

1 Introduction

Calculus Handout

Daniel Frederico Lins Leite

September 2016

Theorem 1.1 (Squeeze theorem).

$$\begin{aligned} & \text{IF } g(x) \leq f(x) \leq h(x) \\ & \text{AND } \lim_{x \rightarrow a} g(x) = \lim_{x \rightarrow a} h(x) = L \\ & \text{THEN } \lim_{x \rightarrow a} f(x) = L \end{aligned}$$

Used by:

1. Theorem 1.2

Theorem 1.2 (Limit of $\sin(x)/x$).

$$\lim_{x \rightarrow 0} \frac{\sin(x)}{x} = 1$$

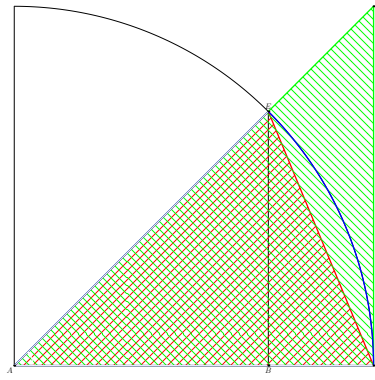
Proof. Given that:

$$\mu_1(\overline{AB}) = \text{LENGTH of } \overline{AB}$$

$$\mu_2(\triangle ABC) = \text{AREA of } \triangle ABC$$

We have from the figure that:

$$\mu_2(\triangle ACE) \leq \mu_2(\widehat{ACE}) \leq \mu_2(\triangle ACD)$$



$$\mu_2(\triangle ACE) = \frac{1}{2} * \mu_1(\overline{AC}) * \mu_1(\overline{BE})$$

$$\mu_2(\triangle ACE) = \frac{1}{2} * 1 * \mu_1(\overline{BE})$$

$$\mu_2(\triangle ACE) = \frac{1}{2} * 1 * |\sin(\theta)|$$

$$\mu_2(\triangle ACE) = \frac{|\sin(\theta)|}{2}$$

$$\mu_2(\widehat{ACE}) = \frac{|\theta|}{2 * \pi} * \pi * r^2$$

$$\mu_2(\widehat{ACE}) = \frac{|\theta|}{2 * \pi} * \pi * 1^2$$

$$\mu_2(\widehat{ACE}) = \frac{|\theta|}{2 * \pi} * \pi * 1$$

$$\mu_2(\widehat{ACE}) = \frac{|\theta|}{2 * \pi} * \pi$$

$$\mu_2(\widehat{ACE}) = \frac{|\theta|}{2}$$

$$\mu_2(\triangle ACD) = \frac{1}{2} * \mu_1(\overline{AC}) * \mu_1(\overline{CD})$$

$$\mu_2(\triangle ACD) = \frac{1}{2} * 1 * \mu_1(\overline{CD})$$

$$\mu_2(\triangle ACD) = \frac{1}{2} * \mu_1(\overline{CD})$$

$$\mu_2(\triangle ACD) = \frac{1}{2} * |\tan(\theta)|$$

$$\mu_2(\triangle ACD) = \frac{|\tan(\theta)|}{2}$$

So we have:

$$\frac{|\sin(\theta)|}{2} \leq \frac{|\theta|}{2} \leq \frac{|\tan(\theta)|}{2}$$

$$|\sin(\theta)| \leq |\theta| \leq |\tan(\theta)|$$

$$|\sin(\theta)| \leq |\theta| \leq \frac{|\sin(\theta)|}{|\cos(\theta)|}$$

$$|\sin(\theta)| * \frac{1}{|\sin(\theta)|} \leq |\theta| * \frac{1}{|\sin(\theta)|} \leq \frac{|\sin(\theta)|}{|\cos(\theta)|} * \frac{1}{|\sin(\theta)|}$$

$$1 \leq \frac{|\theta|}{|\sin(\theta)|} \leq \frac{1}{|\cos(\theta)|}$$

$$1 \geq \frac{|\sin(\theta)|}{|\theta|} \leq |\cos(\theta)|$$

In the domain $\theta \in (\frac{\pi}{2}, -\frac{\pi}{2})$, $\cos(\theta) > 0$ and $\sin(\theta)$ and θ always have the same signal. So we can remove the modulus operator in both cases.

$$1 \geq \frac{\sin(\theta)}{\theta} \leq \cos(\theta)$$

$$\lim_{\theta \rightarrow 0} 1 \geq \lim_{\theta \rightarrow 0} \frac{\sin(\theta)}{\theta} \leq \lim_{\theta \rightarrow 0} \cos(\theta)$$

Given that

$$\lim_{\theta \rightarrow 0} 1 = 1$$

$$\lim_{\theta \rightarrow 0} \cos(\theta) = 1$$

and using the Squeeze Theorem

$$\lim_{\theta \rightarrow 0} \frac{\sin(\theta)}{\theta} = 1$$

□

See More:

1. Proofwiki
https://proofwiki.org/wiki/Limit_of_Sine_of_X_over_X
2. Khan Academy
<https://www.khanacademy.org/math/ap-calculus-ab/ab-derivative-rules/ab-derivative-rules-opt-vids/v/sinx-over-x-as-x-approaches-0>
3. MIT OpenCourse 18-01 Lecture 03 at 15:15
<https://youtu.be/kCPVB1953eY?t=915>

Remarks I do not like to use the Proofwiki proof using the "definition" of $\sin(x)$. We can see that this proof is using Binmore definition, that depends on the derivative of the $\sin(x)$ and $\cos(x)$. And that this limit is used to prove $\sin'(x)$. Creating a circular proof.

"Mathematical Analysis: A Straightforward Approach, 2nd Edition" of K. G. Binmore - Chapter 16.2

"Suppose that the power series $f(x) = \sum_{n=0}^{\infty} a_n x^n$ converges for all real values of x and that $f''(x) + f'(x) = 0$. This recurrence relation is readily solved and we obtain

$$f(x) = a_0 \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n+1)!} = x - \frac{x^3}{3!} \dots$$

We want $\sin(0) = 0$ and $\sin'(0) = \cos(0) = 1$. We therefore define $\sin(x)$ as $\sin(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \dots$ and $\cos(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} = x - \frac{x^2}{2!} + \dots$ "

Theorem 1.3 (Limit of $(\cos(x) - 1)/x$).

$$\lim_{x \rightarrow 0} \frac{\cos(x) - 1}{x} = 0$$

Proof. see proof

□

See More:

1. MIT OpenCourse 18-01 Lecture 03 at 21:07
<https://youtu.be/kCPVB1953eY?t=1267>

Definition 1.3.1 (Derivative Definition).

$$\frac{d}{dx} f(x) = f'(x) = \lim_{\Delta x \rightarrow 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

Theorem 1.4 (Derivative of the scale).

$$\begin{aligned} f(x) &= c g(x) \\ f'(x) &= c g'(x) \end{aligned}$$

Proof.

$$\begin{aligned} f(x) &= c g(x) \\ f'(x) &= \lim_{\Delta x \rightarrow 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} \\ &= \lim_{\Delta x \rightarrow 0} \frac{c g(x + \Delta x) - c g(x)}{\Delta x} \\ &= \lim_{\Delta x \rightarrow 0} \frac{c(g(x + \Delta x) - g(x))}{\Delta x} \\ &= \lim_{\Delta x \rightarrow 0} c * \frac{g(x + \Delta x) - g(x)}{\Delta x} \\ &= c * \lim_{\Delta x \rightarrow 0} \frac{g(x + \Delta x) - g(x)}{\Delta x} \\ &= c g'(x) \end{aligned}$$

□

Theorem 1.5 (Derivative of the sum).

$$\begin{aligned} f(x) &= u(x) + g(x) \\ f'(x) &= u'(x) + g'(x) \end{aligned}$$

Proof.

$$\begin{aligned}f(x) &= u(x) + g(x) \\f'(x) &= \lim_{\Delta x \rightarrow 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} \\&= \lim_{\Delta x \rightarrow 0} \frac{(u(x + \Delta x) + g(x + \Delta x)) - (u(x) + g(x))}{\Delta x} \\&= \lim_{\Delta x \rightarrow 0} \frac{u(x + \Delta x) - u(x) + g(x + \Delta x) - g(x)}{\Delta x} \\&= \lim_{\Delta x \rightarrow 0} \left[\frac{u(x + \Delta x) - u(x)}{\Delta x} + \frac{g(x + \Delta x) - g(x)}{\Delta x} \right] \\&= \lim_{\Delta x \rightarrow 0} \frac{u(x + \Delta x) - u(x)}{\Delta x} + \lim_{\Delta x \rightarrow 0} \frac{g(x + \Delta x) - g(x)}{\Delta x} \\&= u'(x) + g'(x)\end{aligned}$$

□

Theorem 1.6 (Derivative of the multiplication).

$$f(x) = u(x)g(x)$$

$$f'(x) = u'(x)g(x) + u(x)g'(x)$$

Proof.

$$\begin{aligned}
 f(x) &= u(x)g(x) \\
 f'(x) &= \lim_{\Delta x \rightarrow 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} \\
 &= \lim_{\Delta x \rightarrow 0} \frac{u(x + \Delta x)g(x + \Delta x) - u(x)g(x)}{\Delta x} \\
 &= \lim_{\Delta x \rightarrow 0} \frac{u(x + \Delta x)g(x + \Delta x) - u(x)g(x) - u(x + \Delta x)g(x) + u(x + \Delta x)g(x)}{\Delta x} \\
 &= \lim_{\Delta x \rightarrow 0} \frac{u(x + \Delta x)g(x + \Delta x) - u(x)g(x) - u(x + \Delta x)g(x) + u(x + \Delta x)g(x)}{\Delta x} \\
 &= \lim_{\Delta x \rightarrow 0} \frac{[u(x + \Delta x) - u(x)]g(x) + u(x + \Delta x)[g(x + \Delta x) - g(x)]}{\Delta x} \\
 &= \lim_{\Delta x \rightarrow 0} \frac{[u(x + \Delta x) - u(x)]g(x)}{\Delta x} + \lim_{\Delta x \rightarrow 0} \frac{u(x + \Delta x)[g(x + \Delta x) - g(x)]}{\Delta x} \\
 &= \lim_{\Delta x \rightarrow 0} \frac{u(x + \Delta x) - u(x)}{\Delta x} * \lim_{\Delta x \rightarrow 0} g(x) + \lim_{\Delta x \rightarrow 0} u(x + \Delta x) * \lim_{\Delta x \rightarrow 0} \frac{g(x + \Delta x) - g(x)}{\Delta x} \\
 &= \lim_{\Delta x \rightarrow 0} \frac{u(x + \Delta x) - u(x)}{\Delta x} * g(x) + u(x) * \lim_{\Delta x \rightarrow 0} \frac{g(x + \Delta x) - g(x)}{\Delta x} \\
 &= u'(x)g(x) + u(x)g'(x)
 \end{aligned}$$

□

See More:

1. Product Rule:
https://proofwiki.org/wiki/Product_Rule_for_Derivatives
2. Khan Academy:
<https://www.khanacademy.org/math/ap-calculus-ab/ab-derivative-rules/ab-product-rule/a/proving-the-product-rule>

Product Rule and Leibniz Men of Mathematics - Chapter 7 - Master of All Trades

"Instead of the infinitesimals of Leibniz we shall use the rates discussed in the preceding chapter. If u and v are function of x , how shall the rate of change of $u * v$ with respect to x be expressed in terms of the respective rates of change of u and v with respect to x ?

In symbols what is $\frac{d}{dx}(u * v)$ in terms of $\frac{d}{dx}u$ and $\frac{d}{dx}v$?

Leibniz once thought it should be $\frac{d}{dx} u + \frac{d}{dx} v$ which is nothing like the correct

$$\frac{d}{dx}(u * v) = v \frac{d}{dx} u + u \frac{d}{dx} v \quad (1)$$

Theorem 1.7.

$$f(x) = \frac{u(x)}{g(x)}$$

$$f'(x) = \frac{u'(x)g(x) - u(x)g'(x)}{g(x)^2}$$

Proof.

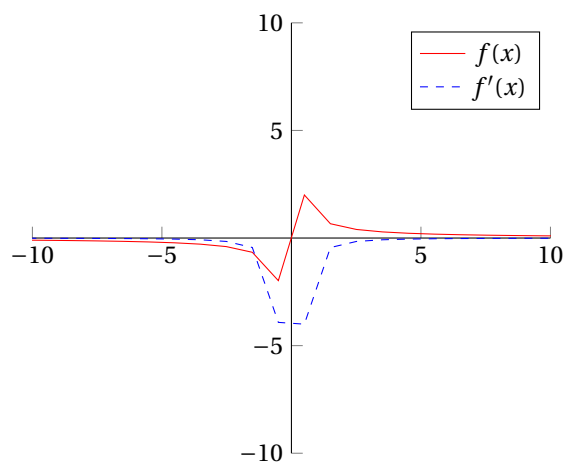
$$\begin{aligned} f(x) &= u(x)g(x) \\ f'(x) &= \lim_{\Delta x \rightarrow 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} \\ &= \lim_{\Delta x \rightarrow 0} \frac{\frac{u(x + \Delta x)}{g(x + \Delta x)} - \frac{u(x)}{g(x)}}{\Delta x} \\ &= \lim_{\Delta x \rightarrow 0} \frac{\frac{u(x + \Delta x) * g(x)}{g(x + \Delta x) * g(x)} - \frac{u(x) * g(x + \Delta x)}{g(x) * g(x + \Delta x)}}{\Delta x} \\ &= \lim_{\Delta x \rightarrow 0} \frac{\frac{u(x + \Delta x) * g(x) - u(x) * g(x + \Delta x)}{g(x) * g(x + \Delta x)}}{\Delta x} \\ &= \lim_{\Delta x \rightarrow 0} \left[\frac{\frac{u(x + \Delta x) * g(x) - u(x) * g(x + \Delta x)}{g(x) * g(x + \Delta x)}}{\Delta x} \right] \\ &= \lim_{\Delta x \rightarrow 0} \left[\frac{1}{g(x) * g(x + \Delta x)} * \frac{u(x + \Delta x) * g(x) - u(x) * g(x + \Delta x)}{\Delta x} \right] \\ &= \lim_{\Delta x \rightarrow 0} \left[\frac{1}{g(x) * g(x + \Delta x)} * \frac{g(x)[u(x + \Delta x) - u(x)] - u(x)[g(x + \Delta x) - g(x)]}{\Delta x} \right] \\ &= \lim_{\Delta x \rightarrow 0} \left[\frac{1}{g(x) * g(x + \Delta x)} * \left(\frac{g(x)[u(x + \Delta x) - u(x)]}{\Delta x} - \frac{u(x)[g(x + \Delta x) - g(x)]}{\Delta x} \right) \right] \\ &= \frac{1}{g(x) * g(x)} * [g(x)u'(x) - u(x)g'(x)] \\ &= \frac{g(x)u'(x) - u(x)g'(x)}{g(x)^2} \\ &= \frac{u'(x)g(x) - u(x)g'(x)}{g(x)^2} \end{aligned}$$

□

Theorem 1.8 (Derivative of $1/x$).

$$f(x) = \frac{1}{x}$$

$$f'(x) = \frac{-1}{x^2}$$



Proof.

$$\begin{aligned}
 f(x) &= \frac{1}{x} \\
 f'(x) &= \lim_{\Delta x \rightarrow 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} \\
 &= \lim_{\Delta x \rightarrow 0} \frac{\frac{1}{x + \Delta x} - \frac{1}{x}}{\Delta x} \\
 &= \lim_{\Delta x \rightarrow 0} \frac{\frac{1 * x}{(x + \Delta x) * x} - \frac{1 * (x + \Delta x)}{x * (x + \Delta x)}}{\Delta x} \\
 &= \lim_{\Delta x \rightarrow 0} \frac{\frac{(1 * x) - [1 * (x + \Delta x)]}{(x + \Delta x) * x}}{\Delta x} \\
 &= \lim_{\Delta x \rightarrow 0} \frac{\frac{x - (x + \Delta x)}{(x + \Delta x) * x}}{\Delta x} \\
 &= \lim_{\Delta x \rightarrow 0} \frac{\frac{x - x - \Delta x}{(x + \Delta x) * x}}{\Delta x} \\
 &= \lim_{\Delta x \rightarrow 0} \frac{\frac{-\Delta x}{(x + \Delta x) * x}}{\Delta x} \\
 &= \lim_{\Delta x \rightarrow 0} \left[\frac{1}{\Delta x} * \frac{-\Delta x}{(x + \Delta x) * x} \right] \\
 &= \lim_{\Delta x \rightarrow 0} \frac{-1}{(x + \Delta x) * x} \\
 &= \frac{-1}{(x + 0) * x} \\
 &= \frac{-1}{x * x} \\
 &= -\frac{1}{x^2}
 \end{aligned}$$

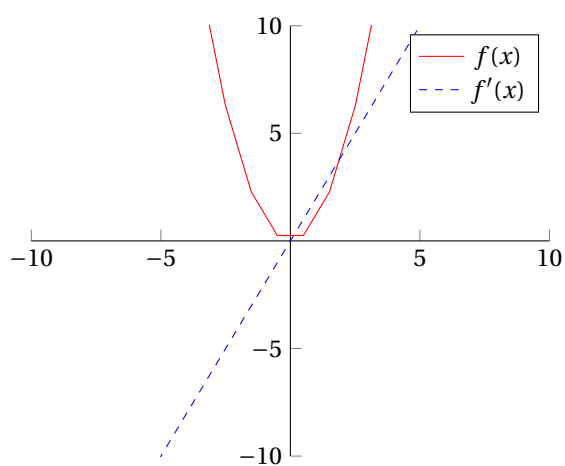
□

Theorem 1.9 (Derivative of x^n).

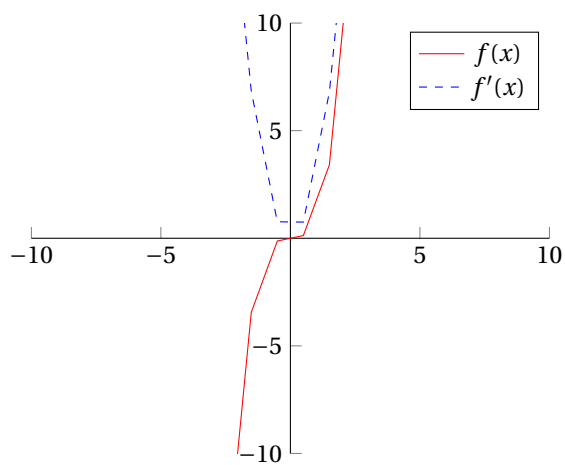
$$f(x) = x^n$$

$$f'(x) = nx^{n-1}$$

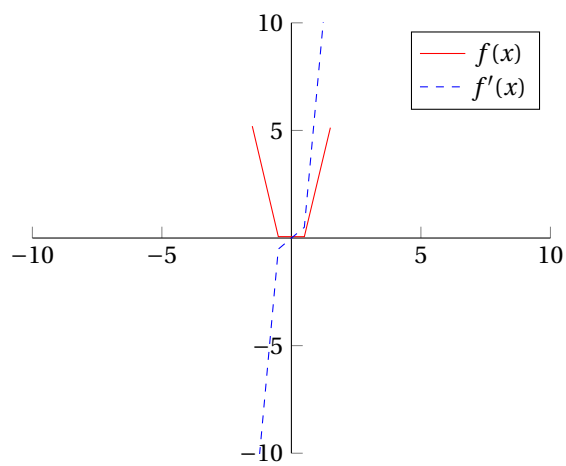
$$f(x) = x^2 \text{ and } f'(x) = 2x$$



$$f(x) = x^3 \text{ and } f'(x) = 3x^2$$



$$f(x) = x^4 \text{ and } f'(x) = 4x^3$$



Proof.

$$\begin{aligned}
 f(x) &= x^n \\
 f'(x) &= \lim_{\Delta x \rightarrow 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} \\
 &= \lim_{\Delta x \rightarrow 0} \frac{(x + \Delta x)^n - x^n}{\Delta x} \\
 &= \lim_{\Delta x \rightarrow 0} \frac{[x^n + nx^{n-1}\Delta x + \mathcal{O}((\Delta x)^2)] - x^n}{\Delta x} \\
 &= \lim_{\Delta x \rightarrow 0} \frac{[nx^{n-1}\Delta x + \mathcal{O}((\Delta x)^2)]}{\Delta x} \\
 &= \lim_{\Delta x \rightarrow 0} \frac{\Delta x[nx^{n-1} + \mathcal{O}(\Delta x)]}{\Delta x} \\
 &= \lim_{\Delta x \rightarrow 0} nx^{n-1} + \mathcal{O}(\Delta x) \\
 &= nx^{n-1}
 \end{aligned}$$

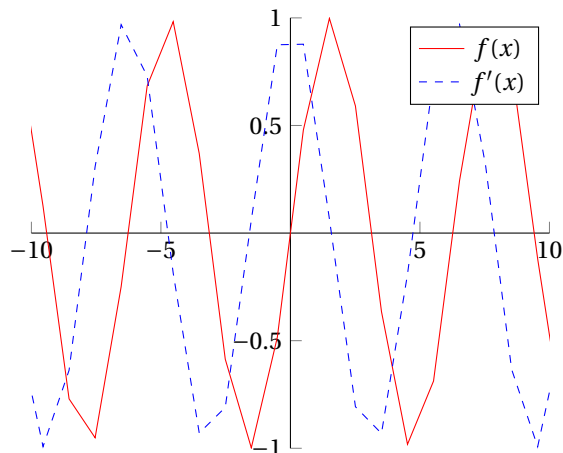
□

Theorem 1.10 (Derivative of $\sin(x)$).

$$f(x) = \sin(x)$$

$$f'(x) = \cos(x)$$

$$f(x) = \sin(x) \text{ and } f'(x) = \cos(x)$$



Proof.

$$f(x) = \sin(x)$$

$$\begin{aligned} f'(x) &= \lim_{\Delta x \rightarrow 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} \\ &= \lim_{\Delta x \rightarrow 0} \frac{\sin(x + \Delta x) - \sin(x)}{\Delta x} \\ &= \lim_{\Delta x \rightarrow 0} \frac{\sin(x)\cos(\Delta x) + \cos(x)\sin(\Delta x) - \sin(x)}{\Delta x} \\ &= \lim_{\Delta x \rightarrow 0} \frac{\sin(x)(\cos(\Delta x) - 1) + \cos(x)\sin(\Delta x)}{\Delta x} \\ &= \lim_{\Delta x \rightarrow 0} \left[\frac{\sin(x)(\cos(\Delta x) - 1)}{\Delta x} + \frac{\cos(x)\sin(\Delta x)}{\Delta x} \right] \\ &= \lim_{\Delta x \rightarrow 0} \left[\sin(x) * \frac{\cos(\Delta x) - 1}{\Delta x} + \cos(x) * \frac{\sin(\Delta x)}{\Delta x} \right] \\ &= \lim_{\Delta x \rightarrow 0} \left[\sin(x) * \frac{\cos(\Delta x) - 1}{\Delta x} \right] + \lim_{\Delta x \rightarrow 0} \left[\cos(x) * \frac{\sin(\Delta x)}{\Delta x} \right] \\ &= \left[\lim_{\Delta x \rightarrow 0} \sin(x) \right] * \left[\lim_{\Delta x \rightarrow 0} \frac{\cos(\Delta x) - 1}{\Delta x} \right] + \left[\lim_{\Delta x \rightarrow 0} \cos(x) \right] * \left[\lim_{\Delta x \rightarrow 0} \frac{\sin(\Delta x)}{\Delta x} \right] \\ &= \left[\lim_{\Delta x \rightarrow 0} \sin(x) \right] * [0] + \left[\lim_{\Delta x \rightarrow 0} \cos(x) \right] * [1] \\ &= \lim_{\Delta x \rightarrow 0} \cos(x) \end{aligned}$$

see 1.3, 1.2

$$= \cos(x)$$

□

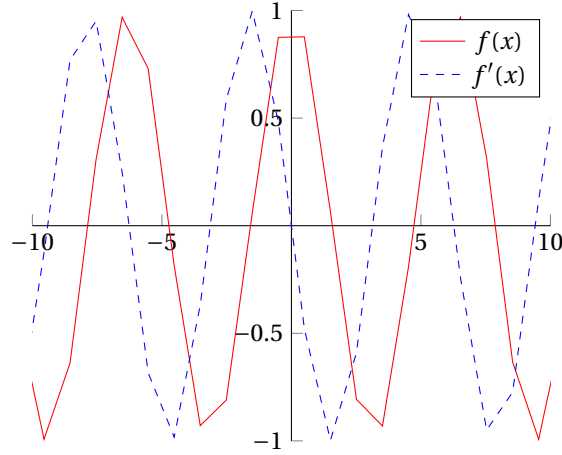
See More:

1. Khan Academy
<https://www.khanacademy.org/math/ap-calculus-ab/ab-derivative-rules/ab-derivative-rules-opt-vids/v/derivative-of-sin-x>
2. MIT OpenCourse 18-01 Lecture 03 at 03:09
<https://youtu.be/kCPVB1953eY?t=189>

Theorem 1.11 (Derivative of $\cos(x)$).

$$f(x) = \cos(x)$$

$$f'(x) = -\sin(x)$$



Proof.

$$f(x) = \cos(x)$$

$$f'(x) = \lim_{\Delta x \rightarrow 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

$$f'(x) = \lim_{\Delta x \rightarrow 0} \frac{\cos(x + \Delta x) - \cos(x)}{\Delta x}$$

$$f'(x) = \lim_{\Delta x \rightarrow 0} \frac{\cos(x)\cos(\Delta x) - \sin(x)\sin(\Delta x) - \cos(x)}{\Delta x}$$

$$f'(x) = \lim_{\Delta x \rightarrow 0} \frac{\cos(x)(\cos(\Delta x) - 1) - \sin(x)\sin(\Delta x)}{\Delta x}$$

$$f'(x) = \lim_{\Delta x \rightarrow 0} \left[\frac{\cos(x)(\cos(\Delta x) - 1)}{\Delta x} - \frac{\sin(x)\sin(\Delta x)}{\Delta x} \right]$$

$$f'(x) = \lim_{\Delta x \rightarrow 0} \frac{\cos(x)(\cos(\Delta x) - 1)}{\Delta x} - \lim_{\Delta x \rightarrow 0} \frac{\sin(x)\sin(\Delta x)}{\Delta x}$$

$$f'(x) = \lim_{\Delta x \rightarrow 0} \left[\cos(x) * \frac{(\cos(\Delta x) - 1)}{\Delta x} \right] - \lim_{\Delta x \rightarrow 0} \left[\sin(x) * \frac{\sin(\Delta x)}{\Delta x} \right]$$

$$f'(x) = \left[\lim_{\Delta x \rightarrow 0} \cos(x) \right] * \left[\lim_{\Delta x \rightarrow 0} \frac{(\cos(\Delta x) - 1)}{\Delta x} \right] - \left[\lim_{\Delta x \rightarrow 0} \sin(x) \right] * \left[\lim_{\Delta x \rightarrow 0} \frac{\sin(\Delta x)}{\Delta x} \right]$$

$$f'(x) = \left[\lim_{\Delta x \rightarrow 0} \cos(x) \right] * [0] - \left[\lim_{\Delta x \rightarrow 0} \sin(x) \right] * [1]$$

see 1.3, 1.2

$$f'(x) = -\lim_{\Delta x \rightarrow 0} \sin(x)$$

$$f'(x) = -\sin(x)$$

□

See More:

1. Khan Academy
<https://www.khanacademy.org/math/ap-calculus-ab/ab-derivative-rules/ab-derivative-rules-opt-vids/v/derivative-of-cos-x>
2. MIT OpenCourse 18-01 Lecture 03 at 08:54
<https://youtu.be/kCPVB1953eY?t=534>