Week3_Lab_Discrete_Time_Markov_Chains

March 18, 2018

MOOC: Understanding queues
Python simulations
Week III: Discrete time Markov chains

In this lab, we consider the Markov chain of the weather forecast example of the course. We check convergence of the probability $\pi(t)$ of the chain at time t to a steady state distribution π^*, independently from the initial distribution $\pi(0)$ of the chain. We solve the load balance equations to get π^*.

Let us consider the Markov chain of the weather forecast example of the course. Recall that its states 1, 2 and 3 represent clear, cloudy and rainy states, and the transition matrix is

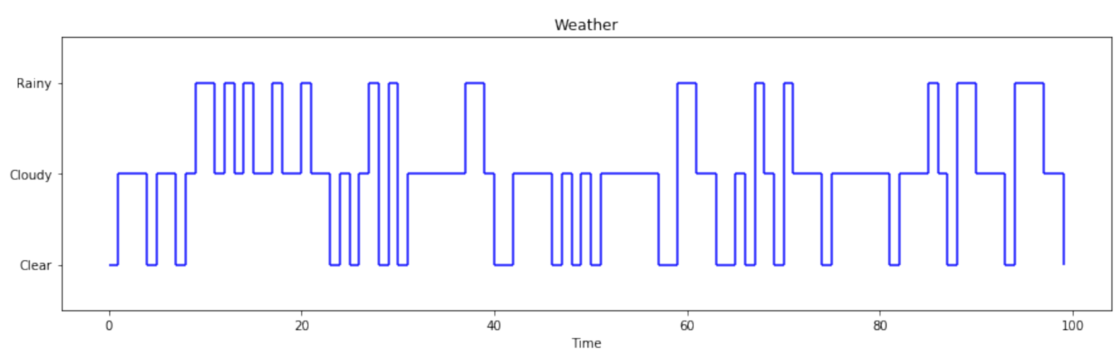
$$P = \begin{pmatrix} 0.7 & 0.3 & 0 \\ 0.3 & 0.5 & 0.2 \\ 0.1 & 0.4 & 0.5 \end{pmatrix}.$$

1) Complete below the code of the function that generates trajectories of the Markov chain. The function inputs are the chain initial state $x0$, the transition matrix P and final time index T. Its output will be a trajectory x of the chain observed between instants 0 and T. Draw a trajectory of the evolution of the weather between time 0 and time $T = 100$.

```
In [5]: %matplotlib inline
        from pylab import *

In [17]: P    = array([[.7, .3, 0], [.3, .5, .2], [.1, .4, .5]])

         def X(x0,P=P,T=100):
             # Function X supplies a trajectory of the discrete Markov chain
             # with initial state x0 and transition matrix P, till time T
             x        = [x0]
             for _ in range(T):
                 #####################
                 # supply the vector p of probabilities to transit to states
                 # 1,2,3 from the last calculated state
                 p = P[ x[len(x)-1]-1 ]
```

```
        ####################
        u = rand()
        if u<p[0]:
            x.append(1)
        elif u<p[0]+p[1]:
            x.append(2)
        else:
            x.append(3)
    return array(x)

V1 = mean(X(x0=1,T=10**4))
```

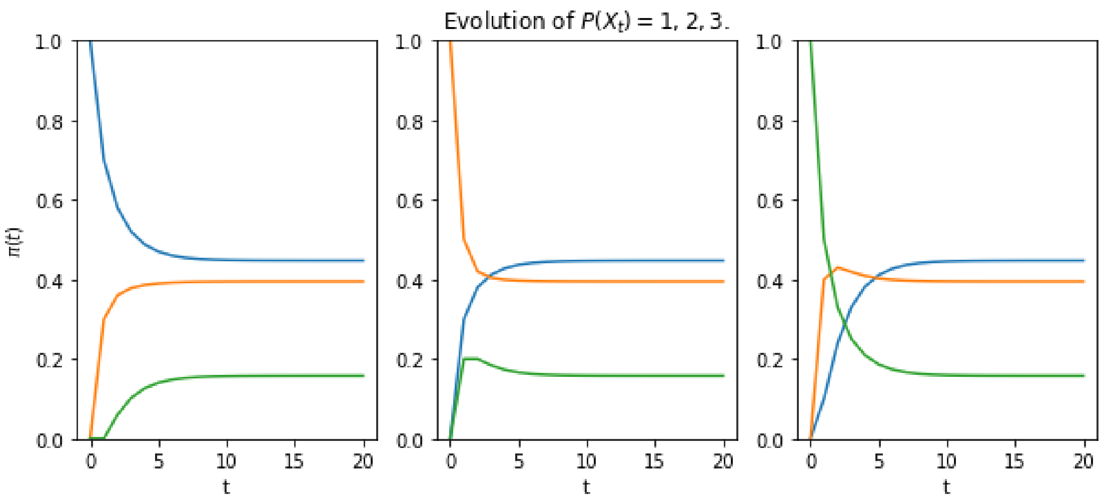
```
In [8]: def step(x,y,Tmax=0,color='b'):
            # step function
            # plots a step function representing the number
            # of clients in the system at each instant
            if Tmax==0:
                Tmax = max(x)
            x = append(x,[Tmax])  # number of clients
            y = append(y,[y[-1]]) # instants of events
            for k in range(len(x)-1):
                vlines(x[k+1],y[k],y[k+1],color=color)
                hlines(y[k],x[k],x[k+1],color=color)

        T = 100
        x = X(x0=1)
        figure(num=None, figsize=(15, 4))
        step(range(T),x)
        axis(ymin=0.5,ymax=3.5)
        xlabel("Time")
        title("Weather")
        yticks([1.0,2.0,3.0], ["Clear","Cloudy","Rainy"]);
```



2) Run the following code that computes recursively the state probability vectors $\pi(t)$ at times $t = 0, \ldots, 100$. The state probability vectors can be computed recursively : $\pi(t+1) = \pi(t)P$.

Check that, when changing the initial state $x0$, $\pi(t)$ still converges rapidly to the same asymptotic vector π^* as t increases.

```
In [9]: T = 20

        def PI(pi0,P=P,T=T):
            # Function PI computes the state probability vectors
            # of the Markov chain until time T
            pi_ = array([pi0])
            for i in range(T):
                pi_ = vstack((pi_,pi_[-1] @ P))
            return pi_

        def plot_PI(x0):
            # subplot(1,3,n+1) of successive states probabilities
            # with initial state x0
            pi_0        = zeros(3)
            pi_0[x0-1] = 1
            pi_   = PI(pi_0)
            subplot(1,3,x0)
            plot(pi_)
            xlabel('t');axis(ymin=0,ymax=1)
            if x0==1: ylabel(r"$\pi(t)$")
            if x0==2: title("Evolution of $P(X_t)=1,2,3$.")

        rcParams["figure.figsize"] = (10., 4.)
        for x0 in range(1,4):
            plot_PI(x0)
```



3) To compute the steady state distribution $\pi^* = [\pi_1^*, \pi_2^*, \pi_3^*]$, we must solve the system of load balance equations $\pi^* = \pi^* P$ with the normalization condition $\pi_1^* + \pi_2^* + \pi_3^* = 1$. The system of equations $\pi^* = \pi^* P$ is redundant : the third equation is a straightforward linear combination of the first two ones. Taking into account the normalization condition $\pi_1^* + \pi_2^* + \pi_3^* = 1$ and discarding the third redundant equation in $\pi^*(P - I_3) = 0$ yields a full rank system of equations. Complete the code below to solve this system and obtain the steady state ditribution. We will use the solve function from the **scipy.linalg** library.

```
In [14]: from scipy.linalg import solve
         ####################
         # complete the code to get the steady state distribution
         # of the discrete time Markov chain
         pi_ = solve([[-.3, .3, 0.1], [.3, -.5, .4], [1, 1, 1]],[0,0,1])
         print("steady state distribution: pi* =",pi_)
         ####################
         V2,V3 = pi_[0],pi_[1]
```

```
steady state distribution: pi* = [ 0.44736842  0.39473684  0.15789474]
```

1 Your answers for the exercise

```
In [18]: print("---------------------------\n"
               +"RESULTS SUPPLIED FOR LAB 3:\n"
               +"---------------------------")
         results = ("V"+str(k) for k in range(1,4))
         for x in results:
             try:
                 print(x+" = {0:.2f}".format(eval(x)))
             except:
                 print(x+": variable is undefined")
```

```
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RESULTS SUPPLIED FOR LAB 3:
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V1 = 1.72
V2 = 0.45
V3 = 0.39
```