Expected Value

X is discrete random variable

$g : \mathbf{R} \to \mathbf{R}$

$\Omega X = Im(X)$

$$E[g(X)] = \sum_{x \in \Omega X} P(X = x) * g(x) \tag{1}$$

Variance

$$\begin{align}
var(g(X)) &= E([g(X) - E(g(X))]^2) \tag{2}\\
&= \sum_{x \in \Omega X} [g(X) - E(g(X))]^2 * P(X = x)] \tag{3}\\
&= \sum_{x \in \Omega X} [g(X)^2 - 2 * g(X) * E(g(X)) + E(g(X))^2] * P(X = x) \tag{4}\\
&= \sum_{x \in \Omega X} g(X)^2 * P(X = x) \tag{5}\\
&- \sum_{x \in \Omega X} 2 * g(X) * E(g(X)) * P(X = x) \tag{6}\\
&+ \sum_{x \in \Omega X} E(g(X))^2 * P(X = x) \tag{7}\\
&= \sum_{x \in \Omega X} g(X)^2 * P(X = x) \tag{8}\\
&- 2 * E(g(X)) * \sum_{x \in \Omega X} g(X) * P(X = x) \tag{9}\\
&+ E(g(X))^2 * \sum_{x \in \Omega X} P(X = x) \tag{10}\\
&= E(g(X)^2) - 2 * E(g(X)) * E(g(X)) + E(g(X))^2 * 1 \tag{11}\\
&= E(g(X)^2) - 2 * E(g(X))^2 + E(g(X))^2 \tag{12}\\
&= E(g(X)^2) - E(g(X))^2 \tag{13}\\
\square \tag{14}
\end{align}$$

Covariance Matrix

$$
\begin{aligned}
cov(X) &= E[(X - E(X)]^2 && (15)\\
&= \sum_{x \in \Omega X} (X - E(X))^2 * P(X = x) && (16)\\
&= \sum_{x \in \Omega X} [X^2 - 2XE(X) + E(X)^2] * P(X = x) && (17)\\
&= \sum_{x \in \Omega X} X^2 P(X = x) && (18)\\
&- \sum_{x \in \Omega X} 2XE(X) * P(X = x) && (19)\\
&+ \sum_{x \in \Omega X} E(X)^2 P(X = x) && (20)\\
&= \sum_{x \in \Omega X} X^2 P(X = x) && (21)\\
&- 2E(X) * \sum_{x \in \Omega X} XP(X = x) && (22)\\
&+ E(X)^2 * \sum_{x \in \Omega X} P(X = x) && (23)\\
&= E(X^2) - 2E(X)E(X) + E(X)^2 * 1 && (24)\\
&= E(X^2) - 2 * E(X)^2 + E(X)^2 && (25)\\
&= E(X^2) - E(X)^2 && (26)\\
&= E(X^t X) - \mu^t \mu && (27)\\
\square & && (28)
\end{aligned}
$$

Variance of the Sample Mean

$$
\begin{aligned}
Var\left(\overline{X}\right) &= Var\left(\frac{1}{n}\sum_{i=1}^{n} X_i\right) && (29)\\
&= \frac{1}{n^2} Var\left(\sum_{i=1}^{n} X_i\right) && (30)\\
&= \frac{1}{n^2}\sum_{i=1}^{n} Var(X_i), \text{ by independence} && (31)\\
&= \frac{1}{n^2}\left[Var(X_1) + Var(X_2) + \ldots + Var(X_n)\right] && (32)\\
&= \frac{1}{n^2}\left[\sigma^2 + \sigma^2 + \ldots + \sigma^2\right], \text{ since the } X_i \text{ are identically distributed} && (33)\\
&= \frac{1}{n^2}(n\sigma^2) && (34)\\
&= \frac{\sigma^2}{n} && (35)
\end{aligned}
$$

Binomial Distribution
PMF

$$
P(X = k) = \binom{n}{k} p^k (1-p)^{n-k} \tag{36}
$$

Expected Value

$E[g(X)]$ when $g(X) = X$.

$$
\begin{aligned}
E(X) &= \sum_{k \geqslant 0} P(x = k) * k && (37)\\
&= \sum_{k \geqslant 0} [\binom{n}{k} p^k (1-p)^{n-k}] * k && (38)\\
& && (39)
\end{aligned}
$$

when

$$
k = 0
$$

, the formula

$$
[\binom{n}{k} p^k (1-p)^{n-k}] * k = [\binom{n}{0} p^k (1-p)^{n}] * 0 = 0
$$

, so the index of the summation can be increased by 1.

$$
\begin{aligned}
E(X) &= \sum_{k\geqslant 1} \binom{n}{k} p^k (1-p)^{n-k} * k && (40)\\
&= \sum_{k\geqslant 1} \frac{n}{k} * \binom{n-1}{k-1} p^k (1-p)^{n-k} * k \qquad \text{see BinomialCoefficient} && (41)\\
&= \sum_{k\geqslant 1} \frac{n*k}{k} * \binom{n-1}{k-1} p^k (1-p)^{n-k} && (42)\\
&= \sum_{k\geqslant 1} n * \binom{n-1}{k-1} p^k (1-p)^{n-k} && (43)\\
&= \sum_{k\geqslant 1} n * p * \binom{n-1}{k-1} p^{k-1} (1-p)^{n-k} && (44)\\
&= np * \sum_{k\geqslant 1} \binom{n-1}{k-1} p^{k-1} (1-p)^{n-k} && (45)\\
u &= n-1 && (46)\\
z &= k-1 && (47)\\
u-z &= (n-1)-(k-1) && (48)\\
&= n-1-k+1 && (49)\\
&= n-k && (50)\\
k>1 &= (z+1)>1 && (51)\\
&= z>0 && (52)\\
&= np * \sum_{z>0} \binom{u}{z} p^z (1-p)^{u-z} && (53)\\
&= np * 1 \qquad see Binomial Distribution Proof Equals 1 && (54)\\
&= np && (55)\\
&\square && (56)
\end{aligned}
$$

Variance

$$
\begin{aligned}
Var(X) &= E(X^2) - E(X)^2 \qquad \text{see Variance} && (57)\\
&= \sum_{k\geqslant 0} [\binom{n}{k} p^k (1-p)^{n-k}] * k^2 - np \qquad \text{see Binomial Expected Value} && (58)
\end{aligned}
$$

when

$$k = 0$$

, the formula

$$[\binom{n}{k}p^k(1-p)^{n-k}] * k = [\binom{n}{0}p^k(1-p)^{n}] * 0 = 0$$

, so the index of the summation can be increased by 1.

$$
\begin{aligned}
&= \sum_{k\geqslant 1} [\binom{n}{k} p^k (1-p)^{n-k}] * k^2 - (np)^2 && (59)\\
&= \sum_{k\geqslant 1} \frac{n}{k} [\binom{n-1}{k-1} p^k (1-p)^{n-k}] * k^2 - (np)^2 && (60)\\
&= \sum_{k\geqslant 1} \frac{n*k^2}{k} [\binom{n-1}{k-1} p^k (1-p)^{n-k}] - (np)^2 && (61)\\
&= \sum_{k\geqslant 1} [nk * \binom{n-1}{k-1} p^k (1-p)^{n-k}] - (np)^2 && (62)\\
&= \sum_{k\geqslant 1} [nkp * \binom{n-1}{k-1} p^{k-1} (1-p)^{n-k}] - (np)^2 && (63)\\
&= np * \sum_{k\geqslant 1} [k * \binom{n-1}{k-1} p^{k-1} (1-p)^{n-k}] - (np)^2 && (64)\\
u &= n-1 && (65)\\
z &= k-1 && (66)\\
u - z &= (n-1) - (k-1) && (67)\\
&= n - 1 - k + 1 && (68)\\
&= n - k && (69)\\
k >= 1 &= (z+1) >= 1 && (70)\\
&= z >= 0 && (71)\\
&= np * \sum_{z\geqslant 0} [(z+1) * \binom{u}{z} p^z (1-p)^{u-z}] - (np)^2 && (72)\\
&= np * [\sum_{z\geqslant 0} [z * \binom{u}{z} p^z (1-p)^{u-z}] + \sum_{z\geqslant 0} [\binom{u}{z} p^z (1-p)^{u-z}]] - (np)^2 && (73)\\
&= np * [\sum_{z\geqslant 0} [z * \frac{u}{z} * \binom{u-1}{z-1} p^z (1-p)^{u-z}] + \sum_{z\geqslant 0} [\binom{u}{z} p^z (1-p)^{u-z}]] - (np)^2 && (74)\\
&= np * [u * \sum_{z\geqslant 0} [\binom{u-1}{z-1} p^z (1-p)^{u-z}] + \sum_{z\geqslant 0} [\binom{u}{z} p^z (1-p)^{u-z}]] - (np)^2 && (75)\\
& && (76)
\end{aligned}
$$

$$= np * [up * \sum_{z \geqslant 0} [\binom{u-1}{z-1} p^{z-1}(1-p)^{u-z}] + \sum_{z \geqslant 0} [\binom{u}{z} p^z (1-p)^{u-z}]] - (np)^2 \tag{77}$$

$$= np * [up * \sum_{z \geqslant 1} [\binom{u-1}{z-1} p^{z-1}(1-p)^{(u-1)-(z-1)}] + \sum_{z \geqslant 0} [\binom{u}{z} p^z (1-p)^{u-z}]] - (np)^2 \tag{78}$$

$$= np * [up * \sum_{z \geqslant 0} [\binom{u}{z} p^z (1-p)^{u-z}] + \sum_{z \geqslant 0} [\binom{u}{z} p^z (1-p)^{u-z}]] - (np)^2 \tag{79}$$

$$= np * [up * (p+q)^{u-1} + (p+q)^u] - (np)^2 \tag{80}$$

$$= np * [(n-1) * p * (p+q)^{n-1-1} + (p+q)^(n-1)] - (np)^2 \tag{81}$$

$$= np * [(n-1) * p * (p+q)^{n-2} + (p+q)^(n-1)] - (np)^2 \tag{82}$$

$$= np * ([(n-1) * p * (p+q)^{n-2} + (p+q)^(n-1)] - np) \tag{83}$$

$$= np * ([(n-1) * p + 1] - np) \quad \text{p+q=1} \tag{84}$$

$$= np * ([(n-1) * p + 1] - np) \tag{85}$$

$$= np * ([np - p + 1] - np) \tag{86}$$

$$= np * (np - p + 1 - np) \tag{87}$$

$$= np * (-p + 1) \tag{88}$$

$$= np * (1-p) \tag{89}$$

$$\square \tag{90}$$

Bernoulli Distribution

The Bernoulli Distribution is a special case of the Binomial Distribution, where

$$n = 1$$

.

PMF

$$P(X = k) = \binom{1}{k} p^k (1-p)^{1-k} \tag{91}$$

$$= p^k (1-p)^{n-k} \tag{92}$$

Expected Value

$$E(x) = \sum_{k \geqslant 1} [\binom{n}{k} p^k (1-p)^{n-k}] * k \tag{93}$$

$$= np \qquad seeBinomialDistribution \tag{94}$$

$$= 1 * p \tag{95}$$

$$= p \tag{96}$$

Variance

$$
\begin{align}
Var(X) &= np * (1-p) \tag{97} \\
&= p * (1-p) \tag{98} \\
& \tag{99}
\end{align}
$$

Likelihood of IID Bernoulli

$$
\begin{align}
x_i \overset{iid}{\sim} Bernoulli(p) \tag{100} \\
L(x_i|p) &= p(x_1, x_2, ..., x_n|p) \tag{101} \\
&= \prod_{n=1}^{n} p(x_i|p) \tag{102} \\
&= p^S * (1-p)^{n-S} \tag{103}
\end{align}
$$

Maximun Likelihood

$$
\begin{aligned}
\frac{d[L(x_i|p)]}{dp} &= \frac{d[p^S * (1-p)^{n-S}]}{dp} && (104)\\
\frac{d[log(L(x_i|p))]}{dp} &= \frac{d[log(p^S * (1-p)^{n-S})]}{dp} && (105)\\
&= \frac{d}{dp} * [log(p^S * (1-p)^{n-S})] && (106)\\
&= \frac{d}{dp}[log(p^S) + log((1-p)^{n-S})] && (107)\\
&= \frac{d}{dp}[S * log(p) + (n-S) * log(1-p)] && (108)\\
&= S * \frac{d}{dp}[log(p)] + (n-S) * \frac{d}{dp}[log(1-p)] && (109)\\
&= S * [\frac{1}{p}] + (n-S) * \frac{d}{dp}[log(1-p)] \qquad \text{chain rule} && (110)\\
&= S * \frac{1}{p} + (n-S) * \frac{1}{p-1} && (111)\\
&= \frac{S}{p} + \frac{n-S}{p-1} && (112)\\
&= \frac{S * (p-1)}{p * (p-1)} + \frac{p * (n-S)}{p * (p-1)} && (113)\\
&= \frac{S * (p-1) + p * (n-S)}{p * (p-1)} && (114)\\
&= \frac{S * p - S + p * n - p * S}{p * (p-1)} && (115)\\
&= \frac{-S + p * n}{p * (p-1)} && (116)\\
0 &= \frac{-S + p * n}{p * (p-1)} && (117)\\
0 * (p * (p-1)) &= -S + p * n && (118)\\
0 &= -S + p * n && (119)\\
S &= p * n && (120)\\
\frac{S}{n} &= p && (121)\\
p &= \frac{S}{n} && (122)\\
& && (123)
\end{aligned}
$$

Normal Distribution Definition

$$pdf(x) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right) \tag{124}$$