$\{H, Z, CZ, CCZ\} \Rightarrow$ T + S

formalution → polynomials

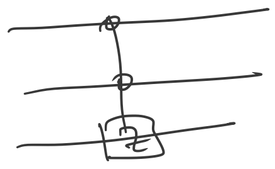
Gate Matrix

$H - \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$ ✓

$Z - \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \rightarrow |0\rangle \rightarrow |0\rangle$, $|1\rangle \rightarrow -|1\rangle$

$CZ - \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \rightarrow CZ|11\rangle \Rightarrow -|11\rangle$

$CCZ -$ ~ $\rightarrow CCZ|111\rangle \Rightarrow -|111\rangle$



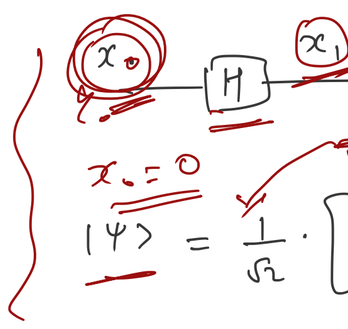
$H|0\rangle = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix}$

① H-gate – $H|0\rangle = \frac{|0\rangle + |1\rangle}{\sqrt{2}} = \frac{|0\rangle + (-1)^{x}|1\rangle}{\sqrt{2}}$, $x = 1$

$H|1\rangle = \frac{|0\rangle - |1\rangle}{\sqrt{2}} = \frac{|0\rangle + (-1)^{x}|1\rangle}{\sqrt{2}}$, $x = -1$

x = initial state $(\{0,1\})$

How?



$|\psi\rangle = \frac{1}{(\sqrt{2})^1}\sum_{x_1 \in \{0,1\}} (-1)^{x_0 x_1}|x_1\rangle$ → output variable

$x_0 = 0$

$|\psi\rangle = \frac{1}{\sqrt{2}} \cdot \left[(-1)^{x_0 \cdot 0}|0\rangle + (-1)^{x_0 \cdot 1}|1\rangle\right]$

$|\psi\rangle = \frac{|0\rangle + (-1)^{x_0}|1\rangle}{\sqrt{2}}$

input variable

in-between

output variable

a new variable ⇒ H gate

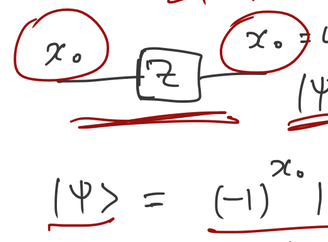
② Z-gate

$Z|0\rangle = |0\rangle = (-1)^{x_0}|w_0\rangle$

$Z|1\rangle = -|1\rangle = (-1)^{x_0}|w_0\rangle$

$Z|x\rangle = (-1)^{x}|x\rangle$

⇒



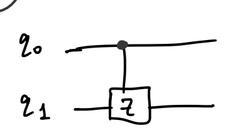
$|\psi\rangle = \frac{1}{(\sqrt{2})^0}(-1)^{x_0}|x_0\rangle$

$|\psi\rangle = (-1)^{x_0}|x_0\rangle$

if $x_0 = 0$ then $|\psi\rangle = |0\rangle$

if $x_0 = 1$ then $|\psi\rangle = -|1\rangle$

③ CZ-gate



$CZ|00\rangle = |00\rangle$

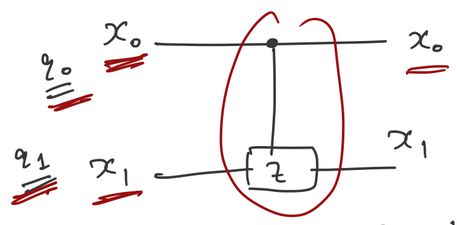
$CZ|01\rangle = |01\rangle$

$CZ|10\rangle = |10\rangle$

$CZ|11\rangle = -|11\rangle$

$(-1)^{x_0 x_1}|w_1 w_0\rangle$

⇒



$|\psi\rangle = \frac{1}{(\sqrt{2})^0}(-1)^{x_0 x_1}|x_0 x_1\rangle$

if initial state is $|11\rangle \Rightarrow |\psi\rangle = -|11\rangle$

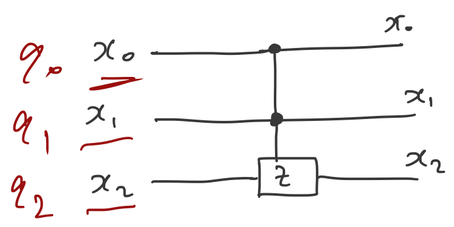
else it remains unchanged.

④ CCZ-gate

when input is $|111\rangle \rightarrow$ output is $-|111\rangle$

else unchanged

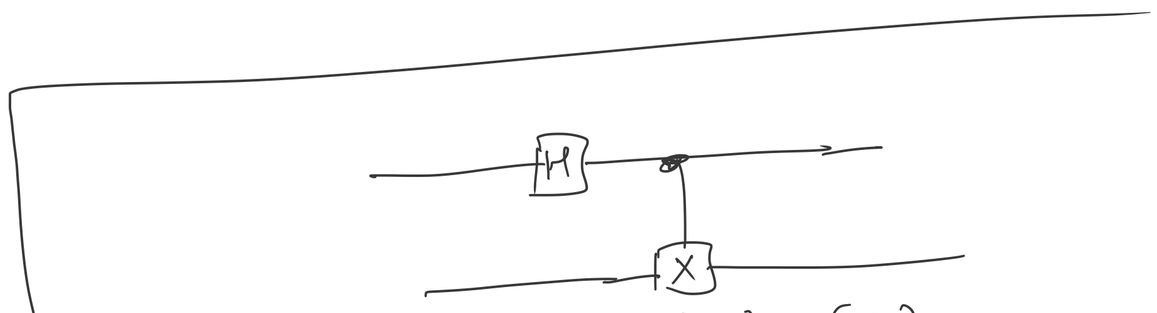
⇒



$|\psi\rangle = \frac{1}{(\sqrt{2})^0}(-1)^{x_0 x_1 x_2}|x_0 x_1 x_2\rangle$

if initial state is $|111\rangle$ then $|\psi\rangle = -|111\rangle$

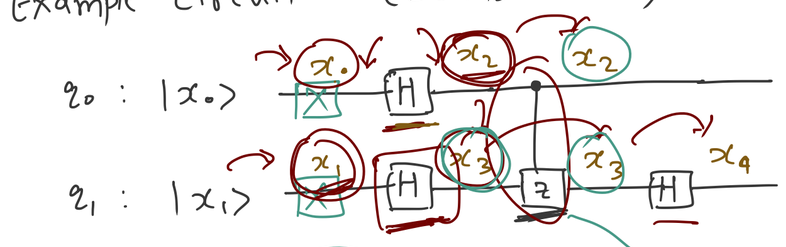
else unchanged.



$CX = (I \otimes H)\,CX\,(I \otimes H)$

$X = HZH$

Example circuit – (Bell's Circuit)



input → x_0/x_1

output → x_2/x_4

x_2 in-betw → x_3 } $|\psi\rangle$

x_4

$f(x) = x_0 x_2 \oplus x_1 x_3 \oplus x_2 x_3 \oplus x_3 x_4$

$x_i \in \{0,1\}$

$|\psi\rangle = \frac{1}{(\sqrt{2})^h}\sum_{x_2, x_3, x_4 \in \{0,1\}^3} (-1)^{f(x)}|x_2 x_4\rangle$

$w_0 w_1 = x_2 x_4$

let $x_0 = 0$ and $x_1 = 0$ (initial state $= |00\rangle$)

then $f(x) = x_2 x_3 \oplus x_3 x_4$

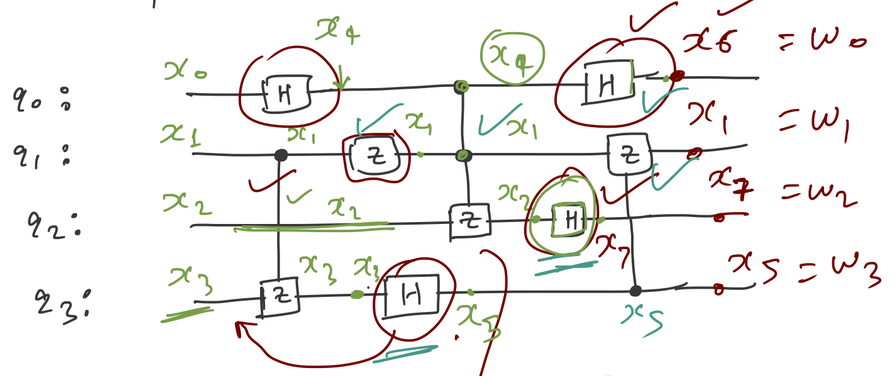
x_2	x_3	x_4	$f(x)$	$(-1)^{f(x)}$	$(-1)^{f(x)}\,\vert x_2 x_4\rangle$ ($w_0 w_1$)
0	0	0	0	1	$1.\vert 00\rangle$
0	0	1	0	1	$1.\vert 01\rangle$
0	1	0	0	1	$1.\vert 00\rangle$
0	1	1	1	-1	$-1.\vert 01\rangle$
1	0	0	0	1	$1.\vert 10\rangle$
1	0	1	0	1	$1.\vert 11\rangle$ ✓
1	1	0	1	-1	$-1.\vert 10\rangle$
1	1	1	2 ≡ 0	1	$1.\vert 11\rangle$

Total sum – $|\psi\rangle = \frac{2|00\rangle + 2|11\rangle}{(\sqrt{2})^3} = \frac{|00\rangle + |11\rangle}{\sqrt{2}}$ ($(\sqrt{2})^3 = 2\sqrt{2}$)

⊕ ✓

Ashley Montanaro | PolyQ

Example ② with CCZ gate – (random circuit)



$f(x) = x_0 \cdot x_4 \oplus x_1 \cdot x_3 \oplus x_1 \oplus x_3 x_5$

H CZ

$\oplus\ x_4 x_1 x_2 \oplus x_2 x_7 \oplus x_4 x_6 \oplus x_5 x_1$

n qubits

$|\psi\rangle = \frac{1}{(\sqrt{2})^h}\sum_{x_n, x_{n+1}, \ldots x_{n+h-1} \in \{0,1\}^h} (-1)^{f(x)}|w_0 w_1 \cdots w_{n-1}\rangle$

n variables

$x_0 \rightarrow x_{n-1}$

H gate → addition of a variable if we have h H gates

x_{n-1+h}