Exercise 1.5.16

(d)

Think about how your subroutines could be used to calculate the inverse of a positive definite matrix. Calculate A^{-1}, where

$$A = \begin{bmatrix} 1 & 1/2 & 1/3 \\ 1/2 & 1/3 & 1/4 \\ 1/3 & 1/4 & 1/5 \end{bmatrix}$$

It turns out that the entries of A^{-1} are all integers. Notice that your computed solution suffers from significant roundoff errors. This is because A is (mildly) ill conditioned. This is the 3x3 member of a famous family of ill-conditioned matrices called Hilbert matrices; the condition gets rapidly worse as the size of the matrix increases. We will discuss ill-conditioned matrices in Chapter 2.

Answers

We can calculate a inverse matrix using Cholesky factorization because:

$$\begin{aligned}
A &= R^T R \\
A * A^{-1} &= R^T R * A^{-1} \\
I &= R^T R * A^{-1} \\
\begin{bmatrix} e_1 \\ e_2 \\ \vdots \\ e_n \end{bmatrix} &= R^T R * A^{-1} \\
\begin{bmatrix} e_1 \\ e_2 \\ \vdots \\ e_n \end{bmatrix} &= R^T R * \begin{bmatrix} a_1^{-1} & a_2^{-1} & ... & a_n^{-1} \end{bmatrix}
\end{aligned}$$

Which can be simplified as:

$$\begin{bmatrix} e_1 = R^T R * a_1^{-1} \\ e_2 = R^T R * a_2^{-1} \\ \vdots \\ e_n = R^T R * a_n^{-1} \end{bmatrix}$$

In this way we can find the inverse of a Matrix with n "solves" using Cholesky. Each Cholesky solve makes one backward substitution and one foward substitution. They have the same complexity: n^2. Which gives us a solution in n^3.