An Algorithm for Takes in Numero

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Abstract

This article describes a brute-force algorithm for determining valid 'takes' in the card game Numero, by generating permutations of a multiset to describe specific takes in the general case, or simply a set in a specific case, then calculating the result of the take, and valid answers can then be written to a file. Some Numero puzzles are included at the end with some solutions, calculated using the algorithm described.

1 What is Numero?

1.1 Some history and background

The card game 'Numero' was created by Frank Drysdale as a product of playing card games with his grandchildren as a way of keeping mentally fit after being diagnosed with Alzheimer's disease[1]. The game was shown to the Mathematics Association of Western Australia in 1993 and quickly grew, with more than 40 000 packs sold by 1998, being a familiar sight is classrooms across WA, including that of my Year 4 class in primary school, where I first encountered the game.

Frank has since passed away, but in 2018 his daughter Julie gained publishing and distribution rights to the game, with plans of continuing the previous success of Numero[1].

1.2 How to play

Each player starts with 5 cards, and there are 5 cards in the centre.

Player 1	**Centre (shared cards)**	**Player 2**
$1\ 10\ 15\ \underset{\text{W}}{\times 2}\ \underset{\text{W}}{\div 3}$	$1\ \underset{\text{W}}{\tfrac{2}{3}}\ 14\ \underset{\text{W}}{-5}\ \underset{\text{W}}{\sqrt{N}}$	$2\ \underset{\text{W}}{-4}\ 6\ \underset{\text{W}}{\tfrac{3}{4}}\ \underset{\text{W}}{N^3}$

[1] From 'History of Numero' on the Numero website https://numero.org/history-of-numero/

Cards can either be ‘number cards’ or ‘wildcards’. Number cards have whole numbers from 1 to 15 inclusive, and wildcards (denoted with a ‘W’ underneath their symbols) perform a variety of different operations, such as multiplication by some number, or subtraction or raising to some power. Note that the fractional wildcards are multiplications, but the multiplication symbols has been omitted for brevity.

Players can perform ‘takes’ using the cards in their hand and with the cards in the centre. To do this, the cards in the centre must be combined in such a way that they equal a number card in the player’s hand. For example with the cards above, the centre cards must be combined to make 7, 10 or 15. When two number cards in the centre are combined, they are added, and wildcards from the hand can be used in addition to those in the centre. One possible take for Player 1 is:

$$15 = 1 + 14$$

The 15 from the player’s hand and the 1 and 14 from the centre are then collected to count for points later, and new cards are used to replace the cards used.

This take used 3 cards, however we use 4 cards by incorporating wildcards.

$$10 = 14 + 1 - 5$$

We can do even better, by noticing that we can take the square root of 1 to give 1 again.

$$10 = 14 + \sqrt{1} - 5$$

Or even better (if you are able to figure it out), you can use all five available wildcards and both number cards in the centre to take using the 1 in our hand.

$$1 = \sqrt{1 \times 2 \times \frac{2}{3} + 14 \div 3 - 5}$$

The used cards are then replaced and Player 2 can make their move.

Player 1	**Centre (shared cards)**	**Player 2**
3 10 15 $\underset{\text{W}}{\frac{1}{2}}$ $\underset{\text{W}}{\sqrt[3]{N}}$	11 6 $\underset{\text{W}}{\frac{3}{2}}$ 13 4	$\underset{\text{W}}{\frac{1}{10}}$ 6 $\underset{\text{W}}{N^3}$ $\underset{\text{W}}{\frac{3}{4}}$ $\underset{\text{W}}{-4}$

Another rule is that multiple groups can be taken with a single number card from a player’s hand. To illustrate this, note the following three ways of making 6, where each card is only used once.

$$\underset{\text{from hand}}{6} = \underset{\text{from centre}}{6} = 4 \times \frac{3}{2} = \left((11 + 13 - 4) \times \frac{1}{10}\right)^3 \times \frac{3}{4}$$

All cards used to make the multiple groups of 6 can be taken together with the 6 from the hand.

Players keep on taking turns, making 'takes', or if unable to, players can discard a card from their hand into the centre. As there are a set number of cards, the game ends when no more takes are possible, at which point the remaining centre cards are given to the last player to make a take, and the score is counted, either by counting the total number of cards taken, or by using a points system.

2 The algorithm

We wish to determine the take/s which use all cards possible, given the cards in a player's hand and those in the centre.

The difficulty with determining designing such an algorithm is firstly with describing possible takes in a convenient way. One method for doing so is as follows.

2.1 Linear takes

We define a linear take as one where all the cards used in the take are used in succession on one particular starting card.

As an example to help distinguish linear and non-linear takes, consider the following arrangement:

Centre:	6	9	$\underset{\text{W}}{\div 3}$	$\underset{\text{W}}{\div 2}$
Hand:	4	6		

One solution is $4 = ((6 \div 2) + 9) \div 3$. Notice that we start with 6 as the base, the successively apply cards onto it to obtain our final answer, and thus this is a linear take. This is in contrast with the solution $6 = (9 \div 3) + (6 \div 2)$, where we must apply the two wildcards onto different bases. Note that although we have two number cards in the centre, they do not necessarily have to be bases, as in the case with the linear take, as it was just used to add to the base, as if just another wildcard.

2.2 An algorithm for linear takes

Enumerating all possible linear takes is relatively easy, as a linear take can be considered an ordering of all the cards, beginning with the base card, then applying cards to the base in the specified order. For example, the linear take $((6 \div 2) + 9) \div 3$ corresponds to the ordering $6, \underset{\text{W}}{\div 2}, 9, \underset{\text{W}}{\div 3}$. We can assign each card a unique index and generate all permutations of the indices which begin with a valid base card, compute the answer corresponding to each take, and mark that take if it is valid. Note that the answers for these takes are not necessarily integers, nor are they limited to whichever number cards a player has in their hand, so it is necessary to filter out answers which do not satisfy these requirements.

For example, we assign indices to cards as follows:

Card:	6	9	$\underset{\text{W}}{\div 3}$	$\underset{\text{W}}{\div 2}$
Index:	1	2	3	4

The following table shows some of the takes corresponding to some of these permutations.

Index permutation	Card ordering	Take	Answer
1234	$6, 9, \underset{\text{W}}{\div 3}, \underset{\text{W}}{\div 2}$	$(6+9) \div 3 \div 2$	$\frac{5}{2}$
1243	$6, 9, \underset{\text{W}}{\div 2}, \underset{\text{W}}{\div 3}$	$(6+9) \div 3 \div 2$	$\frac{5}{2}$
1324	$6, \underset{\text{W}}{\div 3}, 9, \underset{\text{W}}{\div 2}$	$(6 \div 3 + 9) \div 2$	$\frac{11}{2}$
1342	$6, \underset{\text{W}}{\div 3}, \underset{\text{W}}{\div 2}, 9$	$(6 \div 3 \div 2) + 9$	11
1423	$6, \underset{\text{W}}{\div 2}, 9, \underset{\text{W}}{\div 3}$	$(6 \div 2 + 9) \div 3$	4
$\vdots$	$\vdots$	$\vdots$	$\vdots$
2431	$9, \underset{\text{W}}{\div 2}, \underset{\text{W}}{\div 3}, 6$	$(9 \div 2 \div 3) + 6$	$\frac{15}{2}$

The permutation of the indices and their corresponding answers can be stored and the list of possible answers can be checked to see if a take with a particular answer card is possible.

2.3 Branched takes

The case of branched takes (those which are not linear takes) is more complicated, as we cannot enumerate branched takes the same way then permute to obtain all possible takes. However, such a method should also work for linear takes, and we will see that our proposed algorithm does work as such.

We consider only takes which use all available cards, so all possible base cards must be added to each other at some point, however wildcards can be used on these bases before summing these together, and wildcards can even be played on bases which have already been summed. For the remainder of this section we will consider the following scenario:

Centre:	3	4	5	6	15
Hand:	$\underset{\text{W}}{\frac{3}{5}}$	$\underset{\text{W}}{\div 3}$	$\underset{\text{W}}{\frac{1}{8}}$	$\underset{\text{W}}{-4}$	$\underset{\text{W}}{\sqrt{N}}$

We have five base cards (3, 4, 5, 6 and 15). We can add one base to another, and doing so reduces the number of bases we have remaining (e.g. adding 3 to 4 to make 7 would mean we only have four bases left: 7, 5, 6 and 15). As we use all cards eventually, these additions continue until one base remains. For example, we can add 3 to 4 to make 7, then 5 to 15 to make 20, then 7 to 20 to make 27, then 27 to 6 to make 33. However, at any point during this we can also apply wildcards on any base. For example, we could use the $\underset{\mathsf{W}}{\div 3}$ on the 3 at the start, or on the 7, or on 15, or on 27 etc. This is the case for all the wildcards, and wildcards can also be played on the same base in different orders, for example we can apply $\underset{\mathsf{W}}{-4}$ onto 15 then $\underset{\mathsf{W}}{\div 3}$, or vice versa, and any other possible permutation of wildcards. We are now ready to enumerate these branched takes.

First we generate a permutation of the base cards. For example, 6, 4, 15, 5, 3. We then also assign unique indices to the wildcards, as follows:

Wildcards:	$\underset{\mathsf{W}}{\frac{3}{5}}$	$\underset{\mathsf{W}}{\div 3}$	$\underset{\mathsf{W}}{\frac{1}{8}}$	$\underset{\mathsf{W}}{-4}$	$\underset{\mathsf{W}}{\sqrt{N}}$
Indices:	1	2	3	4	5

Consider five spaces separated by commas, each space corresponding to a base.

____, ____, ____, ____, ____

We can place wildcards in these boxes to apply them to the corresponding base. For example

____, 54, ____, ____, ____

corresponds to considering the second base card (4), then applying the 5th then the 4th wildcards ($\underset{\mathsf{W}}{\sqrt{N}}$ then $\underset{\mathsf{W}}{-4}$).

We may also choose not to play any wildcards at this stage. We then choose two of these bases to add together. The process for this will be explained later, however we then have a total of four bases remaining, and can play wildcards on any of the four bases. We can then add two more bases and the process continues until there is only one base remaining and we have had the option of playing wildcards on it. The following diagram shows this, with vertical bars indicating when two bases are added together.

__, __, __, __, __ |__, __, __, __ |__, __, __ |__, __ |__

Applying all wildcards then corresponds to placing wildcards in boxes, with permutations within each box being significant. For example,

__, 5, __, __, __ |342, __, __, __ |__, 1, __ |__, __ |__

where at the first bar we add the second base to the first base (base index 3 is then assigned to be the new index 2, base index 4 is then assigned to be the new index 3, and similarly for index 5 and new index 4). The second bar corresponds to adding base 3 to base 2, the third bar to adding base 2 to base 1, and the fourth bar adding base 2 to base 1 (again, this will be explained later). This corresponds to the take

$$14 = \left(\left(\sqrt{4}+6\right)\times\frac{1}{8}-4\right)\div 3+(15+5)\times\frac{3}{5}+3$$

Finally, we can replace the commas (which delimit between spaces) and the vertical bars (which indicates addition of two bases) with an arbitrary symbol, such as 0. Then the above diagram becomes

$$05000\underset{|}{0}342000\underset{|}{0}010\underset{|}{0}0\underset{|}{0}$$

where the 0's corresponding to additions are indicated with a bar underneath them, for ease of reading. Note that it is always clear which 0's correspond to additions as in this case we start with 5 bases, so the 5th 0 corresponds to an addition, then after that the 4th 0, then the 3rd and so on. With this, we now have a way of enumerating all possible branched takes, up to the bases added in each addition, and producing all possible permutations of the above elements, we can generate all possible takes. The next subsection goes through one way of generating these permutations.

2.4 Permutations of a multiset

Firstly we must realise that the object

$$05000\underset{|}{0}342000\underset{|}{0}010\underset{|}{0}0\underset{|}{0}$$

is not strictly a set, because there are repeated elements. Instead such an object is a multiset, which is like a set but can contain multiples of the same element. Thus, we are actually looking for all permutations of a multiset.

Generating all permutations of a multiset is not a trivial task, however a number of papers provide algorithms for this. The algorithm used here is that proposed by Aaron Williams in his paper 'Loopless Generation of Multiset Permutations by Prefix Shifts'. I give a brief overview of Williams' algorithm here for completeness.

We can begin with any permutation of our multiset, although it is ideal to start with the permutation which is strictly non-increasing (e.g. 32100000), which is called the 'tail'. We define a function $\angle(s)$, where s is a permutation of the multiset, which returns the length of the longest non-increasing sequence, starting of the first element of s. For example, the tail 21000300 is non-increasing until its 5th element, so $\angle(21000300) = 5$. It is clear that $s = 32100000$ uniquely gives the maximum value of 8 for $\angle(s)$. This is useful as our approach will be to use a function to generate a new permutation based on a current permutation. We can then begin with the tail as our permutation, and repeatedly using our function to create new permutations until we end up back with the tail - and this is easy to check as we only need to check that $\angle$ of the permutation is the length of the permutation.

We now also define the prefix shift of a permutation. The prefix shift $\sigma_k(s)$ is a function of a permutation s and an integer k, where the function returns the permutation, but with the kth element of the initial permutation moved to the front of of the permutation, and elements are moved back to accomodate this. For example,

$$\sigma_3(11, 12, 13, 14) = (13, 11, 12, 14)$$

$$\sigma_7(3, 2, 1, 0, 0, 0, 0, 0) = (0, 3, 2, 1, 0, 0, 0, 0)$$

We now define the function

$$\lhd(s) = \begin{cases} \sigma_{\angle(s)+1}(s) & \text{if } i \leq n-2 \text{ and } s_{\angle(s)+2} > s_{\angle(s)} \\ \sigma_{\angle(s)+2}(s) & \text{if } i \leq n-2 \text{ and } s_{\angle(s)+2} \leq s_{\angle(s)} \\ \sigma_n(s) & \text{otherwise (i.e. if } \angle(s) \geq n-1) \end{cases}$$

where n is the length of the permutation. This function $\lhd(s)$ generates all permutations of the multiset given by s exactly once, and this is shown in Williams' paper. However, for our purposes we need only implement the above steps.

2.5 Partitions and the addition of bases

When considering the addition of bases, many 'ways' of adding bases together are equivalent up to permutations of the bases at the start. To illustrate this, consider the bases in order 11, 12, 13, 14 and 15. Each base is assigned an index based on their position in the sequence (i.e. 11 has index 1, 15 has index 5), and upon adding two bases together, the higher index base is added to the lower index base, and indices are moved down accordingly, For example, before and after adding 12 and 14 together,

Index:	1	2	3	4	5
Value before addition:	11	12	13	14	15
Value after addition:	11	26	13	15	

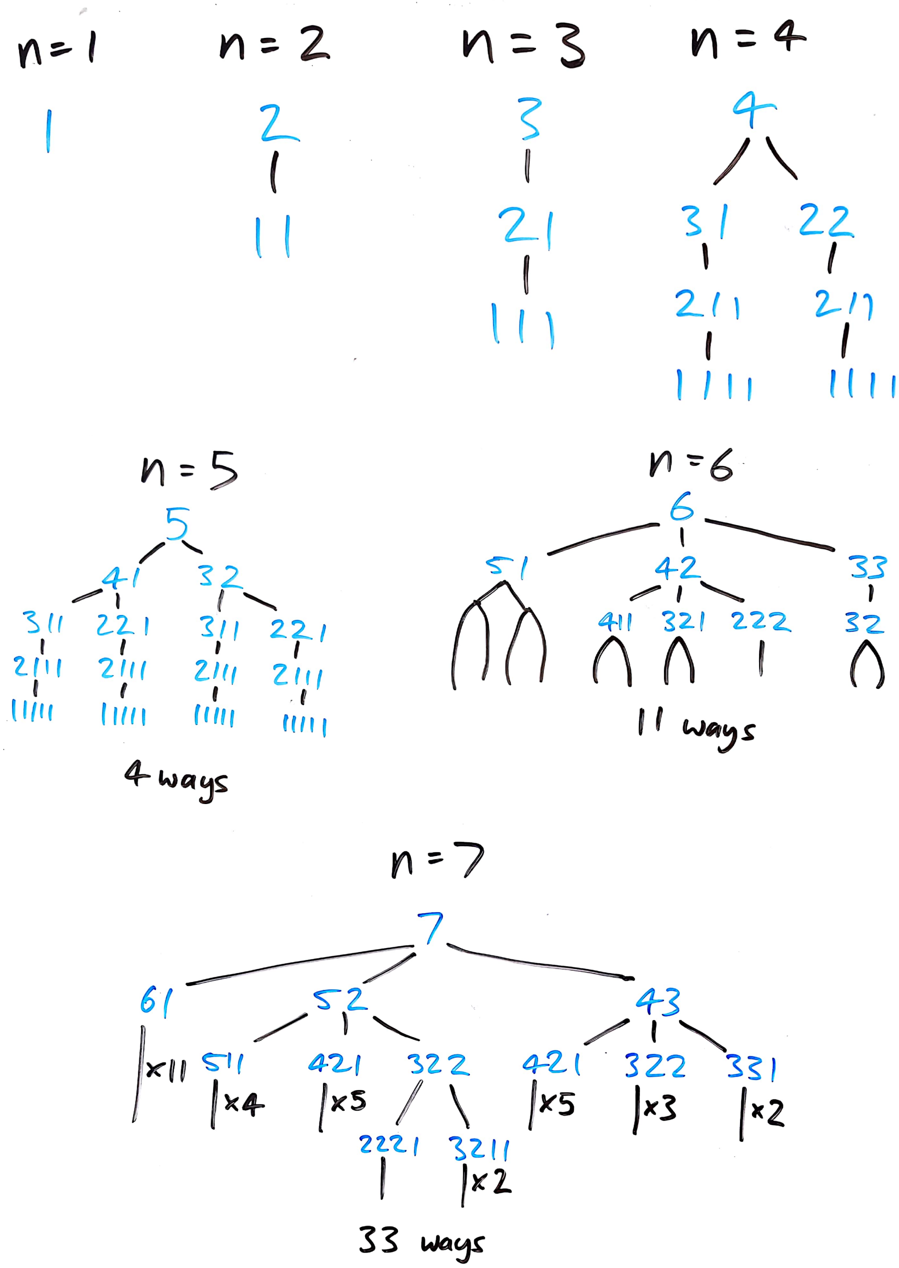
We can denote this addition with the index pair '24', meaning that the number with index 4 is added to the number with index 2 (and other indices are reordered as above). If we consider this addition with the multiset of bases 1, 1, 1, 1, 1, we obtain 1, 2, 1, 1. From here we can perform another addition, for example '12', meaning the 2nd base is added to the 1st, resulting in the bases 3, 1, 1. Now we perform '23', to obtain 3, 2, and then finally we perform '12' to obtain the final sum of 5.

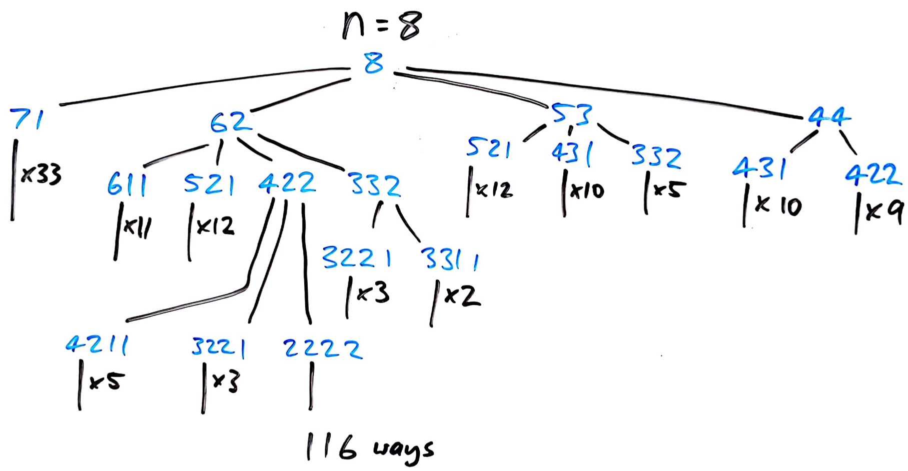
However, this particular way of addition is identical to the additions (in order) of ‘12’, ‘12’, ‘23’, ‘12’, as we can permute the bases at the beginning. The bases at each stage for this are 1, 1, 1, 1, 1, then 2, 1, 1, 1, then 3, 1, 1, then 3, 2, then 5. Note that these are the same combinations as with the first ‘way’ of addition of the bases. In fact, additions of bases are only distinct if the combinations at any point are different, for example with additions ‘12’, ‘12’, ‘12’, ‘12’, which corresponds to the bases 1, 1, 1, 1, 1, then 2, 1, 1, 1, then 3, 1, 1, then 4, 1, then 5. Note that these both correspond to unique ways of continually partitioning the number 5 until all elements are 1. This is also referred to as the number of ways of transforming a set of 5 indistinguishable objects into singletons via a sequence of refinements, according to the Online Encyclopedia of Integer Sequences (OEIS).

Note that a partition of an integer is a combination of integers greater than 0 which sum to that integer. For example, all partitions of 5 are (5), (4,1), (3, 2), (3, 1, 1), (2, 2, 1), (2, 1, 1, 1) and (1, 1, 1, 1, 1). In this article we define a ‘partitioning’ as a sequence of partitions of an integer, where consecutive partitions have exactly the same elements, except for one element, which is partitioned into two elements in the other partition. For example, all partitionings of the number 5 are

5	5	5	5
41	41	32	32
311	221	311	221
2111	2111	2111	2111
11111	11111	11111	11111
12121212	12231212	12122312	12232312

The following diagrams illustrate these unique partitionings for the cases of $n = 1$ to $n = 8$. Open-ended black lines denote obvious continuations (e.g. 222 must follow with 2211, then 21111, then 111111), or structures that have already occurred in that case or in cases of lower n (e.g. with 71, the 1 cannot be partitioned further, so further partitions are the same as the case for just $n = 7$, but with a 1 appended to each).


n=1
1
n=2
2
11
n=3
3
21
111
n=4
4
31
22
211
211
1111
1111
n=5
5
41
32
311
221
311
221
2111
2111
2111
2111
11111
11111
11111
11111
4 ways
n=6
6
51
42
33
411
321
222
321
11 ways
n=7
7
61
52
43
×11
511
×4
421
×5
322
421
×5
322
×3
331
×2
2221
3211
×2
33 ways



The total number of unique partitionings for increasing n starting from $n = 1$ is

$$1, 1, 1, 2, 4, 11, 33, 116, \cdots$$

and is sequence A002846 on the OEIS.

Returning to the addition of bases, the partitionings above correspond to distinct 'ways' of adding together bases, so the previous algorithm for determining takes must be repeated for each distinct partitioning for that number of bases, with the partitioning determining which bases are added given a particular permutation of the bases.

Each partitioning can be summarised in a numerical string which describes which elements to add together. These numerical strings are shown under the line in the previous table showing all partitionings of the number 5. For example, for 12231212 we start from (1,1,1,1,1) and add the 1st and 2nd elements, then the 2nd and 3rd, then the 1st and 2nd, then the 1st and 2nd again. This results in the partitioning shown in the table, but in reverse order.

The following table shows complete sets for numerical strings for up to 6 bases.

Bases	Numerical strings
1	-
2	12
3	1212
4	121212 122312
5	12121212 12231212 12122312 12232312
6	1212121212 1223121212 1212231212 1223231212 1212122312 1223122312 1212231312 1223231312 1223341212 1212232312 1223232312

2.6 Algorithm summary

To summarise the steps of the algorithm,

```
for each base permutation e.g.  (3, 4, 5, 6, 15), until all permutations performed
{
     for each way of adding the bases together
     {
         begin with s = ‘tail’ permutation
         while ∠(s) ≠ n or no permutations calculated yet for this partitioning
         {
             calculate value of the take specified by s, and record the permutation
             and answer if the answer is desirable
             redefine s ← ◁(s)
         }
     }
     use ◁ to create new permutation
}
```

The recorded permutations and answers can then be analysed.

2.7 Complexity

Let n be the number of bases and a be the number of wildcards. We also define $p(n)$ as the number of partitionings of n.

There are $n!$ base permutations (in the worst-case where all bases are distinct) and $p(n)$ ways to add together bases. The number of 0's in the permutation is given by $\frac{n(n+3}{2}$, so the number of permutations is

$$\frac{\left(\frac{n(n+3)}{2}+a\right)!}{\left(\frac{n(n+3)}{2}\right)!}$$

and thus the complexity of the algorithm is

$$O\left(\frac{\left(\frac{n(n+3)}{2}+a\right)!}{\left(\frac{n(n+3)}{2}\right)!}n!p(n)\right)$$

2.8 Further work

There are three major issues with this proposed algorithm as a perfect Numero solver.

Firstly, 'parallel' takes are not accounted for. To give an example of a parallel take, consider the following:

Centre:	6	9	$\underset{\text{W}}{\div 3}$	$\underset{\text{W}}{\div 2}$	6
Hand:	4	6			

A valid move is to make multiple groups of 6, for example performing $9 \div 3 + 6 \div 2$ to yield one group of 6, and using one of the 6's from the centre as the second group, and then using the 6 from the hand to clear both. This is clearly also an ideal take, but additional structure needs to be added to the algorithm to search these types of takes.

Secondly, the numerical bitstrings describing the partitionings were generated by hand using the partitioning diagrams shown, so it is not possible to scale in number of bases unless these are also done by hand, and this become very tedious quickly, with 33 ways with 7 bases, and 116 bases with 8 bases. Although, it can be argued that such cases only rarely occur in actual Numero play, as the only way to have more than 5 base cards is for players to have discarded extra cards to increase the number of base cards in the centre.

Thirdly, the algorithm is a brute-force algorithm, and the number of possible permutations of multisets for increasing numbers of bases grow very quickly.

3 Examples and Selected Results

The following examples were all analysed using the described algorithm, with the added features of counting the number of permutation-partitionings that give a particular answer, and counting only answer without fractional partial answers. Fractional partial answers are when the operations specified by a permutation result in a non-integer result. This is significant as solutions with fractional partial answers are very difficult to determine by hand in general, and such solutions are very common when multiple fractional wildcards are in play.

Note that 'permutation-partitioning' is used instead of 'solution'. This is because, due to arbitrary definitions of what distinct 'solutions' are, different permutation-partitionings can result in the same solution. For example, $2\times2+3$ and $3+2\times2$ are both different permutation-partitionings, but could be considered as the same solution. This is the case any time there are commuting operations, such as also when multiplying and dividing e.g. $4 \times \frac{3}{4} \times \frac{2}{3} \times \frac{1}{2}$ and $4 \times \frac{1}{2} \times \frac{2}{3} \times \frac{3}{4}$ are different permutation-partitionings, but could be considered as the same solution.

3.1 Example 1

Centre:	3	6	7	7	$\underset{\text{W}}{-4}$
Hand:	$\underset{\text{W}}{\frac{3}{2}}$	$\underset{\text{W}}{\frac{1}{3}}$	$\underset{\text{W}}{\frac{1}{4}}$	$\underset{\text{W}}{\sqrt{N}}$	

The wildcards are indexed:

Indices:	1	2	3	4	5
Wildcards:	$\underset{\text{W}}{-4}$	$\underset{\text{W}}{\frac{3}{2}}$	$\underset{\text{W}}{\frac{1}{3}}$	$\underset{\text{W}}{\frac{1}{4}}$	$\underset{\text{W}}{\sqrt{N}}$

Answer	Total PPs	Integer-only PPs
1	25320	22672
2	75490	57362
3	31242	30676
4	161642	131880
5	75398	61330
6	91772	89202
7	43633	33120
8	134960	133280
9	56520	44740
10	106848	98814
11	52858	44552
12	57676	54702
13	53810	44020
14	14844	11462
15	70144	70112

A particular PP for the answer 14 is as follows, with the permutation, numerical bitstring (describing the addition of bases), and then the initial base permutation.

$$00200453100000 \qquad 121212 \qquad 3\,6\,7\,7$$

Wildcard 2 ($\frac{3}{2}$) is applied to the 3rd base (7) to obtain $\frac{21}{2} = 7 \times \frac{3}{2}$. The first two bases, 3 and 6 and then added to make 9, which is the new first base. Then we apply wildcards 4, 5, 3 and 1 to obtain $-\frac{7}{2} = \sqrt{9 \times \frac{1}{4}} \times \frac{1}{3} - 4$. We can then add all the remaining bases together to obtain $14 = \frac{21}{2} - \frac{7}{2} + 7$.

4 Puzzles

1. Find six number cards that can be used to take all the following cards.
   $2 \quad 3 \quad 7 \quad \underset{\mathsf{W}}{\tfrac{1}{3}} \quad \underset{\mathsf{W}}{\tfrac{3}{2}}$

   Harder: Find nine number cards that can be used to take all the above cards.

2. Find all number cards that can be used to take all the following cards.
   $1 \quad 4 \quad 9 \quad \underset{\mathsf{W}}{\times 4} \quad \underset{\mathsf{W}}{\tfrac{2}{5}}$

3. Find all number cards that can be used to take all the following cards.
   $4 \quad 5 \quad 7 \quad \underset{\mathsf{W}}{\div 4} \quad \underset{\mathsf{W}}{N^2}$

4. Find all number cards that can be used to take all the following cards.
   $3 \quad 15 \quad \underset{\mathsf{W}}{-4} \quad \underset{\mathsf{W}}{\tfrac{1}{2}} \quad \underset{\mathsf{W}}{\div 5}$

5. Find all number cards that can be used to take all the following cards.
   $1 \quad 4 \quad \underset{\mathsf{W}}{\times 2} \quad \underset{\mathsf{W}}{\tfrac{1}{8}} \quad \underset{\mathsf{W}}{\tfrac{2}{5}}$

6. Find all number cards that can be used to take all the following cards.
   $12 \quad 14 \quad \underset{\mathsf{W}}{\times 3} \quad \underset{\mathsf{W}}{\tfrac{1}{4}} \quad \underset{\mathsf{W}}{\tfrac{3}{5}}$

7. Find all number cards that can be used to take all the following cards.
   $2 \quad 3 \quad 6 \quad 9 \quad \underset{\mathsf{W}}{\tfrac{3}{4}} \quad \underset{\mathsf{W}}{\sqrt[3]{N}}$

8. Find all number cards that can be used to take all the following cards.
   $2 \quad 4 \quad 9 \quad \underset{\mathsf{W}}{\div 3} \quad \underset{\mathsf{W}}{\tfrac{1}{2}} \quad \underset{\mathsf{W}}{\times 5} \quad \underset{\mathsf{W}}{-5}$

9. Find a take for each number cards 1-15 using all the following cards.
   $1 \quad 5 \quad 9 \quad \underset{\mathsf{W}}{\div 3} \quad \underset{\mathsf{W}}{N^2} \quad \underset{\mathsf{W}}{-5} \quad \underset{\mathsf{W}}{\tfrac{5}{2}}$

10. **Very difficult:** Find a 10-card take. The '?' card is a number card of your choice.

    Centre: $\underset{\mathsf{W}}{\tfrac{2}{5}} \quad 9 \quad 3 \quad 2 \quad \underset{\mathsf{W}}{N^2}$

    Hand: $\underset{\mathsf{W}}{\tfrac{1}{8}} \quad ? \quad \underset{\mathsf{W}}{\tfrac{1}{3}} \quad \underset{\mathsf{W}}{\tfrac{3}{4}} \quad \underset{\mathsf{W}}{\tfrac{3}{5}}$

11. **Very difficult:** Find a 10-card take. The '?' card is a number card of your choice.

Centre: $13 \quad 12 \quad 3 \quad \underset{\mathsf{W}}{\div 3} \quad \underset{\mathsf{W}}{\times 4}$

Hand: $? \quad \underset{\mathsf{W}}{\frac{1}{5}} \quad \underset{\mathsf{W}}{\sqrt{N}} \quad \underset{\mathsf{W}}{\frac{1}{10}} \quad \underset{\mathsf{W}}{\frac{2}{5}}$

Hint: It is possible to take using 1 without any fractional partial answers.

12. **Very difficult:** Find an 11-card take, for each case of the '?' card being 1 to 15 inclusive.

Centre: $3 \quad 4 \quad 5 \quad 6 \quad 15 \quad \underset{\mathsf{W}}{\frac{3}{5}}$

Hand: $? \quad \underset{\mathsf{W}}{\div 3} \quad \underset{\mathsf{W}}{\frac{1}{8}} \quad \underset{\mathsf{W}}{-4} \quad \underset{\mathsf{W}}{\sqrt{N}}$

Hint: It is possible to take using any number 1-15 without any fractional partial answers.

5 Solutions

1. The first six possible numbers involve only integer partial answers

   $6 = (2 + 3 + 7) \times \frac{3}{2} \times \frac{1}{3}$

   $7 = (3 + 7) \times \frac{3}{2} \times \frac{1}{3} + 2$

   $9 = \left((2 + 7) \times \frac{1}{3} + 3\right) \times \frac{3}{2}$

   $11 = 2 \times \frac{3}{2} \times \frac{1}{3} + 3 + 7$

   $14 = \left(3 \times \frac{1}{3} + 7\right) \times \frac{3}{2} + 2$

   $15 = \left(3 \times \frac{1}{3} + 7 + 2\right) \times \frac{3}{2}$

   The remaining three possible numbers involve fractional partial answers

   $8 = \left((3 + 7) \times \frac{1}{3} + 2\right) \times \frac{3}{2}$

   $10 = \left(7 \times \frac{1}{3} + 3\right) \times \frac{3}{2} + 2$

   $13 = \left((2 + 3) \times \frac{1}{3} + 7\right) \times \frac{3}{2}$

2. It is only possible to take with 11
   $11 = (4 \times 4 + 9) \times \frac{2}{5} + 1$

3. There are four possible numbers to take with
   $7 = (4^2 + 5 + 7) \div 4$

   $9 = (5^2 + 4 + 7) \div 4$

   $12 = (5^2 + 7) \div 4 + 4$

   $13 = (4 \div 4)^2 + 5 + 7$

4. It is only possible to take with 1
   $1 = (15 \div 5 + 3 - 4) \times \frac{1}{2}$

5. It is only possible to take with 2
   $2 = \left(4 \times \frac{1}{8} + 1\right) \times \frac{2}{3} \times 2$

6. There are no possible takes involving all the cards.

7. It is only possible to take with 11
   $11 = \sqrt[3]{2 + 6} + (3 + 9) \times \frac{3}{4}$

8. It is possible to take with all number cards.

   $1 = ((4 - 5) \times 5 + 9 + 2) \times \frac{1}{2} \div 3$

$2 = 4 \times \frac{1}{2} - 5 + 9 \div 3 + 2$

$3 = \left(2 \times \frac{1}{2} - 5 + 4\right) \times 5 + 9 \div 3$

$4 = \left(\left(2 \times \frac{1}{2} - 5 + 4\right) \times 5 + 9 \div 3\right)$

$5 = (2 \times 5 + 4) \times \frac{1}{2} + 9 \div 3 - 5$

$6 = (2 - 5) \times 5 \div 3 + 9 + 4 \times \frac{1}{2}$

$7 = \left(2 \times 5 \times \frac{1}{2} - 5\right) + 9 \div 3 + 4$

$8 = \left((9 \times 5 + 4 - 5) \times \frac{1}{2} + 2\right) \div 3$

$9 = \left(2 \times \frac{1}{2} - 5 + 4\right) \times 5 \div 3 + 9$

$10 = \left((9 \times 5 - 5) \times \frac{1}{2} + 4\right) \div 3 + 2$

$11 = \left(2 \times 5 \times \frac{1}{2} - 5\right) \div 3 + 4 + 9$

$12 = ((9 + 4) \times 5 - 5) \div 3 \times \frac{1}{2} + 2$

$13 = (9 \div 3 - 5) \times \frac{1}{2} + 2 \times 5 + 4$

$14 = ((2 + 4) \times 5 - 5 + 9 \div 3) \times \frac{1}{2}$

$15 = (4 \times 5 - 5) \div 3 + 2 \times \frac{1}{2} + 9$

9. Some solutions are:

$1 = \left((9 \div 3 - 5) \times \frac{5}{2}\right)^2 + 1$

$2 = \left(\left((1 - 5) \times \frac{5}{2} + 9\right)^2 + 5\right) \div 3$

$3 = \left(\left(1^2 + 5\right) \times \frac{5}{2} + 9\right) \div 3 - 5$

$4 = (5 - 5)^2 \times \frac{5}{2} + 1 + 9 \div 3$

$5 = \left(\left(1^2 + 9\right) \times \frac{5}{2} + 5\right) \div 3 - 5$

$6 = \left((9 - 5) \times \frac{5}{2} + 5\right) \div 3 + 1^2$

$7 = \left((5 + 9) \times \frac{5}{2} + 1^2\right) \div 3 - 5$

$8 = \left((1 - 5) \times \frac{5}{2} + 5^2 + 9\right) \div 3$

$9 = \left((1 + 5) \times \frac{5}{2} \div 3 - 5\right)^2 + 9$

$10 = (5 - 5)^2 \times \frac{5}{2} \div 3 + 1 + 9$

$11 = \left((5 + 9) \times \frac{5}{2} - 5\right) \div 3 + 1^2$

$12 = \left(5 + 1^2\right) \div 3 + (9 - 5) \times \frac{5}{2}$

$13 = \left(5 + 1^2\right) \times \frac{5}{2} - 5 + 9 \div 3$

$14 = \left((1 - 5) \times \frac{5}{2} + 5^2\right) \div 3 + 9$

$15 = \left((1 + 9) \times \frac{5}{2} - 5 + 5^2\right) \div 3$

10. It is possible to take with numbers 1 through to 13. Some examples are:

$1 = \left((9 + 2) \times \frac{1}{3} + 3\right)^2 \times \frac{2}{5} \times \frac{1}{8} \times \frac{3}{4} \times \frac{3}{5}$

$2 = \left(2^2 \times \frac{3}{4} + 9\right) \times \frac{2}{5} \times \frac{1}{8} \times \frac{1}{3} + 3 \times \frac{3}{5}$

$3 = \left(\left(2^2 \times \frac{1}{8} \times \frac{1}{3} + 9\right) \times \frac{2}{5} + 3\right) \times \frac{3}{4} \times \frac{3}{5}$

$4 = \left(9 \times \frac{3}{4} + 3\right) \times \frac{2}{5} + 2^2 \times \frac{1}{8} \times \frac{1}{3} \times \frac{3}{5}$

$5 = \left(3 \times \frac{1}{8} \times \frac{1}{3} + 9\right) \times \frac{2}{5} + \left(2 \times \frac{3}{4}\right)^2 \times \frac{3}{5}$

$6 = 9 \times \frac{1}{3} \times \frac{3}{4} + 2 \times \frac{1}{8} \times \frac{3}{5} + 3^2 \times \frac{2}{5}$

$7 = (2 + 3)^2 \times \frac{2}{5} \times \frac{1}{8} \times \frac{1}{3} \times \frac{3}{5} + 9 \times \frac{3}{4}$

$8 = 2 \times \frac{3}{5} + 9 \times \frac{3}{4} + \left(3 \times \frac{1}{3}\right)^2 \times \frac{2}{5} \times \frac{1}{8}$

$9 = \left(\left(2 \times \frac{1}{3} \times \frac{3}{5} + 3\right) \times \frac{3}{4} + 9 \times \frac{2}{5} \times \frac{1}{8}\right)^2$

$10 = 3 \times \frac{2}{5} \times \frac{3}{4} + 2^2 \times \frac{1}{8} \times \frac{1}{3} \times \frac{3}{5} + 9$

$11 = (3 + 9)^2 \times \frac{1}{8} \times \frac{3}{5} + 2 \times \frac{2}{5} \times \frac{1}{3} \times \frac{3}{4}$

$12 = \left(3 \times \frac{3}{4} + 2 \times \frac{1}{8} \times \frac{3}{5} + 9 \times \frac{2}{5}\right)^2 \times \frac{1}{3}$

$13 = \left(3 + 9^2 \times \frac{3}{5}\right) \times \frac{1}{3} \times \frac{3}{4} + 2 \times \frac{2}{5} \times \frac{1}{8}$

11. It is possible to take with all number cards except 5 and 11. A solution for 1 without fractional partial answers is

$1 = \left(\sqrt{(13 + 12) \times \frac{2}{5} \times \frac{1}{10}} \times 4 + 3 \times \frac{1}{3}\right) \times \frac{1}{5}$