Exercise 1.4.71

Prove by induction on $n$ that

$$\sum_{i=0}^{n} i^2 = \frac{n(n+\frac{1}{2})(n+1)}{3}$$

Proof by Induction:

Base:

$$\begin{aligned} i &= 0 \\ \sum_{i=0}^{0} i^2 &= 0 \\ \frac{n(n+\frac{1}{2})(n+1)}{3} &= \frac{0(0+\frac{1}{2})(0+1)}{3} \\ &= 0 \end{aligned}$$

Step:

$$\begin{aligned} \sum_{i=0}^{n+1} i^2 &= \frac{(n+1)((n+1)+\frac{1}{2})((n+1)+1)}{3} \\ &= \frac{(n+1)(n+1+\frac{1}{2})(n+1+1)}{3} \\ &= \frac{(n+1)(n+\frac{3}{2})(n+2)}{3} \\ &= \frac{(n+1)(n^2+2n+\frac{3}{2}n+\frac{6}{2})}{3} \\ &= \frac{(n+1)(n^2+2n+\frac{3}{2}n+3)}{3} \\ &= \frac{(n+1)(n^2+2n+\frac{3}{2}n+3)-3(n+1)^2}{3} + (n+1)^2 \\ &= \frac{(n+1)(n^2+2n+\frac{3}{2}n+3)-3(n+1)(n+1)}{3} + (n+1)^2 \\ &= \frac{(n+1)[(n^2+2n+\frac{3}{2}n+3)-3(n+1)]}{3} + (n+1)^2 \\ &= \frac{(n+1)[n^2+2n+\frac{3}{2}n+3-3n-3]}{3} + (n+1)^2 \end{aligned}$$

$$
\begin{aligned}
&= \frac{(n+1)[n^2 + \frac{4}{2}n + \frac{3}{2}n + 3 - \frac{6}{2}n - 3]}{3} + (n+1)^2 \\
&= \frac{(n+1)[n^2 + \frac{1}{2}n + 3 - 3]}{3} + (n+1)^2 \\
&= \frac{(n+1)[n^2 + \frac{1}{2}n]}{3} + (n+1)^2 \\
&= \frac{(n+1)[n(n + \frac{1}{2})]}{3} + (n+1)^2 \\
&= \frac{(n+1)n(n + \frac{1}{2})}{3} + (n+1)^2 \\
&= \frac{n(n + \frac{1}{2})(n+1)}{3} + (n+1)^2 \\
&= \left[\sum_{i=0}^{n} i^2\right] + (n+1)^2
\end{aligned}
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$\square$