

Expected Value
 X is discrete random variable
 $g : \mathbf{R} \rightarrow \mathbf{R}$
 $\Omega X = Im(X)$

$$E[g(X)] = \sum_{x \in \Omega X} P(X = x) * g(x) \quad (1)$$

Variance

$$var(g(X)) = E([g(X) - E(g(X))]^2) \quad (2)$$

$$= \sum_{x \in \Omega X} [g(X) - E(g(X))]^2 * P(X = x) \quad (3)$$

$$= \sum_{x \in \Omega X} [g(X)^2 - 2 * g(X) * E(g(X)) + E(g(X))^2] * P(X = x) \quad (4)$$

$$= \sum_{x \in \Omega X} g(X)^2 * P(X = x) \quad (5)$$

$$- \sum_{x \in \Omega X} 2 * g(X) * E(g(X)) * P(X = x) \quad (6)$$

$$+ \sum_{x \in \Omega X} E(g(X))^2 * P(X = x) \quad (7)$$

$$= \sum_{x \in \Omega X} g(X)^2 * P(X = x) \quad (8)$$

$$- 2 * E(g(X)) * \sum_{x \in \Omega X} g(X) * P(X = x) \quad (9)$$

$$+ E(g(X))^2 * \sum_{x \in \Omega X} P(X = x) \quad (10)$$

$$= E(g(X)^2) - 2 * E(g(X)) * E(g(X)) + E(g(X))^2 * 1 \quad (11)$$

$$= E(g(X)^2) - 2 * E(g(X))^2 + E(g(X))^2 \quad (12)$$

$$= E(g(X)^2) - E(g(X))^2 \quad (13)$$

$$\square \quad (14)$$

Variance of the Sample Mean

$$Var(\bar{X}) = Var\left(\frac{1}{n} \sum_{i=1}^n X_i\right) \quad (15)$$

$$= \frac{1}{n^2} Var\left(\sum_{i=1}^n X_i\right) \quad (16)$$

$$= \frac{1}{n^2} \sum_{i=1}^n Var(X_i), \text{ by independence} \quad (17)$$

$$= \frac{1}{n^2} [Var(X_1) + Var(X_2) + \dots + Var(X_n)] \quad (18)$$

$$= \frac{1}{n^2} [\sigma^2 + \sigma^2 + \dots + \sigma^2], \text{ since the } X_i \text{ are identically distributed} \quad (19)$$

$$= \frac{1}{n^2} (n\sigma^2) \quad (20)$$

$$= \frac{\sigma^2}{n} \quad (21)$$

Binomial Distribution
PMF

$$P(X = k) = \binom{n}{k} p^k (1-p)^{n-k} \quad (22)$$

Expected Value

$E[g(X)]$ when $g(X) = X$.

$$E(X) = \sum_{k \geq 0} P(x = k) * k \quad (23)$$

$$= \sum_{k \geq 0} \left[\binom{n}{k} p^k (1-p)^{n-k} \right] * k \quad (24)$$

$$(25)$$

when

$$k = 0$$

, the formula

$$\left[\binom{n}{k} p^k (1-p)^{n-k} \right] * k = \left[\binom{n}{0} p^k (1-p)^n \right] * 0 = 0$$

, so the index of the summation can be increased by 1.

$$E(X) = \sum_{k \geq 1} \binom{n}{k} p^k (1-p)^{n-k} * k \quad (26)$$

$$= \sum_{k \geq 1} \frac{n}{k} * \binom{n-1}{k-1} p^k (1-p)^{n-k} * k \quad \text{see BinomialCoefficient} \quad (27)$$

$$= \sum_{k \geq 1} \frac{n * k}{k} * \binom{n-1}{k-1} p^k (1-p)^{n-k} \quad (28)$$

$$= \sum_{k \geq 1} n * \binom{n-1}{k-1} p^k (1-p)^{n-k} \quad (29)$$

$$= \sum_{k \geq 1} n * p * \binom{n-1}{k-1} p^{k-1} (1-p)^{n-k} \quad (30)$$

$$= np * \sum_{k \geq 1} \binom{n-1}{k-1} p^{k-1} (1-p)^{n-k} \quad (31)$$

$$u = n - 1 \quad (32)$$

$$z = k - 1 \quad (33)$$

$$u - z = (n - 1) - (k - 1) \quad (34)$$

$$= n - 1 - k + 1 \quad (35)$$

$$= n - k \quad (36)$$

$$k > 1 = (z + 1) > 1 \quad (37)$$

$$= z > 0 \quad (38)$$

$$= np * \sum_{z > 0} \binom{u}{z} p^z (1-p)^{u-z} \quad (39)$$

$$= np * 1 \quad \text{see BinomialDistributionProof Equals 1} \quad (40)$$

$$= np \quad (41)$$

$$\square \quad (42)$$

Variance

$$Var(X) = E(X^2) - E(X)^2 \quad \text{see Variance} \quad (43)$$

$$= \sum_{k \geq 0} \left[\binom{n}{k} p^k (1-p)^{n-k} \right] * k^2 - np \quad \text{see Binomial Expected Value} \quad (44)$$

when

$$k = 0$$

, the formula

$$[\binom{n}{k} p^k (1-p)^{n-k}] * k = [\binom{n}{0} p^k (1-p)^n] * 0 = 0$$

, so the index of the summation can be increased by 1.

$$= \sum_{k \geq 1} \left[\binom{n}{k} p^k (1-p)^{n-k} \right] * k^2 - (np)^2 \quad (45)$$

$$= \sum_{k \geq 1} \frac{n}{k} \left[\binom{n-1}{k-1} p^k (1-p)^{n-k} \right] * k^2 - (np)^2 \quad (46)$$

$$= \sum_{k \geq 1} \frac{n * k^2}{k} \left[\binom{n-1}{k-1} p^k (1-p)^{n-k} \right] - (np)^2 \quad (47)$$

$$= \sum_{k \geq 1} [nk * \binom{n-1}{k-1} p^k (1-p)^{n-k}] - (np)^2 \quad (48)$$

$$= \sum_{k \geq 1} [nkp * \binom{n-1}{k-1} p^{k-1} (1-p)^{n-k}] - (np)^2 \quad (49)$$

$$= np * \sum_{k \geq 1} [k * \binom{n-1}{k-1} p^{k-1} (1-p)^{n-k}] - (np)^2 \quad (50)$$

$$u = n - 1 \quad (51)$$

$$z = k - 1 \quad (52)$$

$$u - z = (n - 1) - (k - 1) \quad (53)$$

$$= n - 1 - k + 1 \quad (54)$$

$$= n - k \quad (55)$$

$$k \geq 1 = (z + 1) \geq 1 \quad (56)$$

$$= z \geq 0 \quad (57)$$

$$= np * \sum_{z \geq 0} [(z + 1) * \binom{u}{z} p^z (1-p)^{u-z}] - (np)^2 \quad (58)$$

$$= np * \left[\sum_{z \geq 0} [z * \binom{u}{z} p^z (1-p)^{u-z}] + \sum_{z \geq 0} \left[\binom{u}{z} p^z (1-p)^{u-z} \right] \right] - (np)^2 \quad (59)$$

$$= np * \left[\sum_{z \geq 0} \left[z * \frac{u}{z} * \binom{u-1}{z-1} p^z (1-p)^{u-z} \right] + \sum_{z \geq 0} \left[\binom{u}{z} p^z (1-p)^{u-z} \right] \right] - (np)^2 \quad (60)$$

$$= np * \left[u * \sum_{z \geq 0} \left[\binom{u-1}{z-1} p^z (1-p)^{u-z} \right] + \sum_{z \geq 0} \left[\binom{u}{z} p^z (1-p)^{u-z} \right] \right] - (np)^2 \quad (61)$$

$$(62)$$

$$= np * [up * \sum_{z \geq 0} [\binom{u-1}{z-1} p^{z-1} (1-p)^{u-z}] + \sum_{z \geq 0} [\binom{u}{z} p^z (1-p)^{u-z}]] - (np)^2 \quad (63)$$

$$= np * [up * \sum_{z \geq 1} [\binom{u-1}{z-1} p^{z-1} (1-p)^{(u-1)-(z-1)}] + \sum_{z \geq 0} [\binom{u}{z} p^z (1-p)^{u-z}]] - (np)^2 \quad (64)$$

$$= np * [up * \sum_{z \geq 0} [\binom{u}{z} p^z (1-p)^{u-z}] + \sum_{z \geq 0} [\binom{u}{z} p^z (1-p)^{u-z}]] - (np)^2 \quad (65)$$

$$= np * [up * (p+q)^{u-1} + (p+q)^u] - (np)^2 \quad (66)$$

$$= np * [(n-1) * p * (p+q)^{n-1-1} + (p+q)^{(n-1)}] - (np)^2 \quad (67)$$

$$= np * [(n-1) * p * (p+q)^{n-2} + (p+q)^{(n-1)}] - (np)^2 \quad (68)$$

$$= np * [(n-1) * p * (p+q)^{n-2} + (p+q)^{(n-1)}] - np \quad (69)$$

$$= np * [(n-1) * p + 1] - np \quad (70)$$

$$= np * [(n-1) * p + 1] - np \quad (71)$$

$$= np * [np - p + 1] - np \quad (72)$$

$$= np * (np - p + 1 - np) \quad (73)$$

$$= np * (-p + 1) \quad (74)$$

$$= np * (1 - p) \quad (75)$$

□

Bernoulli Distribution

The Bernoulli Distribution is a special case of the Binomial Distribution, where

$$n = 1$$

PMF

$$P(X = k) = \binom{1}{k} p^k (1-p)^{1-k} \quad (77)$$

$$= p^k (1-p)^{n-k} \quad (78)$$

Expected Value

$$E(x) = \sum_{k \geq 1} [\binom{n}{k} p^k (1-p)^{n-k}] * k \quad (79)$$

$$= np \quad \text{see Binomial Distribution} \quad (80)$$

$$= 1 * p \quad (81)$$

$$= p \quad (82)$$

p+q=1

Variance

$$Var(X) = np * (1 - p) \quad (83)$$

$$= p * (1 - p) \quad (84)$$

$$(85)$$

Likelihood of IID Bernoulli

$$x_i \stackrel{iid}{\sim} Bernoulli(p) \quad (86)$$

$$L(x_i|p) = p(x_1, x_2, \dots, x_n|p) \quad (87)$$

$$= \prod_{n=1}^n p(x_i|p) \quad (88)$$

$$= p^S * (1 - p)^{n-S} \quad (89)$$

Maximum Likelihood

$$\frac{d[L(x_i|p)]}{dp} = \frac{d[p^S * (1-p)^{n-S}]}{dp} \quad (90)$$

$$\frac{d[\log(L(x_i|p))]}{dp} = \frac{d[\log(p^S * (1-p)^{n-S})]}{dp} \quad (91)$$

$$= \frac{d}{dp} * [\log(p^S * (1-p)^{n-S})] \quad (92)$$

$$= \frac{d}{dp} [\log(p^S) + \log((1-p)^{n-S})] \quad (93)$$

$$= \frac{d}{dp} [S * \log(p) + (n-S) * \log(1-p)] \quad (94)$$

$$= S * \frac{d}{dp} [\log(p)] + (n-S) * \frac{d}{dp} [\log(1-p)] \quad (95)$$

$$= S * \left[\frac{1}{p}\right] + (n-S) * \frac{d}{dp} [\log(1-p)] \quad \text{chain rule} \quad (96)$$

$$= S * \frac{1}{p} + (n-S) * \frac{1}{p-1} \quad (97)$$

$$= \frac{S}{p} + \frac{n-S}{p-1} \quad (98)$$

$$= \frac{S * (p-1)}{p * (p-1)} + \frac{p * (n-S)}{p * (p-1)} \quad (99)$$

$$= \frac{S * (p-1) + p * (n-S)}{p * (p-1)} \quad (100)$$

$$= \frac{S * p - S + p * n - p * S}{p * (p-1)} \quad (101)$$

$$= \frac{-S + p * n}{p * (p-1)} \quad (102)$$

$$0 = \frac{-S + p * n}{p * (p-1)} \quad (103)$$

$$0 * (p * (p-1)) = -S + p * n \quad (104)$$

$$0 = -S + p * n \quad (105)$$

$$S = p * n \quad (106)$$

$$\frac{S}{n} = p \quad (107)$$

$$p = \frac{S}{n} \quad (108)$$

$$(109)$$

Normal Distribution Definition

$$pdf(x) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right) \quad (110)$$