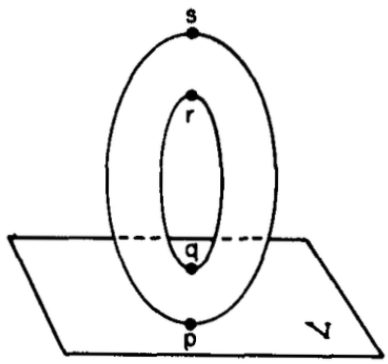
§1 Introduction

Let us consider a Torus M tangent to a plane V:



Let $f : M \to \mathbb{R}$ be the distance of a point to the plane V. For a Number $a \in \mathbb{R}$, let M^a be the set of all points $p \in M$, s.th. $f(p) \leq a$. Then the following are true:

(1) If $a < 0 < f(p)$, then $M^a = \varnothing$

(2) If $f(p) < a < f(q)$, then M^a is homeomorphic to a 2-cell.

(3) If $f(q) < a < f(r)$, then M^a is homeomorphic tp a cylinder.

(4) If $f(q) < a < f(r)$, then M^a is homeomorphic to a compact manifold of genus one with a circle as a boundary.

(5) If $f(s) < a$, then $M^a = M$.

To describe how M^a changes as it passes through the points $f(p)$, $f(q)$, $f(r)$, $f(s)$ it is conveniant to consider homotopy type rather than homeomorphism type.

(1) $\to$ (2) : In case (1), M^a has the same homotopy type as a point, so this step is the attaching of a 0-cell.

(2) $\to$ (3) : Is the operation of attaching a 1-cell.

(3) $\to$ (4) : Again is the operation of attaching a 1-cell.

(4) $\to$ (5) : Is the operation of attaching a 2-cell.

The definition of attaching a k-cell can be given as follows:

Let $S^k = \{x \in \mathbb{R}^{k+1} : \|x\| = 1\}$ be the k-sphere

and $D^k = \{x \in \mathbb{R}^k : \|x\| \leq 1\}$ be the k-disk.

Let M and N be manifolds, then N is created from M by atteching a k-cell, if N is of the same homotopy type as a topological space X s.th. there exists a pushout square in Top

$$\begin{tikzcd} S^{k-1} \arrow[r] \arrow[d] & M \arrow[d] \\ D^k \arrow[r] \arrow[ur, phantom, very near end, "\ulcorner"] & X \end{tikzcd}$$

A pushout square in a category $\mathcal{C}$ is a commutative square

$$\begin{tikzcd} A \arrow[r, "f_0"] \arrow[d, "f_1"] & B \arrow[d, "g_0"] \\ C \arrow[r, "g_0"] & D \end{tikzcd}$$

s.th if there is another commutative diagram

$$\begin{tikzcd} A \arrow[r, "f_0"] \arrow[d, "f_1"] & B \arrow[d, "h_0"] \\ C \arrow[r, "h_1"] & \tilde{D} \end{tikzcd}$$

Then

$$\begin{tikzcd} A \arrow[r, "f_0"] \arrow[d, "f_1"] & B \arrow[d, "g_0"] \arrow[ddr, bend left, "h_0"] & \\ C \arrow[r, "g_1"] \arrow[drr, bend right, "h_1"'] & D & \\ & & \tilde{D} \arrow[ul, dashed, "\exists! h"'] \end{tikzcd}$$

which commutes everywhere.

§2 Definitions and Lemmas

Def. 2.1 (critical Point, non-degenerate critical Point)**.** Let M be a (smooth) manifold and $f : M \to \mathbb{R}$ be a smooth function. Then $p \in M$ is called a *critical point*, if the tangent map $f_* : TM_p \to R$ is not zero.

A critical point is called *non-degenerate*, if for some local coordinates $\varphi = (x_1, ..., x_n)$ the matrix

$$H_p^\varphi f := \left(\frac{\partial^2 f}{\partial x_i \partial x_j}(p)\right)_{1 \leq i,j \leq n}$$

is non-singular, i.e. invertable.

$H_p^\varphi f$ is called the Hessian of f at p (wrt. φ).

Lemma 2.2 (Congruency of Hessians)**.** *Let M be a manifold, $f : M \to \mathbb{R}$, p a critical point of f and $\varphi := (x_1, ..., x_n)$ and $\psi := (y_1, ..., y_n)$ local coordinates aroud p. Let*

$$D_p = \left(\frac{\partial x_i}{\partial y_j}(p)\right)_{1 \leq i,j \leq n}$$

. Then

$$H_p^\psi f = D_p^T H_p^\varphi f D_p$$

Proof. Let M be a manifold, $f : M \to \mathbb{R}$ a smooth function. Let $p \in M$ be a critical point of f and $\varphi = (x_1, ..., x_n)$, $\psi = (y_1, ..., y_n)$ local coordinates in a nbhd. around p. From functoriality of the tangent space, we know that

$$\frac{\partial f}{\partial y_k} = \sum_{i=1}^n \frac{\partial f}{\partial x_i} \cdot \frac{\partial x_i}{\partial y_k} t$$

so

$$\frac{\partial^2 f}{\partial y_k \partial y_l}(p) = \frac{\partial}{\partial y_k}\left(\frac{\partial f}{\partial y_l}\right)(p) = \frac{\partial}{\partial y_k}\left(\sum_{i=1}^n \frac{\partial f}{\partial x_i} \cdot \frac{\partial x_i}{\partial y_l}\right)(p)$$

$$= \sum_{i=1}^n \frac{\partial}{\partial y_k}\left(\frac{\partial f}{\partial x_i}\right)(p) \cdot \frac{\partial x_i}{\partial y_l}(p) + \sum_{i=1}^n \frac{\partial f}{\partial x_i}(p) \cdot \frac{\partial}{\partial y_k}\left(\frac{\partial x_i}{\partial y_l}\right)(p)$$

$$= \sum_{i,j=1}^n \frac{\partial^2 f}{\partial x_i \partial x_j}(p) \cdot \frac{\partial x_j}{\partial y_k}(p) \cdot \frac{\partial x_i}{\partial y_l}(p) + \sum_{i=1}^n \frac{\partial f}{\partial x_i}(p) \cdot \frac{\partial^2 x_i}{\partial y_k \partial y_l}(p)$$

Because p is a critical point, $\frac{\partial f}{\partial x_i}(p) = 0$ for all i, so then

$$\left(H_p^{\psi} f\right)_{k,l} = \sum_{i,j=1}^{n} \frac{\partial^2 f}{\partial x_i \partial x_j}(p) \cdot \frac{\partial x_j}{\partial y_k}(p) \cdot \frac{\partial x_i}{\partial y_l}(p)$$

$$= \left(D_p^T \cdot H_p^{\varphi} f \cdot D_p\right)_{k,l}$$

□

Lemma 2.3 (Invariance of non-degeneracy)**.** *Non-degeneracy does not depend on the chosen local coordinates.*

Proof. Let $f : M \to \mathbb{R}$ be smooth, p a critical point of f and $\varphi = (x_1, ..., x_n)$, $\psi = (y_1, ..., y_n)$ local coordinates around p. Assume that p is non-non degenerate wrt. φ Note that D_p from lemma 2.2 is invertable. Then

$$\det(H_p^{\psi} f) = \det(D_p^T \cdot H_p^{\varphi} \cdot D_p) = \det(D^T) \cdot \det(H_p^{\varphi}) \cdot \det(D_p) \neq 0$$

□

Def. 2.4 (Index)**.** The *index* of a Matrix A is the number of (not necessarily destinct) negative Eigenvalues of A and is denoted by $\operatorname{Index}(A)$.
The *index* of a critical point p of a function $f : M \to \mathbb{R}$ (wrt. a chart φ) is the index of the matrix $H_p^{\varphi} f$.

Lemma 2.5 (Invariance of the Index)**.** *The index of a critical point does not depend on the chosen local coordinates.*

Proof. Let $f : M \to \mathbb{R}$ be a smooth function, p a critical point of f and φ, ψ two charts around p. As seen in lemma 2.2, $H_p^{\varphi} f = D^T \cdot H_p^{\psi} f \cdot D$, i.e. H_p^{φ} is congruent to H_p^{ψ}, because D is invertable. Then by Sylvester's Law,

$$\operatorname{Index}(H_p^{\varphi} f) = \operatorname{Index}(H_p^{\psi} f)$$

□

Theorem 2.6 (Morse's Lemma)**.** *Let M be a manifold, $f : M \to \mathbb{R}$ smooth and p a non-degenerate critical point of f of index k. Then there exist local coordinates $\varphi = (x_1, ..., x_n)$, s.th,*

$$f = f(p) - x_1^2 - ... - x_k^2 + x_{k+1}^2 + ... + x_n^2$$

Proof. **TODO** This is the proof of Morse's Lemma <3 . It's true i swear ☐

§3 The first Deformation Lemma

Def. 3.1 (1-parameter group of diffeomorphisms)**.** A *1-parameter group of diffeomorphisms* of a manifold M is a smooth map

$$\varphi : \mathbb{R} \times M \to M \ ; \ (t, p) \mapsto \varphi_t(p)$$

where φ_t is a diffeomorphism on M, and s.th.

$$\varphi_{t+s} = \varphi_t \circ \varphi_s$$

A vector field X is said to generate a 1-parameter group of diffeomorphisms φ, if for every smooth real valued function the identity

$$df X(p) = \lim_{h \to 0} \frac{f(\varphi_h(p)) - f(p)}{h}$$

holds for all points p in M.

Lemma 3.2 (compactly supported vectorfields generate 1-parameter groups)**.** *A vector field whose support lies in a compact set $K \subseteq M$ generates a unique 1-parameter group of diffeomorphisms on M.*

Proof. Define for a path $c : \mathbb{R} \to M$ the velocity vector

$$\frac{dc}{dt} \in T_{c(t)}M \text{ as } \frac{dc}{dt}(f) = \lim_{h \to 0} \frac{f(c(h+t)) - f(c(t))}{h}$$

Let Now X be a vector field whose support lies in a compact set $K \in M$. Assume there exists a 1-parameter group of diffeomorphisms φ_t that is generated by X. Then the path $t \mapsto \varphi_t(p)$ for some fixed $p \in M$ satisfies the differential equation

$$\frac{d\varphi_t(p)}{dt} = X(\varphi_t(p))$$

with initial condition $\varphi_0(p) = p$. This is true by definition:

$$\frac{d\varphi_t(p)}{dt}(f) = \lim_{h \to 0} \frac{f(\varphi_{h+t}(p)) - f(\varphi_t(p))}{h} = \lim_{h \to 0} \frac{f(\varphi_h(\varphi_t(p))) - f(\varphi_t(p))}{h} = X(\varphi_t(p))(f)$$

So to prove the lemma, one needs to show that such a map exists for all p that depends soomthly on p.

Let $\psi = (x_1, ..., x_n)$ be local coordinates of some open neighborhood U of some point p. Then in local coordinates we get

$$X = X_1 \cdot \frac{\partial}{\partial x_1} + ... + X_n \cdot \frac{\partial}{\partial x_n}$$

and

$$\frac{d\varphi_t(p)}{dt} = \frac{d\varphi_t^1(p)}{dt} \cdot \frac{\partial}{\partial x_1} + ... + \frac{d\varphi_t^n(p)}{dt} \cdot \frac{\partial}{\partial x_n}$$

, so for $u_i = \varphi_t^i(p)$ we get the differential equation

$$\frac{du_i}{dt} = X_i(u_1, ..., u_n)$$

Because X is compactly supported, X_i is bounded, so with Picard-Lindelöf there exists $\varepsilon_i > 0$ s.th. the differential equation has a unique smooth solutionon on the interval $[-\varepsilon_i, \varepsilon_i]$. This is true in every dimension.

So for each point on M, there exists a neighborhood U and a number $\varepsilon > 0$, such that the differential equation

$$\frac{d\varphi_t(p)}{dt} = X(\varphi_t(p))$$

with initial condition

$$\varphi_0(p) = p$$

has a unique solution for $t \in [-\varepsilon, \varepsilon]$ for all $p \in U$, which (apperently) is smooth in $(-\varepsilon, \varepsilon)$. Since K is compact, it can be covered by a finite number of such neighborhoods U. Let $\varepsilon_0 > 0$ be the smallest of the corresponding numbers ε. For $p \notin K$, set $\varphi_t(p) = p$ for all $t \in \mathbb{R}$, then we get a uniquesolution $\varphi_t(p)$ for all $p \in M$ that is smooth in both vairables. Furthermore, we can see that $\varphi_{t+s}(p) = \varphi_t \circ \varphi_s(p)$, provided that $t+s$, t and $s \in (-\varepsilon_0, \varepsilon_0)$. We now need to define $\varphi_t(p)$ for $t \geq \varepsilon_0$. Any Number can be expressed as $t = k \cdot \frac{\varepsilon_0}{2} + r$, where $0 \leq r < \varepsilon_0/2$. Then set

$$\varphi_t = \varphi_{\varepsilon_0/2} \circ \varphi_{\varepsilon_0/2} \circ ... \circ \varphi_r$$

, where φ_{ε_0} is iterated k times. For $t \leq \varepsilon_0$, replace $\varphi_{\varepsilon_0/2}$ with $\varphi_{-\varepsilon_0/2}$. Apperently it is not difficult to verify that this is well defined, smooth, and satisfies the condition

$$\varphi_{t+s} = \varphi_t + \varphi_s$$

. $\square$

Def. 3.3 (Riemannian Metric)**.** A *riemannian metric* g is a smoothly chosen scalar product $g_p : T_pM \times T_pM \to \mathbb{R}$ for every point $p \in M$, s.th. for any vector fields $X, Y : M \to TM$ the map $p \mapsto g_p(X(p), Y(p))$ is smooth.
A different definition can be given follows:
A *riemannian metric* g on a smooth manifold M is a smooth map $g : M \to T^*M \otimes T^*M$; $p \mapsto g_p$, s.th. g_p is a scalar product $T_pM \times T_pM \to \mathbb{R}$.
A manifold together with a riemannian metric is called a *riemannian manifold.*
We write:

$$g_p(x, y) \ =: < x, y >_g =: < x, y >$$

Def. 3.4 (gradient)**.** Let (M, g^{TM}) be a riemannian manifold. The *gradient* of a smooth map $f : M \to \mathbb{R}$ is the vector field which is characterized by the identity

$$< X, \mathrm{grad} f >= X(f), \text{ where } X(f) = \mathrm{d}f X$$

Def. 3.5 (deformation retract)**.** Let X be a topological space. A continuous map $r : X \times [0,1] \to X$ is a *deformation retraction* onto a subspace A, if for every

$$r(\cdot, 0) = \mathrm{id}_X \ , \ r(X, 1) \subseteq A \ , \ r(\cdot, 1)|_A = \mathrm{id}_A$$

$A \subseteq X$ is called a *deformation retract* of M, if there exists a deformation retraction from X onto A. If $X = M$ is a smooth manifold, and there exists a smooth deformation retraction from M onto A, A is called a *smooth deformation retract.*

Lemma 3.6. *If A is a deformation retract of a space X, then the inclusion $A \to X$ is a homotopy equivalence.*

Proof. Let r be the deformation retraction from X onto A. Let $\iota : A \to X$ be the inclusion. Let $f : X \to A$; with $f(x) = r(x, 1)$. This is well defined since $r(\cdot, 1)$ maps into A. Then $f \circ \iota(x) = f(x) = x$. Also, $\iota \circ f(x) = f(x) = r(x, 1)$, so r is a homotopy between $f \circ \iota$ and id_X, so A and X have the same homotopy type. $\square$

Remark. The following is also true:
Let A and B be subspaces of X. Then A and B have the same homotopy type, if and only if they both are deformation retracts of some other subspace C.
"$\Leftarrow$" follows directly from the lemma above, while "$\Rightarrow$" is relatively hard to prove and requires some more topological theory.

Theorem 3.7 (First deformation Lemma, Milnor)**.** *Let M be a manifold, $f : M \to \mathbb{R}$ smooth. Let $a < b \in \mathbb{R}$, s.th. $f^{-1}[a, b]$ is compact and contains no critical points of f. Then M^a is diffeomorphic to M^b.*
Furthermore, M^a is a deformation retract of M^b, s.th. the inclusion $M^a \to M^b$ is a homotopy equivalence.

Proof. Let $\rho : M \to \mathbb{R}$ be a smooth function where

$$\rho(p) = 1/ < \mathrm{grad} f, \mathrm{grad} f >$$

for all $p \in f^{-1}[a, b]$ and which vanishes outside of a compact neighborhood of $f^{-1}[a, b]$, i.e. which is compactly supported. Note ρ is well defined inside $f^{-1}[a, b]$, because there are no critical points in $f^{-1}[a, b]$.

Then the vector field X which is defined by

$$X(p) = \rho(p) \cdot \mathrm{grad} f(p)$$

is compactly supported as well, i.e satisfies the conditions of lemma 3.2, hence X generates a unique 1-parameter group of diffeomorphisms on M

$$\varphi_t : M \to M$$

. For fixed $p \in M$, consider the function $t \mapsto f(\varphi_t(p))$. If $\varphi_t(p)$ lies in $f^{-1}[a, b]$, then

$$\frac{d}{dt} f(\varphi_t(p)) =< d\varphi_t(p), \mathrm{grad} f >=< X, \mathrm{grad} f >= +1$$

. Thus the corrensponderce

$$t \to f(\varphi_t(p))$$

is linear with derivative $+1$, if $p \in \varphi_t^{-1}(f^{-1}[a, b])$. Note that $f \circ \varphi_t$ is "strictly increasing" in the following sense:

$$f(p) > f(q) \Leftrightarrow f(\varphi_t(p)) > f(\varphi_t(p))$$

, as long as p, q, $\varphi_t(p)$ and $\varphi_t(q)$ are in the interior of the support of X. Also the map $t \mapsto f(\varphi_t(p))$ is strictly increasing, as long as p and $\varphi_t(p)$ are in the interior of the support of X. This is true because the paths $t \mapsto \varphi_t(p)$ move parallel to the gradient of f and have velocity > 0 inside the interior of the support of X.

Then the set $f^{-1}(a)$ is diffeomorphically mapped onto $f^{-1}(b)$ by φ_{b-a}:

Let $p \in f^{-1}(a)$. then for $t \in [0, b-a]$, The condition $\varphi_t(p) \in f^{-1}([a,b])$ from above is satisfied, because $t \mapsto \varphi_t(p)$ is strictly increasing in the above interval, so

$$f(\varphi_t(p)) = f(\varphi_0(p)) + t = f(p) + t = a + t \Rightarrow f(\varphi_{b-a}(p)) = b$$

Similarly, if $p \in f^{-1}(b)$, then $f((\varphi_{b-a})^{-1}(p)) = f(\varphi_{a-b}(p)) = a$. Now let $p \in f^{-1}(-\infty, a)$ wlog. let p be in the interior of the support of X, otherwise $\varphi_{b-a}(p) = p \in M^b$.

If we now show that for $p \in f^{-1}(-\infty, a)$, $\varphi_{b-a}(p) \in M^b$, then by the same argument as before, we use the inverse φ_{b-a} and this then implies the first assertion of the theorem. For this, choose such a p. Assume $\varphi_{b-a}(p) \notin M^b$. Then $f(\varphi_{b-a}(p)) > b = f(\varphi_{b-a}(q))$ for some $q \in f^{-1}(a)$. But then $f(p) > f(q) \Rightarrow p \notin M^a$.

Now we still have to show that M^a is a deformation retract of M^b, and that the inclusion $M^a \to M^b$ is a homotopy equivalence.

Take $r : M^b \times \mathbb{R} \to M^b$,

$$r(p,t) = \begin{cases} p & \text{if } f(p) \leq a \\ \varphi_{t(a-f(p))}(p) & \text{if } a \leq f(p) \leq b \end{cases}$$

Then $r(\cdot, 0)$ is the identity and $r(\cdot, 1)$ restricted to M^a is the identity and $r(1,p) \in M^a$ for all $p \in M^b$. $\square$

Remark. Note that for M^a and M^b to be C^r diffeomorphic, f needs only to be C^{r+1}. Also we have proved that the level sets between a and b are all diffeomorphic.

Corollary 3.8 (Hirsch). *Let $f : M \to [a,b]$ be a C^{r+1}-map on a compact manifold with boundary, $1 \leq r \leq \omega$. Suppose f has no critical values and $f(\partial M) = \{a,b\}$. Then there is a C^r diffeomorphism such that the following diagram commutes:*

$$\begin{tikzcd} f^{-1}(a) \times [a,b] \arrow[rr, "F"] \arrow[dr] & & M \arrow[dl, "f"] \\ & {[a,b]} & \end{tikzcd}$$

Proof. Take $F(p,t) = r(p,t)$, where r is the deformation retraction from the proof above.

$\square$

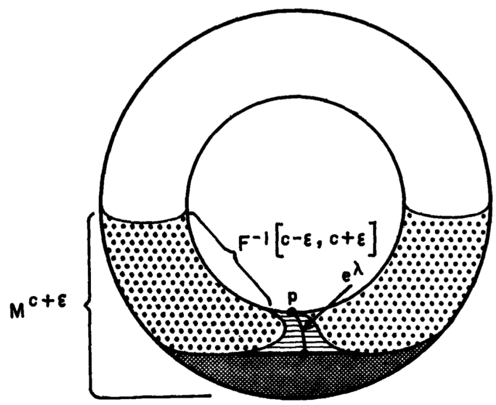
Corollary 3.9 (Hirsch). *Let M be a compact manifold with boundary, such that $\partial M = A \cup B$ and A and B are disjoint. If there exists a C^2 map $f : M \to [0,1]$ with no critical points, such that $f(A) = 0$ and $f(B) = 1$, then M is C^1-diffeomorphic to the cylinders $A \times [0,1]$ and $B \times [0,1]$.*

§4 The second Deformation Lemma

Remark. In the following, whenever talking about homology, it will be the homology with coefficients in $\mathbb{R}$.

Theorem 4.1 (Second deformation Lemma, Milnor). *Let M be a manifold, $f : M \to \mathbb{R}$ smooth and p be a non-degenerate critical point of f of index λ. Let $c := f(p)$ and $\varepsilon > 0$, s.th. $f^{-1}[c-\varepsilon, c+\varepsilon]$ is compact and contains no critical points of f other then p. Then $M^{c+\varepsilon}$ has the homotopy-type of $M^{c-\varepsilon}$ with a λ-cell attached.*

Proof. The idea of the proof is define a new function F, that is equal to f exept for in a small neighborhood of p, there we take $F < f$ slightly. Then we get a situation as in the following diagram, where our manifold is the Torus and the map is the hight map, where $c = f(p)$:



The heavily shaded region is $M^{c-\varepsilon}$. Then $F^{-1}(-\infty, c]$ is the heavily shaded region together with the horizontally shaded region. We Then construct a homotopy-equivalence that "squishes" the horizontally shaded region along the indicated lines, thus only leaving a λ-cell.

By the Morse Lemma 2.6, we can choose local coordinates $\varphi = (u^1, ..., u^n)$ in a neighborhood of p such that

$$f = c - (u^1)^2 - ... - (u^\lambda)^2 + (u^{\lambda+1})^2 + ... + (u^n)^2$$

in a neighborhood U of p. Then for the critical point p we have

$$u^1(p) = ... = u^n(p) = 0$$

Now choose $\varepsilon > 0$ small enough, such that the following two statements hold:

1. $f^{-1}[c - \varepsilon, c + \varepsilon]$ is compact and contains no critical points of f.

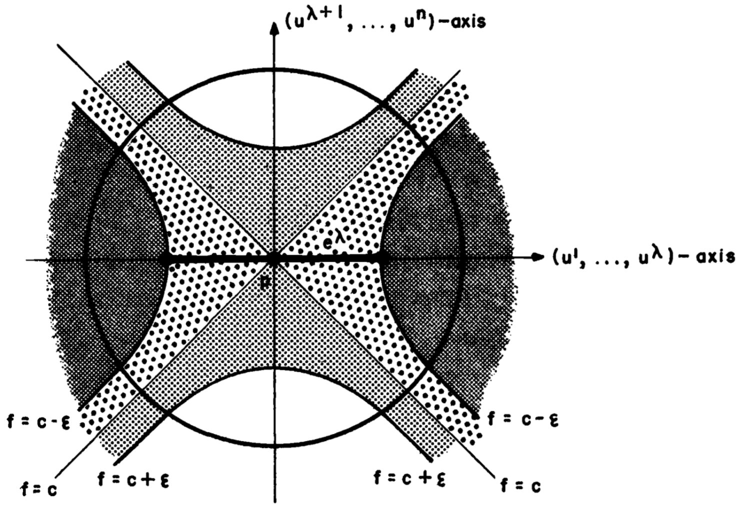
2. $\{x : \|x\| \leq 2\varepsilon\} \subseteq \varphi(U)$

Now choose the λ-cell e^λ to be the points in M with

$$(u^1)^2 + ... + (u^\lambda)^2 \leq \varepsilon \text{ and } u^{\lambda+1} = ... = u^n = 0$$

.

We now have our mainfold parametrized in U in a nice way. The situation is illustrated in the following diagram:



The heavily shaded area is $M^{c-\varepsilon}$, the area shaded with bigger dots is $f^{-1}[c-\varepsilon, c]$ and the area shaded with smaller dots is $f^{-1}[c, c+\varepsilon]$.
The circle is $\{p \in M : (u^1(p))^2 + ... + (u^n(p))^2 = 2\varepsilon\}$. Notice that $e^\lambda \cap M^{c-\varepsilon} = \partial e^\lambda$, so it is attached to $M^{c-\varepsilon}$ as required. We now have to show that $e^\lambda \cup M^{c-\varepsilon}$ is a deformation retract of $M^{c+\varepsilon}$.
Define a C^∞ function $\mu : \mathbb{R} \to \mathbb{R}$ with the following properties:

$$\mu(0) > \varepsilon$$

$$\mu(r) = 0 \text{ for } r \geq 2\varepsilon$$

$$-1 < \mu'(r) \leq 0 \text{ for all } r$$

Now let F equal f outside of U, and let

$$F = f - \mu((u^1)^2 + ... + (u^\lambda)^2 + 2(u^{\lambda+1})^2 + ... + 2(u^n)^2)$$

within U. F is well defined and smooth, because for $((u^1)^2 + ... + (u^n)^2)(p) > 2\varepsilon$, $\mu(p) = 0$ and the closed ball with radius $\sqrt{2\varepsilon}$ is fully contained in U.
Now define

$$\xi, \eta : U \to [0, \infty)$$

by

$$\xi = (u^1)^2 + ... + (u^\lambda)^2$$
$$\eta = (u^{\lambda+1})^2 + ... + (u^n)^2$$

Then $f = c - \xi + \eta$ and $F|_U = f - \mu(\xi + 2\eta) = c - \xi + \eta - \mu(\xi + 2\eta)$

Assertion 1: The region $F^{-1}(-\infty, c+\varepsilon]$ coincides with $M^{c+\varepsilon}$.

For $\xi(p) + 2\eta(p) > 2\varepsilon$, $f(p) = F(p)$. So wlog., let $p \in M$ such that $\xi(p) + 2\eta(p) \leq 2\varepsilon$. Then

$$F(p) \leq f(p) = c + \xi(p) + \eta(p) \leq c + \frac{1}{2}\xi(p) + \eta(p) \leq c + \varepsilon$$

. This prooves the first assertion.

Assertion 2: The critical points of F are the same as those of f.

Note that

$$\frac{\partial F}{\partial \xi} = -1 - \mu'(\xi + 2\eta) < 0$$

and

$$\frac{\partial F}{\partial \eta} = 1 - 2\mu'(\xi + 2\eta) \geq 1$$

, so in particular these derivatives are never 0. Since

$$\mathrm{d}F = \frac{\partial F}{\partial \xi}\mathrm{d}\xi + \frac{\partial F}{\partial \eta}\mathrm{d}\eta$$

And $\mathrm{d}\xi$ and $\mathrm{d}\eta$ are zero only in p, F has no critical points in U other then p. This proves the second assertion.

Assertion 3: The region $F^{-1}(-\infty, c-\varepsilon]$ is a deformation retract of $M^{c+\varepsilon}$.

Consider the region $F^{-1}[c-\varepsilon, c+\varepsilon]$. With assertion 1 and the fact that $F \leq f$, we see that

$$F^{-1}[c-\varepsilon, c+\varepsilon] \subseteq f^{-1}[c-\varepsilon, c+\varepsilon]$$

. But $f^{-1}[c-\varepsilon, c+\varepsilon]$ is compact and $F^{-1}[c-\varepsilon, c+\varepsilon]$ is closed, so it is compact as well.

By assertion 2, it cannot contain any critical points of F exept maybe p, but

$$F(p) = c - \mu(0) < c - \varepsilon$$

, so p is not in the region. Then with the first deformation lemma 3.7, the third assertion is proven.

In the following, H will be the closure of $F^{-1}(-\infty, c-\varepsilon] - M^{c-\varepsilon}$, which we shall call a "handle". So $F^{-1}(-\infty, c-\varepsilon] = M^{c-\varepsilon} \cup H$ will be called "$M^{c-\varepsilon}$ with a handle attached".

Now consider e^λ from above, i.e. the λ-cell consisting of all points q with

$$\xi(q) \leq \varepsilon \text{ and } \eta(q) = 0$$

. Since $\frac{\partial F}{\partial \xi} < 0$, we have

$$F(q) \leq F(p) < c - \varepsilon$$

, but $f(q) \geq c - \varepsilon$ for $q \in e^\lambda$, so e^λ is contained in the handle H.

Assertion 4: $M^{c-\varepsilon} \cup e^\lambda$ is a deformation retract of $M^{c-\varepsilon} \cup H$.

We construct a deformation retraction $r : M^{c-\varepsilon} \cup H \times [0,1] \to M^{c-\varepsilon} \cup H$ for q as follows:

$$r(q,t) = \begin{cases} \varphi^{-1} \circ (u^1, ..., u^\lambda, tu^{\lambda+1}, ..., tu^n)(q) & \text{if } \xi(q) \leq \varepsilon \\ \varphi^{-1} \circ (u^1, ..., u^\lambda, s_t u^{\lambda+1}, ..., s_t u^n)(q) & \text{if } \varepsilon \leq \xi(q) \leq \eta(q) + \varepsilon \\ q & \text{if } \eta(q) + \varepsilon \leq \xi(q) \end{cases}$$

Where

$$s_t = t + (1-t)((\xi - \varepsilon)/\eta)^{1/2}$$

We have:

$\xi(p) \leq \varepsilon$ as case 1,

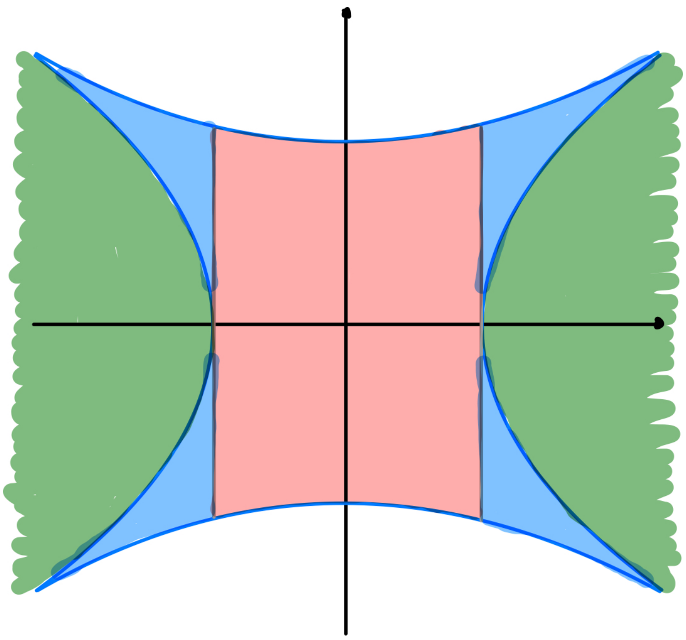
$\varepsilon \leq \xi(q) \leq \eta(q) + \varepsilon$ as case 2 and

$\eta(q) + \varepsilon \leq \xi(q)$ case 3.

Note that

$$\text{case 3, i.e. } \eta(q) + \varepsilon \leq \xi(q) \Leftrightarrow c - \xi(q) + \eta(q) \leq c - \varepsilon \Leftrightarrow q \in M^{c-\varepsilon}$$

, so we have the following situation:



Where the red is case 1, the green is case 3 and the blue is case 2. We now need to verify that

1. r is well definded and continuous

2. $r(\cdot, 1) = \mathrm{id}_{M^{c-\varepsilon} \cup H}$

3. $r(M^{c-\varepsilon} \cup H, 0) \subseteq M^{c-\varepsilon} \cup e^{\lambda}$

4. $r(\cdot, 0)|_{M^{c-\varepsilon} \cup e^{\lambda}} = \mathrm{id}_{M^{c-\varepsilon} \cup e^{\lambda}}$

2. and 4. are easiliy verified: $s_1 = 1$, so in all three cases $r(q, 1) = q$. If q is in e^{λ}, only case 1 will hold for q and then $r(q, 0) = \varphi^{-1} \circ (u^1, ..., u^{\lambda}, 0, ..., 0) \in e^{\lambda}$, and if $q \in M^{c-\varepsilon}$ then only case 3 holds for q and $r(q, 0) = q \in M^{c-\varepsilon}$.

3. is obvious for case 1 and case 3. For q in case 2:

$$s_0(q) = ((\xi(q) - \varepsilon)/\eta(q))^{1/2}$$

so

$$\begin{aligned}
f(r(0,q)) = f(\varphi^{-1})(u^1(q), ..., u^\lambda(q), \left(\frac{\xi(q)-\varepsilon}{\eta(q)}\right)^{1/2} u^{\lambda+1}(q), ..., \left(\frac{\xi(q)-\varepsilon}{\eta(q)}\right)^{1/2} u^n(q)) \\
= c - \xi(q) + \left(\left(\frac{\xi(q)-\varepsilon}{\eta(q)}\right)^{1/2} u^{\lambda+1}\right)^2 + \left(\left(\frac{\xi(q)-\varepsilon}{\eta(q)}\right)^{1/2} u^n(q)\right)^2 \\
= c - \xi(q) + \left(\frac{\xi(q)-\varepsilon}{\eta(q)}\right)\eta(q) \\
= c - \varepsilon
\end{aligned}$$

, so $r(0,q) \in f^{-1}(c-\varepsilon)$.

To check 1. well definedness and continuity, we first need to check continuity on the edge cases:

For $\xi(q) = \varepsilon$: $\quad s_t(q) = t + (1-t)((\varepsilon - \varepsilon)/\eta(q))^{1/2} = t$

For $\eta(q) + \varepsilon = \xi(q)$: $\quad s_t(q) = t + (1-t)((\xi(q) - \varepsilon)/(\xi(q) - \varepsilon)^{1/2} = 1$

Check continuity of s_t: The only problem we may get is if $\eta \to 0$. Note that, because s_t is only defined in case 2, we always have $0 \leq \xi - \varepsilon \leq \eta$. Then we will have to verify that $s_t u^i \to 0$ for $\lambda + 1 \leq i \leq n$, as $\eta \to 0$, because $r(t,q) = \varphi^{-1} \circ (u^1, ..., u^\lambda, 0, ..., 0)$ in the other two cases as $\eta = 0$.

$$\begin{aligned}
\lim_{\eta\to 0} |s_t u^i| &= \lim_{\eta\to 0}(1-t)((\xi-\varepsilon)/\eta)^{1/2}|u^i| \\
&\leq \lim_{\eta\to 0}(1-t)(\eta/\eta)^{1/2}|u^i| \\
&= \lim_{\eta\to 0}(1-t)|u^i| = 0
\end{aligned}$$

And then $\lim_{\eta \to 0} s_t u^i = 0$.

Assertions 3. and 4. together prove the theorem because we get

$$M^{c+\varepsilon} \simeq M^{c-\varepsilon} \cup H$$

by assertion 3 and

$$M^{c-\varepsilon} \cup H \simeq M^{c-\varepsilon} \cup e^{\lambda}$$

by assertion 4, so $M^{c+\varepsilon} \simeq M^{c-\varepsilon} \cup e^{\lambda}$. □

Remark (Milnor). More generally, suppose there are k critical points $p_1, ..., p_k$ with indicies $\lambda_1, ..., \lambda_k$ in $f^{-1}(c)$. Then by a similar proof we get

$$M^{c+\varepsilon} \simeq M^{c-\varepsilon} \cup e^{\lambda_1} \cup ... \cup e^{\lambda_k}$$

Remark (Milnor). A simple modification of the proof above shows that M^c is also a deformation retract $M^{c+\varepsilon}$. In fact M^c is a deformation retract of $F^{-1}(-\infty, c]$, which is a deformation retract of $M^{c+\varepsilon}$. Combinining this fact with The second deformation lemma 4.1, we can easily see that $M^{c-\varepsilon}$ is a deformation retract of M^c.

§5 Some Applications

Theorem 5.1. *If M^n is a compact manifold that admits a morse function f with exactly two critical points, then M is homeomorphic to S^n.*

Theorem 5.2 (CW-Structure of Manifolds)**.** *If M is a manifold that admits a morse function f, such that each M^a is compact, then M has the homotopy-type of a CW-Complex with a λ-cell for each critical point of f of index λ.*

Remark. Theorem 5.1 follows immediately from theorem 5.2.

§6 The Morse Inequalities

Def. 6.1 (Betti number, Euler characteristic)**.** The λ-th Betti number of a space pair $(X, Y) \in \mathsf{Obj}(Top_2)$ if $\dim(\mathrm{H}_\lambda(X, Y; \mathbb{R})) < \infty$ is

$$\mathrm{b}_\lambda(X, Y) = \dim(\mathrm{H}_\lambda(X, Y; \mathbb{R}))$$

The Euler-Characteristic of a space-pair (X, Y) where the dimensions of the homology over $\mathbb{R}$ of (X, Y) in every degree is finite and 0 in almost every degree is

$$\chi(X, Y) = \sum_{\lambda \in \mathbb{Z}} (-1)^\lambda \mathrm{b}_\lambda(X, Y)$$

Lemma 6.2. *If we have an exact sequence of vector spaces*

$$V_0 \xrightarrow{\partial_0} V_1 \xrightarrow{\partial_1} \dots \xrightarrow{\partial n-1} V_n \xrightarrow{\partial_n} 0$$

then

$$\sum_{k=1}^{n} (-1)^k \dim V_k \geq 0$$

.

If $V_0 = 0$, *then*

$$\sum_{k=1}^{n} (-1)^k \dim V_k \geq 0$$

.

Proof. We have $V_k = \ker(\partial_k) \oplus \mathrm{Im}(\partial_k) = \mathrm{Im}(\partial_{k-1}) \oplus \mathrm{Im}(\partial_k)$, so

$$\bigoplus_{k \text{ odd}}^{n} V_k = \bigoplus_{k \text{ even}}^{n} V_k \oplus \mathrm{Im}(\partial_0)$$

, and then since almost all V_k are zero

$$\sum_{k \text{ even}}^{n} \dim V_k - \sum_{k \text{ odd}}^{n} \dim V_k = \dim \mathrm{Im}(\partial_0) \geq 0$$

$$\Leftrightarrow \sum_{k=1}^{n}(-1)^k \dim V_k \geq 0$$

. If $V_0 = 0$, then $\dim \mathrm{Im}(\partial_0) = 0$, so

$$\sum_{k=1}^{n}(-1)^k \dim V_k = 0$$

. $\square$

Def. 6.3 (subadditiv function)**.** A *subadditive function* S is a function from certain space pairs to the integers, s.th. if we have spaces $Z \subseteq Y \subseteq X$, then

$$S(X, Z) \leq S(X, Y) + S(Y, Z)$$

. If equality holds S is called *additive.*

We write $S(X) := S(X, \varnothing)$.

Lemma 6.4. *If S is a subadditive, $X_0 \subseteq ... \subseteq X_n$ admissable for S, then*

$$S(X_n, X_0) \leq \sum_i S(X_i, X_{i-1})$$

with equality if S is additive.

Proof. Proof by induction: For $n = 2$ this is exactly the definition of (sub)additivity. If the statement holds for $n - 1$, then

$$S(X_n, X_0) \leq S(X_n, X_{n-1}) + S(X_{n-1}, X_0) \leq \sum_i S(X_i, X_{i-1})$$

with equality if S is additive. $\square$

Lemma 6.5. b_λ *is subadditive and χ is additive.*

Proof. For any exact sequence of vector spaces

$$... \xrightarrow{\partial_{-1}} V_0 \xrightarrow{\partial_0} V_1 \xrightarrow{\partial_1} V_2 \xrightarrow{\partial_2} ...$$

we get:

$$\begin{aligned}\dim(V_1) &= \dim(\ker(\partial_1)) + \dim(\mathrm{Im}(\partial_1)) \\ &= \dim(\mathrm{Im}(\partial_0)) + \dim(\ker(\partial_2)) \\ &= (\dim(V_0) - \dim(\ker(\partial_0))) + (\dim(V_2) - \dim(\mathrm{Im}(\partial_2))) \\ &\leq \dim(V_0) + \dim(V_2)\end{aligned}$$

For spaces $Z \subseteq Y \subseteq X$ we have the following exact sequence in homology:

$$... \longrightarrow \mathrm{H}_\lambda(Y,Z) \longrightarrow \mathrm{H}_\lambda(X,Z) \longrightarrow \mathrm{H}_\lambda(X,Y) \longrightarrow ...$$

so $\mathrm{b}_\lambda(X,Z) \leq \mathrm{b}_\lambda(X,Y) + \mathrm{b}_\lambda(Y,Z)$.

With the same long exact sequence of space triples and with lemma 6.2, i.e. the fact that the alternating sum of the dimensions in a finite exact sequence of vector spaces equals zero, we get

$$\sum(-1)^\lambda \dim(\mathrm{H}_\lambda(Y,Z)) - \sum(-1)^\lambda \dim(\mathrm{H}_\lambda(X,Z)) + \sum(-1)^\lambda \dim(\mathrm{H}_\lambda(X,Y)) = 0$$

, so $\chi(X,Z) = \chi(Y,Z) + \chi(X,Y)$. □

Theorem 6.6 (Weak Morse Inequalities)**.** *Let $f : M \to \mathbb{R}$ be a smooth map from a compact smooth manifold with only non-degenerate critical points. Let C_λ be the number of critical points of f of degree λ.*

1. $\mathrm{b}_\lambda(M) \leq C_\lambda$
2. $\chi(M) = \sum_\lambda (-1)^\lambda \cdot C_\lambda$

Proof. Wlog., assume f has only isolated critical points. Let $a_0, ..., a_k \in \mathbb{R}$ such that M^{a_i} contains exactly i critical points and $M^{a_k} = M$. Then $M^{a_0} = \varnothing$. Let λ_i be the index of the unique critical point in M^{a_i}. We get

$$\begin{aligned}
\mathrm{H}_\lambda(M^{a_i}, M^{a_{i-1}}) &= \mathrm{H}_\lambda(M^{a_{i-1}} \cup e^{\lambda_i}, M^{a_{i-1}}) \\
&= \mathrm{H}_\lambda(e^{\lambda_i}, \partial e^{\lambda_i}) \text{ (by excision)} \\
&= \mathrm{H}_\lambda(e^{\lambda_i}/\partial e^{\lambda_i}, *) \\
&= \mathrm{H}_\lambda(S^{\lambda_i}, *)
\end{aligned}$$

So

$$\mathrm{H}_\lambda(M^{a_i}, M^{a_{i-1}}) = \begin{cases} \mathbb{R} & \text{if } \lambda = \lambda_i \\ 0 & \text{else} \end{cases}$$

Then because b_λ is subadditive and lemma 6.4:

$$\begin{aligned}
\mathrm{b}_\lambda(M) &\leq \sum_i \mathrm{b}_\lambda(M^{a_i}, M^{a_{i-1}}) \\
&= \sum_i \dim(\mathrm{H}_\lambda(M^{a_i}, M^{a_{i-1}})) \\
&= C_\lambda
\end{aligned}$$

and because χ is additive and lemma 6.4:

$$\begin{aligned}
\chi(M) &= \sum_i \chi(M^{a_i}, M^{a_{i-1}}) \\
&= \sum_i \sum_\lambda (-1)^\lambda \dim \mathrm{H}_\lambda(M^{a_i}, M^{a_{i-1}}) \\
&= \sum_\lambda (-1)^\lambda C_\lambda
\end{aligned}$$

□

Theorem 6.7 (Strong Morse Inequality). *Let M be a compact smooth manifold and $f : M \to \mathbb{R}$ be a smooth function with only non-degenerate critical points. Then let C_λ denote the number of critical points of f with index λ. Then*

$$b_\lambda(M) - b_{\lambda-1}(M) + ... \pm b_0(M) \leq C_\lambda - C_{\lambda-1} ... \pm C_0$$

Proof. We first see that S_λ is subadditive, where

$$S_\lambda(X, Y) = b_\lambda(X, Y) - b_{\lambda-1}(X, Y) + ... \pm b_0$$

.

For this, we again look at the long exact sequence of a space triple $Z \subseteq Y \subseteq X$ in homology. With lemma 6.2, we get

$$\sum_\lambda b_\lambda(Y, Z) - \sum_\lambda b_\lambda(X, Z) + \sum_\lambda b_\lambda(X, Y) \leq 0$$

, and the asserrtion follows. Furthermore, with theorem 6.6:

$$S_\lambda(M^{a_\lambda}) = \chi(M^{a_\lambda}) = \sum_{i=1}^{\lambda} (-1)^i C_i$$

Then with lemma 6.4, we have

$$\begin{aligned} S_\lambda(M) &\leq \sum_{i=1}^{k} (-1)^k S_\lambda(M^{a_i}, M^{a_{i-1}}) \\ &= \sum_{i=1}^{k} b_\lambda(M^{a_i}, M^{a_{i-1}}) - b_{\lambda-1}(M^{a_i}, M^{a_{i-1}}) + ... \pm b_0(M^{a_i}, M^{a_{i-1}}) \end{aligned}$$

, but as we saw above

$$b_\lambda(M^{a_i}, M^{a^{i-1}}) = \begin{cases} 1 & \text{if } \lambda = i \\ 0 & \text{else} \end{cases}$$

So

$$\sum_{i=1}^{k} (-1)^k S_\lambda(M^{a_i}, M^{a_{i-1}}) = C_\lambda - C_{\lambda-1} + ... \pm C_0$$

□