We will transform the system by means of elementary operations of three types:
1. Add a multiple of one equation to another equation.
2. Interchange two equations.
3. Multiply an equation by a nonzero constant.

Exercise 1.7.2:

Proposition 1.7.1
If Ax = b is obtained from Ax = b by an elementary operation of type 1, 2, or 3, then the systems Ax = b and Ax = b are equivalent.

Exercise 1.7.2
Prove Proposition 1.7.1.
Discussion: Suppose the system $Ax = b$ is transformed to $\hat{A}x = \hat{b}$ by an operation of type 1. You must show that
(a) every solution of $Ax = b$ is a solution of $\hat{A}x = \hat{b}$ and
(b) every solution of $\hat{A}x = \hat{b}$ is a solution of $Ax = b$.

Part (a) should be easy. Part (b) becomes easy when you realize that $Ax = b$ can be recovered from $\hat{A}x = \hat{b}$ by an operation of type 1: If $\hat{A}x = b$ was obtained from $Ax = b$ by adding m times the j-th row to the i-th row, then $Ax = b$ can be recovered from $\hat{A}x = b$ by adding $-m$ times the j-th row to the i-th row. Analogous remarks apply to operations of types 2 and 3.

Answers:

(a)

$$\begin{aligned} Ax &= b \\ \forall j \left[\text{row}_j(A) * x = b_j\right] \\ \\ \hat{A} &= A + m * \text{row}_j(A) \\ \hat{b} &= b + m * b_j \end{aligned}$$

$$\begin{aligned} Ax &= b \\ Ax + m * \text{row}_j(A) * x &= b + m * \text{row}_j(A) * x \\ Ax + m * \text{row}_j(A) * x &= b + m * b_j \\ (A + m * \text{row}_j(A)) * x &= b + m * b_j \\ (A + m * \text{row}_j(A)) * x &= \hat{b} \\ \hat{A} * x &= \hat{b} \\ &\square \end{aligned}$$

(b)

$$
\begin{aligned}
Ax &= b \\
\forall j \left[\operatorname{row}_j(A) * x = b_j\right] \\
\\
\hat{A} &= A + m * \operatorname{row}_j(A) \\
\hat{b} &= b + m * b_j \\
\\
\hat{A}x &= \hat{b} \\
(A + m * \operatorname{row}_j(A)) * x &= b + m * b_j \\
A * x + m * \operatorname{row}_j(A) * x &= b + m * b_j \\
b + m * \operatorname{row}_j(A) * x &= b + m * b_j \\
m * \operatorname{row}_j(A) * x &= b + m * b_j - b \\
m * \operatorname{row}_j(A) * x &= m * b_j \\
\operatorname{row}_j(A) * x &= b_j \\
&\square
\end{aligned}
$$