1 Introduction

Calculus Handout

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Theorem 1.1 (Squeeze theorem)**.**

$$
\begin{aligned}
&IF\ g(x) <= f(x) <= h(x)\\
&AND\ \lim_{x->a} g(x) = \lim_{x->a} h(x) = L\\
&THEN\ \lim_{x->a} f(x) = L
\end{aligned}
$$

Used by:

1. Theorem 1.2

Theorem 1.2 (Limit of $sin(x)/x$)**.**

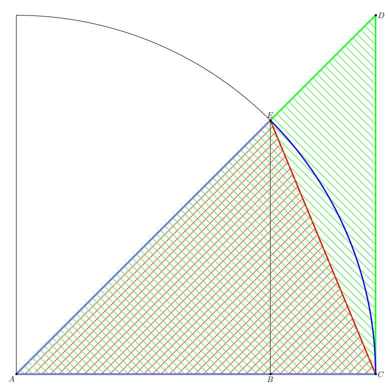
$$\lim_{x\to 0} \frac{sin(x)}{x} = 1$$

Proof. Given that:

$$\mu_1(\overline{A}B) = \text{LENGTH of } \overline{A}B$$
$$\mu_2(\triangle ABC) = \text{AREA of } \triangle ABC$$

We have from the figure that:

$$\mu_2(\triangle ACE) <= \mu_2(\widehat{ACE}) <= \mu_2(\triangle ACD)$$



$$\mu_2(\triangle ACE) = \frac{1}{2} * \mu_1(\overline{AC}) * \mu_1(\overline{BE})$$
$$\mu_2(\triangle ACE) = \frac{1}{2} * 1 * \mu_1(\overline{BE})$$
$$\mu_2(\triangle ACE) = \frac{1}{2} * 1 * |sin(\theta)|$$
$$\mu_2(\triangle ACE) = \frac{|sin(\theta)|}{2}$$

$$\mu_2(\widehat{ACE}) = \frac{|\theta|}{2 * \pi} * \pi * r^2$$

$$\mu_2(\widehat{ACE}) = \frac{|\theta|}{2*\pi} * \pi * 1^2$$
$$\mu_2(\widehat{ACE}) = \frac{|\theta|}{2*\pi} * \pi * 1$$
$$\mu_2(\widehat{ACE}) = \frac{|\theta|}{2*\pi} * \pi$$
$$\mu_2(\widehat{ACE}) = \frac{|\theta|}{2}$$

$$\mu_2(\triangle ACD) = \frac{1}{2} * \mu_1(\overline{AC}) * \mu_1(\overline{CD})$$
$$\mu_2(\triangle ACD) = \frac{1}{2} * 1 * \mu_1(\overline{CD})$$
$$\mu_2(\triangle ACD) = \frac{1}{2} * \mu_1(\overline{CD})$$
$$\mu_2(\triangle ACD) = \frac{1}{2} * |tan(\theta)|$$
$$\mu_2(\triangle ACD) = \frac{|tan(\theta)|}{2}$$

So we have:

$$\frac{|sin(\theta)|}{2} <= \frac{|\theta|}{2} <= \frac{|tan(\theta)|}{2}$$
$$|sin(\theta)| <= |\theta| <= |tan(\theta)|$$
$$|sin(\theta)| <= |\theta| <= \frac{|sin(\theta)|}{|cos(\theta)}$$
$$|sin(\theta)| * \frac{1}{|sin(\theta)|} <= |\theta| * \frac{1}{|sin(\theta)|} <= \frac{|sin(\theta)|}{|cos(\theta)} * \frac{1}{|sin(\theta)|}$$
$$1 <= \frac{|\theta|}{|sin(\theta)|} <= \frac{1}{|cos(\theta)|}$$
$$1 >= \frac{|sin(\theta)|}{|\theta|} <= |cos(\theta)|$$

In the domain $\theta \in (\frac{\pi}{2}, -\frac{\pi}{2})$, $cos(\theta) >= 0$ and $sin(\theta)$ and θ always have the same signal. So we can remove the modulus operator in both cases.

$$1 >= \frac{sin(\theta)}{\theta} <= cos(\theta)$$
$$\lim_{\theta->0} 1 >= \lim_{\theta->0} \frac{sin(\theta)}{\theta} <= \lim_{\theta->0} cos(\theta)$$

Given that

$$\lim_{\theta->0} 1 = 1$$

$$\lim_{\theta->0} cos(\theta) = 1$$

and using the Squeeze Theorem

$$\lim_{\theta->0} \frac{sin(\theta)}{\theta} = 1$$

□

See More:

1. Proofwiki
   `https://proofwiki.org/wiki/Limit_of_Sine_of_X_over_X`

2. Khan Academy
   `https://www.khanacademy.org/math/ap-calculus-ab/ab-derivative-rules/ab-derivtive-rules-opt-vids/v/sinx-over-x-as-x-approaches-0`

3. MIT OpenCourse 18-01 Lecture 03 at 15:15
   `https://youtu.be/kCPVBl953eY?t=915`

Remarks I do not like to use the Proofwiki proof using the "definition" of $sin(x)$. We can see that this proof is using Binmore definition, that depends on the derivative of the $sin(x)$ and $cos(x)$. And that this limit is used to prove $sin'(x)$. Creating a circular proof.

"Mathematical Analysis: A Straightforward Approach, 2nd Edition" of K. G. Binmore - Chapter 16.2
"Suppose that the power series $f(x) = \sum_{n=0}^{\infty} a_n x^n$ converges for all real values of x and that $f''(x) + f'(x) = 0$. This recurrence relation is readily solved and we obtain $f(x) = a_0 \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n+1)!} = x - \frac{x^3}{3!} ...$
We want $sin(0) = 0$ and $sin'(0) = cos(0) = 1$. We therefore define $sin(x)$ as $sin(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + ...$ and $cos(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} = x - \frac{x^2}{2!} + ...$"

Theorem 1.3 (Limit of $(cos(x)-1)/x$).

$$\lim_{x\to 0}\frac{cos(x)-1}{x}=0$$

Proof. see proof □

See More:

1. MIT OpenCourse 18-01 Lecture 03 at 21:07
   `https://youtu.be/kCPVBl953eY?t=1267`

Definition 1.3.1 (Derivative Definition).

$$\frac{d}{dx}f(x)=f'(x)=\lim_{\Delta x\to 0}\frac{f(x+\Delta x)-f(x)}{\Delta x}$$

Theorem 1.4 (Derivative of the scale).

$$f(x)=cg(x)$$
$$f'(x)=cg'(x)$$

Proof.

$$\begin{aligned}
f(x)&=cg(x)\\
f'(x)&=\lim_{\Delta x\to 0}\frac{f(x+\Delta x)-f(x)}{\Delta x}\\
&=\lim_{\Delta x\to 0}\frac{cg(x+\Delta x)-cg(x)}{\Delta x}\\
&=\lim_{\Delta x\to 0}\frac{c(g(x+\Delta x)-g(x))}{\Delta x}\\
&=\lim_{\Delta x\to 0}c*\frac{g(x+\Delta x)-g(x)}{\Delta x}\\
&=c*\lim_{\Delta x\to 0}\frac{g(x+\Delta x)-g(x)}{\Delta x}\\
&=cg'(x)
\end{aligned}$$

□

Theorem 1.5 (Derivative of the sum).

$$f(x)=u(x)+g(x)$$
$$f'(x)=u'(x)+g'(x)$$

Proof.

$$
\begin{aligned}
f(x) &= u(x) + g(x) \\
f'(x) &= \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x} \\
&= \lim_{\Delta x \to 0} \frac{(u(x+\Delta x) + g(x+\Delta x)) - (u(x) + g(x))}{\Delta x} \\
&= \lim_{\Delta x \to 0} \frac{u(x+\Delta x) - u(x) + g(x+\Delta x) - g(x)}{\Delta x} \\
&= \lim_{\Delta x \to 0} \left[\frac{u(x+\Delta x) - u(x)}{\Delta x} + \frac{g(x+\Delta x) - g(x)}{\Delta x} \right] \\
&= \lim_{\Delta x \to 0} \frac{u(x+\Delta x) - u(x)}{\Delta x} + \lim_{\Delta x \to 0} \frac{g(x+\Delta x) - g(x)}{\Delta x} \\
&= u'(x) + g'(x)
\end{aligned}
$$

□

Theorem 1.6 (Derivative of the multiplication)**.**

$$f(x) = u(x)g(x)$$
$$f'(x) = u'(x)g(x) + u(x)g'(x)$$

Proof.

$$\begin{aligned}
f(x) &= u(x)g(x) \\
f'(x) &= \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x} \\
&= \lim_{\Delta x \to 0} \frac{u(x+\Delta x)g(x+\Delta x) - u(x)g(x)}{\Delta x} \\
&= \lim_{\Delta x \to 0} \frac{u(x+\Delta x)g(x+\Delta x) - u(x)g(x) - u(x+\Delta x)g(x) + u(x+\Delta x)g(x)}{\Delta x} \\
&= \lim_{\Delta x \to 0} \frac{u(x+\Delta x)g(x+\Delta x) - u(x)g(x) - u(x+\Delta x)g(x) + u(x+\Delta x)g(x)}{\Delta x} \\
&= \lim_{\Delta x \to 0} \frac{[u(x+\Delta x) - u(x)]g(x) + u(x+\Delta x)[g(x+\Delta x) - g(x)]}{\Delta x} \\
&= \lim_{\Delta x \to 0} \frac{[u(x+\Delta x) - u(x)]g(x)}{\Delta x} + \lim_{\Delta x \to 0} \frac{u(x+\Delta x)[g(x+\Delta x) - g(x)]}{\Delta x} \\
&= \lim_{\Delta x \to 0} \frac{u(x+\Delta x) - u(x)}{\Delta x} * \lim_{\Delta x} g(x) + \lim_{\Delta x \to 0} u(x+\Delta x) * \lim_{\Delta x \to 0} \frac{g(x+\Delta x) - g(x)}{\Delta x} \\
&= \lim_{\Delta x \to 0} \frac{u(x+\Delta x) - u(x)}{\Delta x} * g(x) + u(x) * \lim_{\Delta x \to 0} \frac{g(x+\Delta x) - g(x)}{\Delta x} \\
&= u'(x)g(x) + u(x)g'(x)
\end{aligned}$$

□

See More:

1. Product Rule:
`https://proofwiki.org/wiki/Product_Rule_for_Derivatives`
2. Khan Academy:
`https://www.khanacademy.org/math/ap-calculus-ab/ab-derivative-rules/ab-product-rule/a/proving-the-product-rule`

Product Rule and Leibniz Men of Mathematics - Chapter 7 - Master of All Trades
"Instead of the infinitesimals of Leibniz we shall use the rates discussed in the preceding chapter. If u and v are function of x, how shall the rate of change of $u * v$ with respect to x be expressed in terms of the respective rates of change of u and v with respect to x?
In symbols what is $\frac{d}{dx}(u * v)$ in terms of $\frac{d}{dx}u$ and $\frac{d}{dx}v$?

Leibniz once thought it should be $\frac{d}{dx}u + \frac{d}{dx}v$ which is nothing like the correct

$$\frac{d}{dx}(u * v) = v\frac{d}{dx}u + u\frac{d}{dx}v \tag{1}$$

Theorem 1.7.

$$f(x) = \frac{u(x)}{g(x)}$$
$$f'(x) = \frac{u'(x)g(x) - u(x)g'(x)}{g(x)^2}$$

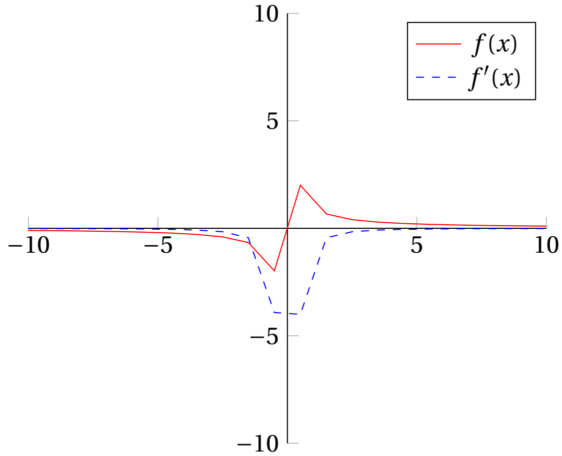
Proof.

$$\begin{aligned}
f(x) &= u(x)g(x) \\
f'(x) &= \lim_{\Delta x\to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x} \\
&= \lim_{\Delta x\to 0} \frac{\frac{u(x+\Delta x)}{g(x+\Delta x)} - \frac{u(x)}{g(x)}}{\Delta x} \\
&= \lim_{\Delta x\to 0} \frac{\frac{u(x+\Delta x)*g(x)}{g(x+\Delta x)*g(x)} - \frac{u(x)*g(x+\Delta x)}{g(x)*g(x+\Delta x)}}{\Delta x} \\
&= \lim_{\Delta x\to 0} \frac{\frac{u(x+\Delta x)*g(x)-u(x)*g(x+\Delta x)}{g(x)*g(x+\Delta x)}}{\Delta x} \\
&= \lim_{\Delta x\to 0} \left[\frac{\frac{u(x+\Delta x)*g(x)-u(x)*g(x+\Delta x)}{g(x)*g(x+\Delta x)}}{\Delta x}\right] \\
&= \lim_{\Delta x\to 0} \left[\frac{1}{g(x)*g(x+\Delta x)} * \frac{u(x+\Delta x)*g(x) - u(x)*g(x+\Delta x)}{\Delta x}\right] \\
&= \lim_{\Delta x\to 0} \left[\frac{1}{g(x)*g(x+\Delta x)} * \frac{g(x)[u(x+\Delta x) - u(x)] - u(x)[g(x+\Delta x) - g(x)]}{\Delta x}\right] \\
&= \lim_{\Delta x\to 0} \left[\frac{1}{g(x)*g(x+\Delta x)} * \left(\frac{g(x)[u(x+\Delta x) - u(x)]}{\Delta x} - \frac{u(x)[g(x+\Delta x) - g(x)]}{\Delta x}\right)\right] \\
&= \frac{1}{g(x)*g(x)} * \left[g(x)u'(x) - u(x)g'(x)\right] \\
&= \frac{g(x)u'(x) - u(x)g'(x)}{g(x)^2} \\
&= \frac{u'(x)g(x) - u(x)g'(x)}{g(x)^2}
\end{aligned}$$

□

Theorem 1.8 (Derivative of $1/x$)**.**

$$f(x) = \frac{1}{x}$$
$$f'(x) = \frac{-1}{x^2}$$



Proof.

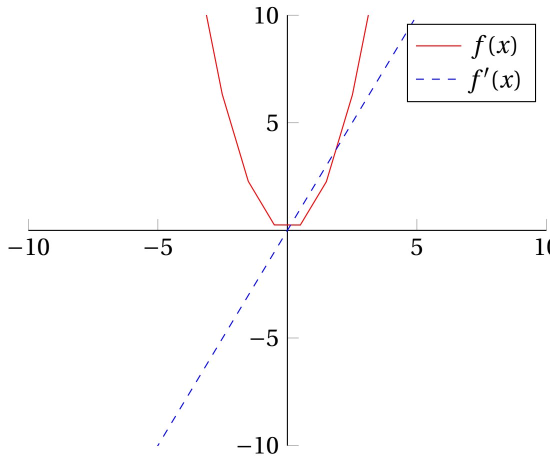
$$
\begin{aligned}
f(x) &= \frac{1}{x} \\
f'(x) &= \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x} \\
&= \lim_{\Delta x \to 0} \frac{\frac{1}{x+\Delta x} - \frac{1}{x}}{\Delta x} \\
&= \lim_{\Delta x \to 0} \frac{\frac{1*x}{(x+\Delta x)*x} - \frac{1*(x+\Delta x)}{x*(x+\Delta x)}}{\Delta x} \\
&= \lim_{\Delta x \to 0} \frac{\frac{(1*x)-[1*(x+\Delta x)]}{(x+\Delta x)*x}}{\Delta x} \\
&= \lim_{\Delta x \to 0} \frac{\frac{x-(x+\Delta x)}{(x+\Delta x)*x}}{\Delta x} \\
&= \lim_{\Delta x \to 0} \frac{\frac{x-x-\Delta x}{(x+\Delta x)*x}}{\Delta x} \\
&= \lim_{\Delta x \to 0} \frac{\frac{-\Delta x}{(x+\Delta x)*x}}{\Delta x} \\
&= \lim_{\Delta x \to 0} \left[\frac{1}{\Delta x} * \frac{-\Delta x}{(x+\Delta x)*x} \right] \\
&= \lim_{\Delta x \to 0} \frac{-1}{(x+\Delta x)*x} \\
&= \frac{-1}{(x+0)*x} \\
&= \frac{-1}{x*x} \\
&= -\frac{1}{x^2}
\end{aligned}
$$

□

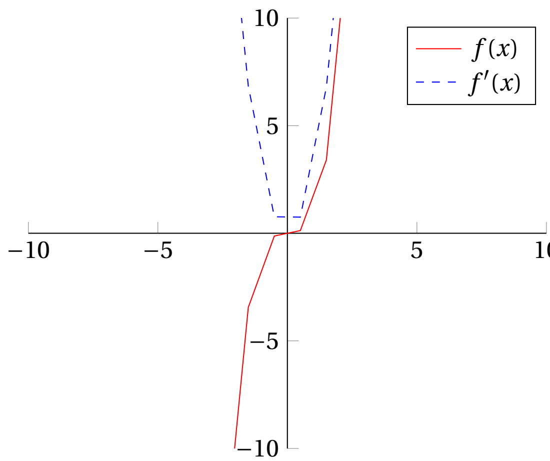
Theorem 1.9 (Derivative of x^n)**.**

$$f(x) = x^n$$
$$f'(x) = nx^{n-1}$$

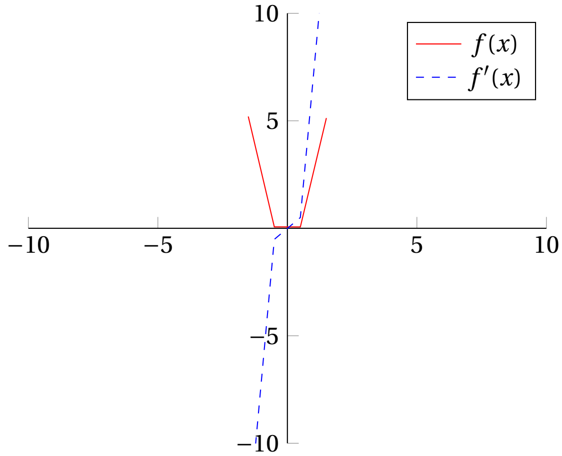
$f(x) = x^2$ and $f'(x) = 2x$



$f(x) = x^3$ and $f'(x) = 3x^2$



$f(x) = x^4$ and $f'(x) = 4x^3$



Proof.

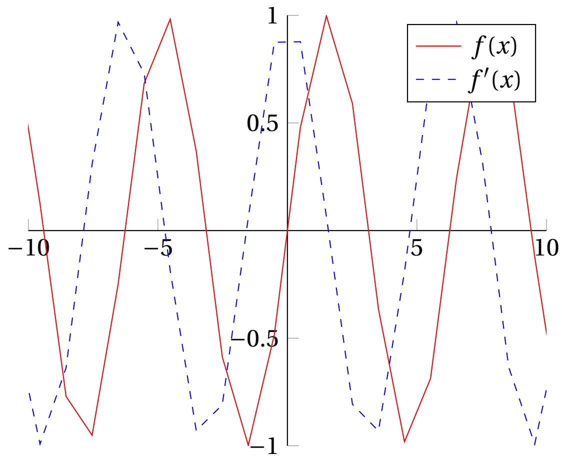
$$
\begin{aligned}
f(x) &= x^n \\
f'(x) &= \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x} \\
&= \lim_{\Delta x \to 0} \frac{(x+\Delta x)^n - x^n}{\Delta x} \\
&= \lim_{\Delta x \to 0} \frac{[x^n + nx^{n-1}\Delta x + \mathcal{O}((\Delta x)^2)] - x^n}{\Delta x} \\
&= \lim_{\Delta x \to 0} \frac{[nx^{n-1}\Delta x + \mathcal{O}((\Delta x)^2)]}{\Delta x} \\
&= \lim_{\Delta x \to 0} \frac{\Delta x[nx^{n-1} + \mathcal{O}(\Delta x)]}{\Delta x} \\
&= \lim_{\Delta x \to 0} nx^{n-1} + \mathcal{O}(\Delta x) \\
&= nx^{n-1}
\end{aligned}
$$

□

Theorem 1.10 (Derivative of $sin(x)$)**.**

$$f(x) = sin(x)$$
$$f'(x) = cos(x)$$

$f(x) = sin(x)$ and $f'(x) = cos(x)$



Proof.

$$
\begin{aligned}
f(x) &= sin(x)\\
f'(x) &= \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}\\
&= \lim_{\Delta x \to 0} \frac{sin(x+\Delta x) - sin(x)}{\Delta x}\\
&= \lim_{\Delta x \to 0} \frac{sin(x)cos(\Delta x) + cos(x)sin(\Delta x) - sin(x)}{\Delta x}\\
&= \lim_{\Delta x \to 0} \frac{sin(x)(cos(\Delta x) - 1) + cos(x)sin(\Delta x)}{\Delta x}\\
&= \lim_{\Delta x \to 0} \left[\frac{sin(x)(cos(\Delta x) - 1)}{\Delta x} + \frac{cos(x)sin(\Delta x)}{\Delta x} \right]\\
&= \lim_{\Delta x \to 0} \left[sin(x) * \frac{cos(\Delta x) - 1}{\Delta x} + cos(x) * \frac{sin(\Delta x)}{\Delta x} \right]\\
&= \lim_{\Delta x \to 0} \left[sin(x) * \frac{cos(\Delta x) - 1}{\Delta x} \right] + \lim_{\Delta x \to 0} \left[cos(x) * \frac{sin(\Delta x)}{\Delta x} \right]\\
&= \left[\lim_{\Delta x \to 0} sin(x) \right] * \left[\lim_{\Delta x \to 0} \frac{cos(\Delta x) - 1}{\Delta x} \right] + \left[\lim_{\Delta x \to 0} cos(x) \right] * \left[\lim_{\Delta x \to 0} \frac{sin(\Delta x)}{\Delta x} \right]\\
&= \left[\lim_{\Delta x \to 0} sin(x) \right] * [0] + \left[\lim_{\Delta x \to 0} cos(x) \right] * [1] && \text{see 1.3, 1.2}\\
&= \lim_{\Delta x \to 0} cos(x)
\end{aligned}
$$

$= cos(x)$

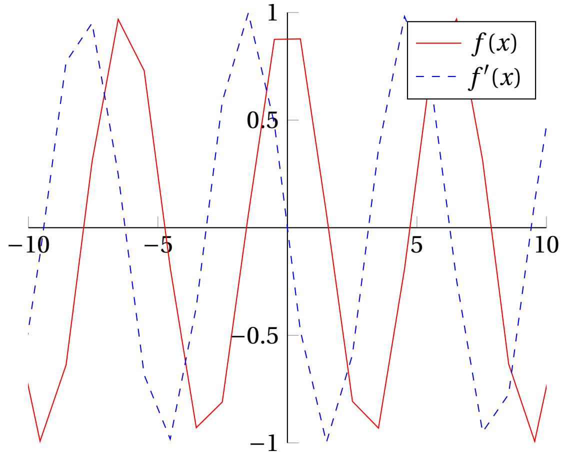
$\square$

See More:

1. Khan Academy
   `https://www.khanacademy.org/math/ap-calculus-ab/ab-derivative-rules/ab-derivtive-rules-opt-vids/v/derivative-of-sin-x`
2. MIT OpenCourse 18-01 Lecture 03 at 03:09
   `https://youtu.be/kCPVBl953eY?t=189`

Theorem 1.11 (Derivative of $cos(x)$)**.**

$$f(x) = cos(x)$$
$$f'(x) = -sin(x)$$



Proof.

$$f(x) = cos(x)$$
$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$$
$$f'(x) = \lim_{\Delta x \to 0} \frac{cos(x+\Delta x) - cos(x)}{\Delta x}$$
$$f'(x) = \lim_{\Delta x \to 0} \frac{cos(x)cos(\Delta x) - sin(x)sin(\Delta x) - cos(x)}{\Delta x}$$
$$f'(x) = \lim_{\Delta x \to 0} \frac{cos(x)(cos(\Delta x) - 1) - sin(x)sin(\Delta x)}{\Delta x}$$
$$f'(x) = \lim_{\Delta x \to 0} \left[\frac{cos(x)(cos(\Delta x) - 1)}{\Delta x} - \frac{sin(x)sin(\Delta x)}{\Delta x} \right]$$
$$f'(x) = \lim_{\Delta x \to 0} \frac{cos(x)(cos(\Delta x) - 1)}{\Delta x} - \lim_{\Delta x \to 0} \frac{sin(x)sin(\Delta x)}{\Delta x}$$
$$f'(x) = \lim_{\Delta x \to 0} \left[cos(x) * \frac{(cos(\Delta x) - 1)}{\Delta x} \right] - \lim_{\Delta x \to 0} \left[sin(x) * \frac{sin(\Delta x)}{\Delta x} \right]$$
$$f'(x) = \left[\lim_{\Delta x \to 0} cos(x) \right] * \left[\lim_{\Delta x \to 0} \frac{(cos(\Delta x) - 1)}{\Delta x} \right] - \left[\lim_{\Delta x \to 0} sin(x) \right] * \left[\lim_{\Delta x \to 0} \frac{sin(\Delta x)}{\Delta x} \right]$$
$$f'(x) = \left[\lim_{\Delta x \to 0} cos(x) \right] * [0] - \left[\lim_{\Delta x \to 0} sin(x) \right] * [1] \qquad \text{see 1.3, 1.2}$$
$$f'(x) = -\lim_{\Delta x \to 0} sin(x)$$
$$f'(x) = -sin(x)$$

□

See More:

1. Khan Academy
   https://www.khanacademy.org/math/ap-calculus-ab/ab-derivative-rules/ab-derivtive-rules-opt-vids/v/derivative-of-cos-x
2. MIT OpenCourse 18-01 Lecture 03 at 08:54
   https://youtu.be/kCPVBl953eY?t=534

Laplace Transform

Theorem 1.12.

$$\mathscr{L}(f(t)) = \int_0^\infty e^{-st} f(t)\,dt$$

Theorem 1.13 (Lagrange of 1)**.**

$$
\begin{aligned}
\mathscr{L}(1) &= \frac{1}{s} \\
\mathscr{L}(1) &= \int_0^\infty e^{-st} 1 dt \\
&= \int_0^\infty e^{-st} dt \\
&= \lim_{x\to\infty} \int_0^x e^{-st} dt \\
&= \lim_{x\to\infty} \frac{e^{-st}}{-s}\Big|_0^x \\
&= \lim_{x\to\infty} \left[\frac{e^{-sx}}{-s} - \frac{e^{-s0}}{-s}\right] \\
&= \lim_{x\to\infty} \left[\frac{e^{-sx}}{-s} - \frac{1}{-s}\right] \\
&= \lim_{x\to\infty} \left[\frac{e^{-sx}}{-s}\right] - \frac{1}{-s} \\
&= 0 - \frac{1}{-s} \\
&= -\frac{1}{-s} \\
&= \frac{1}{s}
\end{aligned}
$$