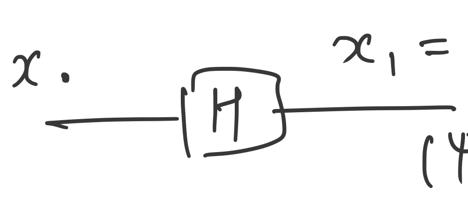
$$H|0\rangle = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

(1) **H-gate** – $H|0\rangle = \frac{|0\rangle + |1\rangle}{\sqrt{2}} = \frac{|0\rangle + (-1)^x |1\rangle}{\sqrt{2}}$

$$H|1\rangle = \frac{|0\rangle - |1\rangle}{\sqrt{2}} = \frac{|0\rangle + (-1)^x |1\rangle}{\sqrt{2}}$$

x = initial state $(\{0,1\})$

How?



$$|\psi\rangle = \frac{1}{(\sqrt{2})^1} \sum_{x_1 \in \{0,1\}} (-1)^{x_0 x_1} |x_1\rangle$$

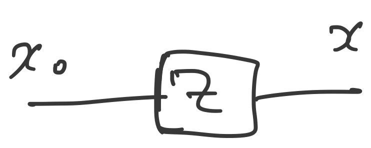
$$|\psi\rangle = \frac{1}{\sqrt{2}} \cdot \left[(-1)^{x_0 \cdot 0} |0\rangle + (-1)^{x_0 \cdot 1} |1\rangle \right]$$

$$|\psi\rangle = \frac{|0\rangle + (-1)^{x_0} |1\rangle}{\sqrt{2}}$$

(2) **Z-gate** $\quad Z|0\rangle = |0\rangle = (-1)^{x_0} |w_0\rangle$

$$Z|1\rangle = -|1\rangle = (-1)^{x} |w_0\rangle$$

$\Rightarrow$

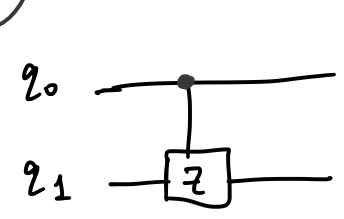


$$|\psi\rangle = \frac{1}{(\sqrt{2})^0} \cancel{\sum} (-1)^{x_0} |x_0\rangle$$

$$|\psi\rangle = (-1)^{x_0} |x_0\rangle$$

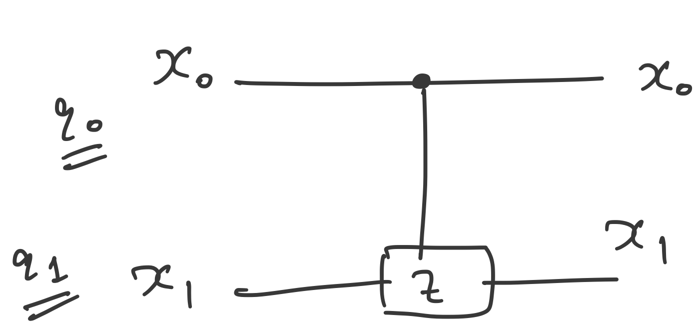
if $x_0 = 0$ then $|\psi\rangle = |0\rangle$
if $x_0 = 1$ then $|\psi\rangle = -|1\rangle$

(3) CZ-gate



$$\begin{aligned} CZ\, |\overset{q_1 q_0}{00}\rangle &= |00\rangle \\ CZ\,|01\rangle &= |01\rangle \\ CZ\,|10\rangle &= |10\rangle \\ CZ\,|11\rangle &= -|11\rangle \end{aligned} \Bigg\} \; (-1)^{x_0 x_1} |w_1 w_0\rangle$$

$\Rightarrow$



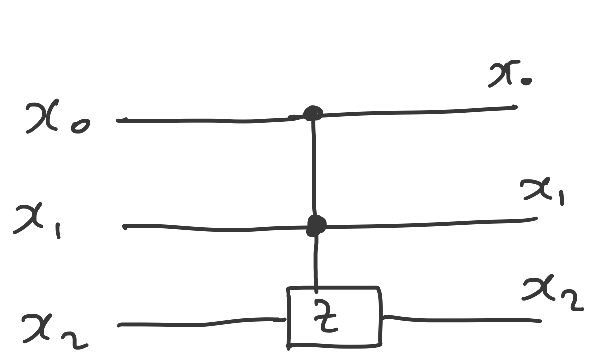
$$|\psi\rangle = \frac{1}{(\sqrt{2})^0} \cancel{\sum} (-1)^{x_0 x_1} |x_0 x_1\rangle$$

if initial state is $|11\rangle \Rightarrow |\psi\rangle = -|11\rangle$
else it remains unchanged.

(4) CCZ-gate

when input is $|111\rangle \rightarrow$ output is $-|111\rangle$
else unchanged

$\Rightarrow$

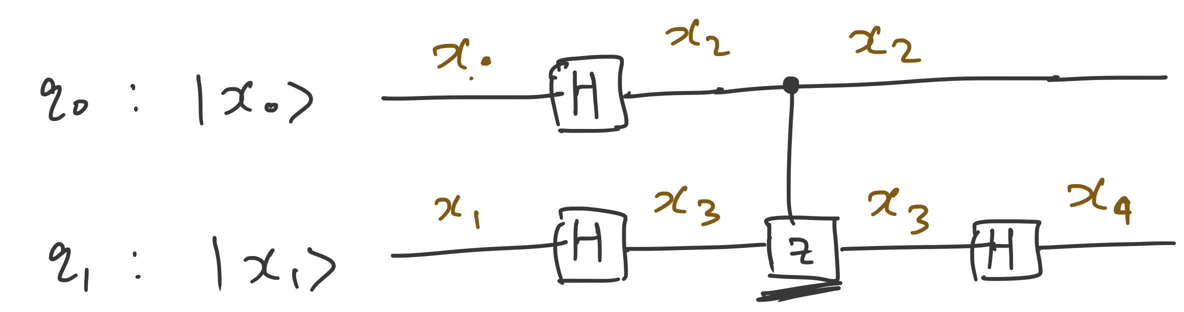


$$|\psi\rangle = \frac{1}{(\sqrt{2})^0} \cancel{\sum} (-1)^{x_0 x_1 x_2} |x_0 x_1 x_2\rangle$$

if initial state is $|111\rangle$ then $|\psi\rangle = -|111\rangle$
else unchanged.

$$X = HZH$$

Example circuit – (Bell's Circuit)



input $\rightarrow x_0 / x_1$
output $\rightarrow x_2 / x_4$
in-betw $\rightarrow x_3$

$x_2, x_4 \} |\psi\rangle$

$$f(x) = x_0 x_2 \oplus x_1 x_3 \oplus x_2 x_3 \oplus x_3 x_4$$

$x_i \in \{0,1\}$

$$|\psi\rangle = \frac{1}{(\sqrt{2})^h} \sum_{x_2, x_3, x_4 \in \{0,1\}^3} (-1)^{f(x)} |x_2 x_4\rangle \rightarrow \begin{matrix} w_0 w_1 \\ = x_2 x_4 \end{matrix}$$

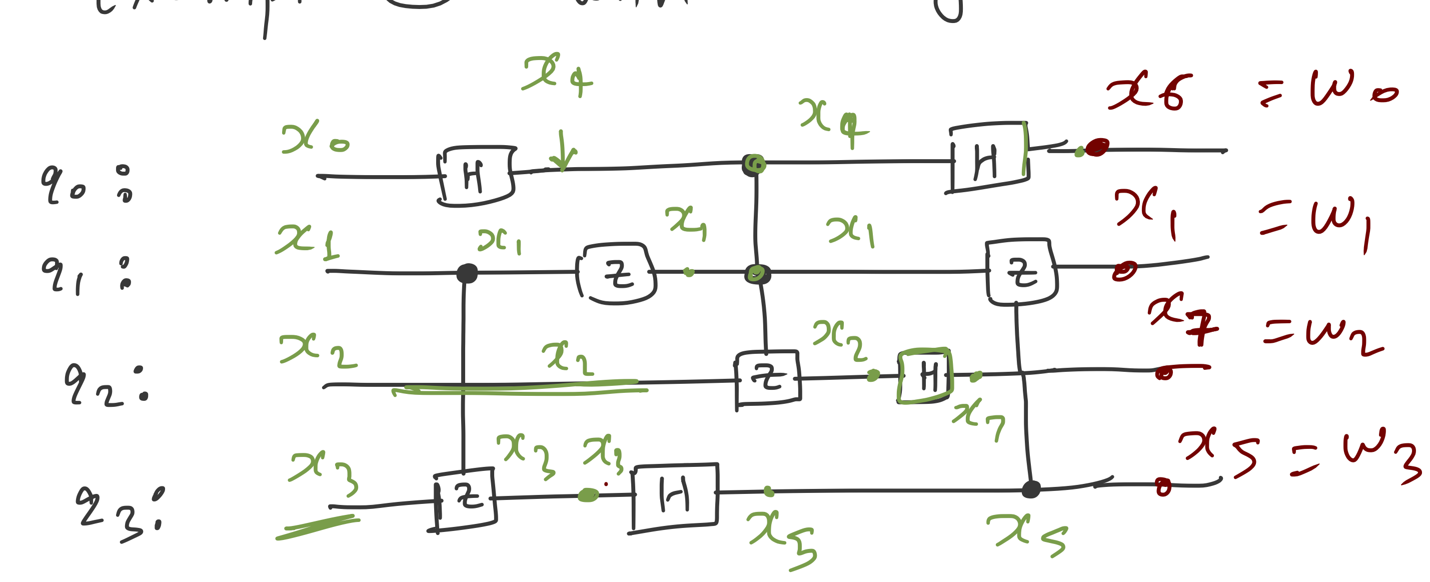
let $x_0 = 0$ and $x_1 = 0$ (initial state $= |00\rangle$)

then $f(x) = x_2 x_3 \oplus x_3 x_4$

x_2	x_3	x_4	$f(x)$	$(-1)^{f(x)}$	$(-1)^{f(x)} \vert x_2 x_4\rangle$
0	0	0	0	1	$1. \vert 00\rangle$
0	0	1	0	1	$1. \vert 01\rangle$
0	1	0	0	1	$1. \vert 00\rangle$
0	1	1	1	-1	$-1. \vert 01\rangle$
1	0	0	0	1	$1. \vert 10\rangle$
1	0	1	0	1	$1. \vert 11\rangle$
1	1	0	1	-1	$-1. \vert 10\rangle$
1	1	1	$2 \equiv 0$	1	$1. \vert 11\rangle$

Total sum – $|\psi\rangle = \frac{2|00\rangle + 2|11\rangle}{(\sqrt{2})^3} = \frac{|00\rangle + |11\rangle}{\sqrt{2}}$

Example (2) with CCZ gate – (random circuit)



$\} |\psi\rangle$

$$f(x) = x_0 \cdot x_4 \oplus x_1 \cdot x_3 \oplus x_1 \oplus x_3 x_5 \oplus x_4 x_1 x_2 \oplus x_2 x_7 \oplus x_4 x_6 \oplus x_5 \cdot x_1$$

$\Rightarrow n$ qubits

$$|\psi\rangle = \frac{1}{(\sqrt{2})^h} \sum_{x_n, x_{n+1}, \ldots, x_{n+h-1} \in \{0,1\}^h} (-1)^{f(x)} |w_0 w_1 \cdots w_{n-1}\rangle$$

n variables

$x_0 — x_{n-1}$

H gate $\rightarrow$ addition of a variable
if we have h H gates

x_{n-1}th