Contour Plots and Directional Derivatives – worksheet 10

1. A contour map of a function $f(x, y)$ is shown with level curves labeled $f(x, y) = 1, 2, 3, 4, 5$. write the direction of the gradient vector $\nabla f(x_0, y_0)$ at the point (x_0, y_0) and explain why it is perpendicular to the contour through that point.

2. The contour plot of a function $f(x, y)$ shows ellipses centered at $(2, 1)$. Determine whether f has a local minimum, maximum, or saddle point at $(2, 1)$ based solely on this contour information.

3. Let
$$f(x, y) = x^2 - xy + 3y^2.$$
Compute the directional derivative $D_{\mathbf{u}}f(1, 2)$ where $\mathbf{u} = \langle 3, 4 \rangle$ is a direction vector.

4. Compute the directional derivative of the function
$$f(x, y) = e^{xy} + x^2 y$$
at the point $(0, 1)$ in the direction of the vector $\mathbf{v} = \langle -6, 8 \rangle$.

5. The contour lines of a function f become closer and closer together as you approach the point $(5, 5)$. Explain what this tells you about the magnitude of the gradient vector $\|\nabla f(5, 5)\|$.

6. A contour map of a function $f(x, y)$ shows level curves spaced very far apart near the point $(1, 4)$. What does this indicate about the magnitude of $\|\nabla f(1, 4)\|$? Explain.

7. A contour plot shows level curves that are vertical lines (parallel to the y-axis). Based on this, determine:

- the direction of the gradient vector at any point,
- whether the function changes more rapidly in the x-direction or the y-direction.

8. Let
$$f(x, y) = \ln(1 + x^2 + y^2).$$
Compute the directional derivative of f at the point $(3, 4)$ in the direction of the vector $\mathbf{v} = \langle -4, 3 \rangle$.

9. A function f has gradient at the point $(2, 1)$ given by
$$\nabla f(2, 1) = \langle 5, -12 \rangle.$$

Find the directional derivative of f at $(2, 1)$ in the direction making a $60°$ angle with the gradient vector.

10. A contour plot of a function shows that:

- contours are circles centered at the origin,
- larger circles correspond to larger function values.

Answer the following:

(a) What is the likely shape of the gradient field?

(b) In which direction does the gradient vector point at any (x, y)?

(c) Would the directional derivative in the radial direction be positive or negative?