Proposition 1.4.53
Let A be positive definite, and consider a partition

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$$

in which A_{11} and A_{22} are square. Then A_{11} and A_{22} are positive definite.
Equation 1.4.57

$$A = \begin{bmatrix} a_{11} & b^T \\ b & \hat{A} \end{bmatrix}$$

Proposition 1.4.51
guarantees that $a_{11} > 0$. Using 1.4.27 and 1.4.28 as a guide, define

$$\begin{aligned} r_{11} &= +\sqrt{a_{11}} \\ s &= r_{11}^{-1} b \\ \tilde{A} &= \hat{A} - ss^T \end{aligned}$$

Then, as one easily checks,

$$A = \begin{bmatrix} r_{11} & 0 \\ s & I \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & \tilde{A} \end{bmatrix} \begin{bmatrix} r_{11} & s^T \\ 0 & I \end{bmatrix}$$

Exercise 1.4.58
Let

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$$

be "Positive Definite", and suppose A_{11} is $j \times j$ and A_{22} is $k \times k$.
By Proposition 1.4.53, A_{11} is "Positive Definite".
Let R_{11} be the Cholesky factor of A_{11}, let $R_{12} = R_{11}^{-T} A_{12}$, and let $\tilde{A}_{22} = A_{22} - R_{12}^T R_{12}$.
The matrix $\tilde{A}_{22}$ is called the "Schur Complement" of A_{11} in A.
(a) Show that $\tilde{A}_{22} = A_{22} - A_{21} A_{11}^{-1} A_{12}$.
(b) Establish a decomposition of A that is similar to 1.4.57 and involves $\tilde{A}_{22}$
(c) Prove that $\tilde{A}_{22}$ is "Positive Definite".

Proof:
(a)

$$\tilde{A}_{22} = A_{22} - R_{12}^T R_{12}$$

$$\begin{aligned} R_{12} &= R_{11}^{-T} A_{12} \\ R_{12}^T &= A_{12}^T R_{11}^{-TT} \\ &= A_{12}^T R_{11}^{-1} \end{aligned}$$

$$\begin{aligned} \tilde{A}_{22} &= A_{22} - (A_{12}^T R_{11}^{-1})(R_{11}^{-T} A_{12}) \\ \tilde{A}_{22} &= A_{22} - A_{12}^T R_{11}^{-1} R_{11}^{-T} A_{12} \end{aligned}$$

$$\begin{aligned} A_{11} &= R_{11}^T R_{11} \\ A_{11}^{-1} &= (R_{11}^T R_{11})^{-1} \\ A_{11}^{-1} &= R_{11}^{-1} R_{11}^{-T} \end{aligned}$$

$$\tilde{A}_{22} = A_{22} - A_{12}^T A_{11}^{-1} A_{12}$$

$$A_{21} = A_{12}^T$$

$$\tilde{A}_{22} = A_{22} - A_{21} A_{11}^{-1} A_{12}$$

$\square$

(b)

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$$

using 1.4.57

$$A = X^T B X$$

$$A = \begin{bmatrix} R_{11}^T & 0 \\ s & I \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & \tilde{A} \end{bmatrix} \begin{bmatrix} R_{11} & s^T \\ 0 & I \end{bmatrix}$$

$$= \begin{bmatrix} R_{11}^T * I + 0 * 0 & R_{11}^T * 0 + 0 * \tilde{A} \\ s * I + I * 0 & s * 0 + I * \tilde{A} \end{bmatrix} \begin{bmatrix} R_{11} & s^T \\ 0 & I \end{bmatrix}$$

$$= \begin{bmatrix} R_{11}^T & 0 \\ s & \tilde{A} \end{bmatrix} \begin{bmatrix} R_{11} & s^T \\ 0 & I \end{bmatrix}$$

$$= \begin{bmatrix} R_{11}^T * R_{11} + 0 * 0 & R_{11}^T * s^T + 0 * I \\ s * R_{11} + \tilde{A} * 0 & s * s^T + \tilde{A} * I \end{bmatrix}$$

$$= \begin{bmatrix} R_{11}^T R_{11} & R_{11}^T s^T \\ s R_{11} & s s^T + \tilde{A} \end{bmatrix}$$

$$
\begin{aligned}
A_{11} &= R_{11}^T R_{11} \\
A_{12} &= R_{11}^T s^T \\
A_{21} &= s R_{11} \\
A_{22} &= s s^T + \tilde{A}
\end{aligned}
$$

$$
\begin{aligned}
A_{21} &= s R_{11} \\
A_{21} R_{11}^{-1} &= s R_{11} R_{11}^{-1} \\
A_{21} R_{11}^{-1} &= s I \\
s &= A_{21} R_{11}^{-1} \\
s^T &= R_{11}^{-T} A_{21}^T
\end{aligned}
$$

$$
\begin{aligned}
A_{22} &= s s^T + \tilde{A} \\
-\tilde{A} &= s s^T - A_{22} \\
\tilde{A} &= -s s^T + A_{22} \\
\tilde{A} &= A_{22} - s s^T \\
\tilde{A} &= A_{22} - (A_{21} R_{11}^{-1})(R_{11}^{-T} A_{21}^T) \\
&\ldots \\
\tilde{A} &= A_{22} - A_{21} A_{11}^{-1} A_{12}
\end{aligned}
$$

$$
\begin{aligned}
A &= \begin{bmatrix} R_{11}^T & 0 \\ A_{21} R_{11}^{-1} & I \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & A_{22} - A_{21} A_{11}^{-1} A_{12} \end{bmatrix} \begin{bmatrix} R_{11} & R_{11}^{-T} A_{21}^T \\ 0 & I \end{bmatrix} \\
&= \begin{bmatrix} R_{11}^T R_{11} & R_{11}^T R_{11}^{-T} A_{21}^T \\ A_{21} R_{11}^{-1} R_{11} & A_{21} R_{11}^{-1} R_{11}^{-T} A_{21}^T + A_{22} - (A_{21} R_{11}^{-1})(R_{11}^{-T} A_{21}^T) \end{bmatrix} \\
&= \begin{bmatrix} R_{11}^T R_{11} & -T A_{21}^T \\ A_{21} & A_{21} R_{11}^{-1} R_{11}^{-T} A_{21}^T + A_{22} - (A_{21} R_{11}^{-1})(R_{11}^{-T} A_{21}^T) \end{bmatrix}
\end{aligned}
$$

(c)

We know that $A = X^T B X$ is "Positive Definite", so if B is "Positive Definite", $\tilde{A}$ will be "Positive Definite". To prove that B is "Positive Definite" we need to prove that X is nonsingular.
So we need to prove that X is nonsingular and that B is "Positive Definite".

(c.1) X is nonsingular

$$
\begin{aligned}
X &= \begin{bmatrix} R_{11} & s^T \\ 0 & I \end{bmatrix} \\
det(X) &= \prod x_{ii} \text{ because X is Upper Triangular} \\
x_{ii} &> 0 \\
det(x) &> 0 \\
\square
\end{aligned}
$$

(c.2) B is "Positive Definite".

$$
\begin{aligned}
A &= X^T B X \\
AX^{-1} &= X^T B X X^{-1} \\
AX^{-1} &= X^T B I \\
AX^{-1} &= X^T B \\
X^{-T} A X^{-1} &= X^{-T} X^T B \\
X^{-T} A X^{-1} &= I B \\
X^{-T} A X^{-1} &= B \\
B &= X^{-T} A X^{-1}
\end{aligned}
$$

$$
\begin{aligned}
Y &= X^{-1} \\
B &= Y^T A Y \\
\square
\end{aligned}
$$