# Space-Filling Curves & Generalizing the Peano Curve

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Senior Honors Thesis
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University of North Carolina at Chapel Hill
4/8/2019

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# Abstract

The theory of space-filling curves will be developed. The two original space-filling curves, those of Peano and Hillbert, will be dissected and analyzed, with an emphasis on geometric constructions and demonstration of analytic properties. The Peano curve will be extended to every odd base, with an analogous geometric construction to the original. New proofs will be shown in order to conclude that these "extended" Peano Curves are indeed space-filling curves, mapping surjectively and continuously from the unit interval to the unit square. Two fairly recent applications of space-filling curves in computer science will also be touched on.

# Acknowledgements

There are many people I own thanks to, first and foremost being my advisor, Professor Idris Assani. Without his guidance, support, and seemingly endless patience, none of this would have been possible. I'd also like to thank my committee members, Professors Prakash Belkale and Ivan Cherednik, for their suggestions and the roles they both played in my mathematical education. A special thanks to Professor David Adalsteinsson for the suggestion of including the application of space-filling curves to the optimization of computational matrix multiplication.

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# Chapter 1

# Introduction

## 1.1 Notations and Preliminaries

### 1.1.1 Notations

Before beginning, some common notations that will be used throughout this document will be established. Take $\mathbb{R}^n$ to be standard $n$-dimensional Euclidean Space, with the usual metric of Euclidean distance $d$:

$$d : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R},\ (x, y) \mapsto \left[ \sum_{i=1}^{n} (x_i - y_i)^2 \right]^{\frac{1}{2}}$$

Let $\mathcal{I} = [0, 1] \subset \mathbb{R},\ \mathcal{Q} = [0, 1]^2 \subset \mathbb{R}^2$.

**Definition 1.1.1:** A **curve** is a function $\gamma : \mathcal{I} \to \mathbb{R}^n$ that is continuous.

It will prove useful to review representing numbers in different bases. To denote some arbitrary $t \in \mathcal{I}$ in its base-$b$ representation, I will write $0._b t_1 t_2 ... t_n ...$ where each $t_i$ is a digit of $t$, and each $t_i \in \{0, 1, 2, ..., b - 1\}$ such that

$$t = 0._b t_1 t_2 ... t_n ... = \frac{t_1}{b} + \frac{t_2}{b^2} + ... + \frac{t_n}{b^n} + ... = \sum_{i=1}^{\infty} \frac{t_i}{b^i}$$

If no base $b$ is specified, then it will be assumed that the number is being represented in base-10. Along with this notation, bars over digits will denote repeating digits, such that

$$0._b \overline{t_1 t_2 t_3} = 0._b t_1 t_2 t_3 t_1 t_2 t_3 ... t_1 t_2 t_3 ...$$

Often to make the distinction between digits more clear, they will be surrounded in brackets:

$$0._b [t_1][t_2][t_3]...[t_n]... = 0._b t_1 t_2 t_3 ... t_n ...$$

While this may seem excessive now, in the future it will be useful, as more complex expressions will be needed to show the values of certain digits in numbers.

Recall that in base$-10$, one can take any number $t = 0.t_1t_2...t_n$ with a terminating representation and replace the least-significant nonzero digit $t_n$ with $t_n - 1$ and represent $t = 0.t_1t_2...[t_n - 1]\overline{9} = 0.t_1t_2...t_n$. Similarly, for an arbitrary $t$ represented in base$-b$, one can take a terminating $0._bt_1t_2...t_n$ and rewrite it as $0._bt_1t_2...[t_n - 1]\overline{[b-1]}$ and maintain equality between the two. This difference in representations of numbers will be important in assuring that the mappings to come are well-defined.

### 1.1.2 Preliminaries

In this document, I will provide geometric intuitions about how space-filling curves are defined, and why they are objects of interest. In addition, some analytic properties will be demonstrated. While some proofs of these concepts are available in [SAG94], I aim to provide them in more complete detail. Following a construction that is similar to the one presented in [SAG94] of the Peano Curve, I will extend the Peano Curve to any odd base, in a similar manner to [COL85]. However, [COL85] tends to focus on finite iterations of curves and their applications to computer science, and constructs them using Gray Codes (a certain method of representing numbers in different bases). I will instead provide both a new, analogous geometric construction of such extended Peano Curves, and new proofs that they are indeed space-filling curves mapping from $\mathcal{I}$ into $\mathcal{Q}$.

Some analytic properties of the Hilbert Curve will also be demonstrated, again using a geometric construction following some constructions in [SAG94]. [SAG94] also stands as a fantastic reference for the analytic construction of the Hilbert Curve. This document will conclude with some modern applications of space-filling curves in the field of computer science.

## 1.2 Bijections between $\mathcal{I}$ and $\mathcal{Q}$

In the late 1800's, Cantor's work on the cardinality of the continuum led to a number of realizations in set theory, one of which is that $\mathbb{R}$ can be put in bijection with $\mathbb{R}^2$. Along with this realization the question arose of whether or not there exists a continuous bijection between the two. However, using some basic ideas of topology, such a function can be ruled out as impossible.

**Definition 1.2.1:** A subset $E$ of a metric space $(X, d)$ is **connected** if there are no subsets of $E$ that are simultaneously open and closed aside from $E$ and $\emptyset$.

**Theorem 1.2.2:** If $(X, d_x)$ and $(Y, d_y)$ are metric spaces and $f : X \to Y$ is a continuous function, then $f$ maps connected sets in $X$ to connected sets in $Y$.

**Proof:** Let $f : X \to Y$ be a continuous function, and take $C$ to be a connected subset of $X$. Consider the restricted function $f|_C : C \to f(C)$, and note that by design $f|_X$ is surjective. Suppose that $D$ is a subset of $f(C)$ that is both open and closed. Since $f$ is continuous we have that $f^{-1}(D)$ is both open and closed in $C$. Since $C$ is connected, one has that $f|_C^{-1}(D)$ is either $\emptyset$ or $C$. Now, since $f|_C$ is surjective, it can be concluded that $D$ is either $\emptyset$ or $f(C)$, and so $f(C)$ is connected. $\square$

**Corollary 1.2.3:** No continuous bijections from $\mathcal{Q}$ to $\mathcal{I}$ exist.

**Proof:** Suppose for the sake of contradiction that $f : \mathcal{Q} \to \mathcal{I}$ is a continuous bijection. Take some $p \in \mathcal{Q}$, and note that $\mathcal{Q} \setminus p$ is connected. However, $\mathcal{I} \setminus f(p)$ is not connected. As such, no such continuous bijection exists. Since $(\mathcal{Q} \setminus p) \subset \mathbb{R}^2$, it follows that there is also not a continuous bijection from $\mathbb{R}^2$ to $\mathbb{R}$. $\square$

With continuous bijections being impossible, mathematicians began to wonder if there are continuous surjections exist.

## 1.3 Space-Filling Curves

Along with Cantor's work in classical set theory came a proof that $\mathbb{R}$ is bijective with $\mathbb{R}^2$, and in particular that $\mathcal{I}$ is bijective with $\mathcal{Q}$. This can be realized with the following theorem:

**Theorem 1.3.1:** (Cantor-Schröder-Bernstein) If there exists $f : X \to Y$ such that $f$ is injective, and $g : Y \to X$ such that $g$ is injective then there exists a bijection between the sets $X$ and $Y$. The history behind this theorem is fairly rich, beginning with Cantor publishing its statement in the late 1880's with no proof, though roughly a decade later he did produce a proof [CAN97].

With this, many mathematicians sought out such a bijection, and it would be interesting if such a bijection were continuous– though in this document it was already shown that a continuous bijection between the two is impossible, at the time this was unknown. In this search, one finds a definition for a space-filling curve:

**Definition 1.3.2:** A **Space-filling curve** is a curve that maps surjectively into its co-domain. Unless otherwise stated, assume that the domain in question is $\mathcal{I}$ and the co-domain is $\mathcal{Q}$.

Space-filling curves were sought after because at the time because they were (and arguably still are) very counter-intuitive constructions. Many found it bewildering that a line with zero "thickness" could manage to intersect every point in $\mathcal{Q}$, entirely filling a square in $\mathbb{R}^2$. In this paper, I will develop the theory behind the two original space-filling curves of Peano and Hilbert, and show an explicit method of generating infinitely many distinct space-filling curves from the construction of Peano.

# Chapter 2

# The Peano Curve

## 2.1 Construction of the Peano Curve

Peano originally defined his eponymous space-filling curve in 1890 using base-3 representations of numbers, with the map

$$f_p : \mathcal{I} \to \mathcal{Q},\ (0._3 t_1 t_2 t_3 t_4 \ldots) \mapsto \begin{pmatrix} \phi_p(t) \\ \psi_p(t) \end{pmatrix} = \begin{pmatrix} 0._3 t_1 (k^{t_2}(t_3))(k^{t_2+t_4}(t_5)) \ldots \\ 0._3 (k^{t_1}(t_2))(k^{t_1+t_3}(t_4)) \ldots \end{pmatrix}$$

where $k(x) = 2 - x$ for $x \in \{0, 1, 2\}$. While it is not immediately obvious that this function defines a space-filling curve, it will first be useful to get some geometric insight of what this map looks like.

In order to get some idea of what the Peano curve looks like, consider some arbitrary $t \in [0, \frac{1}{9}]$. Then in base 3, $t$ takes the form $t = 0._3 00 t_3 t_4 \ldots$. As such, one sees that

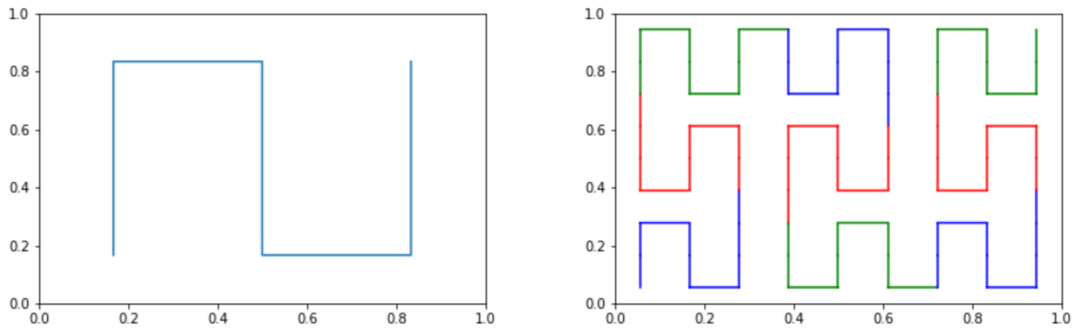
$$f_p(t) = \begin{pmatrix} 0._3 0 \alpha_2 \alpha_3 \ldots \\ 0._3 0 \beta_2 \beta_3 \ldots \end{pmatrix}$$

So one would expect the interval $[0, \frac{1}{9}] \subset \mathcal{I}$ is mapped to the square $[0, \frac{1}{3}]^2 \subset \mathcal{Q}$. By continuing this argument, one finds that

$$\left[\frac{1}{9}, \frac{2}{9}\right] \mapsto \left[0, \frac{1}{3}\right] \times \left[\frac{1}{3}, \frac{2}{3}\right],\ \left[\frac{3}{9}, \frac{4}{9}\right] \mapsto \left[0, \frac{1}{3}\right] \times \left[\frac{2}{3}, 1\right],\ \left[\frac{4}{9}, \frac{5}{9}\right] \mapsto \left[\frac{1}{3}, \frac{2}{3}\right] \times \left[\frac{2}{3}, 1\right]$$

Such a construction provides the inference that $f_p$ maps ninths of the unit interval to a snake-like pattern on the unit square. This provides insight to the Peano curve in two different ways.

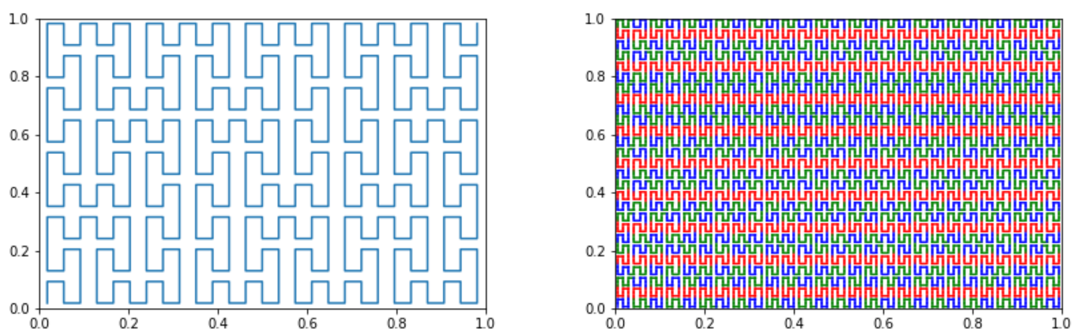
First off, it suggests a method of plotting points of this curve to give a visualization. Fix $i \in \mathbb{N}$, $i > 1$ and break the unit interval into $9^i$ equally spaced intervals, and then trace out the path in $\mathbb{R}^2$ produced by the midpoints of these intervals. For a concrete example, let $i = 1$. Then we'll plot $f_p(\frac{1}{18}), f_p(\frac{3}{18}), \ldots, f_p(\frac{17}{18})$. This gives us the initial "snaking" pattern of the Peano curve. One calls $i$ be the "iteration" number of the Peano curve.




(a) $i = 1$ (b) $i = 2$

Figure 2.1: Plots of the first and second iterations of the Peano curve. The first iteration of the Peano curve, produced by breaking the unit interval into ninths. The second iteration of the Peano curve, produced by breaking the unit interval into 81 intervals equal in length. Note that it consists of nine replicas of the first iteration, although some of them appear flipped.

If this process is applied again to each interval $\left[\frac{j}{9}, \frac{j+1}{9}\right]$ (for $0 \leq j < 9$) then we again find that this snake pattern appears within each interval.




(a) $i = 3$ (b) $i = 4$

Figure 2.2: Notice that as we iterate, we begin to see how the Peano curve fills the unit square.

Now, to truly get the Peano curve out of this iterative process, we'd need to take the limit of these iterations as $i \to \infty$, but this visualization does help formulate ideas about the continuity, surjectivity, injectivity (or rather, the lack thereof), and differentiability of the Peano Curve.

## 2.2 Properties of the Peano Curve

To demonstrate why $f_p$ is a space-filling curve, it needs to be shown that $f_p$ maps to every point in $\mathcal{Q}$ and that $f_p$ is continuous. But even before that, it needs to be shown that $f_p$ is a well-defined map.

**Theorem 2.2.1:** $f_p$ is a space-filling curve from $\mathcal{I}$ to $\mathcal{Q}$. The proof of this will be delivered in several small steps, first demonstrating that $f_p$ is well-defined, and then showing surjectivity and continuity.

**Proposition 2.2.2:** $f_p$ is a well-defined map that is independent of input representation.

**Proof:** We'll need to check that

$$f_p(0._3t_1t_2...t_n) = \alpha,\ f_p(0._3t_1t_2...(t_n-1)\overline{2}) = \beta \implies \alpha = \beta$$

To do this, there are two separate cases to consider: $n = 2k$ and $n = 2k+1$ for $k \in \mathbb{Z}_{\geq 0}$.

**Case 1:** $n = 2k$

With this assumption, note the following:

$$\alpha = \begin{pmatrix} 0._3t_1[k^{t_2}(t_3)]...[k^{t_2+...+t_{2k-2}}(t_{2k-1})]\overline{[k^{t_2+...+t_{2k-2}+t_{2k}}(0)]} \\ 0._3[k^{t_1}(t_2)]...[k^{t_1+t_3+...+t_{2k-1}}(t_{2k})]\overline{[k^{t_1+t_3+...+t_{2k-1}}(0)]} \end{pmatrix}$$

$$\beta = \begin{pmatrix} 0._3t_1[k^{t_2}(t_3)]...[k^{t_2+...+t_{2k-2}}(t_{2k-1})]\overline{[k^{t_2+...+t_{2k-2}+t_{2k}-1}(2)]} \\ 0._3[k^{t_1}(t_2)]...[k^{t_1+t_3+...+t_{2k-1}}(t_{2k}-1)]\overline{[k^{t_1+t_3+...+t_{2k-1}}(2)]} \end{pmatrix}$$

Now, we'll again need to consider four cases: combinations of when the sums of even or odd digits of $t$ are even or odd. Instead of including all four of such cases, two will be presented, noting that the other two follow similarly (and are essentially combinations of these two cases):

**Case 1.i:**

$$\sum_{i=1}^{k} t_{2i} \equiv 0 \mod 2,\ \sum_{i=1}^{k} t_{2i-1} \equiv 0 \mod 2$$

Then we can simplify $\alpha$ and $\beta$ as follows:

$$\alpha = \begin{pmatrix} 0._3t_1[k^{t_2}(t_3)]...[k^{t_2+...+t_{2k-2}}(t_{2k-1})]\overline{0} \\ 0._3[k^{t_1}(t_2)]...[k^{t_1+t_3+...+t_{2k-1}}(t_{2k})]\overline{0} \end{pmatrix} = \begin{pmatrix} 0._3t_1[k^{t_2}(t_3)]...[k^{t_2+...+t_{2k-2}}(t_{2k-1})] \\ 0._3[k^{t_1}(t_2)]...[k^{t_1+t_.3+...+t_{2k-1}}(t_{2k})] \end{pmatrix}$$

$$\beta = \begin{pmatrix} 0._3t_1[k^{t_2}(t_3)]...[k^{t_2+...+t_{2k-2}}(t_{2k-1})]\overline{[k(2)]} \\ 0._3[k^{t_1}(t_2)]...[(t_{2k}-1)]\overline{2} \end{pmatrix} = \begin{pmatrix} 0._3t_1[k^{t_2}(t_3)]...[k^{t_2+...+t_{2k-2}}(t_{2k-1})] \\ 0._3[k^{t_1}(t_2)]...[(t_{2k}-1)]\overline{2} \end{pmatrix}$$

As such, $\alpha = \beta$.

**Case 1.ii:**

$$\sum_{i=1}^{k} t_{2i} \equiv 1 \mod 2, \ \sum_{i=1}^{k} t_{2i-1} \equiv 1 \mod 2$$

As such, the following simplifications can be made:

$$\alpha = \begin{pmatrix} 0._3 t_1[k^{t_2}(t_3)]...[k^{t_2+...+t_{2k-2}}(t_{2k-1})]\overline{2} \\ 0._3[k^{t_1}(t_2)]...[k^{t_1+t_3+...+t_{2k-1}}(t_{2k})]\overline{2} \end{pmatrix} = \begin{pmatrix} 0._3 t_1[k^{t_2}(t_3)]...[k^{t_2+...+t_{2k-2}}(t_{2k-1})]\overline{2} \\ 0._3[k^{t_1}(t_2)]...[2 - t_{2k}]\overline{2} \end{pmatrix}$$

$$\beta = \begin{pmatrix} 0._3 t_1[k^{t_2}(t_3)]...[k^{t_2+...+t_{2k-2}}(t_{2k-1})]\overline{2} \\ 0._3[k^{t_1}(t_2)]...[k^{t_1+...+t_{2k-1}}(t_{2k}-1)][\overline{k(2)}] \end{pmatrix} = \begin{pmatrix} 0._3 t_1...[k^{t_2+...+t_{2k-2}}(t_{2k-1})]\overline{2} \\ 0._3[k^{t_1}(t_2)]...[2 - t_{2k} + 1]\overline{0} \end{pmatrix}$$

Now, note that no matter the value of $t_{2k} \in \{1, 2\}$ that $\alpha = \beta$.

**Case 2:** $n = 2k + 1$

With this assumption, we can write

$$\alpha = \begin{pmatrix} 0._3 t_1[k^{t_2}(t_3)]...[k^{t_2+t_4+...+t_{2k}}(t_{2k+1})][\overline{k^{t_2+...+t_{2k}}(0)}] \\ 0._3[k^{t_1}(t_2)]...[k^{t_1+...+t_{2k-1}}(t_{2k})][\overline{k^{t_1+...+t_{2k-1}+t_{2k+1}}(0)}] \end{pmatrix}$$

$$\beta = \begin{pmatrix} 0._3 t_1[k^{t_2}(t_3)]...[k^{t_2+t_4+...+t_{2k}}(t_{2k+1}-1)][\overline{k^{t_2+...+t_{2k}}(2)}] \\ 0._3[k^{t_1}(t_2)]...[k^{t_1+...+t_{2k-1}}(t_{2k})][\overline{k^{t_1+...+t_{2k-1}+t_{2k+1}-1}(2)}] \end{pmatrix}$$

Again, the parities of the sums of the even and odd digits need to be considered on a case-by-case basis.

**Case 2.i:**

$$\sum_{i=1}^{k-1} t_{2i} \equiv 0 \mod 2, \ \sum_{i=1}^{k} t_{2i+1} \equiv 0 \mod 2$$

Given that both sums are even, the following simplifications hold:

$$\alpha = \begin{pmatrix} 0._3 t_1...[t_{2k+1}]\overline{0} \\ 0._3[k^{t_1}(t_2)]...[k^{t_1+...+t_{2k-1}}(t_{2k})]\overline{0} \end{pmatrix}$$

$$\beta = \begin{pmatrix} 0._3 t_1...[t_{2k+1}-1]\overline{2} \\ 0._3[k^{t_1}(t_2)]...[k^{t_1+...+t_{2k-1}}(t_{2k})][\overline{k(2)}] \end{pmatrix} = \begin{pmatrix} 0._3 t_1...[t_{2k+1}-1]\overline{2} \\ 0._3[k^{t_1}(t_2)]...[k^{t_1+...+t_{2k-1}}(t_{2k})]\overline{0} \end{pmatrix}$$

With this, $\alpha = \beta$

**Case 2.ii:**

$$\sum_{i=1}^{k-1} t_{2i} \equiv 1 \mod 2, \ \sum_{i=1}^{k} t_{2i+1} \equiv 1 \mod 2$$

$$\alpha = \begin{pmatrix} 0._3 t_1 ... [2 - t_{2k-1}]\overline{2} \\ 0._3 k^{t_1}(t_2) ... [k^{t_1 + ... + t_{2k-1}}(t_{2k})\overline{2} \end{pmatrix}$$

$$\beta = \begin{pmatrix} 0._3 t_1 ... [2 - t_{2k+1} + 1]\overline{0} \\ 0._3 k^{t_1}(t_2) ... [k^{t_1 + ... + t_{2k-1}}(t_{2k})]\overline{2} \end{pmatrix}$$

In an exactly analogous argument to case 1.ii, we have that no matter the value of $t_{2k+1}$ that $\alpha = \beta$. Note that one could also have combinations of case 1 and case 2 (namely that these sums have mis-matched parities), but they will boil down to the same argument either way. As such, $f_p$ is well-defined. $\square$

**Proposition 2.2.3:** $f_p$ is surjective

Let $\begin{pmatrix} 0._3 \alpha_1 \alpha_2 \alpha_3 \alpha_4 ... \\ 0._3 \beta_1 \beta_2 \beta_3 \beta_4 ... \end{pmatrix} \in [0,1]^2$ be arbitrary. Using the map $f_p$, one has

$$\alpha_n = k^{t_2 + t_4 + ... + t_{2n-2}}(t_{2n-1}),\ \beta_n = k^{t_1 + t_3 + ... + t_{2n-1}}(t_{2n})$$

One can successively solve for $t_i$ as

$$t_{2n-1} = k^{t_2 + t_4 + ... + t_{2n-2}}(\alpha_n),\ t_{2n} = k^{t_1 + t_3 + ... + t_{2n-1}}(\beta_n)$$

As such, any $p \in \mathcal{Q}$ can be mapped to by some $t \in [0,1]$ $\square$

To show that $f_p$ is indeed a space-filling curve it will also be necessary to show that $f_p$ is a continuous map on $\mathcal{I}$. As such, it will be useful to make the following note:

**Proposition 2.2.4:** $\psi_p(t) = 3\phi_p(\frac{t}{3})$

Letting $t = 0._3 t_1 t_2 t_3 t_4 ....$ one has $\frac{t}{3} = 0._3 0 t_1 t_2 t_3 t_4 ...$ such that

$$3 * \phi_p\left(\frac{t}{3}\right) = 3\left[\phi_p(0._3 0 t_1 t_2 t_3 t_4 ...)\right] = 3\left[0._3 0 (k^{t_1}(t_2))(k^{t_1 + t_3}(t_4)) ...\right] = \psi_p(t) \quad \square$$

**Proposition 2.2.5:** $f_p$ is continuous.

It will be demonstrated that $\phi_p$ is continuous from the right on $[0,1)$ and continuous from the left on $(0,1]$ to say that $\phi_p$ is continuous on $[0,1]$. Continuity of $\psi_p$ will follow from the fact that $\psi_p(t) = 3\phi_p(\frac{t}{3})$.

Now, let $t_0 = 0._3 t_1 t_2 ... t_{2n} t_{2n+1} ...$ be the representation of $t_0$ that does not have infinitely many trailing 2's. Now, let $\delta = 3^{-2n} - 0._3 00..0 t_{2n+1} t_{2n+2} ...$ and note:

$$t_0 + \delta = 0._3 t_1 t_2 ... t_{2n} t_{2n+1} ... + 3^{-2n} - 0._3 00...0 t_{2n+1} t_{2n+2} ... = 0._3 t_1 t_2 ... t_{2n-1} t_{2n} \overline{2}$$

As such, $\forall t \in [t, t+\delta)$ one has that $t$ can be written

$$t = 0._3 t_1 t_2 ... t_{2n-1} t_{2n} \tau_{2n+1} \tau_{2n+2} ...$$

where $\tau_i$ are undetermined. With this, let $\sigma = \sum_{i=1}^{n} t_{2i}$ and one sees:

$$|\phi_p(t) - \phi_p(t_0)| = \left|0._3 t_1[k^{t_2}(t_3)]....[k^{\sigma}(\tau_{2n+1})]... - 0._3 t_1[k^{t_2}(t_3)]...[k^{\sigma}(t_{2n+1})]...\right|$$

$$\leq |k^{\sigma}(\tau_{2n+1}) - k^{\sigma}(t_{2n+1})| * 3^{-(n+1)}$$

$$+ \left|k^{\sigma+\tau_{2n+2}}(\tau_{2n+3}) - k^{\sigma+t_{2n+2}}(t_{2n+3})\right| * 3^{-(n+2)} + ...$$

$$\leq \frac{2}{3^{n+1}} + \frac{2}{3^{n+2}} + ... = \frac{2}{3^n}\sum_{i=1}^{\infty}\frac{1}{3^i} = \frac{1}{3^n}$$

Continuity from the right follows from the fact that $\frac{1}{3^n}$ converges to 0 as $n$ approaches infinity.

To show continuity from the left on $(0, 1]$, again consider $t_0 = 0._3 t_1 t_2 ... t_{2n} t_{2n+1} ...$ and now let $\delta = 0._3 0...0 t_{2n+1} t_{2n+2} ...$ such that $t_0 - \delta = 0._3 t_1 t_2 ... t_{2n-1} t_{2n}$. Then note that $\forall t \in (t_0 - \delta, t_0]$ one has that $t$ must agree with the first $2n$ digits of $t_0$, and now this argument is much the same as the previous one:

$$|\phi_p(t) - \phi_p(t_0)| = \left|0._3 t_1[k^{t_2}(t_3)]....[k^{\sigma}(t_{2n+1})]... - 0._3 t_1[k^{t_2}(t_3)]...[k^{\sigma}(t_{2n+1})]...\right|$$

$$\leq \frac{2}{3^{n+1}} + \frac{2}{3^{n+2}} + ... = \frac{2}{3^{n+1}}\sum_{i=1}^{\infty}\frac{1}{3^i} = \frac{1}{3^n} \to 0 \text{ as } n \to \infty$$

As such, $\phi_p$ is continuous from the right on $[0, 1)$ and continuous from the left on $(0, 1]$, and so it is continuous on $[0, 1]$. Continuity of $\psi_p$ follows as it is the composition of continuous functions.

Now, with continuity of $\phi_p$ proven, one has that (by definition) $\forall \epsilon_1 > 0, \exists \delta_1 : |x - y| < \delta_1 \implies d(\phi(x), \phi(y)) < \epsilon_1$. As such, note that

$$\left|\frac{x}{3} - \frac{y}{3}\right| < \frac{\delta_1}{3} < \delta_1 \implies \left|\phi\left(\frac{x}{3}\right) - \phi\left(\frac{y}{3}\right)\right| < \epsilon_1$$

With this, one has that when $|x - y| < \delta_1$ that

$$d(f_p(x), f_p(y)) = \sqrt{(\phi(x) - \phi(y))^2 + \left(3\phi\left(\frac{x}{3}\right) - 3\phi\left(\frac{y}{3}\right)\right)^2} < \sqrt{10}\epsilon_1$$

As such, $f_p$ is continuous, and this concludes the proof that $f_p$ is a space-filling curve from $\mathcal{I}$ to $\mathcal{Q}$. □

**Lemma 2.2.6:** $f_p$ is non-injective.

Recalling that there exist no continuous bijections from $\mathcal{I}$ to $\mathcal{Q}$, one has that $f_p$ must have at least one point where it is non-injective. Using the difference in representations, a point can be found in $\mathcal{Q}$ whose fibre isn't a singleton. Consider

$$\begin{pmatrix} 1/3 \\ 1/2 \end{pmatrix} = \begin{pmatrix} 0._3 1 \\ 0._3 111\bar{1} \end{pmatrix} = \begin{pmatrix} 0._3 0\bar{2} \\ 0._3 \bar{1} \end{pmatrix}$$

Let $\alpha$ be the representation with a terminating first coordinate, and $\beta$ be the representation with a non-terminating first coordinate. Then, solving for the digits of the preimage of $\alpha$ and $\beta$ respectively, one finds

$$f_p^{-1}(\alpha) = 0._3 1\overline{1210} \neq f_p^{-1}(\beta) = 0._3 0\overline{1012}$$

To expand on the lack of injectivity of the Peano curve, notice that between successive iterations that $f_p$ seems to be converging toward the lines $x = \frac{t}{3^k}$ and $y = \frac{t}{3^k}$ for $t \in \{0, 1, 2\}$ from two different positions. Consider the sets

$$T = \{3^k : k \in \mathbb{Z}_{<0}\},\ \Lambda = \left\{ \begin{pmatrix} a \\ b \end{pmatrix} \in \mathcal{Q} : a \in T \right\}$$

**Lemma 2.2.7:** Each $p \in \Lambda$ can be mapped to by more than one element of $\mathcal{I}$.

To demonstrate this, first let

$$a = 0._3 a_1 a_2 ... a_n = 0._3 00...1$$

$$\alpha = 0._3 \alpha_1 \alpha_2 ... \alpha_n \alpha_{n+1} = 0._3 00...0\bar{2}$$

In other words, $a = \alpha$ where $\alpha$ is the representation of $a$ involving an infinite trail of 2s. Now, we'll look to employ the successive digit-solving algorithm in order to find a $t_\gamma$ and $t_\lambda$ such that $t_\lambda \neq t_\gamma$ but

$$f_p(t_\gamma) = \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} \alpha \\ b \end{pmatrix} = f_p(t_\lambda)$$

So, we'll let $b = 0._3 \beta_1 \beta_2 ...$ and begin by solving for as many digits of $t_\gamma$ as possible :

$$t_\gamma = 0._3 0\beta_1 0\beta_2 ... 0\beta_{k-1} [k^\sigma(1)] t_{\gamma,2n} t_{\gamma,2n+1} ... = 0._3 0\beta_1 0\beta_2 ... 0\beta_{k-1} 1 t_{\gamma,2n} t_{\gamma,2n+1} ...$$

where the $t_{\gamma,i}$ are ternary digits that remain to be determined for $i \geq 2n$. The final equality holds as $\sigma = \sum_{i=1}^{k-1} \beta_i$, but since $k(1) = 1$, no matter the parity of $\sigma$ one has that this digit will be 1.

Now, $t_\lambda$ will be inspected, and we will arrive at a system of constraints that require $t_\gamma$ to differ from $t_\lambda$. Again, solving for digits of $t_\lambda$ yields:

$$t_\lambda = 0._3 0\beta_1 0\beta_2 ... 0\beta_{n-1} [k^\sigma(0)] t_{\lambda,2n} t_{\lambda,2n+1} ...$$

This leads to two cases:

**Case 1:** $\sigma \equiv 0 \mod 2$
If $\sigma$ is even then note that we can further solve

$$t_\lambda = 0._3 0\beta_1 ... 0\beta_{n-1} 0 t_{\lambda,2n} ...$$

Now, since $t_\lambda$ and $t_\gamma$ differ at one place in their ternary representations, for them to be equal it would be required that $t_{\lambda,j} = 2$ for all $j \geq 2n$ and $t_{\gamma,j} = 0$ for all $j \geq 2n$. I'll look to contradict this by further solving for the digits of the two:

$$t_\gamma = 0._3 0\beta_1 ... 0\beta_{n-1} 1[k(\beta_n)] t_{\gamma,2n+1} ... = 0._3 0\beta_1 ... 0\beta_{n-1} 1[2 - \beta_n] t_{\gamma,2n+1} ...$$

$$t_\lambda = 0._3 0\beta_1 ... 0\beta_{n-1} 0[k^0(\beta_n)] t_{\lambda,2n+1} ... = 0._3 0\beta_1 ... 0\beta_{n-1} 0\beta_n t_{\lambda,2n+1} ...$$

As such, for $t_\lambda$ to equal $t_\gamma$, it would be required that $\beta_n = 2$. This allows further deductions:

$$t_\gamma = 0._3 0\beta_1 ... 0\beta_{n-1} 10[k^{\beta_1 + ... + \beta_{n-1} + 0}(0)] t_{\gamma,2n+2} ... = 0._3 0\beta_1 ... 0\beta_{n-1} 100 t_{\gamma,2n+2} ...$$

$$t_\lambda = 0._3 0\beta_1 ... 0\beta_{n-1} 0\beta_n [k^{\beta_1 + ... + \beta_{n-1} + 0}(2)] t_{\lambda,2n+2} ... = 0._3 0\beta_1 ... 0\beta_{n-1} 022 t_{\lambda,2n+2} ...$$

We find that now for equality to hold it must be that $\beta_{n+1}$ will be 2 as well. From this, we note that in order for these two to be equal it must be that $\beta_j = 2$ for all $j \geq n$. However, there are only two possibilities for $b$:

**Case 1.i:** $b$ has an infinite tail of 2s in its ternary representation

If $b$ has an infinite tail of 2s in its ternary representation then one has $b = 0._3\beta_1\beta_2 ... \beta_k\bar{2}$ where $\beta_k \neq 2$ for some $k \in \mathbb{N}$. As such, we can rewrite $b$ as $0._3\beta_1\beta_2 ... \beta_{k-1}[\beta_k + 1]$, and as such at some point a digit of $t_\gamma$ beyond $t_{\gamma,2n}$ will be nonzero, and so $t_\gamma > t_\lambda$.

**Case 1.ii:** $b$ has a non-terminating (possibly irregular) ternary representation not consisting exclusively of 2s

If this is the case, then we again see that at some $t_{\gamma,i}$ where $i > 2n$ that $t_{\gamma,i}$ will be nonzero, and as such $t_\gamma > t_\lambda$.

**Case 2:** $\sigma \equiv 1 \mod 2$

If $\sigma$ is odd then again, we'll start by further solving for the digits of $t_\lambda$ and $t_\gamma$, and then again break off into cases where the representation of $b$ can be exploited to produce distinct values. As such, note:

$$t_\lambda = 0._3 0\beta_. .. 0\beta_{n-1} 2 t_{\lambda,2n} t_{\lambda,2n+1} ...$$

Again, since $t_\lambda$ and $t_\gamma$ differ at one place in their ternary representations, for them to be equal it will be necessary that $t_{\lambda,j} = 0$ and $t_{\gamma,j} = 2$ for all $j \geq 2n$. Solving for further digits of the two yields:

$$t_\gamma = 0._3 0\beta_1 ... 0\beta_{n-1} 1[k(\beta_n)] ... = 0._3 0\beta_1 ... 0\beta_{n-1} 1[2 - \beta_n] ...$$

$$t_\lambda = 0._3 0\beta_1 ... 0\beta_{n-1} 2\beta_n ...$$

So we have that $\beta_n$ must be 0 in order for these two to be equal. Following this assumption, one has:

$$t_\gamma = 0._3 0\beta_1 ... 0\beta_{n-1} 12[k^{\sigma+2}(0)] ... = 0._3 0\beta_1 ... 0\beta_{n-1} 122[k(\beta_{n+1})] ...$$

$$t_\lambda = 0._3 0\beta_1 ... 0\beta_{n-1} 20[k^{\sigma}(2)] ... = 0._3 0\beta_1 ... 0\beta_{n-1} 200[\beta_{n+1}] ...$$

So if $\beta_n = 0$ then $\beta_{n+1}$ must be zero as well. As such, the exact same leveraging of the representation of $b$ in cases 1.i and 1.ii can be employed to conclude that there is a solution such that $t_\lambda > t_\gamma$. And indeed, the entire argument can be repeated for the case in which $b \in \Lambda$ and $a$ is arbitrary, and the method of generating non-equal preimages is again to take advantage of the different representations of numbers.

□

Notice that as we iterate higher and higher, the Peano curve seems to get more and more twisted in $\mathcal{Q}$. This provides some intuition to the claim that $f_p$ is nowhere differentiable:

**Lemma 2.2.8:** $f_p$ is nowhere differentiable.

Again, this claim will be demonstrated for $\phi_p$, and then it will follow for $\psi_p$ and thus $f_p$. Now, let $t = 0._3 t_1 t_2 ... t_{2n} t_{2n+1} ... \in [0, 1]$ and let $t_* = 0._3 t_1 t_2 ... t_{2n} \tau_{2n+1} t_{2n+2} ...$ where $\tau_{2n+1} \equiv t_{2n+1} + 1 \mod 2$. Thus (with $\sigma = \sum_{i=1}^{n} t_{2i}$)

$$|\phi_p(t) - \phi_p(t_*)| = |0._3 t_1 ... t_{2n} t_{2n+1} t_{2n+2} ... - 0._3 t_1 ... t_{2n} \tau_{2n+1} t_{2n+2} ...|$$

$$= \left|0._3 t_1 [k^{t_2}(t_3)] ... - 0._3 t_1 [k^{t_2}(t_3)] ... [k^{\sigma}(\tau_{2n+1})] ...\right| = \frac{1}{3^{n+1}}$$

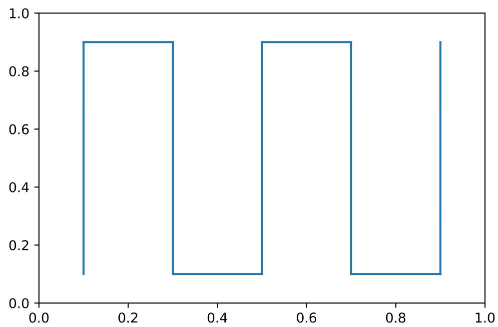
as $\phi_p(t)$ and $\phi_p(t_*)$ only differ in the $n+1$st decimal position. So then

$$\frac{|\phi_p(t) - \phi_p(t_*)|}{|t - t_*|} = \frac{\frac{1}{3^{n+1}}}{\frac{1}{3^{2n+1}}} = 3^n \to \infty \text{ as } n \to \infty$$

So $\phi_p$ is nowhere differentiable, and thus neither is $\psi_p$. □

## 2.3 Extending the Peano Curve

The Peano Curve is heavily dependent on the geometric idea of successive iterations partitioning subintervals into 9 pieces of equal length and then mapping to appropriate subsquares. This raises a number of questions though. First, what would be to stop us from instead partitioning into 16 pieces, or 25, or any square number? Would the function (well-definedness pending) still be a space-filling curve?



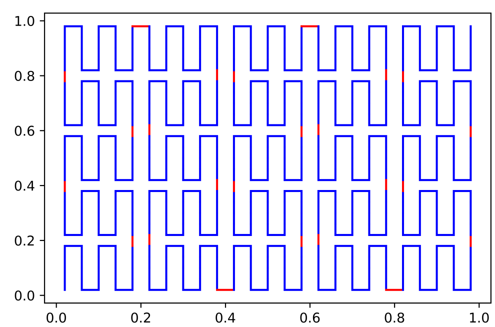



Figure 2.3: Instead of partitioning the unit square into $3^2$ subsquare, one could instead partition it into $5^2$ subsquares and map accordingly.

Now, since using base-3 representations was instrumental in partitioning the unit square into $3^2$ subsquares, one can extend the definition of the Peano Curve to an arbitrary base as follows:

**Definition 2.3.1:** Let $f_{p,3}$ denote the original Peano Curve presented earlier. Define the **base-$b$ Peano Curve** as the mapping $\mathbf{f_{p,b}} : \mathcal{I} \to \mathcal{Q}$ where $k(x) = b - x$ for $x \in \{0, 1, 2, ..., b-1\}$ and

$$0._b t_1 t_2 ... t_n ... \mapsto \begin{pmatrix} 0._b t_1 [k^{t_2}(t_3)][k^{t_2+t_4}(t_5)]... \\ 0._b [k^{t_1}(t_2)][k^{t_1+t_3}(t_4)]... \end{pmatrix} = \begin{pmatrix} \phi_b(t) \\ \psi b(t) \end{pmatrix}$$

This idea was earlier explored in [COL85] with an emphasis on applications in computer science. However, in this paper the author only considers finite iterations of Peano curves and Peano-like curves (as these are generally what is used in the applications of space-filling curves). More specifically, a correspondence is drawn in [COL87] between certain inputs (called "Gray Codes") for an arbitrary odd base $b$ and the vertices of certain iterations of the base-$b$ Peano Curve. I will now develop proofs in order to assert when the base-$b$ Peano Curve is truly a space-filling curve.

So, using this new nomenclature, I will refer to Peano's original Space-filling curve as the base-3 Peano Curve, $f_{p,3}$. This definition seems to be the most natural extension of Peano's Curve, and could provide a way to generate infinitely many unique space-filling curves. Let's first try to determine for what $b$ that $f_{p,b}$ seems to be reasonable.

**Lemma 2.3.2:** $f_{p,4}$ is ill-defined

In order for this to be shown, one just needs to produce some $t \in \mathcal{I}$ such that $t$ maps to distinct elements of $\mathcal{Q}$ based on its representation. So, let

$$t = 0._4 2,\ t_f = 0._4 1\overline{3}$$

Now, since $t = t_f$, it should be the case that their images are the same. However:

$$f_{p,4}(t) = \begin{pmatrix} 0._4 2 \\ 0._4 0 \end{pmatrix}$$

$$f_{p,4}(t_f) = \begin{pmatrix} 0._4 1[k^3(3)]... \\ 0._4[k^1(3)]... \end{pmatrix} = \begin{pmatrix} 0._4 10... \\ 0._4 0... \end{pmatrix}$$

So, as we see $f_{p,4}(t) \neq f_{p,4}(t_f)$, and as such the map is not well-defined. This remark leads to a larger conclusion:

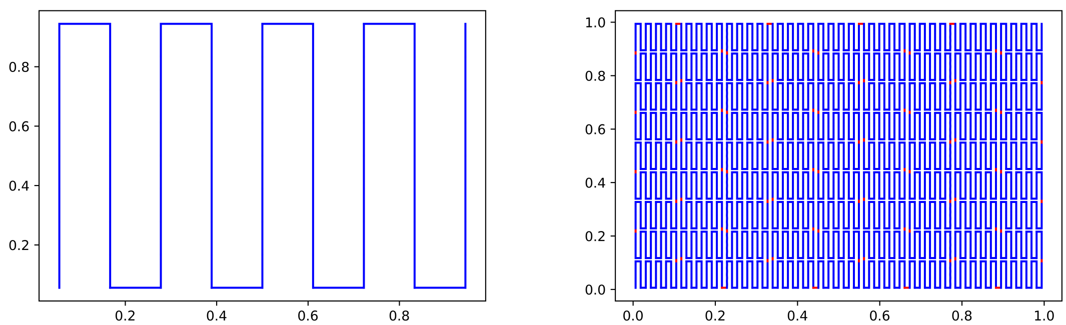



Figure 2.4: The first iterations of plotting $f_{p,9}$. Note that although $9 = 3^2$ is not prime, $f_{p,9}$ still appears upon first glace to be a space-filling curve.

**Theorem 2.3.3:** When $b = 2\ell$ is even, $f_{p,b}$ is ill-defined.

This will be a generalization of the previous argument, and essentially boils down to the fact that $k^2(x) = x$ in every base, and that when working with an even base, by considering representations of numbers with an infinite number of trailing $b-1$'s, the parity of $k$ will be altered. It will be easiest to consider two cases:

**Case 1:** $b = 2$

If $b = 2$ then take $\alpha = 0._2 1$, $\beta = 0._2 0\overline{1}$. Since $\alpha = \beta$, it'd be necessary that $f_{p,2}(\alpha) = f_{p,2}(\beta)$, however:

$$f_{p,2}(\alpha) = \begin{pmatrix} 0._2 1 \\ 0._2 \overline{1} \end{pmatrix} \neq \begin{pmatrix} 0._2 0... \\ 0._2 0... \end{pmatrix} = f_{p,2}(\beta)$$

Note that we only need to calculate the first digits of $f_{p,2}(\beta)$ in order to see the inequality, as no matter what values the next digits of $\psi_{p,2}(\beta)$ take, it will never be equal to $\psi_{p,2}(\alpha)$.

**Case 2:** $b = 2m,\ m \in \mathbb{N} \setminus \{1\}$

If $b = 2m > 2$, then let $\alpha = 0._b2,\ \beta = 0._b1[\overline{b-1}]$.

$$f_{p,b}(\alpha) = \begin{pmatrix} 0._b2 \\ 0._b0 \end{pmatrix} \neq \begin{pmatrix} 0._b10... \\ 0._b0... \end{pmatrix} = f_{p,b}(\beta)$$

So again, although $\alpha = \beta$, $f_{p,b}(\alpha) \neq f_{p,b}(\beta)$. As such, for each even integer $b$, $f_{p,b}$ is not a well-defined map. $\square$

Now that the case of $b = 2m$ has been ruled out, we will examine the case of $b = 2m + 1$ in depth in order to prove the following:

**Theorem 2.3.4:** When $b = 2m + 1$ is an odd integer, $f_{p,b}$ is a space-filling curve mapping from $\mathcal{I}$ to $\mathcal{Q}$.

To demonstrate this, well-definedness, surjectivity, and continuity will be proven as propositions. The arguments presented should feel reminiscent of the arguments presented for $f_{p,3}$.

**Proposition 2.3.5:** $f_{p,b}$ is a well-defined mapping when $b = 2m + 1$ is odd.

For this to be demonstrated, it needs to be shown that the image of a point $t \in \mathcal{I}$ is independent of representation. In other words, it will be necessary that if $t = 0._bt_1t_2...t_n$ then the following equality holds

$$f_{p,b}(0._bt_1t_2...t_n) = f_{p,b}(0._bt_1t_2...[t_n - 1][\overline{b-1}])$$

So, let $\alpha = f_{p,b}((0._bt_1t_2...t_n)$ and $\beta = f_{p,b}(0._5t_1t_2...[t_n - 1][\overline{b-1}])$ and the assertion will be made that $\alpha = \beta$. The trajectory of this proof will be almost exactly the same as the proof presented earlier showing that $f_{p,3}$ is well-defined, but this time all eight subcases will be shown to hold explicitly.

**Case 1:** $n = 2m$ is even

If $n = 2m$, then:

$$\alpha = \begin{pmatrix} 0._bt_1[k^{t_2}(t_3)]...[k^{t_2+...+t_{2m-2}}(t_{2m-1})][\overline{k^{t_2+...+t_{2m-2}+t_{2m}}(0)}] \\ 0._b[k^{t_1}(t_2)]...[k^{t_1+t_3+...+t_{2m-1}}(t_{2m})][\overline{k^{t_1+t_3+...+t_{2m-1}}(0)}] \end{pmatrix}$$

$$\beta = \begin{pmatrix} 0._bt_1[k^{t_2}(t_3)]...[k^{t_2+...+t_{2m-2}}(t_{2m-1})][\overline{k^{t_2+...+t_{2m-2}+t_{2m}-1}(b-1)}] \\ 0._b[k^{t_1}(t_2)]...[k^{t_1+t_3+...+t_{2m-1}}(t_{2m}-1)][\overline{k^{t_1+t_3+...+t_{2m-1}}(b-1)}] \end{pmatrix}$$

So consider the following subcases:

**Case 1.i:**

$$\sum_{i=1}^{m} t_{2i} \equiv 0 \mod 2,\ \sum_{i=1}^{m} t_{2i-1} \equiv 0 \mod 2$$

Then we can simplify $\alpha$ and $\beta$ as follows:

$$\alpha = \begin{pmatrix} 0._bt_1[k^{t_2}(t_3)]...[k^{t_2+...+t_{2m-2}}(t_{2m-1})]\overline{0} \\ 0._b[k^{t_1}(t_2)]...[k^{t_1+t_3+...+t_{2m-1}}(t_{2m})]\overline{0} \end{pmatrix} = \begin{pmatrix} 0._bt_1[k^{t_2}(t_3)]...[k^{t_2+...+t_{2m-2}}(t_{2m-1})] \\ 0._b[k^{t_1}(t_2)]...[k^{t_1+t_3+...+t_{2m-1}}(t_{2m})] \end{pmatrix}$$

$$\beta = \begin{pmatrix} 0._b t_1[k^{t_2}(t_3)]...[k^{t_2+...+t_{2m-2}}(t_{2m-1})]\overline{[k(b-1)]} \\ 0._b[k^{t_1}(t_2)]...[(t_{2m}-1)]\overline{[b-1]} \end{pmatrix}$$

$$= \begin{pmatrix} 0._b t_1[k^{t_2}(t_3)]...[k^{t_2+...+t_{2m-2}}(t_{2m-1})] \\ 0._b[k^{t_1}(t_2)]...[(t_{2m}-1)]\overline{[b-1]} \end{pmatrix}$$

As such, $\alpha = \beta$.

**Case 1.ii:**

$$\sum_{i=1}^{m} t_{2i} \equiv 1 \mod 2, \ \sum_{i=1}^{m} t_{2i-1} \equiv 1 \mod 2$$

$$\implies \alpha = \begin{pmatrix} 0._b t_1[k^{t_2}(t_3)]...[k^{t_2+...+t_{2m-2}}(t_{2m-1})]\overline{[b-1]} \\ 0._b[k^{t_1}(t_2)]...[k^{t_1+t_3+...+t_{2m-1}}(t_{2m})]\overline{[b-1]} \end{pmatrix}$$

$$= \begin{pmatrix} 0._b t_1[k^{t_2}(t_3)]...[k^{t_2+...+t_{2m-2}}(t_{2m-1})]\overline{[b-1]} \\ 0._b[k^{t_1}(t_2)]...[(b-1)-t_{2m}]\overline{[b-1]} \end{pmatrix}$$

$$\beta = \begin{pmatrix} 0._b t_1[k^{t_2}(t_3)]...[k^{t_2+...+t_{2m-2}}(t_{2m-1})]\overline{[b-1]} \\ 0._b[k^{t_1}(t_2)]...[k^{t_1+...+t_{2m-1}}(t_{2m}-1)]\overline{[k(b-1)]} \end{pmatrix}$$

$$= \begin{pmatrix} 0._b t_1...[k^{t_2+...+t_{2m-2}}(t_{2m-1})]\overline{[b-1]} \\ 0._b[k^{t_1}(t_2)]...[(b-1)-(t_{2m}+1)]\overline{0} \end{pmatrix} = \begin{pmatrix} 0._b t_1...[k^{t_2+...+t_{2m-2}}(t_{2m-1})]\overline{[b-1]} \\ 0._b[k^{t_1}(t_2)]...[b-t_{2m}]\overline{0} \end{pmatrix}$$

As such, no matter what value $t_{2m}$ takes in $\{1, ..., b-1\}$, one has $\alpha = \beta$.

**Case 1.iii:**

$$\sum_{i=1}^{m} t_{2i} \equiv 0 \mod 2, \ \sum_{i=1}^{m} t_{2i-1} \equiv 1 \mod 2$$

$$\implies \alpha = \begin{pmatrix} 0._b t_1...[k^{t_2+...+t_{2m-2}}(t_{2m-1})]\overline{0} \\ 0._b[k^{t_1}(t_2)]...[k(t_{2m})]\overline{[b-1]} \end{pmatrix} = \begin{pmatrix} 0._b t_1...[k^{t_2+...+t_{2m-2}}(t_{2m-1})]\overline{0} \\ 0._b[k^{t_1}(t_2)]...[b-1-t_{2m}]\overline{[b-1]} \end{pmatrix}$$

$$\beta = \begin{pmatrix} 0._b t_1...[k^{t_2+...+t_{2m-2}}(t_{2m-1})]\overline{[k(b-1)]} \\ 0._b[k^{t_1}(t_2)]...[k^{t_1+...+t_{2m-1}}(t_{2m}-1)]\overline{[k(b-1)]} \end{pmatrix} = \begin{pmatrix} 0._b t_1...[k^{t_2+...+t_{2m-2}}(t_{2m-1})]\overline{0} \\ 0._b[k^{t_1}(t_2)]...[b-1-t_{2m}+1]\overline{0} \end{pmatrix}$$

$$= \begin{pmatrix} 0._b t_1...[k^{t_2+...+t_{2m-2}}(t_{2m-1})]\overline{0} \\ 0._b[k^{t_1}(t_2)]...[b-t_{2m}]\overline{0} \end{pmatrix}$$

Again, for any $t_{2m} \in \{1, 2, ..., b-1\}$ the equality $\alpha = \beta$ holds.

**Case 1.iv:**

$$\sum_{i=1}^{k} t_{2i} \equiv 1 \mod 2, \ \sum_{i=1}^{k} t_{2i-1} \equiv 0 \mod 2$$

$$\implies \alpha = \begin{pmatrix} 0._b t_1...[k^{t_2+...+t_{2m-2}}(t_{2m-1})]\overline{[k(0)]} \\ 0._b[k^{t_1}(t_2)]...[k^0(t_{2m})]\overline{[k^0(0)]} \end{pmatrix} = \begin{pmatrix} 0._b t_1...[k^{t_2+...+t_{2m-2}}(t_{2m-1})]\overline{b-1} \\ 0._b[k^{t_1}(t_2)]...t_{2m}\overline{0} \end{pmatrix}$$

$$\beta = \begin{pmatrix} 0._b t_1 ... [k^{t_2+...+t_{2m-2}}(t_{2m-1})]\overline{[k^0(b-1)]} \\ 0._b[k^{t_1}(t_2)]...[k^0(t_{2m}-1)\overline{[k^0(b-1)]} \end{pmatrix} = \begin{pmatrix} 0._b t_1 ... [k^{t_2+...+t_{2m-2}}(t_{2m-1})]\overline{b-1} \\ 0._b[k^{t_1}(t_2)]...[t_{2m}-1]\overline{b-1} \end{pmatrix}$$

So again $\alpha = \beta$ for any appropriate $t_{2m}$, and so when $n$ is even then $\alpha = \beta$.

**Case 2:** n=2m+1 is odd.

If $n = 2m + 1$ is odd, then in the most general sense one has that:

$$\alpha = \begin{pmatrix} 0._b t_1[k^{t_2}(t_3)]...[k^{t_2+t_4+...+t_{2m}}(t_{2m+1})]\overline{[k^{t_2+...+t_{2m}}(0)]} \\ 0._b[k^{t_1}(t_2)]...[k^{t_1+...+t_{2k-1}}(t_{2m})]\overline{[k^{t_1+...+t_{2m-1}+t_{2m+1}}(0)]} \end{pmatrix}$$

$$\beta = \begin{pmatrix} 0._b t_1[k^{t_2}(t_3)]...[k^{t_2+t_4+...+t_{2m}}(t_{2m+1}-1)]\overline{[k^{t_2+...+t_{2m}}(b-1)]} \\ 0._b[k^{t_1}(t_2)]...[k^{t_1+...+t_{2m-1}}(t_{2m})]\overline{[k^{t_1+...+t_{2m-1}+t_{2m+1}-1}(b-1)]} \end{pmatrix}$$

Again, no further deductions can be made without considering four potential sub-cases:

**Case 2.i:**

$$\sum_{i=1}^{m} t_{2i} \equiv 0 \mod 2, \ \sum_{i=1}^{m} t_{2i+1} \equiv 0 \mod 2$$

$$\implies \alpha = \begin{pmatrix} 0._b t_1...t_{2m+1}\overline{0} \\ 0._b[k^{t_1}(t_2)]...t_{2m}\overline{0} \end{pmatrix}, \ \beta = \begin{pmatrix} 0._b t_1...[t_{2m}-1]\overline{[b-1]} \\ 0._b[k^{t_1}(t_2)]...t_{2m}\overline{0} \end{pmatrix}$$

**Case 2.ii:**

$$\sum_{i=1}^{m} t_{2i} \equiv 1 \mod 2, \ \sum_{i=1}^{m} t_{2i+1} \equiv 1 \mod 2$$

$$\implies \alpha = \begin{pmatrix} 0._b t_1...[b-1-t_{2m+1}]\overline{[b-1]} \\ 0._b[k^{t_1}(t_2)]...[k^{t_1+...+t_{2m+1}}(t_{2m})]\overline{[b-1]} \end{pmatrix}$$

$$\beta = \begin{pmatrix} 0._b t_1...[b-1-t_{2m+1}+1]\overline{0} \\ 0._b[k^{t_1}(t_2)]...[k^{t_1+...+t_{2m-1}}(t_{2m})]\overline{[b-1]} \end{pmatrix}$$

**Case 2.iii:**

$$\sum_{i=1}^{m} t_{2i} \equiv 0 \mod 2, \ \sum_{i=1}^{m} t_{2i+1} \equiv 1 \mod 2$$

$$\implies \alpha = \begin{pmatrix} 0._b t_1...t_{2m+1}\overline{0} \\ 0._b[k^{t_1}(t_2)]...[k^{t_1+...+t_{2m-1}}(t_{2m})]\overline{[b-1]} \end{pmatrix}$$

$$\beta = \begin{pmatrix} 0._b t_1...[t_{2m+1}-1]\overline{[b-1]} \\ 0._b[k^{t_1}(t_2)]...[k^{t_1+...+t_{2m-1}}(t_{2m})]\overline{[b-1]} \end{pmatrix}$$

**Case 2.iv:**

$$\sum_{i=1}^{m} t_{2i} \equiv 1 \mod 2, \ \sum_{i=1}^{m} t_{2i+1} \equiv 0 \mod 2$$

$$\implies \alpha = \begin{pmatrix} 0._b t_1 ... [b-1-t_{2m+1}]\overline{[b-1]} \\ 0._b[k^{t_1}(t_2)]...[k^{t_1+...+t_{2m-1}}(t_{2m})]\overline{0} \end{pmatrix}$$

$$\beta = \begin{pmatrix} 0._b t_1 ... [b-1-t_{2m+1}+1]\overline{0} \\ 0._b[k^{t_1}(t_2)]...[k^{t_1+...+t_{2m-1}}(t_{2m})]\overline{0} \end{pmatrix}$$

For any $t_{2m+1} \in \{1, ..., b-1\}$ one has that $\alpha = \beta$ in all four subcases of case 2 and thus $f_{p,b}$ is well-defined for odd bases $b$. □

From here on, when speaking about $f_{p,b}$ it will be assumed that $b$ is odd. Now, much of the intuition that was gained from looking at Peano's original curve will serve as a guide to proving things for the general base $b$ Peano curve.

**Proposition 2.3.6:** $f_{p,b}$ is surjective.

Take an arbitrary point $p \in \mathcal{Q}$, such that

$$p = \begin{pmatrix} 0._b \alpha_1 \alpha_2 ... \alpha_n ... \\ 0._b \beta_1 \beta_2 ... \beta_n ... \end{pmatrix}$$

Just as with the original Peano Curve, individual digits of $f_{p,b}^{-1}(p)$ can be found by successively solving for them. Letting $t = 0._b t_1 t_2 ... t_n ... = f_{p,b}^{-1}(p)$ one has

$$\alpha_n = k^{t_2+t_4+...+t_{2n-2}}(t_{2n-1}),\ \beta_n = k^{t_1+t_3+...+t_{2n-1}}(t_{2n})$$

One can successively solve for $t_i$ as

$$t_{2n-1} = k^{t_2+t_4+...+t_{2n-2}}(\alpha_n),\ t_{2n} = k^{t_1+t_3+...+t_{2n-1}}(\beta_n)$$

As such, any $p \in \mathcal{Q}$ can be mapped to by some $t \in [0,1]$ □

In demonstrating continuity, it will again be useful to show the relationship between $\phi_{p,b}$ and $\psi_{p,b}$ (the coordinate functions of $f_{p,b}$).

**Proposition 2.3.7:** $\psi_b(t) = b\phi_b(t/b)$

Take some $t = 0._b t_1 t_2 ... t_n ... \in \mathcal{I}$. Then $t/b = 0._b 0 t_1 t_2 ...$ so that:

$$b\phi_b\left(\frac{t}{b}\right) = b\left(0._b 0[k^{t_1}(t_2)][k^{t_1+t_3}(t_4)]...[k^{t_1+...+t_{2n-1}}(t_{2n})]...\right)$$

$$= 0._b[k^{t_1}(t_2)][k^{t_1+t_3}(t_4)]... = \psi_b(t)$$

**Proposition 2.3.8:** Given $b = 2m+1$ is an odd integer, $f_{p,b}$ is continuous.

Again, the gluing lemma will be used in order to show that $\phi_b$ and $\psi_b$ are continuous, and continuity of $f_{p,b}$ will follow. As such, let $t_0 = 0._b t_1 t_2 ... t_{2n} t_{2n+1} ...$ be the representation of $t \in \mathcal{I}$ that doesn't have infinitely many trailing $b-1$'s, and let $\delta = b^{-2n} - 0._b 0 ... 0 t_{2n+1} t_{2n+2} ....$ Then

$$t_0 + \delta = 0._b t_1 t_2 ... t_{2n} t_{2n+1} ... + b^{-2n} - 0._b 00...0 t_{2n+1} t_{2n+2} ... = 0._b t_1 t_2 ... t_{2n-1} t_{2n} \overline{[b-1]}$$

As such, any $t \in [t_0, t_0 + \delta)$ can be written $t = 0._b t_1 ... t_{2n} \tau_{2n+1} \tau_{2n+2} ...$ for some suitable $\tau_i$. For shorthand, let $\sigma = \sum_{i=0}^{n} t_{2i}$:

$$|\phi_b(t) - \phi_b(t_0)| = \left| 0._b t_1 [k^{t_2}(t_3)] .... [k^{\sigma}(\tau_{2n+1})] ... - 0._b t_1 [k^{t_2}(t_3)] ... [k^{\sigma}(t_{2n+1})] ... \right|$$

$$\leq |k^{\sigma}(\tau_{2n+1}) - k^{\sigma}(t_{2n+1})| * b^{-(n+1)}$$

$$+ \left| k^{\sigma + \tau_{2n+2}}(\tau_{2n+3}) - k^{\sigma + t_{2n+2}}(t_{2n+3}) \right| * b^{-(n+2)} ...$$

$$\leq \frac{b-1}{b^{n+1}} + \frac{b-1}{b^{n+2}} + ... = \frac{b-1}{b^n} \sum_{i=1}^{\infty} \frac{1}{b^i} = \frac{1}{b^n}$$

So, as $n \to \infty$ one has that $|\phi_b(t) - \phi_b(t_0)| \to 0$, and as such $\phi_b$ is continuous from the right on $[0, 1)$.

Continuity from the left on $(0, 1]$ comes with a similar defense. Letting $t_0$ remain the same and $\delta = 0._b 0...0 t_{2n+1} t_{2n+2} ...$ then $t_0 - \delta = 0._b t_1 t_2 ... t_{2n-2} t_{2n}$, so $\forall t \in (t_0 - \delta, t_0]$ one has:

$$|\phi_b(t) - \phi_b(t_0)| = \left| 0._b t_1 [k^{t_2}(t_3)] .... [k^{\sigma}(t_{2n+1})] ... - 0._b t_1 [k^{t_2}(t_3)] ... [k^{\sigma}(t_{2n+1})] ... \right|$$

$$\leq \frac{b-1}{b^{n+1}} + \frac{b-1}{b^{n+2}} + ... = \frac{b-1}{b^{n+1}} \sum_{i=1}^{\infty} \frac{1}{b^i} = \frac{1}{b^n}$$

So $\phi_b$ is continuous on $[0, 1]$. As such, $\forall \epsilon_1 > 0, \exists \delta_1 : |x - y| < \delta_1 \implies |\phi_b(x) - \phi_b(y)| < \epsilon_1$ and so

$$\left| \frac{x}{b} - \frac{y}{b} \right| < \frac{\delta_1}{b} < \delta_1 \implies \left| \phi_b\left(\frac{x}{b}\right) - \phi_b\left(\frac{y}{b}\right) \right| < \epsilon_1$$

So, if $|x - y| < \delta_1$ then

$$d(f_{p,b}(x), f_{p,b}(y)) = \sqrt{(\phi_b(x) - \phi_b(y))^2 + \left(b\phi_b\left(\frac{x}{b}\right) - b\phi_b\left(\frac{y}{b}\right)\right)^2} < \epsilon_1 \sqrt{b^2 + 1}$$

So $f_{p,b}$ is continuous on $[0, 1]$. As such, when $b$ is an odd integer, $f_{p,b}$ is a space-filling curve. □

# Chapter 3

# The Hilbert Curve

## 3.1 The Hilbert Curve

Whereas Peano's curve emerged with a nice, closed form map, Hilbert's curve first begins with an idea concerning how a map that is a space-filling curve must handle intervals. Hilbert supposed that if $\mathcal{I}$ were to be mapped surjectively to $\mathcal{Q}$, then when partitioned into subintervals, each subinterval should map to an equivalent subsquare.

To show how Hilbert did this, first take the subintervals $[0, \frac{1}{4}], [\frac{1}{4}, \frac{1}{2}], [\frac{1}{2}, \frac{3}{4}]$, and $[\frac{3}{4}, 1]$. Reasonably, one would expect each of these four subintervals to map to four equivalent subsquares of $\mathcal{Q}$, where the overlapping endpoints of the subintervals would correspond to overlapping boundaries of the subsquares. Since it's necessary that the map be continuous, it'd be necessary that nearby subintervals map to nearby subsquares. In other words, it'd be desirable that adjacent subintervals map to adjacent subsquares.

Now, this process will be applied again to each of the initial subintervals, again breaking them down into 4 more subintervals, yielding 16 in total. As we iterate through this process, at iteration $n$ there will be $2^{2n}$ subintervals, each one of length $2^{-2n}$ and $2^{2n}$ subsquares, each one having a side length of $1/2^i$.

An intuitive way to do this would be to map the first subinterval to the bottom leftmost subsquare (containing the origin of the plane):

$$\mathbb{R} \supset \left[0, \frac{1}{4}\right] \mapsto \left[0, \frac{1}{2}\right]^2 \subset \mathbb{R}^2$$

From there, on the first iteration there are two possibilities for mapping the next subinterval: the subsquare $[0, \frac{1}{2}] \times [\frac{1}{2}, 1]$ lying above $[0, \frac{1}{2}]^2$, or the subsquare $[\frac{1}{2}, 1] \times [0, \frac{1}{2}]$ directly to the right. As it turns out, Hilbert dictated that the curve would map

$$\left[\frac{1}{4}, \frac{1}{2}\right] \mapsto \left[0, \frac{1}{2}\right] \times \left[\frac{1}{2}, 1\right]$$

With these stipulations, the images of the other two subintervals are constrained

$$\left[\frac{1}{2}, \frac{3}{4}\right] \mapsto \left[\frac{1}{2}, 1\right]^2, \ \left[\frac{3}{4}, 1\right] \mapsto \left[\frac{1}{2}, 1\right] \times \left[0, \frac{1}{2}\right]$$

These are constrained as we still require that adjacent subintervals in $\mathbb{R}$ map to adjacent subsquares in $\mathbb{R}^2$.

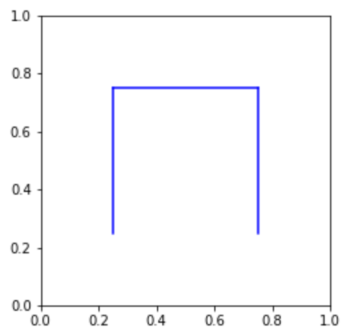
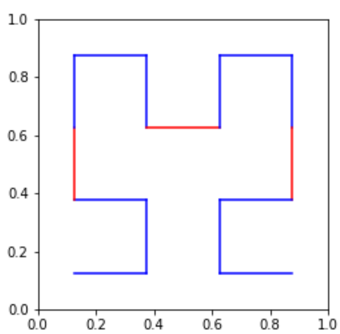
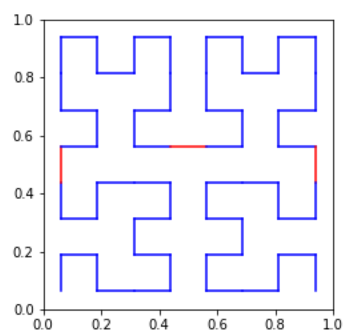



Figure 3.1: The first iteration of the Hilbert Curve

As with the Peano Curve, we'll generate the Hilbert Curve by appling this process ad infinitum, with the exception that we will need to permute copies of the previous iteration to produce the following iteration.




(a) The second iteration of the Hilbert Curve.




(b) The third iteration.

Figure 3.2: The second and third iterations of the Hilbert Curve. Notice that the portions in blue are exact copies of the previous iteration up to some amount of rotation, in order to preserve continuity of the mapping

In [SAG94], Sagan offers the following definition for the Hilbert Curve:

**Definition 3.1.1:** Every $t \in \mathcal{I}$ is determined uniquely by a sequence of nested closed intervals (generated by the successive partitioning described above), the lengths of which shrink to zero. With this sequence corresponds a unique sequence of nested closed squares, the diagonals of which shrink into a point, and which define a unique point in $\mathcal{Q}$. We will call this map $f_h : \mathcal{I} \to \mathcal{Q}$ the **Hilbert Curve** where

$$f_h(t) = \begin{pmatrix} \psi_h(t) \\ \phi_h(t) \end{pmatrix}$$

From the way that the Hilbert curve is constructed, a geometric argument can be formed to demonstrate that it is indeed a space-filling curve:

**Theorem 3.1.2:** The Hilbert Curve is a space-filling curve.

As before, this theorem will be demonstrated by showing continuity and surjectivity of the Hilbert Curve.

**Proposition 3.1.3:** The Hilbert Curve is continuous.

**Proof:** Let $t_1, t_2 \in \mathcal{I}$ be arbitrary. Then for some $n \in \mathbb{N}$, one has that $|t_1 - t_2| < 4^{-n}$. It will be useful to think of this $n$ as the "refinement" of the curve in some sense. For this $n$, one has that the length of the subintervals of $\mathcal{I}$ will be $4^{-n}$, and as such, the interval $[t_1, t_2]$ lies in at most two subintervals. These subintervals must be adjacent, as otherwise it would be the case that $|t_1 - t_2| > 4^{-n}$.

Now, by the way the Hilbert Curve has been constructed, it's guaranteed that adjacent subintervals get mapped to adjacent subsquares of side length $2^{-n}$. For any two of adjacent subsquares, a rectangle $\mathcal{R}$ of side lengths $2^{-n}$-by-$2^{-n+1}$ can be formed containing both subsquares. Using the Pythagorean Theorem, the length $c$ of the diagonal of $\mathcal{R}$ can be solved for as follows:

$$c^2 = \left(2^{-n}\right)^2 + \left(2^{-n+1}\right)^2 = 2^{-2n} + 2^{-2n+2} = 2^{-2n}\left(1 + 2^2\right) \implies c = 2^{-n}\sqrt{5}$$

As such, we can put a bound on the distance of $f_h(t_1)$ and $f_h(t_2)$ in $\mathcal{Q}$, summarized nicely by

$$|t_1 - t_2| < 4^{-n} \implies |f_h(t_1) - f_h(t_2)| < 2^{-n}\sqrt{5}$$

Continuity of $f_h$ follows immediately from this. □

**Proposition 3.1.4:** The Hilbert Curve is surjective

**Proof:** Take some $p \in \mathcal{Q}$ with the intent of finding its preimage in $\mathcal{I}$. Note that since $p$ is in the unit square, one can find a sequence of nested subsquares $D_1, D_2, ...$ such that $D_i$ is generated in the $i$-th iteration of the construction of the Hilbert Curve. As such, $D_i$ will be the image of some closed interval $L_i \subset \mathcal{I}$ of length $2^{-2i}$, and will have side lengths of $2^{-i}$. Note that the length of the diagonals of $D_i$ will tend to zero as $i$ grows. Further, since for each $i$ one has that $L_{i+1} \subset L_i$, by the

Nested Interval Property of the real numbers, one has that

$$\exists t_p \in \mathcal{I} : t_p \in \bigcap_{j=1}^{\infty} L_j$$

Now, since the diagonals of the sequence of $D_i$'s shrinks to zero, one has that $f_h(t_p) = p$, and as such $f_h$ is surjective. This allows the conclusion to be made that the Hilbert Curve is a space-filling curve mapping $\mathcal{I}$ onto $\mathcal{Q}$. $\square$

This proof also provides some insight to the non-injectivity of the Hilbert Curve– if at one of the entries $D_j$ in the sequence $\{D_i\}$ our point $p$ lies on the boundary of $D_j$, then there will be two potential intervals $L_j \subset \mathcal{I}$ which could potentially used in the sequence $\{L_i\}$, yielding two distinct preimages of $p$. As such, we see that the points $p \in \mathcal{Q}$ of non-injectivity of $f_h$ are points such that at least one coordinate can be written as $2^{-k}$ for some $k \in \mathbb{N}$.

The proof of continuity of $f_h$ also provides some insight to the question of differentiability as well:
**Lemma 3.1.5:** The Hilbert Curve is nowhere differentiable.

**Proof:** This will be done by considering the coordinate functions individually. Take any $t \in \mathcal{I}$. Then $\exists t_0 \in \mathcal{I}$ such that $|t - t_0| < 4^{-n} \implies |f_h(t) - f_h(t_0)| < 2^{-n} \implies |\phi_h(t) - \phi_h(t_0)| < 2^{-n}\sqrt{5}$. So then

$$\frac{|\phi_h(t) - \phi_h(t_0)|}{|t - t_0|} \leq \frac{2^{-n}\sqrt{5}}{4^{-n}} = 2^n\sqrt{5} \to \infty \text{ as } n \to \infty$$

So $\phi_h$ is nowhere differentiable. The exact same argument holds to demonstrate that $\psi_h$ is nowhere differentiable, and as such $f_h$ is nowhere differentiable. $\square$

# Chapter 4

# Conclusion

## 4.1 Applications of Space-filling Curves

As is often the case, this work did not have immediate applications outside of the implications it yielded in mathematics for some time. Eventually some practical uses in computer science were also found. These uses range from more efficient algorithms for performing numerical calculations to better geographical mapping. Two such examples will be discussed briefly.

### 4.1.1 Matrix Multiplication

One use for space-filling curves comes from the problem of computational matrix multiplication. Essentially all modern computers have memory that is composed linearly, where memory addresses follow one after another. When a 2-dimensional matrix is stored in the memory of a computer, it would be ideal if the addresses of nearby elements of said array are near one another in memory. However, there's no "natural" way of doing this, as there's not necessarily one method of storage that's obviously computationally better than another.

In [REI14], the authors investigate the computational benefits of using a pseudo space-filling curve to map nearby entries of the matrix to nearby entries in memory– "pseudo" as the matrices in question are finite, and so one only needs some finite number of iterations to be able to uniquely hit every element of the matrix at least once with a space-filling curve. The reasoning behind doing this is that modern processors are extremely optimized to traverse memory in a linear fashion, and so by cleverly encoding the matrix into memory, one can access its elements more quickly and hopefully reap the reward.

The authors of [REI14] found that when using the Hilbert Curve for matrix multiplication that the costs outweighed the benefits – the speed increase was marginal in comparison to the required energy input. However, using a curve called the "Morton Space-filling Curve", the authors found a competitive alternative against their test case. They also believe that by creating hardware that is more fine-tuned for the operations required, the benefits could be largely increased.

### 4.1.2 Geographic Maps

Space-filling curves can be used to improve the loading time of geographic maps. If a person pulls up a real-time map of their position, then it's important that the map show up quickly. This collection of nearby 2-dimensional points on a map will correspond to a collection of 1-dimensional addresses in memory, and ideally one wants nearby points on the map to correspond to nearby points in memory. A naïve approach would be to regularly place points on lines of latitude and longitude and map each line of latitude into memory. Unfortunately, this ignores longitudinal proximity, so this method would likely result in the map data loading very slowly.

For mapping, space-filling curves can be used to improve spatial locality of stored memory addresses, and can also be used to provide a way to dynamically place points on the map. If one considers the naïve approach above, then a fixed refinement is required across the mapping area, which may not be necessary, leading to inefficiencies. However, using space-filling curves, a more clever method of placing points can be employed when considering a geographic object, as described in [BAD16].

## 4.2 Summary

In this document, a brief historical overview of the search for space-filling curves is provided, and then two of the most well-know space-filling curves are explored and their properties are demonstrated. Common properties demonstrated are continuity, surjectivity, lack of injectivity, and nowhere differentiability. Though the work here may suggest otherwise, there are space-filling curves which are differentiable on some subset of the domain. Lebesgue is believed to be the first to have constructed an example of this, employing the Cantor set to define a space-filling curve from $\mathcal{I}$ to $\mathcal{Q}$ [SAG94]. In addition, a method presented in [COL85] for generating infinitely many unique space-filling curves is also presented. Arguments are developed defending the claim that these curves are indeed space-filling curves.

# Chapter 5

# References

## 5.1 References

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