Gameplan: Ascending/Descending $\to N = A \cap D/\mathbb{R}$ is manifold $\to$ orientation on N $\to$ Morse Homology

§1 Morse Functions

Def. 1.1 (Critical Point)**.** Let M be a manifold and $f : M \to \mathbb{R}$ be a smooth function. A critical point of f is a point $p \in M$, such that $df(p) = 0$.

Remark. If M is compact, f always has critical points.

Def. 1.2 (Poisson Bracket)**.** Let X and Y be vector fields on a manifold M. Then for a point $p \in M$ define the Poisson Bracket by

$$[X, Y]f := XYf - YXf$$

The Poisson Bracket is a Lie Bracket on the $\mathbb{R}$-vectorspace of vector fields on M. Also, if p is a critical point of a function f, then $[X, Y]f(p) = 0$.

Def. 1.3 (Hessian)**.** In $\mathbb{R}^n$, we are used to defining the Hessian of a function, that is the second order derivative. On a manifold, this is not so easy, since the second order derivative will always depend on the local coordinates. We will have to do with defining the second order derivative just on the critical points of a function: For $p \in M$ a critical point of a function f, $x, y \in T_p(M)$, choose X, Y vectorfields extending x and y locally, i.e. with $Y(p) = y$ and $X(p) = x$. Then define

$$\mathrm{d}^2 f(x, y)(p) = XYf(p)$$

Lemma 1.4. *$d^2 f(\cdot, \cdot)(p)$ is a symmetrical bilinear form if p is a critical point of f.*

Proof. Note that
$$d^2 f(x, y)(p) - d^2 f(y, x)(p) = [X, Y]f(p) = 0$$

so $XYf(p)$ is symmetrical at any critical point and

$$XYf(p) = X(p)Yf(p) = xYf(p)$$

for any choice of X and Y, so $d^2 f(\cdot, \cdot)(p)$ does not depend on X, and by the same argument because it is symmetrical it does not depend on Y. Then $d^2 f(x, y)(p)$ is a well defined, symmetrical bilinear form. □

Def. 1.5 (non-degeneracy, index)**.** We call a critical point of a function $f : M \to \mathbb{R}$ *non-degenerate*, if the bilinear form $d^f(\cdot, \cdot)(p)$ is non-degenerate.
We define the index of p as the maximal dimension of negative definite subspaces, on which

$d^2 f(\cdot,\cdot)(p)$ is negative-definite.
Note that with local coordinates $(x_1, ..., x_n)$ around p, we get an induced Basis of the tangent space $T_p(M)$ as $B = (\frac{\partial}{\partial x^1}, ..., \frac{\partial}{\partial x^n})$, and then if Φ_B is the coordinate isomorphism, we get a Matrix A, such that

$$d^2 f(x,y)(p) = \varphi_B(x)^T A \varphi_B(y)$$

A can be given by

$$A = \left(\frac{\partial^2 f}{\partial x_i \partial x_j}(p) \right)_{1 \leq i,j \leq n}$$

Then the index of p is the number of ngative eingenvalues of A.

Def. 1.6. A smooth function $f : M \to \mathbb{R}$ is called a *Morse Function*, if all its critical points are non-degenerate.

Lemma 1.7. *Let $M \subseteq \mathbb{R}^n$ be a submanifold. Then for almost every point $p/in\mathbb{R}^n$, the function*

$$\begin{aligned} f : M &\to \mathbb{R} \\ q &\mapsto ||q-p||^2 \end{aligned}$$

is a Morse Function.

Proof. □

Remark. Note that by Whitney's embedding theorem, there exist Morse-Functions on any Manifold.

Theorem 1.8. *Any smooth function $f : M \to \mathbb{R}$ and all its derivatives can be uniformly approximated by a Morse Function.*

Proof. □

Theorem 1.9. *Let M be a compact manifold. Then the set of Morse functions is dense in $\mathcal{C}^\infty(M)$.*

Proof. □

Theorem 1.10 (Morse Lemma)**.** *Let $f : M \to \mathbb{R}$ be a smooth function. Let p be a non-degenerate critical point of index k of f. Then there exist local coordinates $(x_1, ..., x_n)$ in a neighburhood U of p, such that*

$$f = f(p) - x_1^2 - ... - x_k^2 + x_{k+1}^2 + ... + x_n^2$$

and

$$x_1(p) = ... = x_n(p) = 0$$

U is called Morse chart.

Proof. □

Corollary 1.11. *Non-degenerate critical points are isolated.*

Remark. A critical point of Index k of f is a critical point of index $n-k$ of $-f$.

§2 Pseudo-Gradients

Def. 2.1 (Riemannian Metric, Gradient)**.** A *Riemannian metric* g on a manifold M is a choice of scalar products $g_p : T_pM \times T_pM \to \mathbb{R}$ for every point $p \in M$, such that for any vectorfields X and Y, the map

$$p \mapsto g_p(X(p), Y(p))$$

is smooth. For $x, y \in T_pM$, we write

$$\langle x, y \rangle := g_p(x, y)$$

and

$$||x|| = \sqrt{g_p(x, x)}$$

If $f : M \to \mathbb{R}$ is a smooth map, then the gradient of f is a vectorfield ∇f, such that for any vectorfield X the identity

$$\langle X, \nabla f \rangle = \mathrm{d}fX$$

holds.

Def. 2.2 (Pseudo-Gradient)**.** Let $f : M \to \mathbb{R}$ be a smooth function. A pseudo-gradient of f is a vectorfield X on M, such that

1. $\mathrm{d}f(p)X(p) \geq 0$, where equality holds if and only if p is a critical point of f,

2. in a Morse chart neighborhood of a critical point of f the vectorfield X coincides with $-\nabla f$.

Lemma 2.3 (Existence of Pseudo Gradients)**.** *For any smooth function $f : M \to \mathbb{R}$ there exists a pseudo-gradient of f.*

Proof. This follows directly from the fact that on any manifold there exists a Riemannian metric. □

Def. 2.4 (Stable and unstable Manifolds)**.** Let p be a critical point of a smooth function $f : M \to \mathbb{R}$. Denote by φ_s the flow of a pseudo-gradient of f. Then We define the *stable manifold* to be

$$W^s(p) = \left\{ q \in M : \lim_{s \to \infty} \varphi_s = p \right\}$$

and the *unstable manifold*

$$W^u(p) = \left\{ q \in M : \lim_{s \to -\infty} \varphi_s = p \right\}$$

Proposition 2.5. *Let p be a critical point of index k of a smooth function $f : M \to \mathbb{R}$. Then $W^s(p)$ is diffeomoric to the open disk of dimension k and $W^u(p)$ is diffeomorphic to the open disc of dimension $n - k$.*

Proof. □

Proposition 2.6. *Assume that M is a compact manifold Let X be a pseudo-gradient vectorfield of some smooth function $f : M \to \mathbb{R}$ and γ be a trajectory of X. Then there exist critical points p and q of f, such that*

$$\lim_{t \to \infty} \gamma(t) = p \textit{ and } \lim_{t \to -\infty} \gamma(t) = q$$

Proof. □

Theorem 2.7 (Fist Deformation Lemma)**.**

Proof. □

Theorem 2.8 (Second Deformation Lemma)**.**

Proof. □

Def. 2.9 (Transversality)**.** Let $U, V \subseteq M$ be submanifolds. Then U and V are said to meet *transversly*, if for all $p \in U \cap V$ we have

$$T_pU \oplus T_pV = T_pM$$

If U and V meet transversly, we write

$$U \pitchfork V$$

A vectorfield X on M is *transversal* to a submanifold $U \in M$ of dimension $n - 1$, if for all $p \in U$ we have

$$X(p) \notin T_p(U)$$

Note that this is similar to the first definition, because then we have

$$\langle X(p)\rangle \oplus T_pU = T_pM$$

for all $p \in U$. We write

$$X \pitchfork U$$

Def. 2.10 (Smale Condition)**.** A pseudo gradient vectorfield is said to satisfy the *Smale condition*, if all its stable and unstable manifolds meet transversly, i.e if for all critical points p and q of f, we have

$$W^u(p) \pitchfork W^s(q)$$

§3 The Morse Complex

§4 Morse Homology