Exercise 1.4.15
Let

$$A = \begin{bmatrix} 4 & 0 \\ 0 & 9 \end{bmatrix}$$

(a) Prove that A is positive definite,
(b) Calculate the Cholesky factor of A,
(c) Find three other upper triangular matrices R such that $A = R^T R$,
(d) Let A be any $n \times n$ positive definite matrix. How many upper-triangular matrices R such that $A = R^T R$ are there?

Based on the work of `https://en.wikipedia.org/wiki/Andr%C3%A9-Louis_Cholesky`

Proofs:
(a)

$$\begin{aligned}
x^T A x &= \begin{bmatrix} x_{11} & x_{21} \end{bmatrix} * \begin{bmatrix} 4 & 0 \\ 0 & 9 \end{bmatrix} * \begin{bmatrix} x_{11} \\ x_{21} \end{bmatrix} \\
&= \begin{bmatrix} x_{11} * 4 + x_{21} * 0 & x_{11} * 0 + x_{21} * 9 \end{bmatrix} * \begin{bmatrix} x_{11} \\ x_{21} \end{bmatrix} \\
&= \begin{bmatrix} x_{11} * 4 & x_{21} * 9 \end{bmatrix} * \begin{bmatrix} x_{11} \\ x_{21} \end{bmatrix} \\
&= \begin{bmatrix} x_{11} * 4 * x_{11} & x_{21} * 9 * x_{21} \end{bmatrix} \\
&= \begin{bmatrix} 4x_{11}^2 & 9x_{21}^2 \end{bmatrix} \text{ necessarily} > 0
\end{aligned}$$

(b)

$$R = \begin{bmatrix} r_{11} & r_{12} \\ 0 & r_{22} \end{bmatrix}$$

$$A = \begin{bmatrix} 4 & 0 \\ 0 & 9 \end{bmatrix}$$

$$R^T R = A$$

$$\begin{bmatrix} r_{11} & 0 \\ r_{12} & r_{22} \end{bmatrix} \begin{bmatrix} r_{11} & r_{12} \\ 0 & r_{22} \end{bmatrix} = \begin{bmatrix} 4 & 0 \\ 0 & 9 \end{bmatrix}$$

$$\begin{bmatrix} r_{11} * r_{11} + 0 * 0 & r_{11} * r_{12} + 0 * r_{22} \\ r_{12} * r_{11} + r_{22} * 0 & r_{12} * r_{12} + r_{22} * r_{22} \end{bmatrix} = \begin{bmatrix} 4 & 0 \\ 0 & 9 \end{bmatrix}$$

$$r_{11} * r_{11} + 0 * 0 = 4$$
$$r_{11} * r_{12} + 0 * r_{22} = 0$$
$$r_{12} * r_{11} + r_{22} * 0 = 0$$
$$r_{12} * r_{12} + r_{22} * r_{22} = 9$$

$$r_{11}^2 = 4$$
$$r_{11} * r_{12} = 0$$
$$r_{12} * r_{11} = 0$$
$$r_{12}^2 + r_{22}^2 = 9$$

$$r_{11} = 2$$
$$r_{11} * r_{12} = 0$$
$$r_{12} * r_{11} = 0$$
$$r_{12}^2 + r_{22}^2 = 9$$

$$r_{11} = 2$$
$$2 * r_{12} = 0$$
$$r_{12} * 2 = 0$$
$$r_{12}^2 + r_{22}^2 = 9$$

$$
\begin{aligned}
r_{11} &= 2 \\
r_{12} &= 0 \\
r_{12} &= 0 \\
r_{12}^2 + r_{22}^2 &= 9
\end{aligned}
$$

$$
\begin{aligned}
r_{11} &= 2 \\
r_{12} &= 0 \\
r_{12} &= 0 \\
0^2 + r_{22}^2 &= 9
\end{aligned}
$$

$$
\begin{aligned}
r_{11} &= 2 \\
r_{12} &= 0 \\
r_{12} &= 0 \\
r_{22}^2 &= 9
\end{aligned}
$$

$$
\begin{aligned}
r_{11} &= 2 \\
r_{12} &= 0 \\
r_{12} &= 0 \\
r_{22} &= 3
\end{aligned}
$$

$$
R = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}
$$

(c)

The Cholesky Factor must have all its diagonal entries greater than zero. That is what allowed us in (b) to choose 2 and 3 instead of -2 and -3 when finding the squares roots.
But we can still find other matrices that satisfy the proposition $A = R^T R$ using this negative roots.

$$R = \begin{bmatrix} -2 & 0 \\ 0 & 3 \end{bmatrix}$$

$$R = \begin{bmatrix} 2 & 0 \\ 0 & -3 \end{bmatrix}$$

$$R = \begin{bmatrix} -2 & 0 \\ red0 & -3 \end{bmatrix}$$

All of them when "squared" generate the ogirinal matrix.

(d) We can genralize this idea. For $A = n \times n$ matrix we can always find N matrices where each one of them came from a permutation of the "signals". Given that we have n elements in the diagonal and each of then can have 2 possible signals, we have 2^n possible matrices.