# Topological and simplicial features in reservoir computing networks

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**Abstract.** Reservoir computing is a framework which uses the non-linear internal dynamics of a recurrent neural network to perform complex non-linear transformations of the input. This enables reservoirs to carry out a variety of tasks involving the processing of time-dependent or sequential-based signals. Reservoirs are particularly suited for tasks that require memory or the handling of temporal sequences, common in areas such as speech recognition, time series prediction, and signal processing. Learning is restricted to the output layer and can be thought of as "reading out" or "selecting from" the states of the reservoir. With all but the output weights fixed they do not have the costly and difficult training associated with deep neural networks. However, while the reservoir computing framework shows a lot of promise in terms of efficiency and capability, it can be unreliable. Existing studies show that small changes in hyperparameters can markedly affect the network's performance. Here we studied the role of network topologies in reservoir computing in the carrying out of three conceptually different tasks: working memory, perceptual decision making, and chaotic time-series prediction. We implemented three different network topologies (ring, lattice, and random) and tested reservoir network performances on the tasks. We then used algebraic topological tools of directed simplicial cliques to study deeper connections between network topology and function, making comparisons across performance and linking with existing reservoir research.

**Keywords:** reservoirs, recurrent neural networks, network topologies, directed simplicial cliques

## 1 Introduction

The link between structure and function in networks of complex systems has been demonstrated theoretically and empirically, but a clear and comprehensive understanding of it is nevertheless elusive [2]. Networks have many applications, but wherever they are applied, their structure has important consequences for their function or behaviour. In reservoir computing [7], the structure and connectivity of the reservoir — a recurrent neural network — has received much interest, as it impacts upon task performance [8, 24, 14]. The parameters of the reservoir network components have vital implications for the performance of the reservoir. These parameters include input signal scaling factor, connection sparsity of the reservoir, spectral radius of the weight matrix (the maximum of the absolute values of its eigenvalues), and the choice of non-linear activation function in the network nodes. However, even very small changes in the hyperparameters can markedly affect the network's performance [12, 24]. The performance differences even with similar distributions of parameters indicate that the heart of the matter may lie in the topology of the network [8].

Answers to the structure-function question in reservoir networks could lead to increased computational ability and efficiency. Knowing *a priori* which network features might optimise functionality would reduce a large amount of the uncertain, inefficient, and costly "groping in the dark" for suitable networks, and *post hoc* adjustments to fine-tune the setup. A better knowledge of the structure-function relationship in reservoir computing would not only be of theoretical interest, but also of great potential practical use.

## 2 Review: Reservoir Computing and Network Structure

In reservoir computing a recurrent neural network (RNN) is used as a fixed, random, large-scale dynamical system, called the reservoir, which is used to process time-dependent or sequential-based signals. It is particularly suited for tasks that require memory or the handling of temporal sequences, common in areas such as speech recognition, time series prediction, and signal processing [7, 21]. The key elements of the framework (see Fig. 1) are: the input layer with randomised fixed weights $\boldsymbol{W}_{\text{in}}$; the reservoir with fixed internal weights $\boldsymbol{W}$ of a desired sparsity and/or node degree, with weights often uniformly sampled between -1 and 1 and scaled by the spectral radius; and the output layer with weight matrix $\boldsymbol{W}_{\text{out}}$, which is typically trained by regularised linear regression.

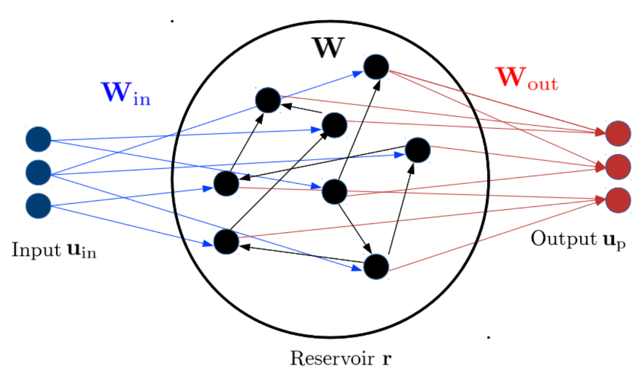



Fig. 1: Adapted [24] illustration of a reservoir setup, with fixed input weights $W_{\text{in}}$, fixed reservoir $W$, and trained output layer $W_{\text{out}}$.

The recurrent neural networks in reservoirs have rich dynamics allowing complex non-linear transformations of their input. The dynamics are driven by the equation

$$\boldsymbol{x}(t+1) = (1-\alpha)\boldsymbol{x}(t) + \alpha f(\boldsymbol{W}\boldsymbol{x}(t) + \boldsymbol{W}_{\text{in}}\boldsymbol{i}(t))$$

where $\alpha$ is the leaky coefficient, $f$ is the non-linear activation function, $\boldsymbol{x}$ is the state vector, and $\boldsymbol{i}$ is the input data. By non-linearly embedding the input into a higher dimensional feature space, the problem is more likely to be linearly separable and therefore solvable (Cover's theorem [6]). Since the training of the system is restricted to the output layer, reservoirs do not have the costly and challenging training associated with deep neural networks. This makes them particularly desirable from an efficiency point of view. However, as mentioned above, it is not fully known how the connectivity and layout of nodes within the reservoir affect performance. Reservoirs with ostensibly equivalent characteristics can show drastically varied performance depending on their initialisations.

There is a connection between reservoir performance and general architectural factors such as modularity [27], small-world qualities [14], scale-free characteristics [10], and even specific topologies [8]. For instance, node degree $k$ (the number of in and out connections a node within the network has) has been studied and shown to be related to the reservoir behaviour. A common suggested heuristic is that low-connectivity of node degree promotes a richer reservoir response [4]. On the other hand, while greater sparsity and low-connectivity has been linked with greater richness of response, it is also the case that lowering the average degree of connectivity shows decreased sensitivity, network strength, and persistence of the signal [17]. The modularity of a network's architecture has also been linked to reservoir performance [27], specifically in the case of memory capacity and recall tasks. An optimal modularity is found with a balance between local cohesion and global connectivity (Fig. 2), which optimises neural dynamics and memory performance. However, this is in the case of threshold neurons.

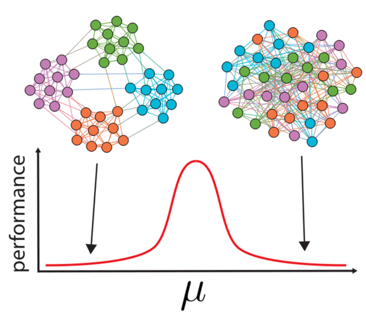
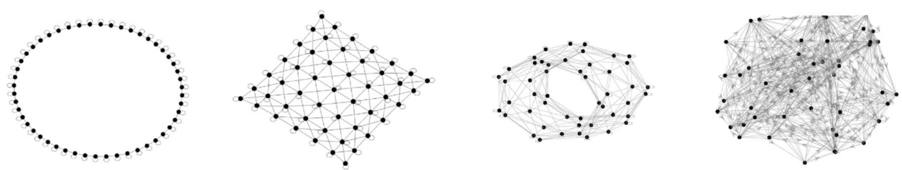



Fig. 2: Illustration (adapted from [27]) where reservoir performance increases when a balance between the local cohesion of communities and the global connectivity of bridges is met.

Another general network property that has been studied in relation to reservoirs is that of small-world characteristics. Small-world characteristics are those which possess high degrees of clustering, and low average shortest path length [31]. Their properties have been linked with better performance in feed-forward [11] and Hopfield [3] networks, as well as the reservoir computing context [14]. Small-world qualities in reservoir networks often increase task performances and provide greater robustness in parameter selection [14, 8]. Specifically, the small-world topology appears to enhance reservoir learning performance in non-linear time-series prediction compared to networks with random topologies, due to its promotion of efficient signal propagation and enhancement of the echo state property.



(a) ring (b) lattice (c) torus (d) Fully-connected

Fig. 3: Different topologies used in reservoir networks [8] within the CHARC framework.

Some previous work has studied the general performance properties of several explicit topologies — including ring and lattice (Fig. 3) — for computing substrates using a CHARC (CHAracterisation of Reservoir Computers) framework [8]. This study examines the theoretical qualities of these topologies across different network sizes (from 16 to 400 nodes), and consistently find that, while the fully-connected structure performs with the highest quality of dynamical behaviour (and the ring performs poorest), the lattice and torus achieve comparably high performances to the fully-connected network. Their results demonstrate that networks with specific topologies and nuanced connectivity exhibit similar quality of behaviours to larger and higher connected structures. However, they only investigated three *generic* and *theoretical* properties of reservoirs in these topologies, namely, the separation property, the generalisation property, and the echo state property. These characteristics are highly important for reservoir dynamics, but do not fully capture all dynamical properties and functional capacities of a reservoir. Thus they do not specifically address the potential performances of reservoir in defined tasks.

A further aspect of network property and structure which is connected to the characteristics outlined above — but which remains to be explored in the context of reservoir computing — is that of directed simplicial complexes [1, 20]. These are concepts from algebraic topology which extend the notion of connectivity beyond pairwise interactions to higher-order relationships, allowing for the modelling of more complex, multi-dimensional connectivity patterns within a network [20]. These structural motifs have been seen to play a crucial role in a network's functioning, influencing its synchronisation and dynamics [19]. Furthermore, higher-than-random proportions of high dimensional directed simplicial cliques have been identified in biological neural networks [25] (Fig. 4). These are suggested to guide the emergence of correlated network activity, and are crucial to the processing and integration of information in the brain [25]. However, despite recent work uncovering the abundance and importance of simplicial complexes in network performance, there remains a gap in research studying these characteristics in the case of reservoir computing.

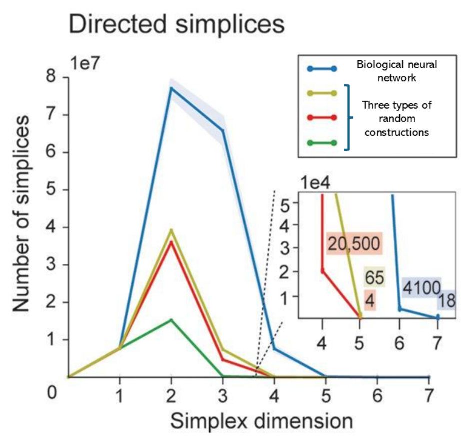



Fig. 4: The number of simplices in each dimension in the biological network network and in three types of random control networks (adapted from [25]).

Here in this work we examine the roles of network connectivity and topology on the performance of reservoir networks in *specific performance contexts*. We identify topological fingerprints of reservoir networks in three conceptually and practically different tasks: working memory, perceptual decision making, and chaotic time-series prediction. The working memory task measures the degree to which the reservoir can reproduce sequential random input data after time delay; the perceptual decision making task tests the capacity to process motion data and discern direction coherence; and the time series prediction task measures how long the reservoir can accurately predict the Lorenz system. We first use reservoirs of differing explicit network topologies (ring, lattice, and random) and test their performances on the three tasks mentioned above. We test topology performance on *specific* tasks and functions. We then make use of concepts from algebraic topology to more deeply probe into the roles of higher-dimensional connectivity in reservoir function. This involves developing reservoirs with differing dimensions of directed simplicial cliques [1] and analysing resulting task-performances.

## 3 Methodology

Here we outline the specifics of the three tasks on which we test reservoir performance, followed by the various topological concepts and tools which will be implemented.

### 3.1 Reservoir Tasks

**Working Memory** This working memory task is inspired by other reservoir work [9, 13, 27]. In this task paradigm, a random input sequence $X(t)$ is presented to the network through an input neuron. The network independently learns delayed versions of the input, producing multiple outputs. Each output $Y_\tau$ predicts the input $X(t)$ delayed by $\tau$ time steps, i.e., $Y_\tau(t) = X(t-\tau)$. The input values are randomly drawn from a uniform distribution, $X(t) \sim \text{Uniform}(-0.5, 0.5)$. The networks are trained for 4000 time steps and tested on the subsequent 1000. Each output is trained independently, and the performance, referred to as Working Memory Capacity, is calculated as the cumulative squared Pearson correlation coefficient ($\rho$) across all outputs:

$$\text{Working Memory Capacity} = \sum_{\tau} \rho^2(y_i, \hat{y}_i)$$

where $y_i$ and $\hat{y}_i$ denote the true and predicted values, respectively. Although such a simple input-output delay is a trivial task from an engineering point of view, we consider it a valuable benchmark task for reservoir networks, because any complicated task on time series data will need to be able to temporarily store information from the past in order to combine it with the present input.

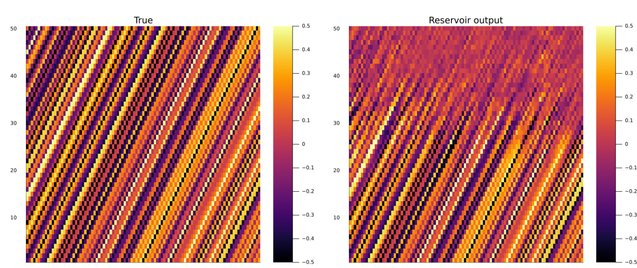



Fig. 5: Working memory task; the left indicates the true/target data, the right shows the reservoir's output. Notice the blurring of reservoir performance.

**Perceptual Decision Making** The random dot motion task is a widely used paradigm in perceptual decision-making experiments involving human and animal participants [29, 28]. In this task, participants are presented with a display consisting of randomly moving dots on a screen. The dots move in different directions and speeds, creating a noisy visual stimulus. The task is typically to detect or discriminate the overall direction of motion among the dots, which may be either coherent (moving in the same direction) or incoherent (moving randomly). By varying the parameter of coherence level (the proportion of dots moving coherently), we manipulate the difficulty of the task. The random dot motion task provides insight into how systems integrate sensory information to make perceptual decisions amidst uncertainty, making it a valuable tool for studying visual perception and decision-making processes [15, 29]. In this task, we create dot motion data, where a specified number of dots move left or right with a certain degree of coherence. The reservoir must learn the coherence level and "decide" the correct direction of the dot motion, providing this as output. We use the reciprocal mean squared error as a measure of performance:

$$\text{Decision Making Performance} = \left( \frac{1}{N} \sum_{i=1}^{N} (y_i - \hat{y}_i)^2 \right)^{-1}$$

where $N$ is the number of data points, $y_i$ the true value, and $\hat{y}_i$ predicted value.

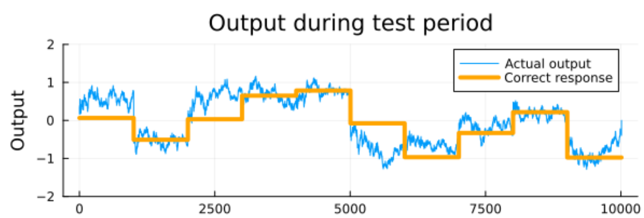



Fig. 6: Example reservoir performance on the perceptual decision-making task; the yellow line indicates correct coherence, while the blue indicates the reservoir's output.

**Chaotic Time-Series Prediction** The system is described by the equations

$$\frac{dx}{dt} = \sigma(y - x) \tag{1}$$

$$\frac{dy}{dt} = x(\rho - z) - y \tag{2}$$

$$\frac{dz}{dt} = xy - \beta z \tag{3}$$

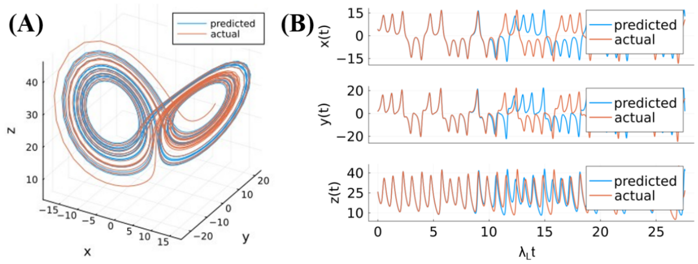



Fig. 7: Illustration of the true and reservoir prediction of the Lorenz system.

The canonical example of a reservoir computing task is prediction of the chaotic time-series data from the Lorenz system [22, 8, 7]. The Lorenz system is notable for having chaotic solutions for certain parameter values and initial conditions. We train the reservoir on a set of Lorenz data, and then provide the system with a random point on the Lorenz trajectory, and measure the valid time for which the reservoir can accurately predict the true trajectory. We calculate the normalised mean squared error [22],

$$\text{NMSE} = \frac{\|\boldsymbol{u}(t) - \tilde{\boldsymbol{u}}(t)\|}{\sqrt{\langle \|\boldsymbol{u}(t)\|^2 \rangle}}$$

where $\boldsymbol{u}(t)$ is the true value and $\tilde{\boldsymbol{u}}(t)$ is the predicted value. We measure, in Lyapunov time, the valid time of the predicted trajectory, that is, the point at which the predicted trajectory exceeds a specified NMSE threshold, set at 0.4 in this study.

### 3.2 Topological Tools

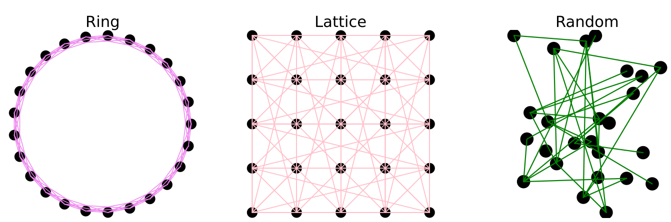



Fig. 8: Three topologies used in reservoir networks.

**Explicit Network Topologies** We studied networks with three distinct topologies: ring, lattice, and random (Fig. 8). The ring topology has the lowest degree of complexity. It connects each node to a fixed number of nearest neighbours in a circular arrangement, with a random subset of nodes also having recurrent connections to themselves. The lattice topology has a greater connection complexity. With this structure, we arrange nodes in a square grid where each node is connected to its immediate neighbouring nodes, with a randomised subset of nodes also having recurrent connections to themselves. These two networks, along with the random network, were all initialised with the same number of nodes, equivalent sparsity, and equal spectral radius. The three types of network structure were implemented in the reservoir framework on the three tasks defined above. The means and standard deviations of the performance on the three tasks using each type of network were used to compare between the three topologies' capacities. With three groups for comparison, we conducted a non-parametric Kruskal-Wallis rank sum test due to lack of homogeneity in data variance. This was to determine if there were statistically significant differences in the means of performances. Post-hoc pairwise tests were used to find pairwise comparisons in significantly different measures.

**Directed Simplicial Cliques** A simplex is the generalisation of a tetrahedron to higher dimensions [20]; in a directed simplex the direction of transmission is taken into account [1]. Directed simplices are network cliques containing a single **source** neuron and a single **sink** neuron (Fig. 9), reflecting a specific motif of connectivity [30, 23]. The presence of simplicial cliques and cavities correlates with functional roles of the network, such as the processing and integration of information in the brain [25]. The activity of nodes within the network is dependent on the number and dimension of the cliques that the nodes belongs to, and their specific positions in directed cliques. This concept has been useful in analysing and modelling networks [19, 25]; however, little has been done to study the implication these properties have for reservoir function.

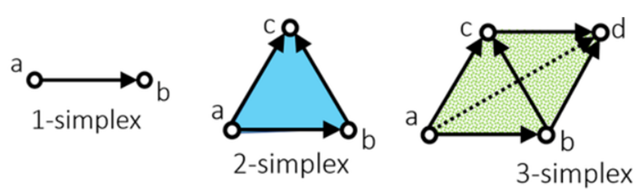



Fig. 9: Example directed simplices (with a source and a sink) of 3 different dimensions.

We constructed enhanced networks with higher-than-average numbers of high dimensional directed simplicial cliques. This process involved initialising a graph with a specified number nodes and creating dense, highly interconnected subgraphs (cliques) by systematically connecting the nodes in these subgraphs with structured edges. Subsequently, the graph is pruned to achieve a desired level of sparsity. Fig. 10 demonstrates the difference in numbers of directed simplicial cliques between a random network and our enhanced networks.

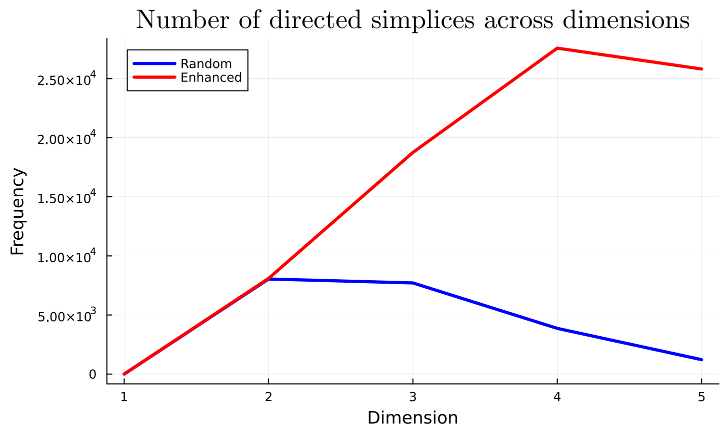



Fig. 10: Comparison of number of directed simplicial cliques of various dimensions in a random and enhanced network.

As in the case of the explicit topologies, we examined the differences in performance across the three tasks. In this case, we only had two groups for comparison, where data variance was non-homogenous. We therefore carried out a non-parametric Wilcoxon Signed-Rank test to determine if there was a statistically significant difference between the performances.

**Eigenspectral Analyses** In addition to the topological concepts employed in this work, we also studied the eigenspectra of the various networks used as reservoirs, calculating spectral measures such as spectral gap and participation ratio. The spectral gap is the difference between the moduli of the two largest eigenvalues. This measure gives information about a graph's potential for information propagation, resilience to disruptions, and overall efficiency in various network dynamics [5, 18]. A large spectral gap may be associated with high network performance, such as facilitated synchronisation and rapid convergence. The participation ratio formula used is:

$$\text{Participation ratio} = \frac{(\sum_i \lambda_i)^2}{\sum_i \lambda_i^2}$$

where $\lambda_i$ are the eigenvalues of the weight matrix. Participation ratio is a measure of dimensionality; it provides a quantitative measure of how broadly or narrowly the network's connectivity or activity is distributed. It helps assess the extent to which certain nodes dominate or share influence within the network, offering insights into the heterogeneity of timescales in the network dynamics [16].

### 3.3 Parameters

Following parameter sweeps on baseline random reservoirs, we set the following parameters constant throughout the study: reservoir size = 400; spectral radius = 0.9; input scaling = 0.1; sparsity = 0.9; regularisation coefficient = $1 \times 10^6$. A reservoir size of 400 nodes ensures a balance between computational efficiency and the ability to capture complex dynamics; prior experimentation indicates that reservoirs of this size provide sufficient representational capacity for a variety of tasks. The spectral radius, set to 0.9, is a crucial parameter related to the echo state property of the reservoir [7]. A spectral radius close to but less than 1 has been suggested to promote stability while ensuring that the reservoir retains memory of past inputs, a characteristic that is essential for effective temporal processing [13]. An input scaling factor of 0.1 was chosen to control the magnitude of the inputs entering the reservoir and to avoid the saturation of the activation functions. The sparsity, set at 0.9 — which mimics the sparse connectivity found in biological neural networks — ensures a diversity of dynamical responses while keeping computational costs manageable [26].

It is important to note that while these parameter choices are informed by existing literature and empirical evidence from experience, our findings in this study are inherently limited to this specific set of parameters. Therefore, the conclusions drawn from this work should be interpreted within the context of these settings. Future work could explore variations in these parameters to assess their impact on the performance and robustness of the reservoir computing models.

## 4 Results

### 4.1 Ring, Lattice, and Random Topologies

The performance of the reservoirs with different topologies varied across the tasks (Fig. 11).

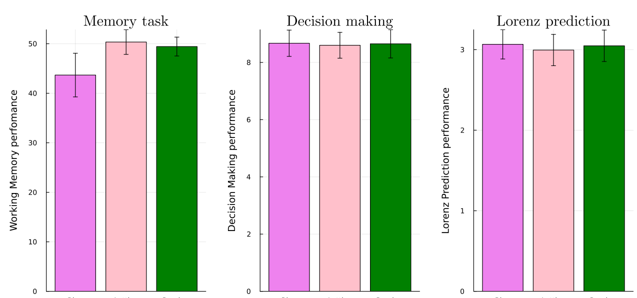



Fig. 11: Performances (means and standard deviations) of different topologies on 3 tasks.

Interestingly, the only significant difference (at the $< 1\%$ confidence level) in performance is seen in the working memory task, where the ring topology was outperformed by both the lattice and random topologies. The other two tasks showed no statistically significant differences (at the $< 1\%$ confidence level). Post-hoc tests confirmed that the ring topology performance was significantly different to the other two, but that there was no significant difference between the lattice and randomised topology performances (not shown).

The unique feature of the working memory task is its requirement of a longer term memory. The decision making (dot motion) and time-series prediction (Lorenz) tasks do not require the ability to remember far into the past; they primarily require only current and very recent historical data to accurately compute the next output. The performance of the ring topology in the working memory task indicates that the simplicity and naivety of its structure make it less suitable for long-term memory. For the other tasks, however, the simple ring topology appears to perform equally well.

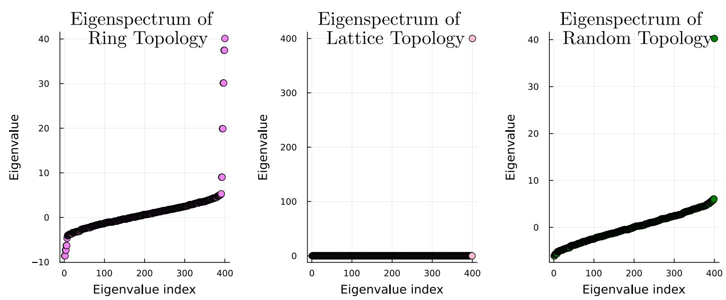



Fig. 12: Eigenspectra of different topologies.

| | Spectral Gap | Participation Ratio |
|---|---|---|
| Ring | 5.36 | 0.0229 |
| Lattice | 400.00 | 0.2506 |
| Random | 34.12 | 0.0033 |

Table 1: Spectral measures of networks with differing topologies.

Examining the eigenspectra (Fig. 12), along with the spectral gap and participation ratio measures (Table 1) of the different topologies gives a clearer picture and insight into their underlying differences. The ring topology stands out for its very small spectral gap, indicating a weaker graph modularity, and lower degree of expansion and information transmission. The other two topologies (especially the lattice) exhibit higher spectral gaps, indicating stronger modularity and concomitant network synchronisability. These are possibly important features when it comes to the ability to internalise longer term historical inputs, as in the case of the working memory task.

### 4.2 Directed Simplicial Clique Enhancement

In the resulting performances we found that across the three different tasks, as in the case of the three explicit topologies, a significant difference (at 1% level) was found in the memory task, where random reservoirs outperform those with higher degrees of simplicial cliques (Fig. 13).

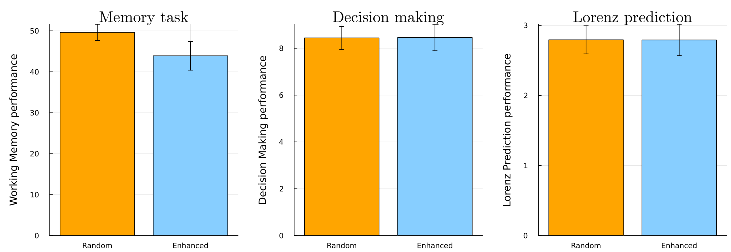



Fig. 13: Performances (means and standard deviations) of random and enhanced networks.

Examining the eigenspectra of the random and "enhanced" topologies reveals an interesting fact: the eigenspectrum of the enhanced topology closely resembles that of the ring topology in the preceding section (Fig. 14 ). It furthermore also demonstrates a smaller spectral gap (Table 2), indicating that the same potential issues inherent to the ring topology may be present with the simplicially enhanced networks.

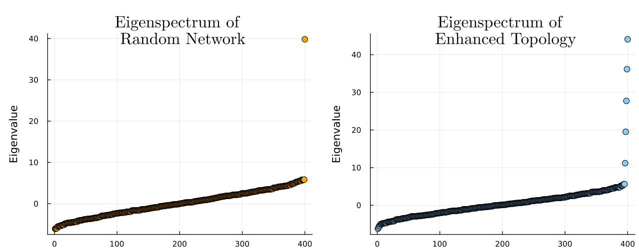



Fig. 14: Eigenspectra of networks with differing directed simplicial dimensions.

| | Spectral gap | Participation ratio |
|---|---|---|
| Random networks | 34.06 | 0.0028 |
| Enhanced networks | 8.83 | 0.0028 |

Table 2: Spectral measures across simplicially differing networks.

## 5 Discussion

We examined the role of explicitly defined network topologies and enhanced topological features on specific computational tasks in reservoir computing. Our primary finding is that reservoir performance can be surprisingly robust to topological differences in the reservoir connectivity.

Indeed, we found that the only statistically significant difference between the network architectures lies with the ring topology and in the memory task. The lack of significant difference between the lattice and random graph may simply be due to the fact that they both have higher degrees of recurrent connections, despite their eigenspectra being qualitatively different. The ring topology is highly regular and simple, with the correspondingly smallest spectral gap. As indicated previously, the decision making and time-series prediction require only a relatively little degree of memory, whereas the working memory task by definition requires longer term storage of historical input. This kind of task may place demands upon the internal recurrent dynamics of the reservoir, which are highly dependent on the network connectivity. The simple and regular connectivity of the ring topology perhaps make it unsuitable for producing the complex, non-linear, recurrent dynamics required for internalising long-term memory. The ring topology has been previously identified as exhibiting more linear and ordered dynamics, and theoretically possessing limitations of memory [8]. Our work has corroborated this in a particular use-case.

In order to probe deeper into the question of network topology and specific task performance, we developed a framework for constructing "enhanced" networks with high-dimensional directed simplices. In comparison with randomly initialised networks, these enhanced networks possess greater amounts of high dimensional topological features: characteristics which have been observed in complex network data, and posited to be important for network function [25]. However, the findings of our research indicate that simple enhancement of higher dimensional directed simplices does not significantly improve the performance of reservoirs on the three specific computational tasks of interest to this study. The only statistically significant outcome was that the random reservoirs outperform the enhanced networks in the memory task. Once again, this can possibly be traced to the much larger spectral gap of the random reservoirs, indicating a higher degree of network synchronisation, and information transmission. This may be vital for the internal dynamics necessary for longer term memory.

Our work carries some limitations. First, we implemented the topological and simplicial features within a very narrow context. A greater investigation of how these network features vary with contexts is necessary, such as how their roles develop over different reservoir sizes, spectral radii, sparsities, etc. Second, we did not expand upon the possibilities of defining particular distributions of simplicial dimensions; the specificity and complexity of these directed simplicial cliques could be extended. Therefore how distinct higher simplicial dimensions and motifs impact performance and network efficiency is yet to be understood.

## 6 Conclusion

Here we tested the network-function relationship question using topology and reservoir computing. The promise of effective and efficient computation through reservoir networks is to be made more certain through a better understanding of the underlying topological features of recurrent neural networks internal to the reservoir. This work has studied the role of three explicit topologies on performance in three tasks. It demonstrated that, in the context of the parameter and task choices in this work, the specific network topologies of ring and lattice do not *generally* have significant impact on task performance in comparison with random network topologies. However, the ring topology significantly underperformed in the memory task. This indicates that certain topological features may underpin performance in tasks requiring longer-term memory. This work further developed a "directed simplicial clique" enhanced framework for implementing reservoirs with higher-dimensional connectivity, which also produced a significant difference from randomly initialised networks in the working memory task, but not in the other two. Overall this suggests that, apart from memory-based use cases, the reservoir computing framework may be highly robust to high-level topological alterations in the internal reservoir.

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