Exercise 12:
How many corners does a cube have in 4 dimensions? How many 3D faces? How many edges?

1 Vertices

A typical corner is $(0, 0, 1, 0)$. A typical edge goes to $(0, 1, 0, 0)$.

In 3D, all cube corners are:
$(0, 0, 0)$
$(0, 0, 1)$
$(0, 1, 0)$
$(0, 1, 1)$
$(1, 0, 0)$
$(1, 0, 1)$
$(1, 1, 0)$
$(1, 1, 1)$

Which follows the permutation logic, and we know that in a set with three members (3D space), each with two possibilities (0 or 1) we have $2^3 = 8$ permutations.

2 Edges

In a 4D space we will have $2^4 = 16$ permutations, so we will have 16 corners. If we use the technique of imagining a 4D-Cube as a moving 3D Cube, we will have:
1 - 12 edges from the "starting" cube;
2 - 12 edges from the "final" cude;
3 - 8 edges from the "moving vertices"

What will give us 32 edges on a 4D-Cube.

The recursive formula is: $f(N) = 2 * f(N-1) + 2^{N-1}$

$$\begin{aligned} f(N) &= 2f(N-1) + 2^{N-1} \\ f(N+1) &= 2f(N) + 2^{N} \\ f(N+1) - 2f(N) &= 2^{N} \\ (E-2)f &= 2^{N} \\ (E-2)(E-2)f &= (E-2) * 2^{N} \\ (E-2)^2 f &= 0 \end{aligned}$$

$$(\alpha n + \beta)2^n$$

$$\begin{aligned} (1\alpha + \beta)2^1 &= 1 \\ (2\alpha + \beta)2^2 &= 4 \\ (3\alpha + \beta)2^3 &= 12 \end{aligned}$$

$$\begin{aligned} 2\alpha + 2\beta &= 1 \\ 8\alpha + 4\beta &= 4 \\ 24\alpha + 8\beta &= 12 \end{aligned}$$

$$\begin{aligned} -2(2\alpha + 2\beta) &= -2 * 1 \\ 8\alpha + 4\beta &= 4 \end{aligned}$$

$$\begin{aligned} -4\alpha - 4\beta &= -2 \\ 8\alpha + 4\beta &= 4 \end{aligned}$$

$$\begin{aligned} -4\alpha - 4\beta &= -2 \\ 8\alpha + 4\beta &= 4 \end{aligned}$$

$$4\alpha + 0\beta = 2$$

$$4\alpha = 2$$
$$\alpha = \frac{2}{4}$$
$$\alpha = \frac{1}{2}$$

$$2\frac{1}{2} + 2\beta = 1$$
$$\frac{2}{2} + 2\beta = 1$$
$$1 + 2\beta = 1$$
$$2\beta = 1 - 1$$
$$2\beta = 0$$
$$\beta = 0$$
$$(\frac{n}{2})2^n$$
$$\frac{n * 2^n}{2}$$

Which is a "closed form" for the quantities of edges of "cube" in any dimension:

$$\text{edges}(n) = \frac{n * 2^n}{2}$$

Which can be interpreted as the dimension times half the vertices.

Reference:

Beyond the Third Dimension
by
Thomas F. Banchoff

> It is helpful to think of cubes as generated by lower-dimensional cubes in motion. A point in motion generates a segment; a segment in motion generates a square; a square in motion generates a cube; and so on. From this progression, a pattern develops, which we can exploit to predict the numbers of vertices and edges.
> Each time we move a cube to generate a cube in the next higher dimension, the number of vertices doubles. That is easy to see since we have an initial position and a final position, each with the same number of vertices. Using this information we can infer an explicit formula for the number of vertices of a cube in any dimension, namely 2 raised to that power.
> What about the number of edges? A square has 4 edges, and as it moves from one position to the other, each of its 4 vertices traces out an edge. Thus we have 4 edges on the initial square, 4 on the final square, and 4 traced out by the moving vertices for a total of 12. That basic pattern repeats itself. If we move a figure in a straight line, then the number of edges in the new figure is twice the original number of edges plus the number of moving vertices. Thus the number of edges in a four-cube is 2 times 12 plus 8 for a total of 32. Similarly we find $32 + 32 + 16 = 80$ edges on a five-cube and $80 + 80 + 32 = 192$ edges on a six-cube.
> This presentation definitely suggests a pattern, namely that the number of edges of a hypercube of a given dimension is the dimension multiplied by half the number of vertices in that dimension. Once we notice a pattern like this, it can be proved to hold in all dimensions by mathematical induction.

`http://www.math.brown.edu/~banchoff/Beyond3d/chapter4/section05.html`

3 Faces