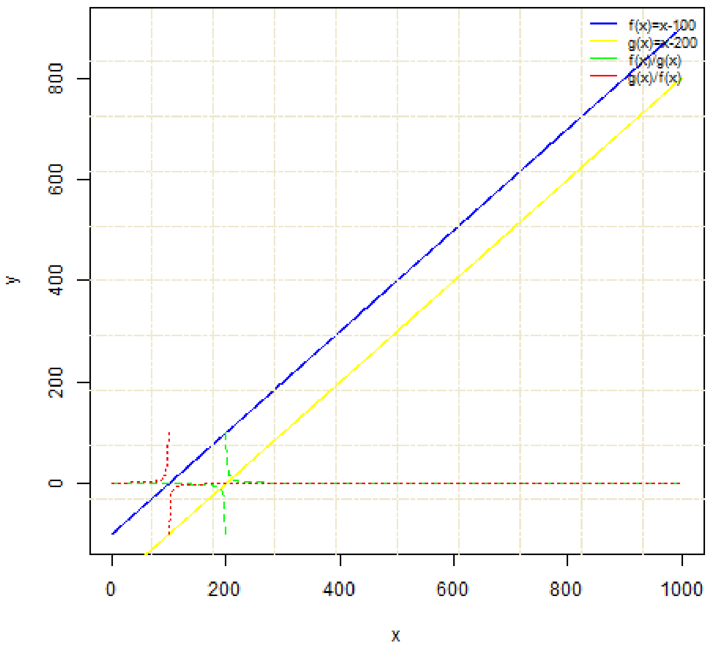
Dasgupta "Algorithms" book: Exercises: 0.a

$$f(n) = n - 100$$
$$g(x) = n - 200$$



Is $f(n) = O(g(n))$?

$$\frac{n-100}{n-200} = c$$
$$n - 100 = c * (n - 200)$$
$$n - 100 = cn - 200c$$
$$n = cn - 200c + 100$$
$$n - cn = -200c + 100$$
$$n(1-c) = -200c + 100$$
$$n = \frac{-200c + 100}{1-c}$$
$$n = \frac{100(-2c+1)}{1-c}$$
$$n = 100\frac{-2c+1}{1-c}$$

$$
\begin{aligned}
n * -1 &= 100\frac{-2c+1}{1-c} * -1 \\
-n &= 100\frac{2c-1}{-1+c} \\
-n &= 100\frac{2c-1}{c-1} \\
-n &= 100\frac{2(c-1)}{c-1} \\
-n &= 200\frac{c-1}{c-1} \\
-n &= 200 \\
n &= -200
\end{aligned}
$$

So

$$
\begin{aligned}
&\frac{f(n)}{g(n)} <= -200 * \frac{g(n)}{g(n)} \\
&f(n) = O(g(n)) \\
&\blacksquare
\end{aligned}
$$

0.2 Show that, if c is a positive real number, then $g(n) = 1 + c + c^2 + ... + c^n$ is:
(a) $\theta(1)$ if $c < 1$.
(b) $\theta(n)$ if $c = 1$.
(c) $\theta(cn)$ if c ¿ 1.
The moral: in big-θ terms, the sum of a geometric series is simply the first term if the series is strictly decreasing, the last term if the series is strictly increasing, or the number of terms if the series is unchanging.

$$
\begin{aligned}
f(n) &= 1 + c + c^2 + ... + c^n \\
f(n) &= \sum_{i=0}^{n} c^i <= \sum_{i=0}^{\infty} c^i \\
&<= \frac{1}{1-c}
\end{aligned}
$$

In the case of $c < 1$ we have

$$f(n) <= \frac{1}{1-c}$$
$$f(n) <= z \qquad\qquad z > 0$$
$$\frac{f(n)}{g(n)} <= k * \frac{g(n)}{g(n)}$$
$$\frac{f(n)}{1} <= k * \frac{1}{1} \qquad\qquad \text{choose } k = z$$

$$f(n) = O(1)$$

In the case of $c = 1$ we have

$$f(n) = 1 + 1^2 + ... + 1^n$$
$$f(n) = \sum_{i=0}^{n} 1 = n$$
$$\frac{f(n)}{g(n)} <= k * \frac{g(n)}{g(n)}$$
$$\frac{n}{n} <= k * \frac{n}{n} \qquad\qquad \text{choose } k = 1$$
$$f(n) = O(n)$$

In the case of $c > 1$ we have

$$f(n) = 1 + c^2 + ... + c^n$$
$$f(n) = \sum_{i=0}^{n} c^i <= \sum_{i=0}^{\infty} c^i$$
$$= \sum_{i=0}^{\infty} c^n$$
$$= c^n * \sum_{i=0}^{\infty} 1$$
$$= n * c^n$$
$$f(n) <= n * c^n$$
$$\frac{f(n)}{g(n)} <= k * \frac{g(n)}{g(n)}$$
$$\frac{f(n)}{c^n} <= k * \frac{c^n}{c^n} \qquad\qquad \text{choose } k = n$$
$$f(n) = O(c^n)\blacksquare$$

0.3 The Fibonacci numbers F 0 ,F 1 ,F 2 ,..., are defined by the rule F 0 = 0, F 1 = 1, F n = F n?1 + F n?2 . In this problem we will confirm that this sequence grows exponentially fast and obtain some bounds on its growth. (a) Use induction to prove that F n ? 2 0.5n for n ? 6. (b) Find a constant c ¡ 1 such that F n ? 2 cn for all n ? 0. Show that your answer is correct. (c) What is the largest c you can find for which F n = ?(2 cn)?

(a) missing bases steps

$$F_n >= 2^{n/2}$$

$$F_n = F_{n-1} + F_{n-2}$$

$$>= 2^{(n-1)/2} + 2^{(n-2)/2}$$

$$= (X + 2^{(n-2)/2}) + 2^{(n-2)/2} \qquad X > 0$$

$$= 2 * 2^{(n-2)/2}$$

$$>= 2 * 2^{(n-2)/2}$$

$$= 2^{\frac{n-2}{2}+1}$$

$$= 2^{\frac{n-2}{2}+\frac{2}{2}}$$

$$= 2^{\frac{n}{2}}$$

$$>= 2^{n/2}$$

■

(B)

$$\begin{aligned}
F_n &<= 2^{cn} \\
F_{n-2} + F_{n-1} &<= 2^{cn} \\
2^{c(n-2)} + 2^{c(n-1)} &<= 2^{cn} \\
2^{cn-2c} + 2^{cn-c} &<= 2^{cn} \\
\frac{2^{cn}}{2^{2c}} + \frac{2^{cn}}{2^{c}} &<= 2^{cn} \\
\frac{(2^c)^n}{(2^c)^2} + \frac{(2^c)^n}{2^c} &<= (2^c)^n \\
\frac{a^n}{a^2} + \frac{a^n}{a} &<= a^n \\
\frac{aa^n}{aa^2} + \frac{a^2a^n}{a^2a} &<= a^n \\
\frac{a^{n+1}}{a^3} + \frac{a^{n+2}}{a^3} &<= a^n \\
\frac{a^{n+1}}{a^3} + \frac{a^{n+2}}{a^3} &<= a^n \\
\frac{a^{n+1} + a^{n+2}}{a^n} &<= a^3 \\
\frac{aa^n + a^2a^n}{a^n} &<= a^3 \\
\frac{a^n(a + a^2)}{a^n} &<= a^3 \\
a + a^2 &<= a^3 \\
2^c + 2^{2c} &<= 2^{3c} \\
\log_2\left(2^c + 2^{2c}\right) &<= \log_2\left(2^{3c}\right)
\end{aligned}$$

Fib 0 1 1 2 3 5 8 13 21 34 55 89 144 233 377 610 987 1597 2584 4181 6765