Exercise 1.3.23

Prove that if G is triangular, then $det(G) = g_{11} * g_{22} * ... * g_{NN}$

Definition:

$$S_N = \text{All permutations of 1..N}$$

$$\det(A) = \sum_{\sigma \in S_N} \left(\operatorname{sgn}(\sigma) \prod_{i=1}^{n} a_{i,\sigma_i} \right)$$

Case 1: Lower Triangular

$$G = \begin{bmatrix} x_{11} & 0 & 0 & \dots & 0 \\ x_{21} & x_{22} & 0 & \dots & 0 \\ x_{31} & x_{32} & x_{33} & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ x_{N1} & x_{N2} & x_{N3} & \dots & x_{dN} \end{bmatrix}$$

$$det(G) = \sum_{\sigma \in S_N} \left(\operatorname{sgn}(\sigma) \prod_{i=1}^{n} g_{i,\sigma_i} \right)$$

but

$$A_{ij} = 0 \text{ if } j > i \text{ (Lower Triangular definition)}$$

$$\forall(\sigma_i > i) \prod_{i=1}^{n} g_{i,\sigma_i} = 0$$

So we can remove these elements of S_n because they will nullify the product.

$$\begin{aligned} \omega &= S_n - \{\sigma | \sigma_i > i\} \\ &= \{[1...N]\} \end{aligned}$$

so

$$\begin{aligned} det(G) &= \sum_{\sigma \in \omega} \left(\operatorname{sgn}(\sigma) \prod_{i=1}^{n} g_{i,\sigma_i} \right) \\ &= \left(\operatorname{sgn}([1...N]) \prod_{i=1}^{n} g_{i,i} \right) \\ &= \left(1 * \prod_{i=1}^{n} g_{i,i} \right) \\ &= g_{11} * g_{22} * ... * g_{NN} \end{aligned}$$

Case 2: Upper Triangular
Exactly the same as the Lower Triangular.