1 Introduction

Calculus Handout

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Theorem 1.1 (Limit of $sin(x)/x$).

$$\lim_{x \to 0} \frac{sin(x)}{x} = 1$$

Proof. see proof $\square$

Theorem 1.2 (Limit of $(cos(x) - 1)/x$).

$$\lim_{x \to 0} \frac{cos(x) - 1}{x} = 0$$

Proof. see proof $\square$

Definition 1.2.1 (Derivative Definition).

$$\frac{d}{dx} f(x) = f'(x) = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

Theorem 1.3 (Derivative of the scale).

$$f(x) = cg(x)$$
$$f'(x) = cg'(x)$$

Proof.

$$\begin{aligned}
f(x) &= cg(x) \\
f'(x) &= \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} \\
&= \lim_{\Delta x \to 0} \frac{cg(x + \Delta x) - cg(x)}{\Delta x} \\
&= \lim_{\Delta x \to 0} \frac{c(g(x + \Delta x) - g(x))}{\Delta x} \\
&= \lim_{\Delta x \to 0} c * \frac{g(x + \Delta x) - g(x)}{\Delta x} \\
&= c * \lim_{\Delta x \to 0} \frac{g(x + \Delta x) - g(x)}{\Delta x} \\
&= cg'(x)
\end{aligned}$$

$\square$

Theorem 1.4 (Derivative of the sum)**.**

$$f(x) = u(x) + g(x)$$
$$f'(x) = u'(x) + g'(x)$$

Proof.

$$\begin{aligned}
f(x) &= u(x) + g(x) \\
f'(x) &= \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} \\
&= \lim_{\Delta x \to 0} \frac{(u(x + \Delta x) + g(x + \Delta x)) - (u(x) + g(x))}{\Delta x} \\
&= \lim_{\Delta x \to 0} \frac{u(x + \Delta x) - u(x) + g(x + \Delta x) - g(x)}{\Delta x} \\
&= \lim_{\Delta x \to 0} \left[\frac{u(x + \Delta x) - u(x)}{\Delta x} + \frac{g(x + \Delta x) - g(x)}{\Delta x} \right] \\
&= \lim_{\Delta x \to 0} \frac{u(x + \Delta x) - u(x)}{\Delta x} + \lim_{\Delta x \to 0} \frac{g(x + \Delta x) - g(x)}{\Delta x} \\
&= u'(x) + g'(x)
\end{aligned}$$

$\square$

Theorem 1.5 (Derivative of the multiplication)**.**

$$f(x) = u(x)g(x)$$
$$f'(x) = u'(x)g(x) + u(x)g'(x)$$

Proof.

$$\begin{aligned}
f(x) &= u(x)g(x) \\
f'(x) &= \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} \\
&= \lim_{\Delta x \to 0} \frac{u(x + \Delta x)g(x + \Delta x) - u(x)g(x)}{\Delta x} \\
&= \lim_{\Delta x \to 0} \frac{u(x + \Delta x)g(x + \Delta x) + [u(x + \Delta x) - u(x)]g(x) - u(x + \Delta x)g(x)}{\Delta x} \\
&= \lim_{\Delta x \to 0} \frac{[u(x + \Delta x) - u(x)]g(x) + u(x + \Delta x)[g(x + \Delta x) - g(x)]}{\Delta x} \\
&= \lim_{\Delta x \to 0} \frac{[u(x + \Delta x) - u(x)]g(x)}{\Delta x} + \lim_{\Delta x \to 0} \frac{u(x + \Delta x)[g(x + \Delta x) - g(x)]}{\Delta x} \\
&= \lim_{\Delta x \to 0} \frac{[u(x + \Delta x) - u(x)]g(x)}{\Delta x} + \lim_{\Delta x \to 0} \frac{u(x)[g(x + \Delta x) - g(x)]}{\Delta x} \\
&= u'(x)g(x) + u(x)g'(x)
\end{aligned}$$

$\square$

Theorem 1.6.

$$f(x) = \frac{u(x)}{g(x)}$$
$$f'(x) = \frac{u'(x)g(x) - u(x)g'(x)}{g(x)^2}$$

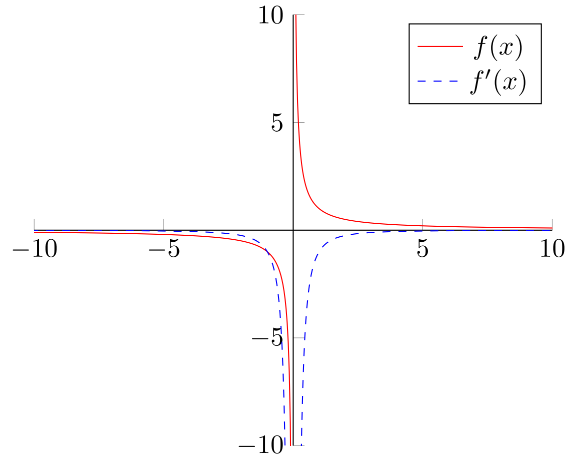
Proof.

$$\begin{aligned}
f(x) &= u(x)g(x) \\
f'(x) &= \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x} \\
&= \lim_{\Delta x \to 0} \frac{\frac{u(x+\Delta x)}{g(x+\Delta x)} - \frac{u(x)}{g(x)}}{\Delta x} \\
&= \lim_{\Delta x \to 0} \frac{\frac{u(x+\Delta x)*g(x)}{g(x+\Delta x)*g(x)} - \frac{u(x)*g(x+\Delta x)}{g(x)*g(x+\Delta x)}}{\Delta x} \\
&= \lim_{\Delta x \to 0} \frac{\frac{u(x+\Delta x)*g(x) - u(x)*g(x+\Delta x)}{g(x)*g(x+\Delta x)}}{\Delta x} \\
&= \lim_{\Delta x \to 0} \left[\frac{\frac{u(x+\Delta x)*g(x) - u(x)*g(x+\Delta x)}{g(x)*g(x+\Delta x)}}{\Delta x} \right] \\
&= \lim_{\Delta x \to 0} \left[\frac{1}{g(x)*g(x+\Delta x)} * \frac{u(x+\Delta x)*g(x) - u(x)*g(x+\Delta x)}{\Delta x} \right] \\
&= \lim_{\Delta x \to 0} \left[\frac{1}{g(x)*g(x+\Delta x)} * \frac{g(x)[u(x+\Delta x) - u(x)] - u(x)[g(x+\Delta x) - g(x)]}{\Delta x} \right] \\
&= \lim_{\Delta x \to 0} \left[\frac{1}{g(x)*g(x+\Delta x)} * \left(\frac{g(x)[u(x+\Delta x) - u(x)]}{\Delta x} - \frac{u(x)[g(x+\Delta x) - g(x)]}{\Delta x} \right) \right] \\
&= \frac{1}{g(x)*g(x)} * [g(x)u'(x) - u(x)g'(x)] \\
&= \frac{g(x)u'(x) - u(x)g'(x)}{g(x)^2} \\
&= \frac{u'(x)g(x) - u(x)g'(x)}{g(x)^2}
\end{aligned}$$

$\square$

Theorem 1.7 (Derivative of $1/x$)**.**

$$f(x) = \frac{1}{x}$$
$$f'(x) = \frac{-1}{x^2}$$



Proof.

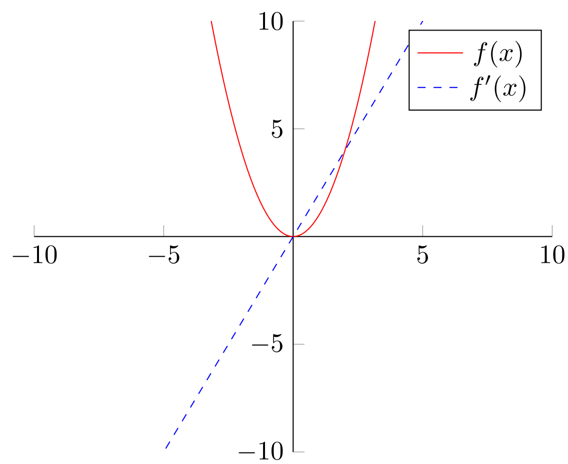
$$
\begin{aligned}
f(x) &= \frac{1}{x} \\
f'(x) &= \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x} \\
&= \lim_{\Delta x \to 0} \frac{\frac{1}{x+\Delta x} - \frac{1}{x}}{\Delta x} \\
&= \lim_{\Delta x \to 0} \frac{\frac{1*x}{(x+\Delta x)*x} - \frac{1*(x+\Delta x)}{x*(x+\Delta x)}}{\Delta x} \\
&= \lim_{\Delta x \to 0} \frac{\frac{(1*x)-[1*(x+\Delta x)]}{(x+\Delta x)*x}}{\Delta x} \\
&= \lim_{\Delta x \to 0} \frac{\frac{x-(x+\Delta x)}{(x+\Delta x)*x}}{\Delta x} \\
&= \lim_{\Delta x \to 0} \frac{\frac{x-x-\Delta x}{(x+\Delta x)*x}}{\Delta x} \\
&= \lim_{\Delta x \to 0} \frac{\frac{-\Delta x}{(x+\Delta x)*x}}{\Delta x} \\
&= \lim_{\Delta x \to 0} \left[\frac{1}{\Delta x} * \frac{-\Delta x}{(x+\Delta x)*x}\right] \\
&= \lim_{\Delta x \to 0} \frac{-1}{(x+\Delta x)*x} \\
&= \frac{-1}{(x+0)*x} \\
&= \frac{-1}{x*x} \\
&= -\frac{1}{x^2}
\end{aligned}
$$

□

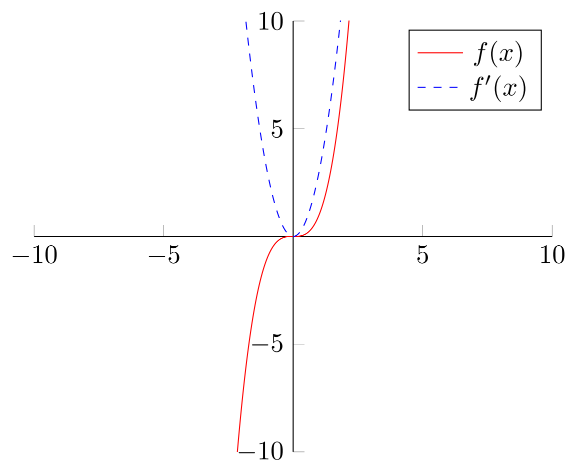
Theorem 1.8 (Derivative of x^n)**.**

$$f(x) = x^n$$
$$f'(x) = nx^{n-1}$$

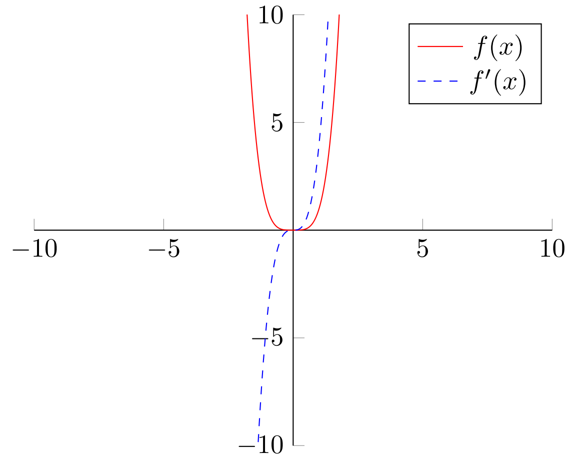
$f(x) = x^2$ and $f'(x) = 2x$



$f(x) = x^3$ and $f'(x) = 3x^2$



$f(x) = x^4$ and $f'(x) = 4x^3$



Proof.

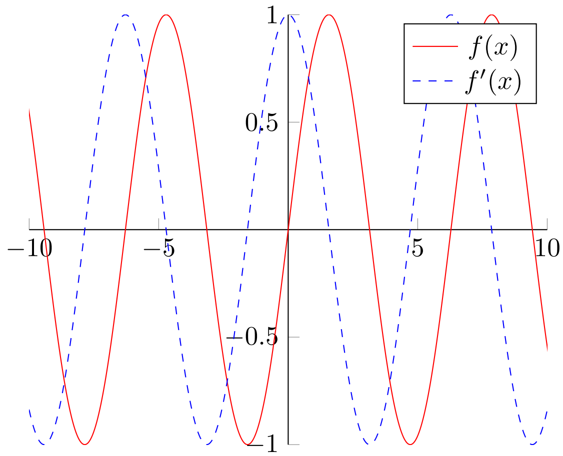
$$
\begin{aligned}
f(x) &= x^n \\
f'(x) &= \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x} \\
&= \lim_{\Delta x \to 0} \frac{(x+\Delta x)^n - x^n}{\Delta x} \\
&= \lim_{\Delta x \to 0} \frac{[x^n + nx^{n-1}\Delta x + \mathcal{O}((\Delta x)^2)] - x^n}{\Delta x} \\
&= \lim_{\Delta x \to 0} \frac{[nx^{n-1}\Delta x + \mathcal{O}((\Delta x)^2)]}{\Delta x} \\
&= \lim_{\Delta x \to 0} \frac{\Delta x[nx^{n-1} + \mathcal{O}(\Delta x)]}{\Delta x} \\
&= \lim_{\Delta x \to 0} nx^{n-1} + \mathcal{O}(\Delta x) \\
&= nx^{n-1}
\end{aligned}
$$

□

Theorem 1.9 (Derivative of $sin(x)$)**.**

$$
\begin{aligned}
f(x) &= sin(x) \\
f'(x) &= cos(x)
\end{aligned}
$$

$f(x) = sin(x)$ and $f'(x) = cos(x)$



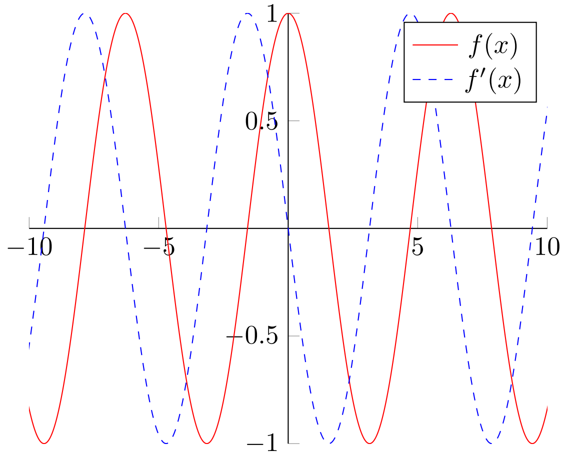
Proof.

$$
\begin{aligned}
f(x) &= sin(x) \\
f'(x) &= \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x} \\
&= \lim_{\Delta x \to 0} \frac{sin(x+\Delta x) - sin(x)}{\Delta x} \\
&= \lim_{\Delta x \to 0} \frac{sin(x)cos(\Delta x) + cos(x)sin(\Delta x) - sin(x)}{\Delta x} \\
&= \lim_{\Delta x \to 0} \frac{sin(x)(cos(\Delta x) - 1) + cos(x)sin(\Delta x)}{\Delta x} \\
&= \lim_{\Delta x \to 0} \left[\frac{sin(x)(cos(\Delta x) - 1)}{\Delta x} + \frac{cos(x)sin(\Delta x)}{\Delta x} \right] \\
&= \lim_{\Delta x \to 0} \left[sin(x) * \frac{cos(\Delta x) - 1}{\Delta x} + cos(x) * \frac{sin(\Delta x)}{\Delta x} \right] \\
&= \lim_{\Delta x \to 0} \left[sin(x) * \frac{cos(\Delta x) - 1}{\Delta x} \right] + \lim_{\Delta x \to 0} \left[cos(x) * \frac{sin(\Delta x)}{\Delta x} \right] \\
&= \lim_{\Delta x \to 0} [sin(x) * 0] + \lim_{\Delta x \to 0} [cos(x) * 1] && \text{see 1.2, 1.1} \\
&= \lim_{\Delta x \to 0} cos(x) \\
&= cos(x)
\end{aligned}
$$

□

Theorem 1.10 (Derivative of $cos(x)$)**.**

$$
\begin{aligned}
f(x) &= cos(x) \\
f'(x) &= -sin(x)
\end{aligned}
$$



Proof.

$$f(x) = sin(x)$$
$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$$
$$f'(x) = \lim_{\Delta x \to 0} \frac{cos(x+\Delta x) - cos(x)}{\Delta x}$$
$$f'(x) = \lim_{\Delta x \to 0} \frac{cos(x)cos(\Delta x) - sin(x)sin(\Delta x) - cos(x)}{\Delta x}$$
$$f'(x) = \lim_{\Delta x \to 0} \frac{cos(x)(cos(\Delta x) - 1) - sin(x)sin(\Delta x)}{\Delta x}$$
$$f'(x) = \lim_{\Delta x \to 0} \left[\frac{cos(x)(cos(\Delta x) - 1)}{\Delta x} - \frac{sin(x)sin(\Delta x)}{\Delta x}\right]$$
$$f'(x) = \lim_{\Delta x \to 0} \frac{cos(x)(cos(\Delta x) - 1)}{\Delta x} - \lim_{\Delta x \to 0} \frac{sin(x)sin(\Delta x)}{\Delta x}$$
$$f'(x) = \lim_{\Delta x \to 0} \left[cos(x) * \frac{(cos(\Delta x) - 1)}{\Delta x}\right] - \lim_{\Delta x \to 0} \left[sin(x) * \frac{sin(\Delta x)}{\Delta x}\right]$$
$$f'(x) = \lim_{\Delta x \to 0} [cos(x) * 0] - \lim_{\Delta x \to 0} [sin(x) * 1] \qquad \text{see 1.2, 1.1}$$
$$f'(x) = -\lim_{\Delta x \to 0} sin(x)$$
$$f'(x) = -sin(x)$$

$\square$