# CS 698S — Homework Assignment 1

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## Question 1

Given $x$ is scalar r.v., $x \sim N(x|0,\eta)$
and, $\eta \sim Exp(\eta|r^2/2)$; $r > 0$
where $Exp(x|\lambda) = \lambda \exp(-\lambda x)$

We have to derive marginal distribution of $x$, i.e;

$$p(x|r) = \int p(x|\eta)\, p(\eta|r)\, d\eta$$

Since this is a hard integral, we use moment generating function to calculate this.

$$p(x|r) = \int_{0}^{\infty} \frac{r^2}{2} \exp\left\{\frac{r^2\eta}{2}\right\} \cdot \frac{1}{\sqrt{2\pi\eta}} \exp\left\{\frac{-x^2}{2\eta}\right\} d\eta$$

[limit is from 0 to $\infty$ because $\eta$ takes values $> 0$]

$$mgf\,(p(x|r)) = \int e^{tx}\left(\int \frac{r^2}{2\sqrt{2\pi\eta}} \exp\left\{\frac{-x^2}{2\eta}\right\} \exp\left\{-\frac{r^2\eta}{2}\right\} d\eta\right) dx$$

$$= \int_{-\infty}^{\infty}\int_{0}^{\infty} \frac{r^2}{2\sqrt{2\pi\eta}} \exp\left\{\frac{-x^2}{2\eta} + tx - \frac{r^2\eta}{2}\right\} d\eta\, dx.$$

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$$= \int_{0}^{\infty}\int_{-\infty}^{\infty} \frac{r^2}{2\sqrt{2\pi\eta}} \exp\left\{\frac{-x^2 + 2tx\eta - r^2\eta^2}{2\eta}\right\} dx\, d\eta$$

$$= \int_{0}^{\infty}\underbrace{\int_{-\infty}^{\infty} \frac{r^2}{2\sqrt{2\pi\eta}} \exp\left\{\frac{-(x-t\eta)^2}{2\eta}\right\}}_{\text{this gets integrated to 1}} \exp\left\{\frac{(r^2-t^2)\eta^2}{2\eta}\right\} dx\, d\eta$$

$$= \frac{r^2}{2}\int_{0}^{\infty} \exp\left\{\frac{(r^2-t^2)}{2}\eta\right\} d\eta = \frac{2}{-(t^2-r^2)} \cdot \frac{r^2}{2} = \frac{1}{1-\frac{t^2}{r^2}} = \frac{e^{\mu t}}{1-t^2b^2}$$

We observe that this is the mgf of the Laplace distribution where $p(x|r) = \mathcal{L}(0, \frac{1}{r})$

The marginalised distribution $p(x|r)$ is a Laplace distribution, with

$$\mathcal{L}(\mu, b) = \frac{1}{b}\exp\left\{-\frac{|x-\mu|}{b}\right\},$$

non-differentiable at its mean (in this case, 0)