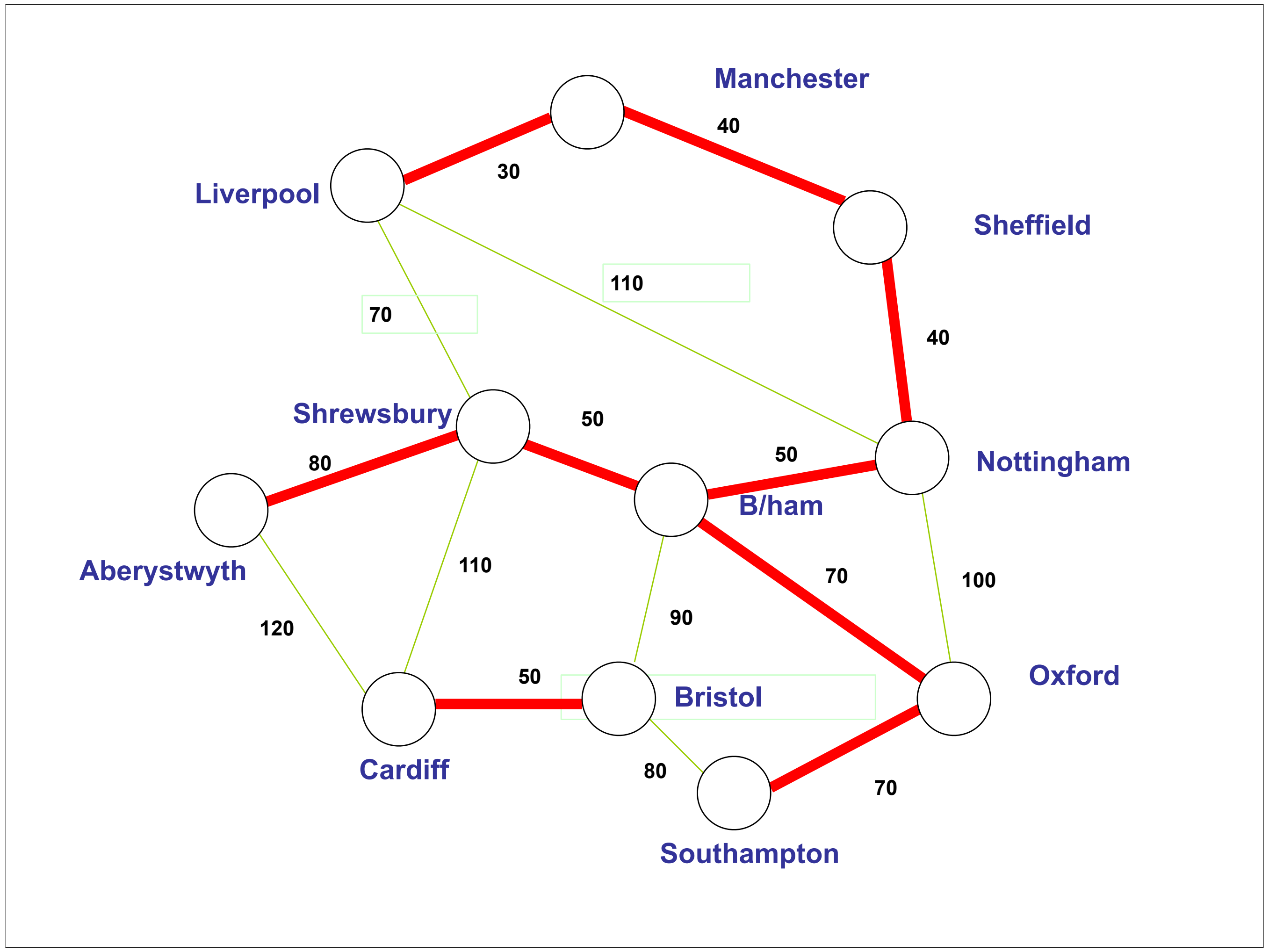
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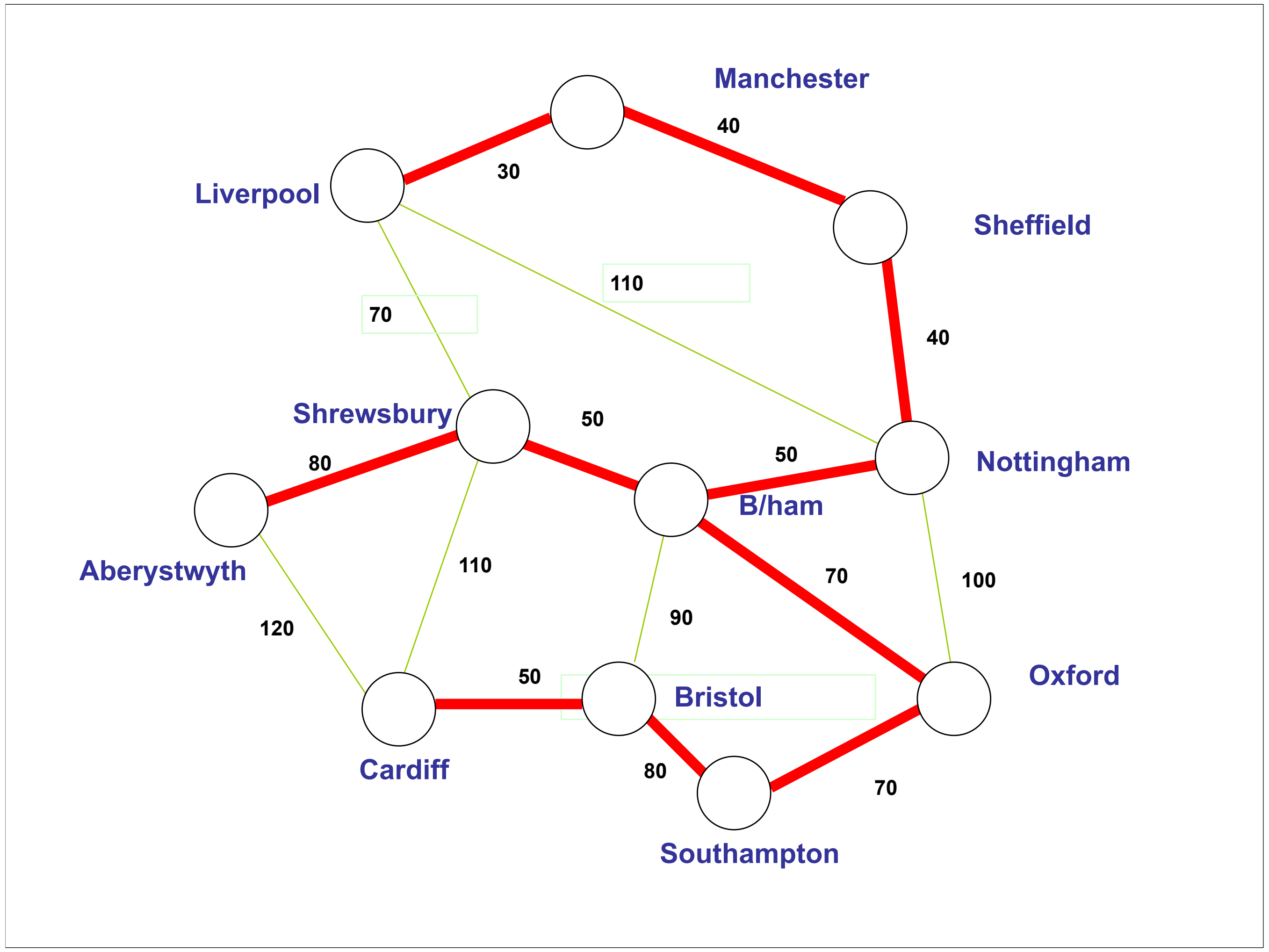

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Prim's Algorithm

```
MST-Prim(G, w, r)
    Q = V[G];
    for each u ∈ Q
        key[u] = ∞;
    key[r] = 0;
    p[r] = NULL;
    while (Q not empty)
        u = ExtractMin(Q);
        for each v ∈ Adj[u]
            if (v ∈ Q and w(u,v) < key[v])
                p[v] = u;
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Prim's Algorithm

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            if (v ∈ Q and w(u,v) < key[v])
                p[v] = u;
                key[v] = w(u,v);
```
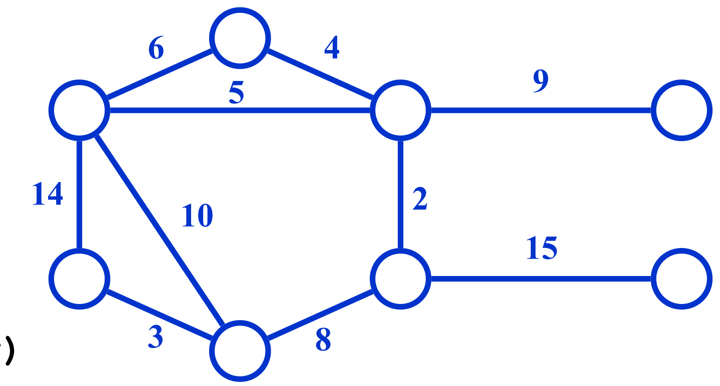
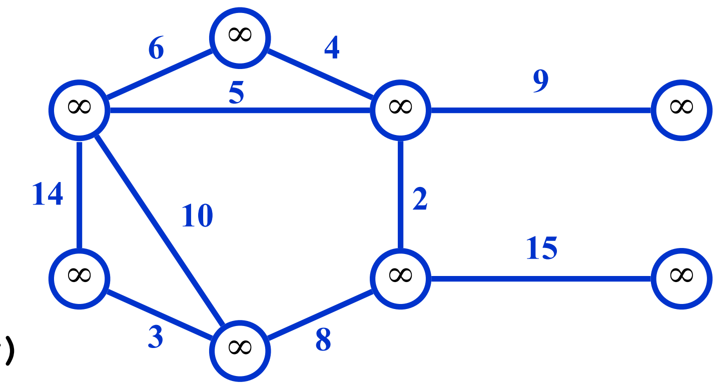



Run on example graph

Prim's Algorithm

```
MST-Prim(G, w, r)
    Q = V[G];
    for each u ∈Q
        key[u] = ∞;
    key[r] = 0;
    p[r] = NULL;
    while (Q not empty)
        u = ExtractMin(Q);
        for each v ∈Adj[u]
            if (v ∈Q and w(u,v) < key[v])
                p[v] = u;
                key[v] = w(u,v);
```



Run on example graph

Prim's Algorithm

```
MST-Prim(G, w, r)
    Q = V[G];
    for each u ∈Q
        key[u] = ∞;
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        for each v ∈Adj[u]
            if (v ∈Q and w(u,v) < key[v])
                p[v] = u;
                key[v] = w(u,v);
```
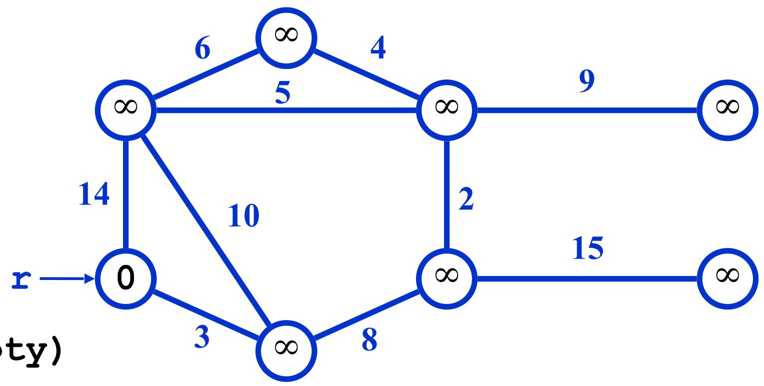
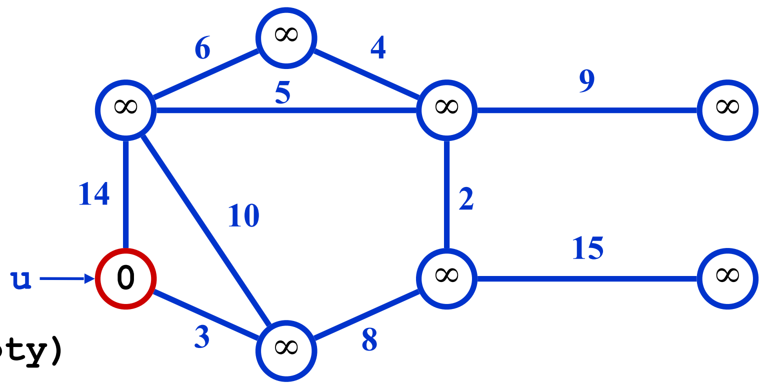



Pick a start vertex r

Prim's Algorithm

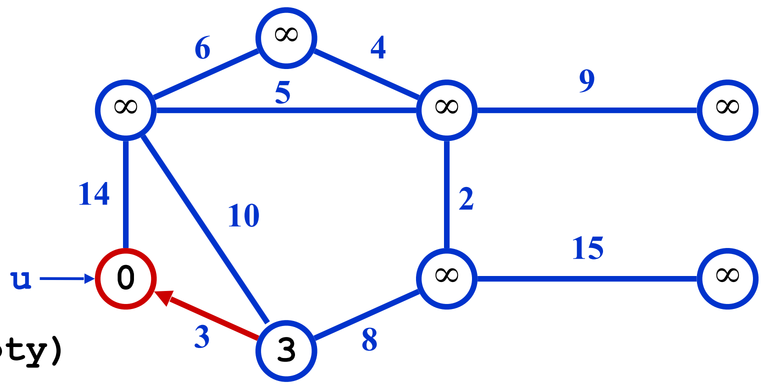
```
MST-Prim(G, w, r)
    Q = V[G];
    for each u ∈ Q
        key[u] = ∞;
    key[r] = 0;
    p[r] = NULL;
    while (Q not empty)
        u = ExtractMin(Q);
        for each v ∈ Adj[u]
            if (v ∈ Q and w(u,v) < key[v])
                p[v] = u;
                key[v] = w(u,v);
```



Red vertices have been removed from Q

Prim's Algorithm

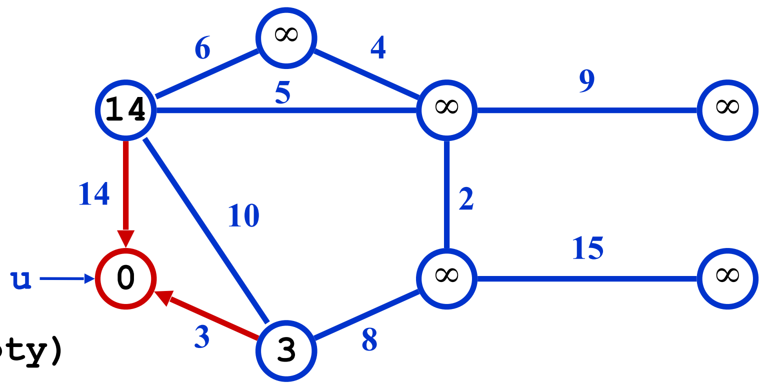
```
MST-Prim(G, w, r)
    Q = V[G];
    for each u ∈ Q
        key[u] = ∞;
    key[r] = 0;
    p[r] = NULL;
    while (Q not empty)
        u = ExtractMin(Q);
        for each v ∈ Adj[u]
            if (v ∈ Q and w(u,v) < key[v])
                p[v] = u;
                key[v] = w(u,v);
```



Red arrows indicate parent pointers

Prim's Algorithm

```
MST-Prim(G, w, r)
    Q = V[G];
    for each u ∈ Q
        key[u] = ∞;
    key[r] = 0;
    p[r] = NULL;
    while (Q not empty)
        u = ExtractMin(Q);
        for each v ∈ Adj[u]
            if (v ∈ Q and w(u,v) < key[v])
                p[v] = u;
                key[v] = w(u,v);
```



Prim's Algorithm

```
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    Q = V[G];
    for each u ∈ Q
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        u = ExtractMin(Q);
        for each v ∈ Adj[u]
            if (v ∈ Q and w(u,v) < key[v])
                p[v] = u;
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```

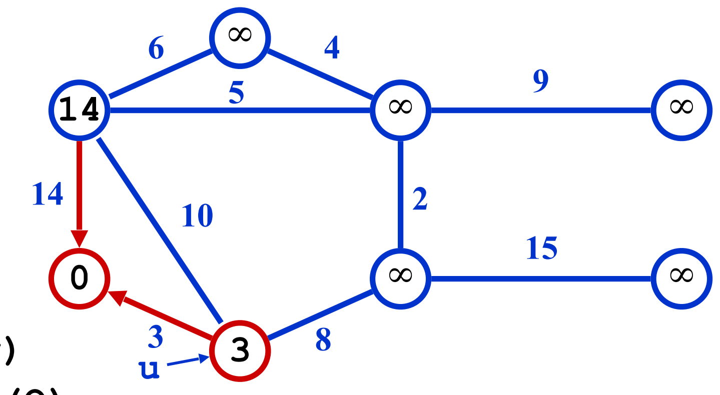

Prim's Algorithm

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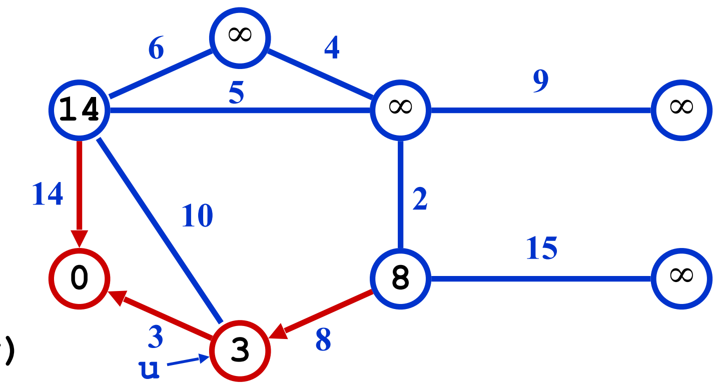
            if (v ∈ Q and w(u,v) < key[v])
                p[v] = u;
                key[v] = w(u,v);
```



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```

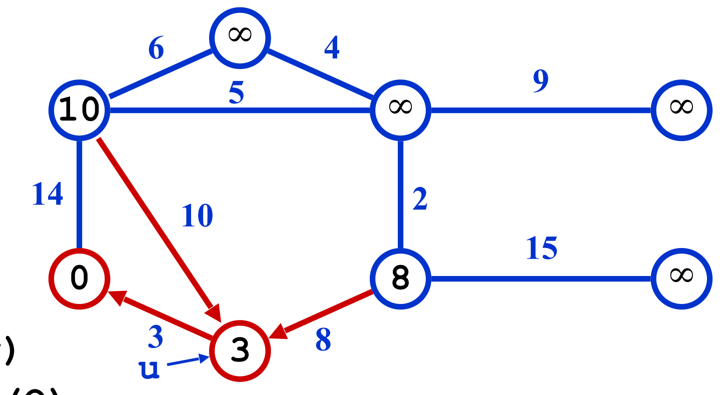

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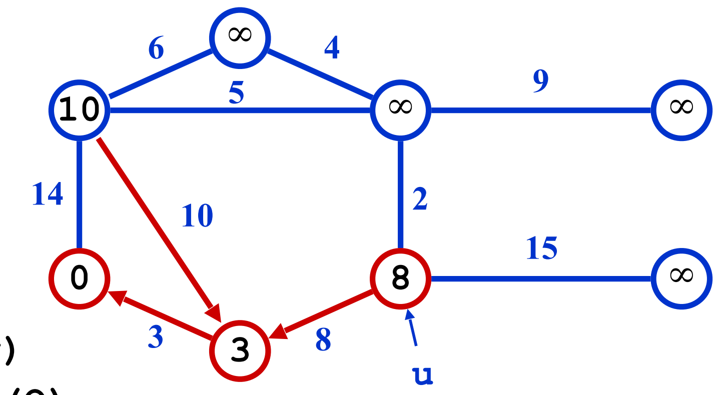

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        for each v ∈Adj[u]
            if (v ∈Q and w(u,v) < key[v])
                p[v] = u;
                key[v] = w(u,v);
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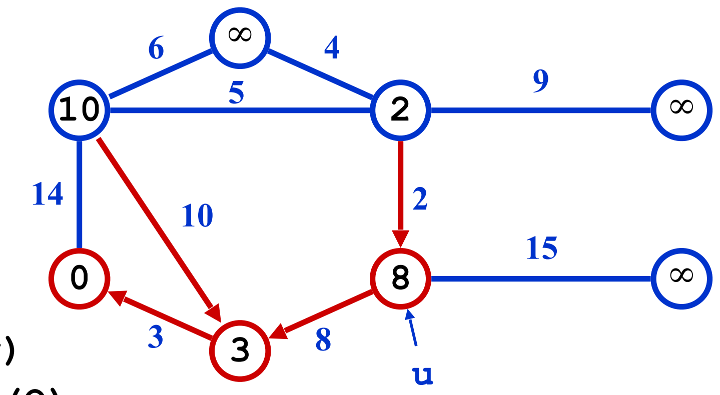

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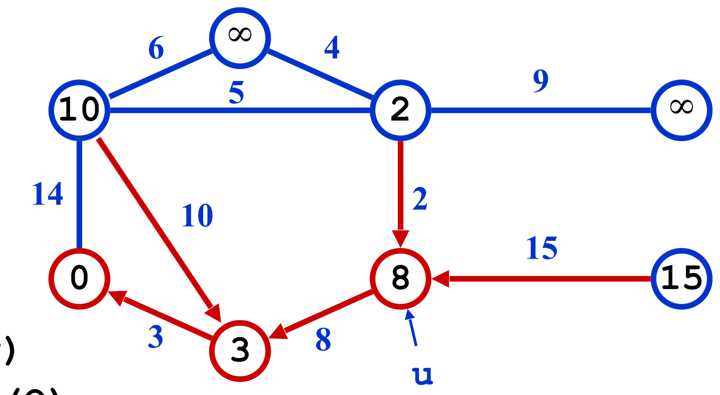

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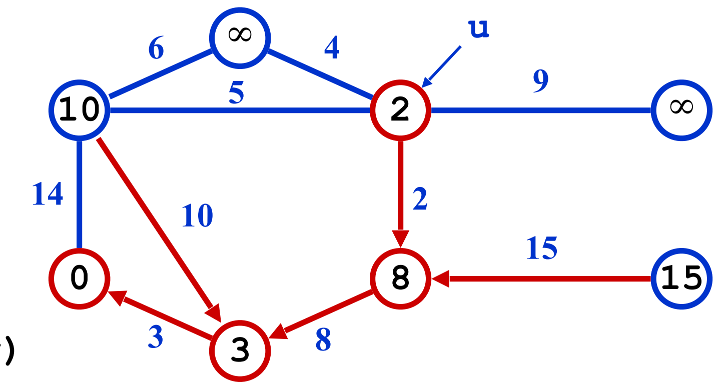

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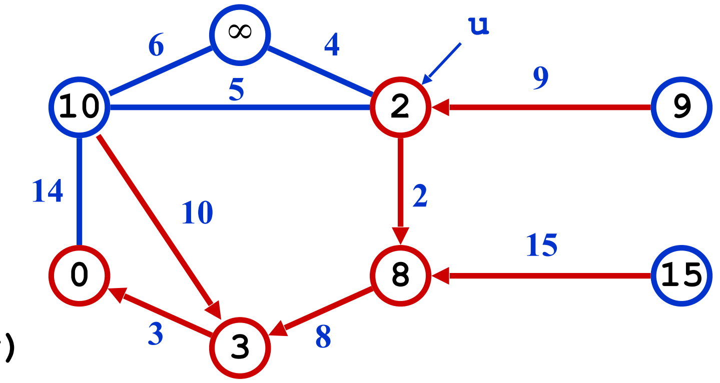

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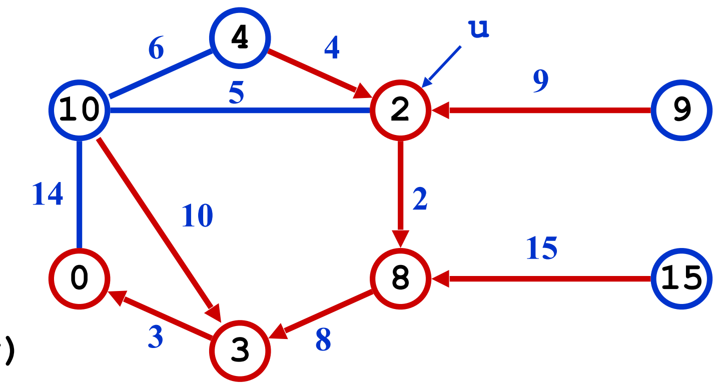

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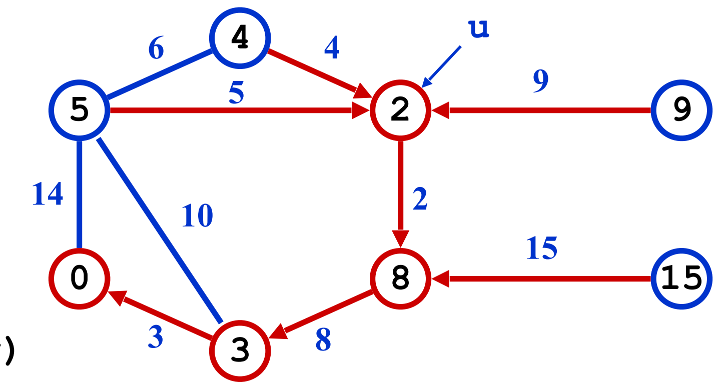

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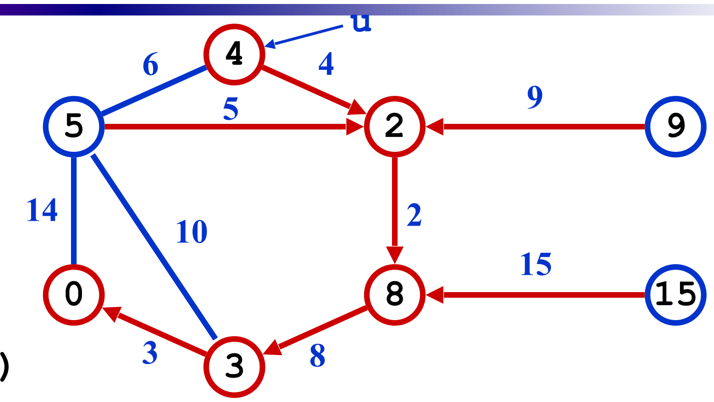

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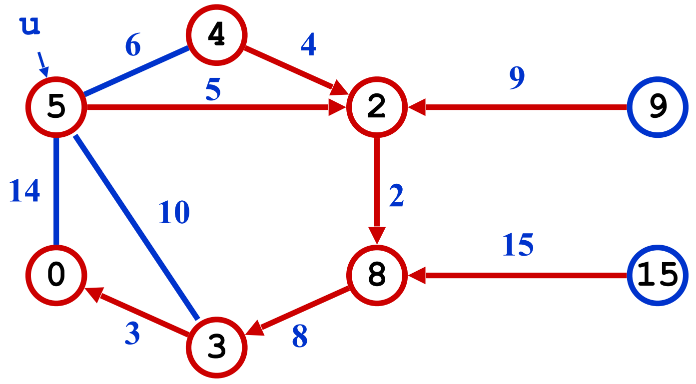

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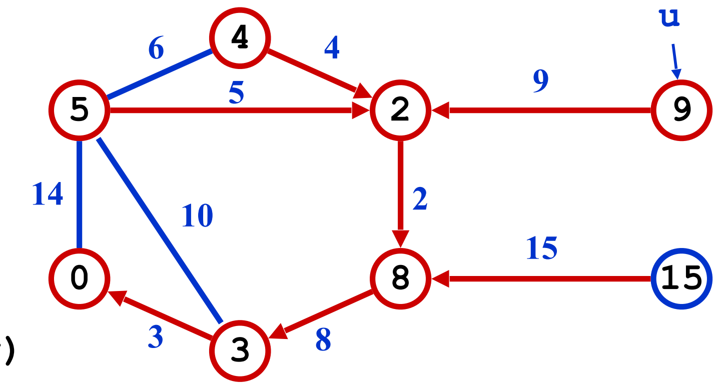

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                p[v] = u;
                key[v] = w(u,v);
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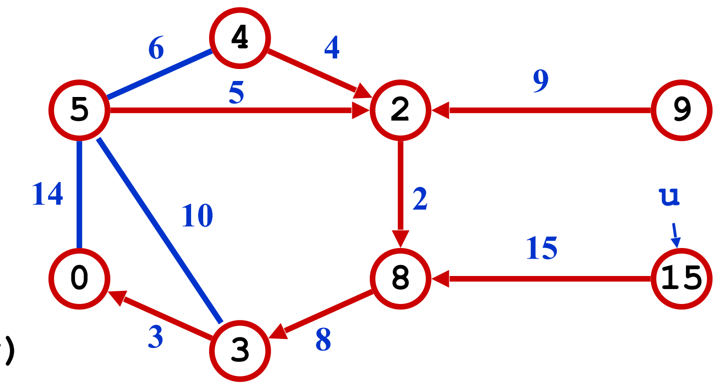
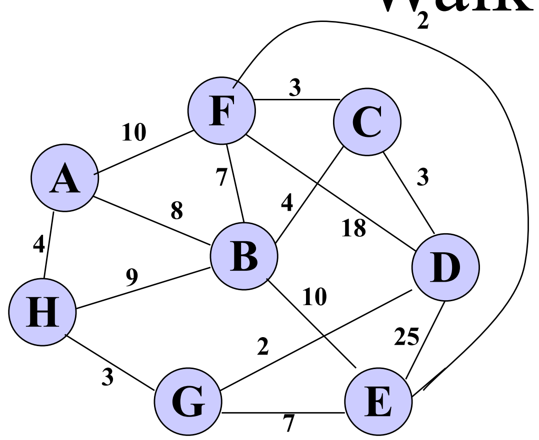
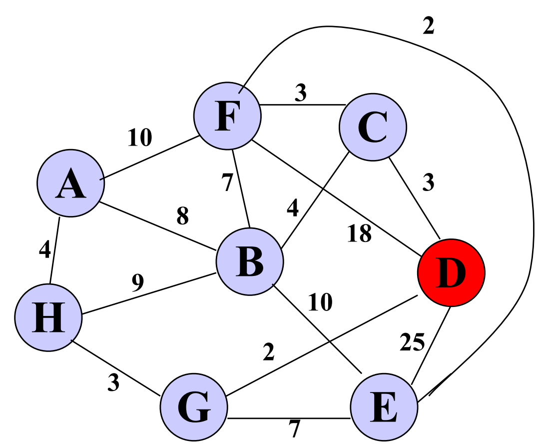


Walk-Through



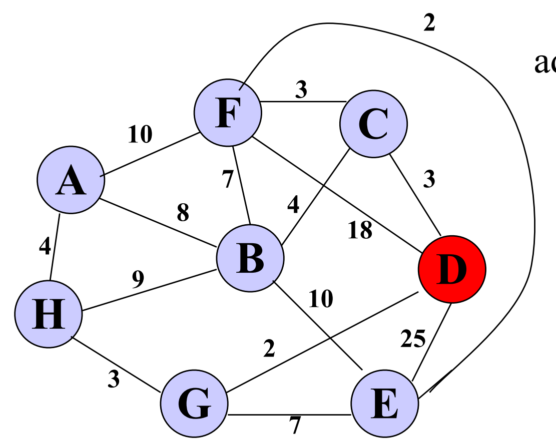
Initialize array

	K	d_v	p_v
A	F	∞	−
B	F	∞	−
C	F	∞	−
D	F	∞	−
E	F	∞	−
F	F	∞	−
G	F	∞	−
H	F	∞	−



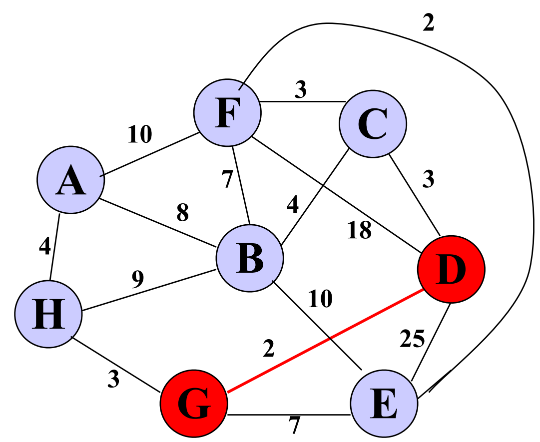
Start with any node, say D

	K	d_v	p_v
A			
B			
C			
D	T	0	−
E			
F			
G			
H			



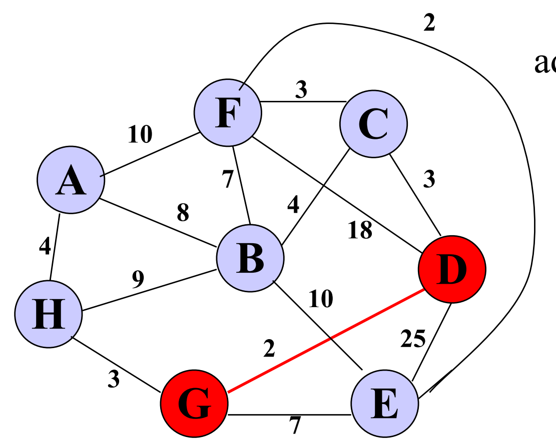
Update distances of
adjacent, unselected nodes

	K	d_v	p_v
A			
B			
C		3	D
D	T	0	–
E		25	D
F		18	D
G		2	D
H			



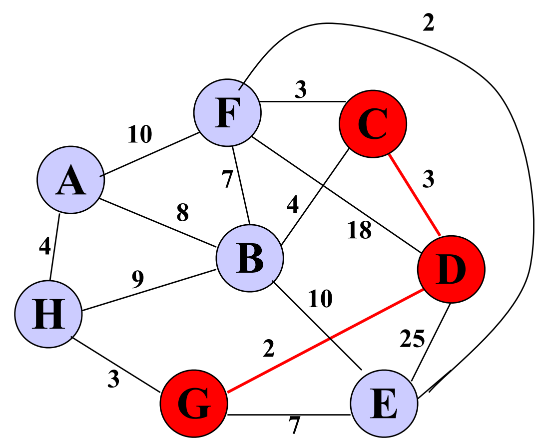
Select node with minimum distance

	K	d_v	p_v
A			
B			
C		3	D
D	T	0	−
E		25	D
F		18	D
G	T	2	D
H			



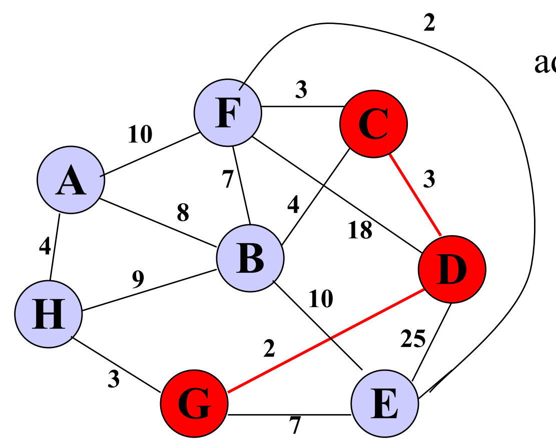
Update distances of adjacent, unselected nodes

	K	d_v	p_v
A			
B			
C		3	D
D	T	0	−
E		7	G
F		18	D
G	T	2	D
H		3	G



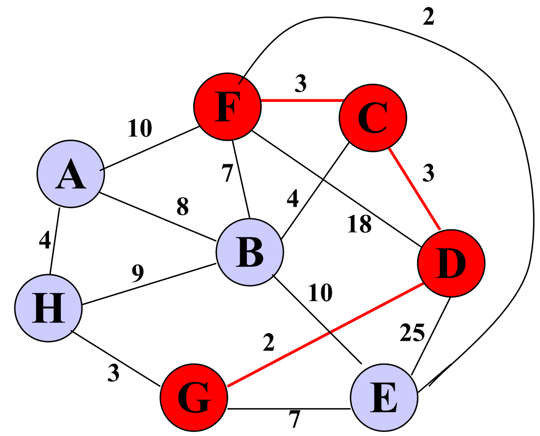
Select node with minimum distance

	K	d_v	p_v
A			
B			
C	T	3	D
D	T	0	−
E		7	G
F		18	D
G	T	2	D
H		3	G



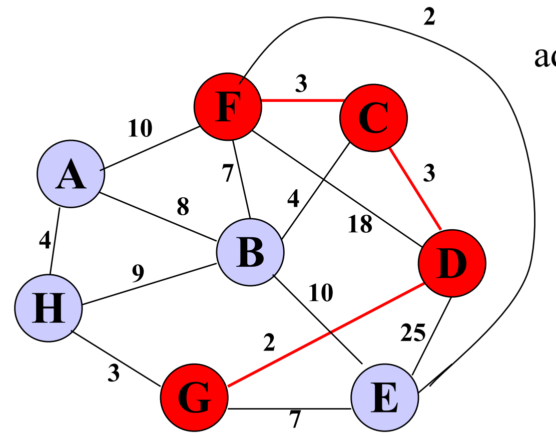
Update distances of adjacent, unselected nodes

	K	d_v	p_v
A			
B		4	C
C	T	3	D
D	T	0	–
E		7	G
F		3	C
G	T	2	D
H		3	G



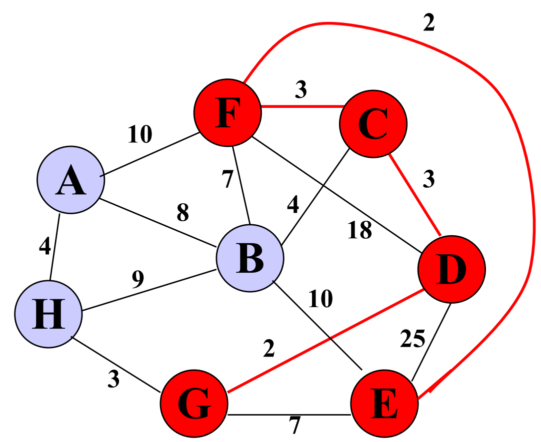
Select node with minimum distance

	K	d_v	p_v
A			
B		4	C
C	T	3	D
D	T	0	−
E		7	G
F	T	3	C
G	T	2	D
H		3	G



Update distances of
adjacent, unselected nodes

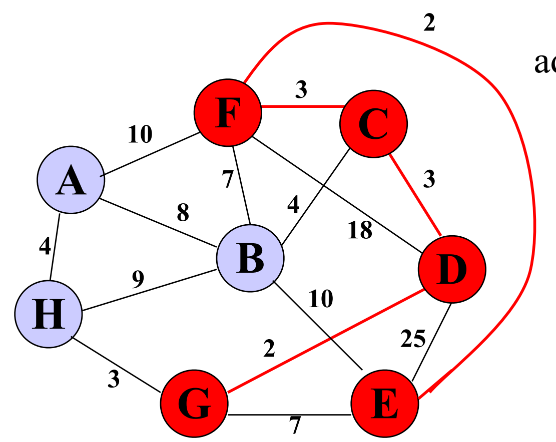
	K	d_v	p_v
A		10	F
B		4	C
C	T	3	D
D	T	0	–
E		2	F
F	T	3	C
G	T	2	D
H		3	G



Select node with minimum distance

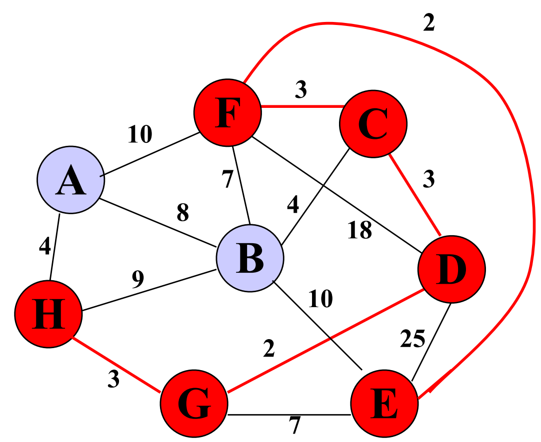
	K	d_v	p_v
A		10	F
B		4	C
C	T	3	D
D	T	0	−
E	T	2	F
F	T	3	C
G	T	2	D
H		3	G

Update distances of adjacent, unselected nodes



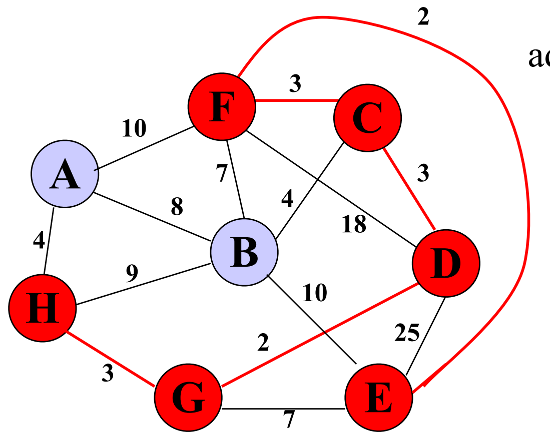
	K	d_v	p_v
A		10	F
B		4	C
C	T	3	D
D	T	0	–
E	T	2	F
F	T	3	C
G	T	2	D
H		3	G

Table entries unchanged



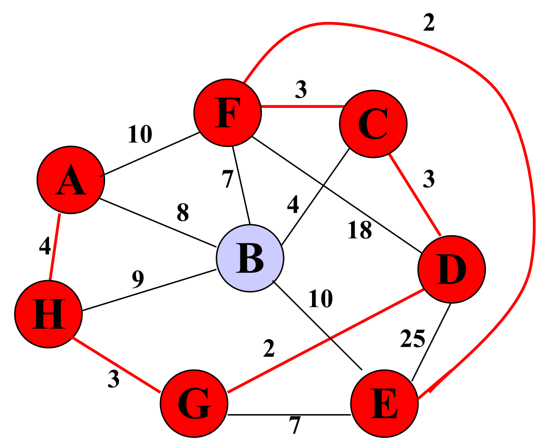
Select node with
minimum distance

	K	d_v	p_v
A		10	F
B		4	C
C	T	3	D
D	T	0	−
E	T	2	F
F	T	3	C
G	T	2	D
H	T	3	G



Update distances of adjacent, unselected nodes

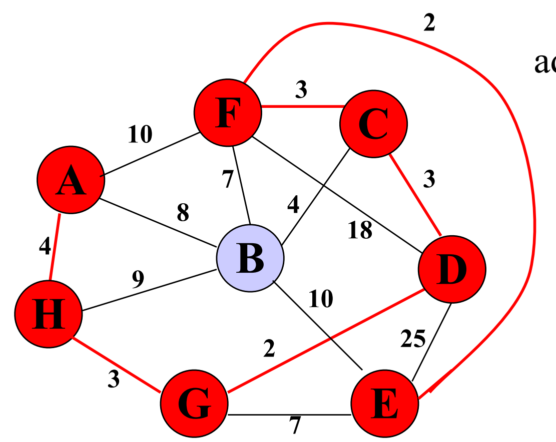
	K	d_v	p_v
A		4	H
B		4	C
C	T	3	D
D	T	0	–
E	T	2	F
F	T	3	C
G	T	2	D
H	T	3	G



Select node with minimum distance

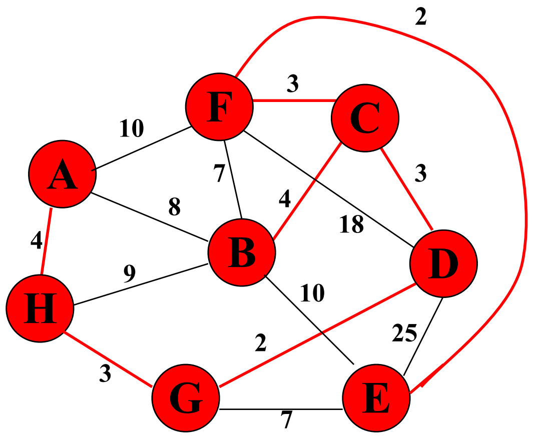
	K	d_v	p_v
A	T	4	H
B		4	C
C	T	3	D
D	T	0	−
E	T	2	F
F	T	3	C
G	T	2	D
H	T	3	G

Update distances of adjacent, unselected nodes



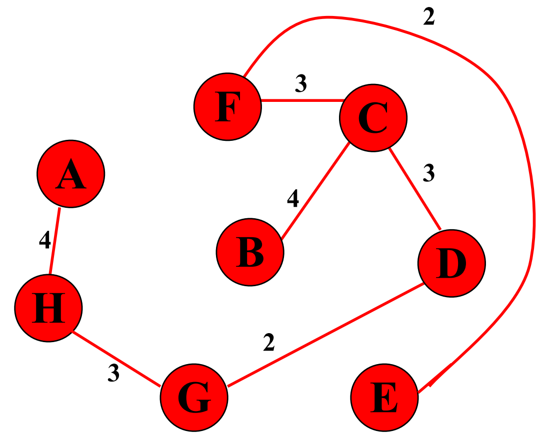
	K	d_v	p_v
A	T	4	H
B		4	C
C	T	3	D
D	T	0	−
E	T	2	F
F	T	3	C
G	T	2	D
H	T	3	G

Table entries unchanged



Select node with minimum distance

	K	d_v	p_v
A	T	4	H
B	T	4	C
C	T	3	D
D	T	0	–
E	T	2	F
F	T	3	C
G	T	2	D
H	T	3	G



Cost of Minimum Spanning Tree = $\Sigma\ d_v = 21$

	K	d_v	p_v
A	T	4	H
B	T	4	C
C	T	3	D
D	T	0	–
E	T	2	F
F	T	3	C
G	T	2	D
H	T	3	G

Done

Complexity of Prim's Algorithm

- Initialize a spanning tree S containing a single vertex, chosen arbitrarily from the graph
- Until n-1 edges have been added
  - find the edge $e = \{x, y\}$ such that
    - x is in S and y is not in S
    - e is the smallest weighted edge left
  - Add e and y to S

O(1)

O(n)

O(m)

O(1)

- $O(1) + O(n(m+1)) = O(nm) = O(n^3)$ in the worst case

A Faster Prim's Algorithm

- To make Prim's Algorithm faster, we need a way to find the edge e faster.
- Can we avoid looking through all edges in each iteration?
  - We can if we sort them first and then make a sorted list of incident edges for each node.
  - In the initialization step this takes $O(m\log m)$ to sort and $O(m)$ to make lists for each node — $O(m\log m) = O(n^2\log n^2)$ in the worst case.
  - Now for each of the $n-1$ iterations, we find the edge $\{x, y\}$ by looking only at the first edge of at most n lists — $O(n^2)$ over all iterations. We must, of course, discard edges on these lists as we build S, so that the edge we want will be first, but with appropriate links, this is $O(1)$.
  - Thus, the sort in the initialization dominates, which makes the algorithm $O(m\log m) = O(n^2\log n^2) =$ O(n2log) in the worst case.

Prims alg.

- Just find the smallest edge by searching the adjacency list of the vertices in V. In this case, each iteration costs O(m) time, yielding a total running time of O(mn).
- By using binary heaps, the algorithm runs in O(m log n).
- By using Fibonacci heaps, the algorithm runs in O(m + n log n) time.

Complexity of Kruskal's Algorithm

- Sort edges by weight (smallest first)
- For each edge $e=\{x, y\}$, until $n-1$ edges added if nodes x and y are not in the same connected component, add e

- $O(m\log m) + O(nm) = O(n^2\log n^2) + O(n^3) = O(n^3)$
- How can Kruskal's algorithm be improved?
  - Often already sorted (omit first step)
  - Find a faster way to check "in same connected component"

$O(m\log m)$ $O(m)$ $O(n)$, i.e. using DFS* $O(1)$

*O(n) — not O(m) — because the edges for the DFS check are the edges added to the spanning tree so far, and there can never be more than n−1 edges in the spanning tree.

Complexity of Kruskal's Algorithm

- Assume edges sorted by weight in descending order.
- Initialize pointers to trees of height 0
- For each edge $e=\{x, y\}$, until n−1 edges added

    if x and y are not in the same tree (i.e. don't have the same root),

    add e

    merge smaller tree (of x or y) into larger tree

O(1) O(1)

O(n) O(m) O(log$n$) — search tree

- $O(n) + O(m \log n) = O(n^2 \log n)$

Which is better ?

- sparse graph:
  - Prim = $O(N^2)$
  - Kruskal = $O(N \log(N))$
- dense graph:
  - Prim = $O(N^2)$
  - Kruskal = $O(N^2 * \log(N))$
- So in dense graphs Prim is better
- In Sparse graphs, kruskal is better