Expected Value

X is discrete random variable

$g : \mathbf{R} \to \mathbf{R}$

$\Omega X = Im(X)$

$$E[g(X)] = \sum_{x \in \Omega X} P(X = x) * g(x) \tag{1}$$

Variance

$$\begin{aligned}
var(g(X)) &= E([g(X) - E(g(X))]^2) && (2)\\
&= \sum_{x \in \Omega X} [g(X) - E(g(X))]^2 * P(X = x)] && (3)\\
&= \sum_{x \in \Omega X} [g(X)^2 - 2 * g(X) * E(g(X)) + E(g(X))^2] * P(X = x) && (4)\\
&= \sum_{x \in \Omega X} g(X)^2 * P(X = x) && (5)\\
&- \sum_{x \in \Omega X} 2 * g(X) * E(g(X)) * P(X = x) && (6)\\
&+ \sum_{x \in \Omega X} E(g(X))^2 * P(X = x) && (7)\\
&= \sum_{x \in \Omega X} g(X)^2 * P(X = x) && (8)\\
&- 2 * E(g(X)) * \sum_{x \in \Omega X} g(X) * P(X = x) && (9)\\
&+ E(g(X))^2 * \sum_{x \in \Omega X} P(X = x) && (10)\\
&= E(g(X)^2) - 2 * E(g(X)) * E(g(X)) + E(g(X))^2 * 1 && (11)\\
&= E(g(X)^2) - 2 * E(g(X))^2 + E(g(X))^2 && (12)\\
&= E(g(X)^2) - E(g(X))^2 && (13)\\
\square & && (14)
\end{aligned}$$

Variance of the Sample Mean

$$
\begin{aligned}
Var\left(\overline{X}\right) &= Var\left(\frac{1}{n}\sum_{i=1}^{n} X_i\right) && (15)\\
&= \frac{1}{n^2} Var\left(\sum_{i=1}^{n} X_i\right) && (16)\\
&= \frac{1}{n^2}\sum_{i=1}^{n} Var(X_i), \text{ by independence} && (17)\\
&= \frac{1}{n^2}\left[Var(X_1) + Var(X_2) + \ldots + Var(X_n)\right] && (18)\\
&= \frac{1}{n^2}\left[\sigma^2 + \sigma^2 + \ldots + \sigma^2\right], \text{ since the } X_i \text{ are identically distributed} && (19)\\
&= \frac{1}{n^2}(n\sigma^2) && (20)\\
&= \frac{\sigma^2}{n} && (21)
\end{aligned}
$$

Binomial Distribution
PMF

$$P(X = k) = \binom{n}{k} p^k (1-p)^{n-k} \tag{22}$$

Expected Value

$E[g(X)]$ when $g(X) = X$.

$$
\begin{aligned}
E(X) &= \sum_{k \geqslant 0} P(x = k) * k && (23)\\
&= \sum_{k \geqslant 0} [\binom{n}{k} p^k (1-p)^{n-k}] * k && (24)\\
& && (25)
\end{aligned}
$$

when

$$k = 0$$

, the formula

$$[\binom{n}{k} p^k (1-p)^{n-k}] * k = [\binom{n}{0} p^k (1-p)^{n}] * 0 = 0$$

, so the index of the summation can be increased by 1.

$$
\begin{aligned}
E(X) &= \sum_{k\geqslant 1}\binom{n}{k}p^k(1-p)^{n-k}*k && (26)\\
&= \sum_{k\geqslant 1}\frac{n}{k}*\binom{n-1}{k-1}p^k(1-p)^{n-k}*k \qquad \text{see BinomialCoefficient} && (27)\\
&= \sum_{k\geqslant 1}\frac{n*k}{k}*\binom{n-1}{k-1}p^k(1-p)^{n-k} && (28)\\
&= \sum_{k\geqslant 1}n*\binom{n-1}{k-1}p^k(1-p)^{n-k} && (29)\\
&= \sum_{k\geqslant 1}n*p*\binom{n-1}{k-1}p^{k-1}(1-p)^{n-k} && (30)\\
&= np*\sum_{k\geqslant 1}\binom{n-1}{k-1}p^{k-1}(1-p)^{n-k} && (31)\\
u &= n-1 && (32)\\
z &= k-1 && (33)\\
u-z &= (n-1)-(k-1) && (34)\\
&= n-1-k+1 && (35)\\
&= n-k && (36)\\
k>1 &= (z+1)>1 && (37)\\
&= z>0 && (38)\\
&= np*\sum_{z>0}\binom{u}{z}p^z(1-p)^{u-z} && (39)\\
&= np*1 \qquad see Binomial Distribution Proof Equals 1 && (40)\\
&= np && (41)\\
&\square && (42)
\end{aligned}
$$

Variance

$$
\begin{aligned}
Var(X) &= E(X^2)-E(X)^2 \qquad \text{see Variance} && (43)\\
&= \sum_{k\geqslant 0}[\binom{n}{k}p^k(1-p)^{n-k}]*k^2-np \qquad \text{see Binomial Expected Value} && (44)
\end{aligned}
$$

when

$$k = 0$$

, the formula

$$[\binom{n}{k}p^k(1-p)^{n-k}] * k = [\binom{n}{0}p^k(1-p)^{n}] * 0 = 0$$

, so the index of the summation can be increased by 1.

$$
\begin{aligned}
&= \sum_{k\geqslant 1} [\binom{n}{k} p^k (1-p)^{n-k}] * k^2 - (np)^2 && (45)\\
&= \sum_{k\geqslant 1} \frac{n}{k} [\binom{n-1}{k-1} p^k (1-p)^{n-k}] * k^2 - (np)^2 && (46)\\
&= \sum_{k\geqslant 1} \frac{n * k^2}{k} [\binom{n-1}{k-1} p^k (1-p)^{n-k}] - (np)^2 && (47)\\
&= \sum_{k\geqslant 1} [nk * \binom{n-1}{k-1} p^k (1-p)^{n-k}] - (np)^2 && (48)\\
&= \sum_{k\geqslant 1} [nkp * \binom{n-1}{k-1} p^{k-1} (1-p)^{n-k}] - (np)^2 && (49)\\
&= np * \sum_{k\geqslant 1} [k * \binom{n-1}{k-1} p^{k-1} (1-p)^{n-k}] - (np)^2 && (50)
\end{aligned}
$$

$$
\begin{aligned}
u = n-1 & && (51)\\
z = k-1 & && (52)\\
u - z &= (n-1) - (k-1) && (53)\\
&= n - 1 - k + 1 && (54)\\
&= n - k && (55)\\
k >= 1 &= (z+1) >= 1 && (56)\\
&= z >= 0 && (57)\\
&= np * \sum_{z\geqslant 0} [(z+1) * \binom{u}{z} p^z (1-p)^{u-z}] - (np)^2 && (58)\\
&= np * [\sum_{z\geqslant 0} [z * \binom{u}{z} p^z (1-p)^{u-z}] + \sum_{z\geqslant 0} [\binom{u}{z} p^z (1-p)^{u-z}]] - (np)^2 && (59)\\
&= np * [\sum_{z\geqslant 0} [z * \frac{u}{z} * \binom{u-1}{z-1} p^z (1-p)^{u-z}] + \sum_{z\geqslant 0} [\binom{u}{z} p^z (1-p)^{u-z}]] - (np)^2 && (60)\\
&= np * [u * \sum_{z\geqslant 0} [\binom{u-1}{z-1} p^z (1-p)^{u-z}] + \sum_{z\geqslant 0} [\binom{u}{z} p^z (1-p)^{u-z}]] - (np)^2 && (61)\\
& && (62)
\end{aligned}
$$

$$= np * [up * \sum_{z \geqslant 0} [\binom{u-1}{z-1} p^{z-1} (1-p)^{u-z}] + \sum_{z \geqslant 0} [\binom{u}{z} p^z (1-p)^{u-z}]] - (np)^2 \tag{63}$$

$$= np * [up * \sum_{z \geqslant 1} [\binom{u-1}{z-1} p^{z-1} (1-p)^{(u-1)-(z-1)}] + \sum_{z \geqslant 0} [\binom{u}{z} p^z (1-p)^{u-z}]] - (np)^2 \tag{64}$$

$$= np * [up * \sum_{z \geqslant 0} [\binom{u}{z} p^z (1-p)^{u-z}] + \sum_{z \geqslant 0} [\binom{u}{z} p^z (1-p)^{u-z}]] - (np)^2 \tag{65}$$

$$= np * [up * (p+q)^{u-1} + (p+q)^u] - (np)^2 \tag{66}$$

$$= np * [(n-1) * p * (p+q)^{n-1-1} + (p+q)^(n-1)] - (np)^2 \tag{67}$$

$$= np * [(n-1) * p * (p+q)^{n-2} + (p+q)^(n-1)] - (np)^2 \tag{68}$$

$$= np * ([(n-1) * p * (p+q)^{n-2} + (p+q)^(n-1)] - np) \tag{69}$$

$$= np * ([(n-1) * p + 1] - np) \qquad \text{p+q=1} \tag{70}$$

$$= np * ([(n-1) * p + 1] - np) \tag{71}$$

$$= np * ([np - p + 1] - np) \tag{72}$$

$$= np * (np - p + 1 - np) \tag{73}$$

$$= np * (-p + 1) \tag{74}$$

$$= np * (1-p) \tag{75}$$

$$\square \tag{76}$$

Bernoulli Distribution

The Bernoulli Distribution is a special case of the Binomial Distribution, where

$$n = 1$$

.

PMF

$$P(X = k) = \binom{1}{k} p^k (1-p)^{1-k} \tag{77}$$

$$= p^k (1-p)^{n-k} \tag{78}$$

Expected Value

$$E(x) = \sum_{k \geqslant 1} [\binom{n}{k} p^k (1-p)^{n-k}] * k \tag{79}$$

$$= np \qquad see Binomial Distribution \tag{80}$$

$$= 1 * p \tag{81}$$

$$= p \tag{82}$$

Variance

$$
\begin{aligned}
Var(X) &= np * (1-p) && (83)\\
&= p * (1-p) && (84)\\
& && (85)
\end{aligned}
$$

Likelihood of IID Bernoulli

$$
\begin{aligned}
x_i \overset{iid}{\sim} Bernoulli(p) & && (86)\\
L(x_i|p) &= p(x_1, x_2, ..., x_n|p) && (87)\\
&= \prod_{n=1}^{n} p(x_i|p) && (88)\\
&= p^S * (1-p)^{n-S} && (89)
\end{aligned}
$$

Maximun Likelihood

$$
\begin{aligned}
\frac{d[L(x_i|p)]}{dp} &= \frac{d[p^S * (1-p)^{n-S}]}{dp} && (90)\\
&= \frac{d}{dp} * [log(p^S * (1-p)^{n-S})] && (91)\\
&= \frac{d}{dp}[log(p^S) + log((1-p)^{n-S})] && (92)\\
&= \frac{d}{dp}[S * log(p) + (n-S) * log(1-p)] && (93)\\
&= S * \frac{d}{dp}[log(p)] + (n-S) * \frac{d}{dp}[log(1-p)] && (94)\\
&= S * \frac{d}{dp}[\frac{1}{x}] + (n-S) * \frac{d}{dp}[log(1-p)] && \text{chain rule} \quad (95)\\
&= S * \frac{1}{p} + (n-S) * \frac{1}{p-1} && (96)\\
&= \frac{S}{p} + \frac{n-S}{p-1} && (97)\\
0 &= \frac{S}{p} + \frac{n-S}{p-1} && (98)\\
\frac{S}{p} &= \frac{n-S}{p-1} && (99)\\
& && (100)
\end{aligned}
$$

Normal Distribution Definition

$$
pdf(x) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right) \qquad (101)
$$