1 Introduction

Calculus Handout

Daniel Frederico Lins Leite

September 2016

Theorem 1.1 (Squeeze theorem)**.**

$$
\begin{aligned}
&IF\ g(x) <= f(x) <= h(x)\\
&AND\ \lim_{x->a} g(x) = \lim_{x->a} h(x) = L\\
&THEN\ \lim_{x->a} f(x) = L
\end{aligned}
$$

Used by:

1. Theorem 1.2

Theorem 1.2 (Limit of $sin(x)/x$)**.**

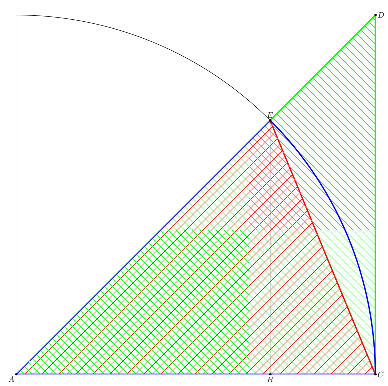
$$\lim_{x \to 0} \frac{sin(x)}{x} = 1$$

Proof. Given that:

$$\mu_1(\overline{AB}) = \text{LENGTH of } \overline{AB}$$
$$\mu_2(\triangle ABC) = \text{AREA of } \triangle ABC$$

We have from the figure that:

$$\mu_2(\triangle ACE) <= \mu_2(\widehat{ACE}) <= \mu_2(\triangle ACD)$$



$$\mu_2(\triangle ACE) = \frac{1}{2} * \mu_1(\overline{AC}) * \mu_1(\overline{BE})$$
$$\mu_2(\triangle ACE) = \frac{1}{2} * 1 * \mu_1(\overline{BE})$$
$$\mu_2(\triangle ACE) = \frac{1}{2} * 1 * |sin(\theta)|$$
$$\mu_2(\triangle ACE) = \frac{|sin(\theta)|}{2}$$

$$\mu_2(\widehat{ACE}) = \frac{|\theta|}{2 * \pi} * \pi * r^2$$

$$\mu_2(\widehat{ACE}) = \frac{|\theta|}{2*\pi}*\pi*1^2$$
$$\mu_2(\widehat{ACE}) = \frac{|\theta|}{2*\pi}*\pi*1$$
$$\mu_2(\widehat{ACE}) = \frac{|\theta|}{2*\pi}*\pi$$
$$\mu_2(\widehat{ACE}) = \frac{|\theta|}{2}$$

$$\mu_2(\triangle ACD) = \frac{1}{2}*\mu_1(\overline{AC})*\mu_1(\overline{CD})$$
$$\mu_2(\triangle ACD) = \frac{1}{2}*1*\mu_1(\overline{CD})$$
$$\mu_2(\triangle ACD) = \frac{1}{2}*\mu_1(\overline{CD})$$
$$\mu_2(\triangle ACD) = \frac{1}{2}*|tan(\theta)|$$
$$\mu_2(\triangle ACD) = \frac{|tan(\theta)|}{2}$$

So we have:

$$\frac{|sin(\theta)|}{2} <= \frac{|\theta|}{2} <= \frac{|tan(\theta)|}{2}$$
$$|sin(\theta)| <= |\theta| <= |tan(\theta)|$$
$$|sin(\theta)| <= |\theta| <= \frac{|sin(\theta)|}{|cos(\theta)}$$
$$|sin(\theta)|*\frac{1}{|sin(\theta)|} <= |\theta|*\frac{1}{|sin(\theta)|} <= \frac{|sin(\theta)|}{|cos(\theta)}*\frac{1}{|sin(\theta)|}$$
$$1 <= \frac{|\theta|}{|sin(\theta)|} <= \frac{1}{|cos(\theta)|}$$
$$1 >= \frac{|sin(\theta)|}{|\theta|} <= |cos(\theta)|$$

In the domain $\theta \in (\frac{\pi}{2}, -\frac{\pi}{2})$, $cos(\theta) >= 0$ and $sin(\theta)$ and θ always have the same signal. So we can remove the modulus operator in both cases.

$$1 >= \frac{sin(\theta)}{\theta} <= cos(\theta)$$
$$\lim_{\theta->0} 1 >= \lim_{\theta->0} \frac{sin(\theta)}{\theta} <= \lim_{\theta->0} cos(\theta)$$

Given that

$$\lim_{\theta->0} 1 = 1$$

$$\lim_{\theta->0} cos(\theta) = 1$$

and using the Squeeze Theorem

$$\lim_{\theta->0} \frac{sin(\theta)}{\theta} = 1$$

□

See More:

1. Proofwiki
   https://proofwiki.org/wiki/Limit_of_Sine_of_X_over_X
2. Khan Academy
   https://www.khanacademy.org/math/ap-calculus-ab/ab-derivative-rules/ab-derivtive-rules-opt-vids/v/sinx-over-x-as-x-approaches-0

Theorem 1.3 (Limit of $(cos(x)-1)/x$)**.**

$$\lim_{x\to 0} \frac{cos(x)-1}{x} = 0$$

Proof. see proof □

Definition 1.3.1 (Derivative Definition)**.**

$$\frac{d}{dx} f(x) = f'(x) = \lim_{\Delta x\to 0} \frac{f(x+\Delta x)-f(x)}{\Delta x}$$

Theorem 1.4 (Derivative of the scale)**.**

$$f(x) = cg(x)$$
$$f'(x) = cg'(x)$$

Proof.

$$\begin{aligned}
f(x) &= cg(x) \\
f'(x) &= \lim_{\Delta x\to 0} \frac{f(x+\Delta x)-f(x)}{\Delta x} \\
&= \lim_{\Delta x\to 0} \frac{cg(x+\Delta x)-cg(x)}{\Delta x} \\
&= \lim_{\Delta x\to 0} \frac{c(g(x+\Delta x)-g(x))}{\Delta x} \\
&= \lim_{\Delta x\to 0} c * \frac{g(x+\Delta x)-g(x)}{\Delta x} \\
&= c * \lim_{\Delta x\to 0} \frac{g(x+\Delta x)-g(x)}{\Delta x} \\
&= cg'(x)
\end{aligned}$$

□

Theorem 1.5 (Derivative of the sum)**.**

$$f(x) = u(x) + g(x)$$
$$f'(x) = u'(x) + g'(x)$$

Proof.

$$
\begin{aligned}
f(x) &= u(x) + g(x) \\
f'(x) &= \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x} \\
&= \lim_{\Delta x \to 0} \frac{(u(x+\Delta x) + g(x+\Delta x)) - (u(x) + g(x))}{\Delta x} \\
&= \lim_{\Delta x \to 0} \frac{u(x+\Delta x) - u(x) + g(x+\Delta x) - g(x)}{\Delta x} \\
&= \lim_{\Delta x \to 0} \left[\frac{u(x+\Delta x) - u(x)}{\Delta x} + \frac{g(x+\Delta x) - g(x)}{\Delta x} \right] \\
&= \lim_{\Delta x \to 0} \frac{u(x+\Delta x) - u(x)}{\Delta x} + \lim_{\Delta x \to 0} \frac{g(x+\Delta x) - g(x)}{\Delta x} \\
&= u'(x) + g'(x)
\end{aligned}
$$

□

Theorem 1.6 (Derivative of the multiplication)**.**

$$f(x) = u(x)g(x)$$
$$f'(x) = u'(x)g(x) + u(x)g'(x)$$

Proof.

$$\begin{aligned}
f(x) &= u(x)g(x) \\
f'(x) &= \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x} \\
&= \lim_{\Delta x \to 0} \frac{u(x+\Delta x)g(x+\Delta x) - u(x)g(x)}{\Delta x} \\
&= \lim_{\Delta x \to 0} \frac{u(x+\Delta x)g(x+\Delta x) - u(x)g(x) - u(x+\Delta x)g(x) + u(x+\Delta x)g(x)}{\Delta x} \\
&= \lim_{\Delta x \to 0} \frac{\color{red}{u(x+\Delta x)g(x+\Delta x)} \color{blue}{- u(x)g(x)} \color{red}{- u(x+\Delta x)g(x)} \color{blue}{+ u(x+\Delta x)g(x)}}{\Delta x} \\
&= \lim_{\Delta x \to 0} \frac{\color{blue}{[u(x+\Delta x) - u(x)]g(x)} + \color{red}{u(x+\Delta x)[g(x+\Delta x) - g(x)]}}{\Delta x} \\
&= \lim_{\Delta x \to 0} \frac{[u(x+\Delta x) - u(x)]g(x)}{\Delta x} + \lim_{\Delta x \to 0} \frac{u(x+\Delta x)[g(x+\Delta x) - g(x)]}{\Delta x} \\
&= \lim_{\Delta x \to 0} \frac{u(x+\Delta x) - u(x)}{\Delta x} * \lim_{\Delta x} g(x) + \lim_{\Delta x \to 0} u(x+\Delta x) * \lim_{\Delta x \to 0} \frac{g(x+\Delta x) - g(x)}{\Delta x} \\
&= \lim_{\Delta x \to 0} \frac{u(x+\Delta x) - u(x)}{\Delta x} * g(x) + u(x) * \lim_{\Delta x \to 0} \frac{g(x+\Delta x) - g(x)}{\Delta x} \\
&= u'(x)g(x) + u(x)g'(x)
\end{aligned}$$

□

See More:

1. Product Rule:
   `https://proofwiki.org/wiki/Product_Rule_for_Derivatives`
2. Khan Academy:
   `https://www.khanacademy.org/math/ap-calculus-ab/ab-derivative-rules/ab-product-rule/a/proving-the-product-rule`

Product Rule and Leibniz Men of Mathematics - Chapter 7 - Master of All Trades "Instead of the infinitesimals of Leibniz we shall use the rates discussed in the preceding chapter. If u and v are function of x, how shall the rate of change of $u * v$ with respect to x be expressed in terms of the respective rates of change of u and v with respect to x?
In symbols what is $\frac{d}{dx}(u * v)$ in terms of $\frac{d}{dx}u$ and $\frac{d}{dx}v$?

Leibniz once thought it should be $\frac{d}{dx}u + \frac{d}{dx}v$ which is nothing like the correct

$$\frac{d}{dx}(u * v) = v\frac{d}{dx}u + u\frac{d}{dx}v \tag{1}$$

Theorem 1.7.

$$f(x) = \frac{u(x)}{g(x)}$$
$$f'(x) = \frac{u'(x)g(x) - u(x)g'(x)}{g(x)^2}$$

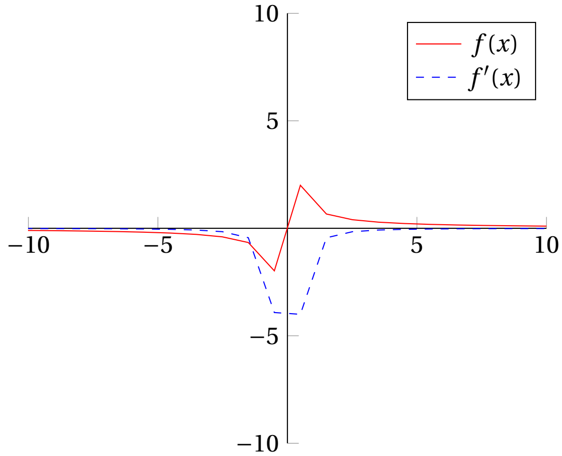
Proof.

$$\begin{aligned}
f(x) &= u(x)g(x) \\
f'(x) &= \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x} \\
&= \lim_{\Delta x \to 0} \frac{\frac{u(x+\Delta x)}{g(x+\Delta x)} - \frac{u(x)}{g(x)}}{\Delta x} \\
&= \lim_{\Delta x \to 0} \frac{\frac{u(x+\Delta x)*g(x)}{g(x+\Delta x)*g(x)} - \frac{u(x)*g(x+\Delta x)}{g(x)*g(x+\Delta x)}}{\Delta x} \\
&= \lim_{\Delta x \to 0} \frac{\frac{u(x+\Delta x)*g(x) - u(x)*g(x+\Delta x)}{g(x)*g(x+\Delta x)}}{\Delta x} \\
&= \lim_{\Delta x \to 0} \left[\frac{\frac{u(x+\Delta x)*g(x) - u(x)*g(x+\Delta x)}{g(x)*g(x+\Delta x)}}{\Delta x} \right] \\
&= \lim_{\Delta x \to 0} \left[\frac{1}{g(x)*g(x+\Delta x)} * \frac{u(x+\Delta x)*g(x) - u(x)*g(x+\Delta x)}{\Delta x} \right] \\
&= \lim_{\Delta x \to 0} \left[\frac{1}{g(x)*g(x+\Delta x)} * \frac{g(x)[u(x+\Delta x) - u(x)] - u(x)[g(x+\Delta x) - g(x)]}{\Delta x} \right] \\
&= \lim_{\Delta x \to 0} \left[\frac{1}{g(x)*g(x+\Delta x)} * \left(\frac{g(x)[u(x+\Delta x) - u(x)]}{\Delta x} - \frac{u(x)[g(x+\Delta x) - g(x)]}{\Delta x} \right) \right] \\
&= \frac{1}{g(x)*g(x)} * \left[g(x)u'(x) - u(x)g'(x) \right] \\
&= \frac{g(x)u'(x) - u(x)g'(x)}{g(x)^2} \\
&= \frac{u'(x)g(x) - u(x)g'(x)}{g(x)^2}
\end{aligned}$$

□

Theorem 1.8 (Derivative of $1/x$)**.**

$$f(x) = \frac{1}{x}$$
$$f'(x) = \frac{-1}{x^2}$$



Proof.

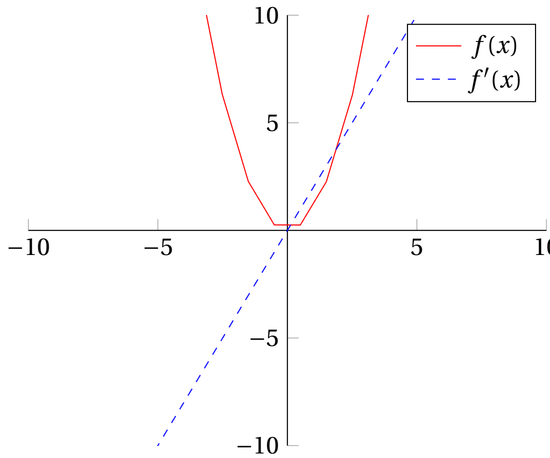
$$
\begin{aligned}
f(x) &= \frac{1}{x} \\
f'(x) &= \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x} \\
&= \lim_{\Delta x \to 0} \frac{\frac{1}{x+\Delta x} - \frac{1}{x}}{\Delta x} \\
&= \lim_{\Delta x \to 0} \frac{\frac{1*x}{(x+\Delta x)*x} - \frac{1*(x+\Delta x)}{x*(x+\Delta x)}}{\Delta x} \\
&= \lim_{\Delta x \to 0} \frac{\frac{(1*x)-[1*(x+\Delta x)]}{(x+\Delta x)*x}}{\Delta x} \\
&= \lim_{\Delta x \to 0} \frac{\frac{x-(x+\Delta x)}{(x+\Delta x)*x}}{\Delta x} \\
&= \lim_{\Delta x \to 0} \frac{\frac{x-x-\Delta x}{(x+\Delta x)*x}}{\Delta x} \\
&= \lim_{\Delta x \to 0} \frac{\frac{-\Delta x}{(x+\Delta x)*x}}{\Delta x} \\
&= \lim_{\Delta x \to 0} \left[\frac{1}{\Delta x} * \frac{-\Delta x}{(x+\Delta x)*x} \right] \\
&= \lim_{\Delta x \to 0} \frac{-1}{(x+\Delta x)*x} \\
&= \frac{-1}{(x+0)*x} \\
&= \frac{-1}{x*x} \\
&= -\frac{1}{x^2}
\end{aligned}
$$

$\square$

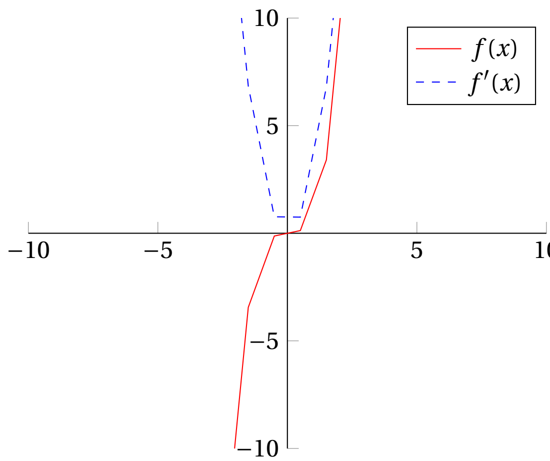
Theorem 1.9 (Derivative of x^n).

$$f(x) = x^n$$
$$f'(x) = nx^{n-1}$$

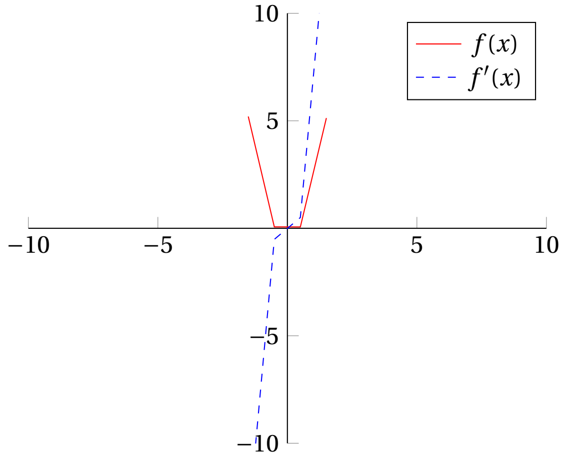
$f(x) = x^2$ and $f'(x) = 2x$



$f(x) = x^3$ and $f'(x) = 3x^2$



$f(x) = x^4$ and $f'(x) = 4x^3$



Proof.

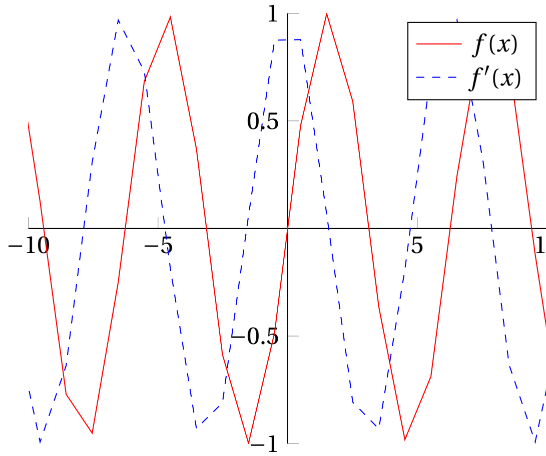
$$
\begin{aligned}
f(x) &= x^n \\
f'(x) &= \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x} \\
&= \lim_{\Delta x \to 0} \frac{(x+\Delta x)^n - x^n}{\Delta x} \\
&= \lim_{\Delta x \to 0} \frac{[x^n + nx^{n-1}\Delta x + \mathcal{O}((\Delta x)^2)] - x^n}{\Delta x} \\
&= \lim_{\Delta x \to 0} \frac{[nx^{n-1}\Delta x + \mathcal{O}((\Delta x)^2)]}{\Delta x} \\
&= \lim_{\Delta x \to 0} \frac{\Delta x[nx^{n-1} + \mathcal{O}(\Delta x)]}{\Delta x} \\
&= \lim_{\Delta x \to 0} nx^{n-1} + \mathcal{O}(\Delta x) \\
&= nx^{n-1}
\end{aligned}
$$

□

Theorem 1.10 (Derivative of $sin(x)$)**.**

$$
\begin{aligned}
f(x) &= sin(x) \\
f'(x) &= cos(x)
\end{aligned}
$$

$f(x) = sin(x)$ and $f'(x) = cos(x)$



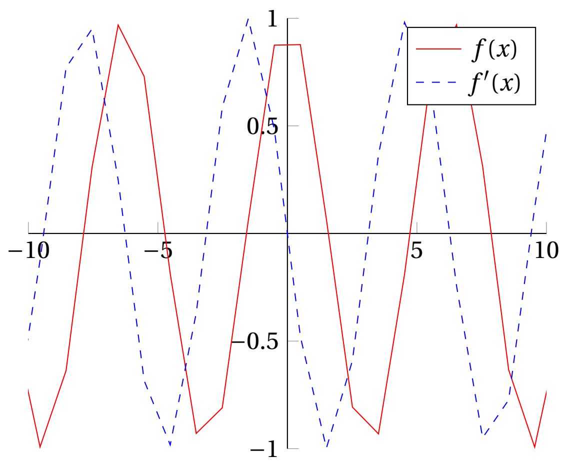
Proof.

$$
\begin{aligned}
f(x) &= sin(x) \\
f'(x) &= \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x} \\
&= \lim_{\Delta x \to 0} \frac{sin(x+\Delta x) - sin(x)}{\Delta x} \\
&= \lim_{\Delta x \to 0} \frac{sin(x)cos(\Delta x) + cos(x)sin(\Delta x) - sin(x)}{\Delta x} \\
&= \lim_{\Delta x \to 0} \frac{sin(x)(cos(\Delta x) - 1) + cos(x)sin(\Delta x)}{\Delta x} \\
&= \lim_{\Delta x \to 0} \left[\frac{sin(x)(cos(\Delta x) - 1)}{\Delta x} + \frac{cos(x)sin(\Delta x)}{\Delta x} \right] \\
&= \lim_{\Delta x \to 0} \left[sin(x) * \frac{cos(\Delta x) - 1}{\Delta x} + cos(x) * \frac{sin(\Delta x)}{\Delta x} \right] \\
&= \lim_{\Delta x \to 0} \left[sin(x) * \frac{cos(\Delta x) - 1}{\Delta x} \right] + \lim_{\Delta x \to 0} \left[cos(x) * \frac{sin(\Delta x)}{\Delta x} \right] \\
&= \lim_{\Delta x \to 0} [sin(x) * 0] + \lim_{\Delta x \to 0} [cos(x) * 1] && \text{see 1.3, 1.2} \\
&= \lim_{\Delta x \to 0} cos(x) \\
&= cos(x)
\end{aligned}
$$

□

Theorem 1.11 (Derivative of $cos(x)$)**.**

$$
\begin{aligned}
f(x) &= cos(x) \\
f'(x) &= -sin(x)
\end{aligned}
$$



Proof.

$$f(x) = cos(x)$$
$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$$
$$f'(x) = \lim_{\Delta x \to 0} \frac{cos(x+\Delta x) - cos(x)}{\Delta x}$$
$$f'(x) = \lim_{\Delta x \to 0} \frac{cos(x)cos(\Delta x) - sin(x)sin(\Delta x) - cos(x)}{\Delta x}$$
$$f'(x) = \lim_{\Delta x \to 0} \frac{cos(x)(cos(\Delta x) - 1) - sin(x)sin(\Delta x)}{\Delta x}$$
$$f'(x) = \lim_{\Delta x \to 0} \left[\frac{cos(x)(cos(\Delta x) - 1)}{\Delta x} - \frac{sin(x)sin(\Delta x)}{\Delta x} \right]$$
$$f'(x) = \lim_{\Delta x \to 0} \frac{cos(x)(cos(\Delta x) - 1)}{\Delta x} - \lim_{\Delta x \to 0} \frac{sin(x)sin(\Delta x)}{\Delta x}$$
$$f'(x) = \lim_{\Delta x \to 0} \left[cos(x) * \frac{(cos(\Delta x) - 1)}{\Delta x} \right] - \lim_{\Delta x \to 0} \left[sin(x) * \frac{sin(\Delta x)}{\Delta x} \right]$$
$$f'(x) = \lim_{\Delta x \to 0} [cos(x) * 0] - \lim_{\Delta x \to 0} [sin(x) * 1] \qquad \text{see 1.3, 1.2}$$
$$f'(x) = -\lim_{\Delta x \to 0} sin(x)$$
$$f'(x) = -sin(x)$$

□