

For gate set $\{H, Z, CZ, CCZ\}$ -

Since Z, CZ, CCZ gates are rotations in Z -basis.
so for any state $|x\rangle$, it will output (phase factor) $^{2x}|x\rangle$
So all these gates commute and add up; so do their terms
In \mathbb{F}_2 domain, addition is modulo 2 addition.

So, the polynomial function for this gate set is defined as:

$$C(x) : \mathbb{F}_2^n \rightarrow \mathbb{F}_2$$

But H gate is different, ... $\frac{x_i}{H} \frac{x_j}{x_j} |\Psi\rangle = \frac{1}{\sqrt{2}} \sum_{x_j} (-1)^{x_i x_j} |x_j\rangle$

By a new variable (x_j), we mean to introduce a new qubit in the circuit, an ancillary qubit and then apply all the gates of the previous qubit on the introduced ancillary qubit. But we do not show this ancillary qubit, we just assume it for simulation purpose.
The circuit equivalence and its proof is as following -

eg: (a) $|x\rangle \xrightarrow{\text{U}} \frac{x_i}{H} \frac{x_j}{V} |\Psi\rangle = \sqrt{H} U |x\rangle$ ---

(b) $|x\rangle \xrightarrow{\text{U}} \frac{x_i}{H} \frac{x_j}{x_j} \xrightarrow{\text{H}} |0\rangle \rightarrow \text{post selection}$
 $|0\rangle \xrightarrow{\text{H}} \frac{x_j}{x_j} \xrightarrow{\text{U}} \sqrt{H} U |x\rangle$ ---
ancillary qubit 'a'

So the H gate is replaced by a new CZ gate, an ancillary qubit, two H gates and post selection.

The proof is as following -

$$\Rightarrow (a) \Rightarrow |\Psi\rangle = \sqrt{H} U |x\rangle$$

$$(b) \Rightarrow |\Psi\rangle = |x\rangle \otimes |0\rangle$$

$$\text{apply } U \otimes I$$

$$|\Psi_1\rangle = (U|x\rangle) \otimes |0\rangle$$

$$\text{apply } I \otimes H$$

$$|\Psi_2\rangle = (U|x\rangle) \otimes \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$$

$$\text{apply } CZ \quad (\text{let } U|x\rangle = a|0\rangle + b|1\rangle)$$

$$|\Psi_2\rangle = (a|0\rangle + b|1\rangle) \otimes \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$$

$$|\Psi_3\rangle = a|0\rangle \otimes \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) + b|1\rangle \otimes \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$$

$$= \frac{1}{\sqrt{2}}(a|0\rangle + b|1\rangle) \otimes |0\rangle + \frac{1}{\sqrt{2}}(a|0\rangle - b|1\rangle) \otimes |1\rangle$$

$$\text{apply } H \otimes I$$

$$|\Psi_4\rangle = (a|0\rangle + b|1\rangle) \otimes \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$$

$$+ (a|0\rangle - b|1\rangle) \otimes \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$$

$$|\Psi_5\rangle = |0\rangle |\Psi_4\rangle = a \frac{1}{2}(|0\rangle + |1\rangle) + b \cdot \frac{1}{2}(|0\rangle - |1\rangle)$$

which is equal to -

$$\Rightarrow \frac{1}{\sqrt{2}} H (a|0\rangle + b|1\rangle)$$

$$|\Psi_5\rangle = \frac{1}{\sqrt{2}} H U |x\rangle$$

$$\text{apply } V -$$

$$|\Psi\rangle = \frac{1}{\sqrt{2}} \sqrt{H} U |x\rangle$$

\Rightarrow This also proves the normalisation factor of $\frac{1}{(\sqrt{2})^h}$ where h is number of H gates.

Introduction of T & S gate in PolyQ:

$$S \Rightarrow \begin{bmatrix} 1 & 0 \\ 0 & i \end{bmatrix} \Rightarrow \frac{\pi}{2} \text{ phase shift}$$

$$T \Rightarrow \begin{bmatrix} 1 & 0 \\ 0 & e^{i\frac{\pi}{4}} \end{bmatrix} \Rightarrow \frac{\pi}{4} \text{ phase shift.}$$

$$\Rightarrow \frac{x_i}{S} \frac{x_i}{x_i} \quad S|0\rangle = |0\rangle \quad S|1\rangle = i|1\rangle$$

$$\text{in general: } S|x_i\rangle = (e^{i\frac{\pi}{4}})^{2x_i} |x_i\rangle$$

$$\text{Similarly } \frac{x_i}{T} \frac{x_i}{x_i} \quad T|0\rangle = |0\rangle \quad T|1\rangle = e^{i\frac{\pi}{4}} |1\rangle$$

$$\text{in general: } T|x_i\rangle = (e^{i\frac{\pi}{4}})^{2x_i} |x_i\rangle$$

$$\text{Also, } \frac{x_i}{H} \frac{x_j}{x_j} \quad H|x_i\rangle = \frac{1}{\sqrt{2}} \sum_{x_j \in \{0,1\}} (e^{i\frac{\pi}{4}})^{x_i x_j} |x_j\rangle$$

We have switched to \mathbb{F}_8 for the possible values of $C(x)$.

So the new $C(x)$ is defined as following -

$$C(x) : \mathbb{F}_2^n \rightarrow \mathbb{F}_8$$

and now terms of polynomial have integer coefficients which we call weights.

The general format of final state is given by -

$$|\Psi\rangle = \frac{1}{(\sqrt{2})^h} \sum_{x_n, x_{n+1}, \dots, x_{n+h-1} \in \{0,1\}} (e^{i\frac{\pi}{4}})^{C(x) \% 8} |w_0 w_1 \dots w_{n-1}\rangle$$

Where w_i 's are the variable at the end of circuit or simply output variables.

In total, the polynomial eqn $C(x)$ can be constructed using previously discussed rules and new weights -

Gate	Weight	Term introduced
T	1	Linear $ x_i\rangle$
S	2	Linear $2x_i$
T^\dagger	7	Linear $7x_i$
S^\dagger	6	Linear $6x_i$
Z	4	Linear $4x_i$
H	4	Quadratic $4x_i x_j$
CZ	4	Quadratic $4x_i x_j$
CCZ	4	Cubic $4x_i x_j x_k$

Also, T & T^\dagger and S & S^\dagger are conjugate to each other and therefore their weights 1 & 7 and 2 & 6 are also conjugate in \mathbb{F}_8 domain.

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Circuit Example -



$$C(x) = 4x_0 x_2 + \frac{4}{\sqrt{2}} x_2 + 2x_1 + 4x_2 x_1 + 1 \cdot x_2 + 6x_2 + 4x_1 x_3 + 7x_3$$

$$|\Psi\rangle = \frac{1}{(\sqrt{2})^2} \sum_{\{x_2, x_3 \in \{0,1\}\}} (e^{i\frac{\pi}{4}})^{C(x) \% 8} |x_2 x_3\rangle$$

$$\text{Let } x_0 = 0 \text{ & } x_1 = 1, \text{ i.e. } |\Psi\rangle = |01\rangle$$

$$\Rightarrow C(x) = 2 + 4x_2 + 4x_2 + x_2 + 6x_2 + 4x_3 + 7x_3$$

$$= 2 + 15x_2 + 11x_3$$

Iterate over all possible combinations of variables except the input variables -

x_2	x_3	$C(x)$	$C(x) \% 8$	$(e^{i\frac{\pi}{4}})^{C(x) \% 8}$
0	0	2	2	$e^{i\frac{\pi}{4} \cdot 2} = j$
0	1	13	5	$e^{i\frac{\pi}{4} \cdot 5} = -\frac{1}{\sqrt{2}}(1+j)$
1	0	17	1	$e^{i\frac{\pi}{4} \cdot 1} = \frac{1}{\sqrt{2}}(1+j)$
1	1	28	4	$e^{i\frac{\pi}{4} \cdot 4} = -1$

$$|\Psi\rangle = \frac{1}{2} \left[j|00\rangle - \frac{1}{\sqrt{2}}(1+j)|01\rangle + \frac{1}{\sqrt{2}}(1+j)|10\rangle - 1|11\rangle \right]$$

$$= j|00\rangle - \frac{1}{\sqrt{2}}(1+j)|01\rangle + \frac{1}{\sqrt{2}}(1+j)|10\rangle - |11\rangle$$