Relativity in Global Positioning System

Project Report

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Abstract

The Global Positioning System (GPS) remains one of the modern technology that we make use of everywhere. In this paper, we review the physics behind GPS, reviewing the essence of both special and general theory of relativity in the system.

1 Introduction

A navigation system is a computing system that aids in navigation. The navigation can be on the vessel or vehicle the system is controlling or be located elsewhere using transmission signals to control the vessel or vehicle. A navigation system is capable of determining geographic position through sensors, maps or other sources. This process of estimating the location of an object is called geopositioning. A satellite navigation system uses satellites to provide geopositioning. Navigation devices use time signals transmitted along line of sight by radio from satellites and give latitude, longitude, and altitude to the user. Such systems with global coverage are called global navigation satellite system (GNSS). As of present, four global systems are operational: Global Positioning System (United States), Global Navigation Satellite System (Russia), BeiDou (China), and Galileo (European Union). In this paper, we look at the Global Positioning System, famously known as GPS, and see how relativistic effects are cruicial in its functioning.

2 The Global Positioning System (GPS)

Global Positioning System (GPS) is one of the four global navigation satellite system launched by the United States in 1973. It is operated by the United States Space Force and provides positioning, navigation, and timing (PNT) services. It has three segments: Space Segment, Control Segment and User Segment. The *Space Segment* is a constellation of 24 operating satellites, which give signals to give current GPS satellite position and time. The *Control Segment* is a worldwide monitor with control stations. It is used to maintain the satellites in proper orbits as well as adjust satellite clocks. The *User Segment* receives the signal from the GPS satellites and calculates the user's three dimensional position and time.

2.1 Measuring the Position

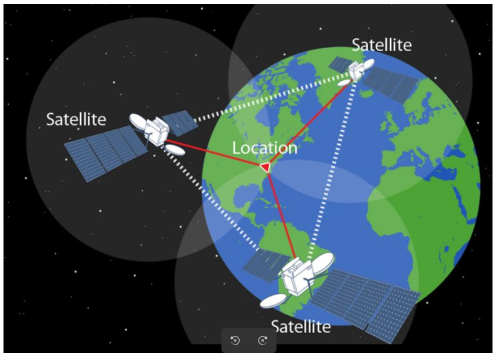

Figure 1: A schematic of geopositioning using triangulation [4]

The satellites are around 12500 miles above the earth's surface, with a 12 hours orbital period. They transmit radio signals in two frequencies $L1 = 1575.42$ MHz and $L2 = 1227.60$ MHz. As the satellites transmit radio signals, how are we able to locate ourselves? The basic idea lies on triangulation. Say a person is at some location on the earth and receives a signal from the satellite. Then, you can draw a circle about that radius with the satellite being the center. The person can be anywhere in the circumference. Now, take signal from another satellite and make similar analogy. The intersection of the two circles should give you two points which can be the person's location. You add one more circle, you'll get the exact position at the intersection point of the three circles. And, this is how GPS measures your position. A fourth satellite is needed for accurate time measurement.

3 Accuracy and Errors in GPS

Each GPS satellite has highly precise atomic clocks such that there is transmission of signals at a precise time. At the receiver end are also atomic clocks, that measure the three spatial coordinates of position. Say we have four satellites, then we have four equations in four unknowns: three spatial coordinates and one time reference. As they transmit radio signals, how accurate they should be? Radio signals travel around 9 m in 30 ns (nanoseconds), which means clock times need to be accurate to 30 ns if we require an accuracy of around 10 m. An error of 0.001 s turns out to be huge in measurements, as it gives around 186 miles positional error.

Several factors affect the accuracy in GPS. Due to refraction from the ionosphere, it might be necessary to set a masking angle for measurements. This leads to errors by atmospheric interference. One also needs to account for the same signal coming from multiple paths, such as reflection from buildings. The relative coordinates of the receiver and satellite is given by a mathematical function called Dilution of Precision and a typical value is 2 for receiver processing signals from 4 satellites. If this dilution increases, i.e., satellites are dense in the sky, the measurements might be less accurate. So, the receiving devices should be embedded with these information to calculate the location accurately. More importantly, there are errors in clock times. As satellite orbits, the clock can not maintain an accuracy of 30 ns for more than a few days. To account for this ground station updates the information broadcast by each satellite once a day. So, to not have offset in position measurements, you need clock accuracy, and for that you need to take relativistic effects into account.

4 Essence of Relativity in GPS

As the satellites are orbiting around the earth, relativity will apply to it, since a moving clock runs slower (time dilation). Moreover, in a potential well, as you go further, the clock runs faster. This is an effect from general relativity and is called gravitational redshift. The clocks are thus synchronized globally only after making the relativistic corrections, which are discussed in detail below.

4.1 Special Relativity

Take s to be the proper time measured by the moving clock, t the time in the reference frame of stationary clock, v the clock velocity, and c the speed of light, then using special relativity we can write the relation,

$$\delta s = \left(1 - \frac{v^2}{c^2}\right)^{1/2} \delta t \tag{1}$$

Taking a second order approximation, we get a result for second order Doppler shift,

$$\delta s \approx \left(1 - \frac{v^2}{2c^2}\right) \delta t \tag{2}$$

However, the above results are valid only for non-rotating frame of reference. For rotating frame of reference, the second order Doppler shift causes time transfer to be non-integrable. It means that if a clock is transferred around a close loop, its reading at the end of the transfer lags that of stationary clock kept at the end points of the loop. This is called *Sagnac Effect* [1] and needs to be taken into account since the Earth is rotating. If ω is the angular velocity of frame rotation and A the surface area of the loop (projected into a place perpendicular to the axis of rotation), then the transfer lag is by the amount

$$\delta s \approx \frac{2\omega A}{c^2} \tag{3}$$

4.2 General Relativity

In a gravitational potential well, the closer the clock is to the source of gravitation, slower the time passes. The Gravitational time dilation equation is given as,

$$\frac{t}{t_o} = \left(1 - \frac{2GM}{rc^2}\right)^{1/2} \tag{4}$$

where t is the time at a distance r from the center of the earth and t_o is the time for a far away observer. Then, for $GM/rc^2 < 1$, and using the gravitational potential $\phi = -GM/r$, we can approximate,

$$\frac{t}{t_o} = 1 + \frac{\phi}{c^2} \tag{5}$$

As the satellites are at a higher gravitational potential than the receiver at the earth's surface, the clock ticks faster in the satellites. Using the above, the fraction by which the satellites' clock tick faster than stationary clock is,

$$\frac{\delta t}{t} \approx \frac{\phi_o}{c^2} - \frac{\phi_E}{c^2} \tag{6}$$

Here, ϕ_o is the gravitational potential at the satellites' position and ϕ_E at the earth's surface.

4.3 Calculations

How do these effects from relativity add up when correcting the clock times? As modern clock are accurate to few ns/day, all these effects should be taken into account. The GPS satellites move around 3874 m/s relative to Earth's center, then the fraction by which satellites' clock ticks slower than the stationary clocks is given as $\delta t = v^2/2c^2 \approx 8.349 \times 10^{-11}$. Then, in a day, the satellite clocks tick $\delta t \times 86400 \times 10^9 \approx 7214$ ns slower. The Sagnac effect amounts to 207 ns of clock transfer around the equator of the rotating earth. We now calculate the general relativistic effects. Take radius of earth $r_E = 6.4 \times 10^6$ m, orbit radius $r_o = 2.6 \times 10^7$ m and earth's mass $M = 5.974 \times 10^{24}$. Then, using equation 6, we see that the satellite clocks tick $5.304 \times 10^{-10} \times 86400 \times 10^9 \approx 45850$ ns faster. These effects are added together, which gives $\approx$ 38640 ns gain for satellite clocks per day. Hence, to compensate for this gain, the GPS clock's frequency should be slowed down by the fraction $5.304 \times 10^{-10} - 8.349 \times 10^{-11} \approx 4.469 \times 10^{-10}$. If the pre-adjusted clock frequency is 10.23 MHz, then the frequency should be slowed down to

$(1 - 4.469 \times 10^{-10}) \times 10.23 = 10.22999999543$ MHz [2].

5 Conclusion

In this paper, we see why relativistic effects are necessary to measure the clock times in GPS precisely and accurately. Due to time dilation in special relativity, Sagnac effect in rotating frame, and gravitational time dilation in a potential well that derives from general relativity, we have discussed the corrections that need to be made for the clock frequency in GPS. The clock frequency should be slowed down from 10.23 Hz to 10.22999999543 Hz to take the relativistic corrections into account. However, besides this, the errors due to ephemeris, multipath signals, and atmospheric interference also need to corrected for more accurate measurements.

References

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[2] *Error analysis for the Global Positioning System*, Wikipedia

[3] Ashby N., *Relativity in the Global Positioning System*, University of Colorado Boulder, 2003.

[4] *Triangulation*, National Geographic Society, 2023.