

1 Introduction

Calculus Handout

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Theorem 1.1 (Limit of $\sin(x)/x$).

$$\lim_{x \rightarrow 0} \frac{\sin(x)}{x} = 1$$

Proof. see proof

□

Theorem 1.2 (Limit of $(\cos(x) - 1)/x$).

$$\lim_{x \rightarrow 0} \frac{\cos(x) - 1}{x} = 0$$

Proof. see proof

□

Definition 1.2.1 (Derivative Definition).

$$\frac{d}{dx} f(x) = f'(x) = \lim_{\Delta x \rightarrow 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

Theorem 1.3 (Derivative of the scale).

$$f(x) = cg(x)$$

$$f'(x) = cg'(x)$$

Proof.

$$\begin{aligned} f(x) &= cg(x) \\ f'(x) &= \lim_{\Delta x \rightarrow 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} \\ &= \lim_{\Delta x \rightarrow 0} \frac{cg(x + \Delta x) - cg(x)}{\Delta x} \\ &= \lim_{\Delta x \rightarrow 0} \frac{c(g(x + \Delta x) - g(x))}{\Delta x} \\ &= \lim_{\Delta x \rightarrow 0} c * \frac{g(x + \Delta x) - g(x)}{\Delta x} \\ &= c * \lim_{\Delta x \rightarrow 0} \frac{g(x + \Delta x) - g(x)}{\Delta x} \\ &= cg'(x) \end{aligned}$$

□

Theorem 1.4 (Derivative of the sum).

$$\begin{aligned} f(x) &= u(x) + g(x) \\ f'(x) &= u'(x) + g'(x) \end{aligned}$$

Proof.

$$\begin{aligned} f(x) &= u(x) + g(x) \\ f'(x) &= \lim_{\Delta x \rightarrow 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} \\ &= \lim_{\Delta x \rightarrow 0} \frac{(u(x + \Delta x) + g(x + \Delta x)) - (u(x) + g(x))}{\Delta x} \\ &= \lim_{\Delta x \rightarrow 0} \frac{u(x + \Delta x) - u(x) + g(x + \Delta x) - g(x)}{\Delta x} \\ &= \lim_{\Delta x \rightarrow 0} \left[\frac{u(x + \Delta x) - u(x)}{\Delta x} + \frac{g(x + \Delta x) - g(x)}{\Delta x} \right] \\ &= \lim_{\Delta x \rightarrow 0} \frac{u(x + \Delta x) - u(x)}{\Delta x} + \lim_{\Delta x \rightarrow 0} \frac{g(x + \Delta x) - g(x)}{\Delta x} \\ &= u'(x) + g'(x) \end{aligned}$$

□

Theorem 1.5 (Derivative of the multiplication).

$$\begin{aligned} f(x) &= u(x)g(x) \\ f'(x) &= u'(x)g(x) + u(x)g'(x) \end{aligned}$$

Proof.

$$\begin{aligned} f(x) &= u(x)g(x) \\ f'(x) &= \lim_{\Delta x \rightarrow 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} \\ &= \lim_{\Delta x \rightarrow 0} \frac{u(x + \Delta x)g(x + \Delta x) - u(x)g(x)}{\Delta x} \\ &= \lim_{\Delta x \rightarrow 0} \frac{u(x + \Delta x)g(x + \Delta x) + [u(x + \Delta x) - u(x)]g(x) - u(x + \Delta x)g(x)}{\Delta x} \\ &= \lim_{\Delta x \rightarrow 0} \frac{[u(x + \Delta x) - u(x)]g(x) + u(x + \Delta x)[g(x + \Delta x) - g(x)]}{\Delta x} \\ &= \lim_{\Delta x \rightarrow 0} \frac{[u(x + \Delta x) - u(x)]g(x)}{\Delta x} + \lim_{\Delta x \rightarrow 0} \frac{u(x + \Delta x)[g(x + \Delta x) - g(x)]}{\Delta x} \\ &= \lim_{\Delta x \rightarrow 0} \frac{[u(x + \Delta x) - u(x)]g(x)}{\Delta x} + \lim_{\Delta x \rightarrow 0} \frac{u(x)[g(x + \Delta x) - g(x)]}{\Delta x} \\ &= u'(x)g(x) + u(x)g'(x) \end{aligned}$$

□

Theorem 1.6.

$$f(x) = \frac{u(x)}{g(x)}$$

$$f'(x) = \frac{u'(x)g(x) - u(x)g'(x)}{g(x)^2}$$

Proof.

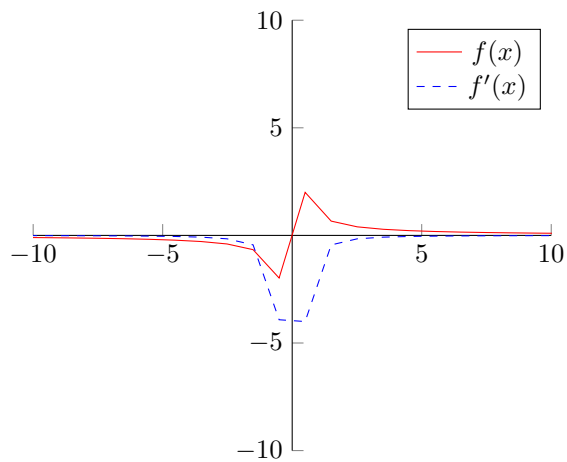
$$\begin{aligned}
f(x) &= u(x)g(x) \\
f'(x) &= \lim_{\Delta x \rightarrow 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} \\
&= \lim_{\Delta x \rightarrow 0} \frac{\frac{u(x + \Delta x)}{g(x + \Delta x)} - \frac{u(x)}{g(x)}}{\Delta x} \\
&= \lim_{\Delta x \rightarrow 0} \frac{\frac{u(x + \Delta x) * g(x)}{g(x + \Delta x) * g(x)} - \frac{u(x) * g(x + \Delta x)}{g(x) * g(x + \Delta x)}}{\Delta x} \\
&= \lim_{\Delta x \rightarrow 0} \frac{\frac{u(x + \Delta x) * g(x) - u(x) * g(x + \Delta x)}{g(x) * g(x + \Delta x)}}{\Delta x} \\
&= \lim_{\Delta x \rightarrow 0} \left[\frac{\frac{u(x + \Delta x) * g(x) - u(x) * g(x + \Delta x)}{g(x) * g(x + \Delta x)}}{\Delta x} \right] \\
&= \lim_{\Delta x \rightarrow 0} \left[\frac{1}{g(x) * g(x + \Delta x)} * \frac{u(x + \Delta x) * g(x) - u(x) * g(x + \Delta x)}{\Delta x} \right] \\
&= \lim_{\Delta x \rightarrow 0} \left[\frac{1}{g(x) * g(x + \Delta x)} * \frac{g(x)[u(x + \Delta x) - u(x)] - u(x)[g(x + \Delta x) - g(x)]}{\Delta x} \right] \\
&= \lim_{\Delta x \rightarrow 0} \left[\frac{1}{g(x) * g(x + \Delta x)} * \left(\frac{g(x)[u(x + \Delta x) - u(x)]}{\Delta x} - \frac{u(x)[g(x + \Delta x) - g(x)]}{\Delta x} \right) \right] \\
&= \frac{1}{g(x) * g(x)} * [g(x)u'(x) - u(x)g'(x)] \\
&= \frac{g(x)u'(x) - u(x)g'(x)}{g(x)^2} \\
&= \frac{u'(x)g(x) - u(x)g'(x)}{g(x)^2}
\end{aligned}$$

□

Theorem 1.7 (Derivative of $1/x$).

$$f(x) = \frac{1}{x}$$

$$f'(x) = \frac{-1}{x^2}$$



Proof.

$$\begin{aligned}
 f(x) &= \frac{1}{x} \\
 f'(x) &= \lim_{\Delta x \rightarrow 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} \\
 &= \lim_{\Delta x \rightarrow 0} \frac{\frac{1}{x + \Delta x} - \frac{1}{x}}{\Delta x} \\
 &= \lim_{\Delta x \rightarrow 0} \frac{\frac{1 * x}{(x + \Delta x) * x} - \frac{1 * (x + \Delta x)}{x * (x + \Delta x)}}{\Delta x} \\
 &= \lim_{\Delta x \rightarrow 0} \frac{\frac{(1 * x) - [1 * (x + \Delta x)]}{(x + \Delta x) * x}}{\Delta x} \\
 &= \lim_{\Delta x \rightarrow 0} \frac{\frac{x - (x + \Delta x)}{(x + \Delta x) * x}}{\Delta x} \\
 &= \lim_{\Delta x \rightarrow 0} \frac{\frac{x - x - \Delta x}{(x + \Delta x) * x}}{\Delta x} \\
 &= \lim_{\Delta x \rightarrow 0} \frac{\frac{-\Delta x}{(x + \Delta x) * x}}{\Delta x} \\
 &= \lim_{\Delta x \rightarrow 0} \left[\frac{1}{\Delta x} * \frac{-\Delta x}{(x + \Delta x) * x} \right] \\
 &= \lim_{\Delta x \rightarrow 0} \frac{-1}{(x + \Delta x) * x} \\
 &= \frac{-1}{(x + 0) * x} \\
 &= \frac{-1}{x * x} \\
 &= -\frac{1}{x^2}
 \end{aligned}$$

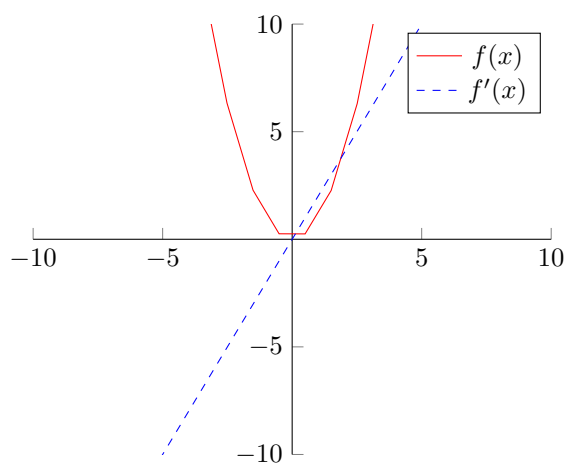
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Theorem 1.8 (Derivative of x^n).

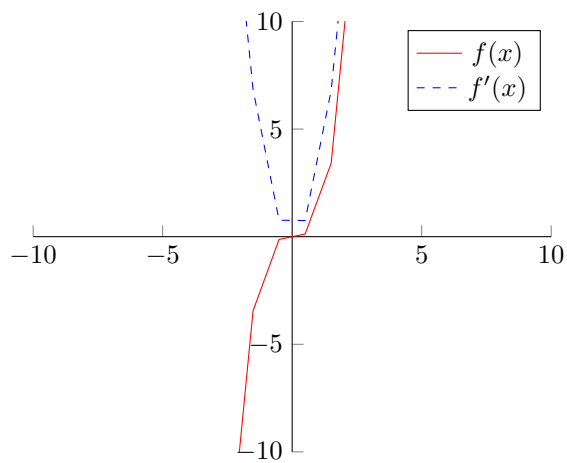
$$f(x) = x^n$$

$$f'(x) = nx^{n-1}$$

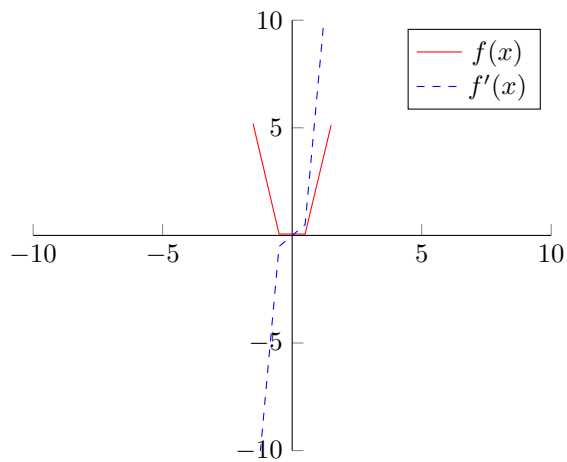
$$f(x) = x^2 \text{ and } f'(x) = 2x$$



$$f(x) = x^3 \text{ and } f'(x) = 3x^2$$



$$f(x) = x^4 \text{ and } f'(x) = 4x^3$$



Proof.

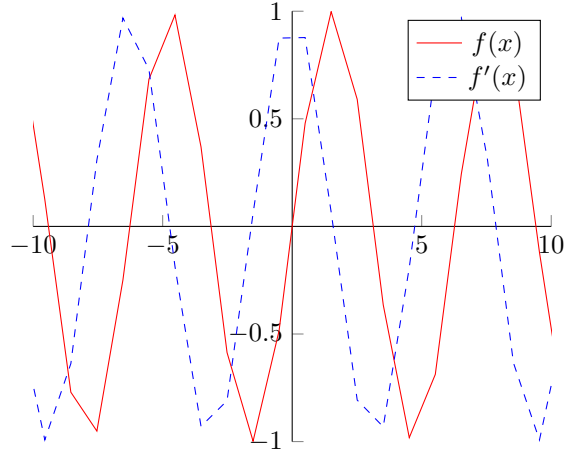
$$\begin{aligned}
 f(x) &= x^n \\
 f'(x) &= \lim_{\Delta x \rightarrow 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} \\
 &= \lim_{\Delta x \rightarrow 0} \frac{(x + \Delta x)^n - x^n}{\Delta x} \\
 &= \lim_{\Delta x \rightarrow 0} \frac{[x^n + nx^{n-1}\Delta x + \mathcal{O}((\Delta x)^2)] - x^n}{\Delta x} \\
 &= \lim_{\Delta x \rightarrow 0} \frac{[nx^{n-1}\Delta x + \mathcal{O}((\Delta x)^2)]}{\Delta x} \\
 &= \lim_{\Delta x \rightarrow 0} \frac{\Delta x[nx^{n-1} + \mathcal{O}(\Delta x)]}{\Delta x} \\
 &= \lim_{\Delta x \rightarrow 0} nx^{n-1} + \mathcal{O}(\Delta x) \\
 &= nx^{n-1}
 \end{aligned}$$

□

Theorem 1.9 (Derivative of $\sin(x)$).

$$\begin{aligned}
 f(x) &= \sin(x) \\
 f'(x) &= \cos(x)
 \end{aligned}$$

$$f(x) = \sin(x) \text{ and } f'(x) = \cos(x)$$



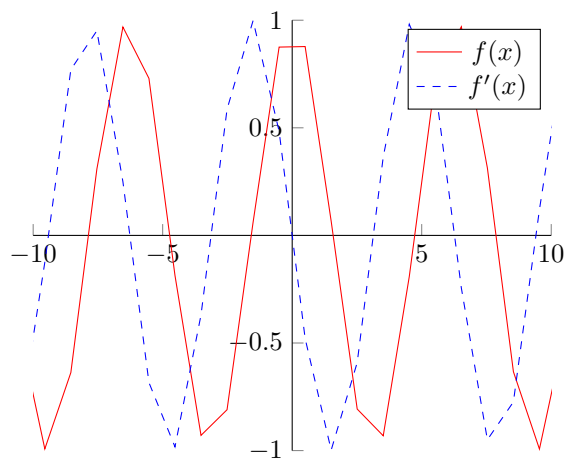
Proof.

$$\begin{aligned}
 f(x) &= \sin(x) \\
 f'(x) &= \lim_{\Delta x \rightarrow 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} \\
 &= \lim_{\Delta x \rightarrow 0} \frac{\sin(x + \Delta x) - \sin(x)}{\Delta x} \\
 &= \lim_{\Delta x \rightarrow 0} \frac{\sin(x)\cos(\Delta x) + \cos(x)\sin(\Delta x) - \sin(x)}{\Delta x} \\
 &= \lim_{\Delta x \rightarrow 0} \frac{\sin(x)(\cos(\Delta x) - 1) + \cos(x)\sin(\Delta x)}{\Delta x} \\
 &= \lim_{\Delta x \rightarrow 0} \left[\frac{\sin(x)(\cos(\Delta x) - 1)}{\Delta x} + \frac{\cos(x)\sin(\Delta x)}{\Delta x} \right] \\
 &= \lim_{\Delta x \rightarrow 0} \left[\sin(x) * \frac{\cos(\Delta x) - 1}{\Delta x} + \cos(x) * \frac{\sin(\Delta x)}{\Delta x} \right] \\
 &= \lim_{\Delta x \rightarrow 0} \left[\sin(x) * \frac{\cos(\Delta x) - 1}{\Delta x} \right] + \lim_{\Delta x \rightarrow 0} \left[\cos(x) * \frac{\sin(\Delta x)}{\Delta x} \right] \\
 &= \lim_{\Delta x \rightarrow 0} [\sin(x) * 0] + \lim_{\Delta x \rightarrow 0} [\cos(x) * 1] \quad \text{see 1.2, 1.1} \\
 &= \lim_{\Delta x \rightarrow 0} \cos(x) \\
 &= \cos(x)
 \end{aligned}$$

□

Theorem 1.10 (Derivative of $\cos(x)$).

$$\begin{aligned}
 f(x) &= \cos(x) \\
 f'(x) &= -\sin(x)
 \end{aligned}$$



Proof.

$$f(x) = \sin(x)$$

$$f'(x) = \lim_{\Delta x \rightarrow 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

$$f'(x) = \lim_{\Delta x \rightarrow 0} \frac{\cos(x + \Delta x) - \cos(x)}{\Delta x}$$

$$f'(x) = \lim_{\Delta x \rightarrow 0} \frac{\cos(x)\cos(\Delta x) - \sin(x)\sin(\Delta x) - \cos(x)}{\Delta x}$$

$$f'(x) = \lim_{\Delta x \rightarrow 0} \frac{\cos(x)(\cos(\Delta x) - 1) - \sin(x)\sin(\Delta x)}{\Delta x}$$

$$f'(x) = \lim_{\Delta x \rightarrow 0} \left[\frac{\cos(x)(\cos(\Delta x) - 1)}{\Delta x} - \frac{\sin(x)\sin(\Delta x)}{\Delta x} \right]$$

$$f'(x) = \lim_{\Delta x \rightarrow 0} \frac{\cos(x)(\cos(\Delta x) - 1)}{\Delta x} - \lim_{\Delta x \rightarrow 0} \frac{\sin(x)\sin(\Delta x)}{\Delta x}$$

$$f'(x) = \lim_{\Delta x \rightarrow 0} \left[\cos(x) * \frac{(\cos(\Delta x) - 1)}{\Delta x} \right] - \lim_{\Delta x \rightarrow 0} \left[\sin(x) * \frac{\sin(\Delta x)}{\Delta x} \right]$$

$$f'(x) = \lim_{\Delta x \rightarrow 0} [\cos(x) * 0] - \lim_{\Delta x \rightarrow 0} [\sin(x) * 1] \quad \text{see 1.2, 1.1}$$

$$f'(x) = - \lim_{\Delta x \rightarrow 0} \sin(x)$$

$$f'(x) = -\sin(x)$$

□