Gameplan: Ascending/Descending $\to N = A \cap D/\mathbb{R}$ is manifold $\to$ orientation on N $\to$ Morse Homology

§1 Morse Functions

Def. 1.1 (Critical Point)**.** Let M be a manifold and $f : M \to \mathbb{R}$ be a smooth function. A critical point of f is a point $p \in M$, such that $df(p) = 0$.

Remark. If M is compact, f always has critical points.

Def. 1.2 (Poisson Bracket)**.** Let X and Y be vector fields on a manifold M. Then for a point $p \in M$ define the Poisson Bracket by

$$[X, Y]f := XYf - YXf$$

The Poisson Bracket is a Lie Bracket on the $\mathbb{R}$-vectorspace of vector fields on M. Also, if p is a critical point of a function f, then $[X, Y]f(p) = 0$.

Def. 1.3 (Hessian)**.** In $\mathbb{R}^n$, we are used to defining the Hessian of a function, that is the second order derivative. On a manifold, this is not so easy, since the second order derivative will always depend on the local coordinates. We will have to do with defining the second order derivative just on the critical points of a function: For $p \in M$ a critical point of a function f, $x, y \in T_p(M)$, choose X, Y vectorfields extending x and y locally, i.e. with $Y(p) = y$ and $X(p) = x$. Then define

$$\mathrm{d}^2 f(x, y)(p) = XYf(p)$$

Lemma 1.4. *$d^2 f(\cdot, \cdot)(p)$ is a symmetrical bilinear form if p is a critical point of f.*

Proof. Note that

$$d^2 f(x, y)(p) - d^2 f(y, x)(p) = [X, Y]f(p) = 0$$

so $XYf(p)$ is symmetrical at any critical point and

$$XYf(p) = X(p)Yf(p) = xYf(p)$$

for any choice of X and Y, so $d^2 f(\cdot, \cdot)(p)$ does not depend on X, and by the same argument because it is symmetrical it does not depend on Y. Then $d^2 f(x, y)(p)$ is a well defined, symmetrical bilinear form. □

Def. 1.5 (non-degeneracy, index)**.** We call a critical point of a function $f : M \to \mathbb{R}$ *non-degenerate*, if the bilinear form $d^2 f(\cdot, \cdot)(p)$ is non-degenerate.
We define the index of p as the maximal dimension of subspaces, on which $d^2 f(\cdot, \cdot)(p)$ is

negative definite.
Note that with local coordinates $(x_1, ..., x_n)$ around p, we get an induced Basis of the tangent space $T_p(M)$ as $B = (\frac{\partial}{\partial x_1}, ..., \frac{\partial}{\partial x_n})$, and then

$$d^2 f\left(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}\right)(p) = \frac{\partial}{\partial x_i}(p)\frac{\partial}{\partial x_j}f(p) = \frac{\partial^2 f}{\partial x_i \partial x_j}(p)$$

Then the index of p is the number of negative eingenvalues of

$$\left(\frac{\partial^2 f}{\partial x_i \partial x_j}(p)\right)_{1 \leq i,j \leq n}$$

and p is non degenerate iff the matrix is invertable.

Def. 1.6. A smooth function $f : M \to \mathbb{R}$ is called a *Morse Function*, if all its critical points are non-degenerate.

Lemma 1.7. *Let $M \subseteq \mathbb{R}^n$ be a submanifold. Then for almost every point $p \in \mathbb{R}^n$, the function*

$$f : M \to \mathbb{R}$$
$$q \mapsto ||q - p||^2$$

is a Morse Function.

Proof. □ TODO

Remark. Note that by Whitney's embedding theorem, there exist Morse-functions on any Manifold.

Theorem 1.8. *Any smooth function $f : M \to \mathbb{R}$ and all its derivatives can be uniformly approximated by a Morse Function.*

Proof. □ TODO

Theorem 1.9. *Let M be a compact manifold. Then the set of Morse functions is dense in $\mathcal{C}^\infty(M)$.*

Proof. □ TODO

Theorem 1.10 (Morse Lemma)**.** *Let $f : M \to \mathbb{R}$ be a smooth function. Let p be a non-degenerate critical point of index k of f. Then there exist local coordinates $(x_1, ..., x_n)$ in a neighburhood U of p, such that*

$$f = f(p) - x_1^2 - ... - x_k^2 + x_{k+1}^2 + ... + x_n^2$$

and

$$x_1(p) = ... = x_n(p) = 0$$

U is called Morse chart.

Proof. □ TODO

Corollary 1.11. *Non-degenerate critical points are isolated.*

Remark. A critical point of Index k of f is a critical point of index $n - k$ of $-f$.

§2 Pseudo-Gradients

Def. 2.1 (Riemannian Metric, Gradient)**.** A *Riemannian metric* g on a manifold M is a choice of scalar products $g_p : T_pM \times T_pM \to \mathbb{R}$ for every point $p \in M$, such that for any vectorfields X and Y, the map

$$p \mapsto g_p(X(p), Y(p))$$

is smooth. For $x, y \in T_pM$, we write

$$\langle x, y \rangle := g_p(x, y)$$

and

$$||x|| = \sqrt{g_p(x, x)}$$

If $f : M \to \mathbb{R}$ is a smooth map, then the gradient of f is a vectorfield ∇f, such that for any vectorfield X the identity

$$\langle X, \nabla f \rangle = \mathrm{d}fX$$

holds.

Def. 2.2 (Pseudo-Gradient)**.** Let $f : M \to \mathbb{R}$ be a smooth function. A pseudo-gradient of f is a vectorfield X on M, such that

1. $\mathrm{d}f(p)X(p) \geq 0$, where equality holds if and only if p is a critical point of f,
2. For every critical point of f, there exists a Morse chart neighborhood in which X coincides with $-\nabla f$

Lemma 2.3 (Existence of Pseudo Gradients)**.** *For any smooth function $f : M \to \mathbb{R}$ there exists a pseudo-gradient of f.*

Proof. This follows from the fact that every Manifold can be equipped with a Riemannian metric, then the gradient is a pseudo-gradient. □

Def. 2.4 (Stable and unstable Manifolds)**.** Let p be a critical point of a smooth function $f : M \to \mathbb{R}$. Denote by φ_s the flow of a pseudo-gradient of f. Then We define the *stable manifold* to be

$$W^s(p) = \left\{ q \in M : \lim_{s \to \infty} \varphi_s = p \right\}$$

and the *unstable manifold*

$$W^u(p) = \left\{ q \in M : \lim_{s \to -\infty} \varphi_s = p \right\}$$

Proposition 2.5. *Let p be a critical point of index k of a smooth function $f : M \to \mathbb{R}$. Then $W^s(p)$ is diffeomoric to the open disk of dimension k and $W^u(p)$ is diffeomorphic to the open disc of dimension $n-k$.*

Def. 2.6. Before the proof, we fix some notation and examine the situation in a specific real case: We set

$$\begin{aligned} x_- &= (x_1, ..., x_k) : \mathbb{R}^n \to \mathbb{R}^k \\ x_+ &= (x_{k+1}, ..., x_n) : \mathbb{R}^n \to \mathbb{R}^{n-k} \end{aligned}$$

and then

$$Q = -||x_-||^2 + ||x_+||^2$$

This is a map $\mathbb{R}^n \to \mathbb{R}$, so we can utilize the gradient that we are used to. We get:

$$-\nabla Q(x_-, x_+) = 2(x_-, -x_+)$$

Further for some $\varepsilon, \eta \in \mathbb{R}$ we set

$$U(\varepsilon, \eta) = \left\{ x \in \mathbb{R}^n : -\varepsilon < Q(x) < \varepsilon \text{ and } ||x_-||^2 ||x_+||^2 \leq \eta(\varepsilon + \eta) \right\}$$

We also define

$$\begin{aligned} \partial_\pm &= \left\{ x \in U : Q(x) = \pm\varepsilon \text{ and } ||x_\mp||^2 \leq \eta \right\} \\ \partial_0 &= \left\{ x \in \partial U : ||x_-||^2 ||x_+||^2 = \eta(\varepsilon + \eta) \right\} \end{aligned}$$

Then we have

$$\partial U = \partial_+ \cup \partial_- \cup \partial_0$$

We also fix $V_-, V_+ \subseteq \mathbb{R}^n$ to be the subspaces on which Q is negative and positive definite respectively. We get

$$\partial U \cap V_\pm \subseteq \partial_\pm U$$

Proof. With the Morse-lemma, we obtain a Morse-neighborhood $U = U(\varepsilon, \eta)$ with local coordinates $h^{-1} = (x_1, ..., x_n)$. We have

$$\tilde{f} = f \circ h : U \to \mathbb{R} \text{ with } \tilde{f} = f(p) + Q$$

The only critical point of $\tilde{f}$ is 0. We have

$$W^s(0) = U \cap V_+ \text{ and } W^u = U \cap V_-$$

We also get a smooth embedding

$$(h(\partial_+ U \cap V_+) \times (-\infty, \infty])/(h(\partial_+ U \cap V_+) \times \{\infty\}) \to M; (x, s) \mapsto \varphi_s(x)$$

onto $W^s(p)$. $\partial_+ U \cap V_+$ is a sphere of dimension $n - k - 1$, and h is a diffeomorphism, so $(h(\partial_+ U \cap V_+) \times (-\infty, \infty])/(h(\partial_+ U \cap V_+) \times \{\infty\})$ is diffeomorphic to the open disk of dimension $n - k$, and then $W^s(p)$ is as well. Similarly $W^u(p)$ is diffeomorphic to the open disk of dimension k. □

Proposition 2.7. *Assume that M is a compact manifold Let X be a pseudo-gradient vectorfield of some smooth function $f : M \to \mathbb{R}$ and γ be a trajectory of X. Then there exist critical points p and q of f, such that*

$$\lim_{t \to \infty} \gamma(t) = p \textit{ and } \lim_{t \to -\infty} \gamma(t) = q$$

Proof. We show that $\gamma(t)$ has a limit as t tends to $+\infty$, and that this limit is a critical point p of f. This is the case if at some point the trajectory enters $S_+(p) := \partial_+ \Omega(p) \cap W^s(p)$. Suppose that this is not true. Then every time the trajectory enters a morse neighborhood, it must also leave it again and never return to it, because f is decreasing along γ. Let t_0 be the time that γ leaves the last of the morse neighborhoods, i.e. the finite union

$$\Omega = \bigcup_{q \in \mathrm{Crit}(f)} \Omega(p)$$

Because $df(x)X(x)$ is zero iff x is a critical point of f, and $dfX \leq 0$, there exists an $\varepsilon_0 > 0$, such that

$$\forall x \in V - \Omega, df(x)X(x) \leq -\varepsilon_0$$

Then for every $t \geq t_0$, we have

$$\begin{aligned} f(\gamma(t)) - f(\gamma(t_0)) &= \int_{t_0}^{t} \frac{\mathrm{d}(f \circ \gamma)}{\mathrm{d}u} \mathrm{d}u \\ &= \int_{t_0}^{t} \mathrm{d}f(\gamma(u))X(\gamma(u))du \\ &\leq -\varepsilon(t - t_0) \end{aligned}$$

And then

$$\lim_{t \to +\infty} f(\gamma(t)) = -\infty$$

which is absurd. $\square$

Theorem 2.8 (Fist Deformation Lemma). TODO

Proof. $\square$ TODO

Theorem 2.9 (Second Deformation Lemma). TODO

Proof. $\square$ TODO

Def. 2.10 (Transversality)**.** Let $U, V \subseteq M$ be submanifolds. Then U and V are said to meet *transversly*, if for all $p \in U \cap V$ we have

$$T_pU + T_pV = T_pM$$

If U and V meet transversly, we write

$$U \pitchfork V$$

A vectorfield X on M is *transversal* to a submanifold $U \in M$ of dimension $n-1$, if for all $p \in U$ we have

$$X(p) \notin T_p(U)$$

Note that this is similar to the first definition, because then we have

$$\langle X(p) \rangle + T_pU = T_pM$$

for all $p \in U$. We write

$$X \pitchfork U$$

Def. 2.11 (Smale Condition)**.** A pseudo gradient vectorfield is said to satisfy the *Smale condition*, if all its stable and unstable manifolds meet transversly, i.e if for all critical points p and q of f, we have

$$W^u(p) \pitchfork W^s(q)$$

Proposition 2.12. *Let (f, X) be a Smale pair. For critical points p and q of f define*

$$\mathcal{M}(p,q) = W^s \cap W^u = \left\{ r \in M : \lim_{s\to\infty} \varphi_s(r) = p \text{ and } \lim_{s\to-\infty} \varphi_s(r) = q \right\}$$

Proposition 2.13. ***Why does the action not need to be proper?*** *If $p \neq q$ are critical points of f, then $\mathbb{R}$ acts freely on $\mathcal{M}(p,q)$ via*

$$\begin{aligned} g : \mathbb{R} \times \mathcal{M}(p,q) &\to \mathcal{M}(p,q) \\ (t,p) &\mapsto \varphi_t(p) \end{aligned}$$

We define $\mathcal{L}(p,q) := \mathcal{M}(p,q)/\mathbb{R}$. Consequently $\mathcal{L}(p,q)$ is a manifold with

$$\dim \mathcal{L}(p,q) = \operatorname{ind}(p) - \operatorname{ind}(q) - 1$$

Proof. Note that $\mathbb{R}$ with addition is a Lie-group, that acts freely on $\mathcal{L}(p,q)$. This is easy to see: For any $x \in \mathcal{M}(p,q)$, the function $t \mapsto f(\varphi_t(x))$ is strictly decreasing, so if there are t, t' s.th. $\varphi_t(x) = \varphi_{t'}(x)$, then $t = t'$. With the quotient manifold theorem, $\mathcal{L}(p,q)$ is a manifold of dimension $\dim \mathcal{M}(p,q) - \dim \mathbb{R} = \operatorname{ind}(p) - \operatorname{ind}(q) - 1$. □

Remark. The most convenient way to consider the quotient is the following. If a is a **regular** value of f lying between $f(p)$ and $f(q)$, then $\mathcal{M}(p,q)$ is transversal to the level set $f^{-1}(a)$. This level-set has codimension 1 and the vector field X is transversal to it. All trajectories starting at p meet this level set at exactly one point, so $\mathcal{L}(p,q)$ can be identified by $\mathcal{M}(p,q) \cap f^{-1}(a)$.

Hence, if p and q are two distinct critical points and if the gradient used satisfies the Smale-condition, then for $\mathcal{M}(p,q)$ or $\mathcal{L}(p,q)$ to be non-empty, we must have

$$\operatorname{ind} p \geq \operatorname{ind} q$$

In other words, the index decreases along gradient lines.

§3 The Morse Complex

Proposition 3.1. *Define $C_k(M,f)$ as the $\mathbb{Z}/2$-Module generated by the critical points of f of index k and let $n_X(p,q) = \#\mathcal{L}(p,q) \mod 2$. Then*

$$\partial_X : C_k(M,f) \to C_{k-1}(M,f)$$

such that if p is a critical point of f of index k, we have

$$\partial_X(p) = \sum_{\text{ind}(p)=\text{ind}(q)+1} n_X(p,q)q$$

Then $(C_*(M,f), \partial_X)$ is a chain complex.

To proof that this is a chain complex we first have to examin the so called *space of broken trajectories*:

Def. 3.2. The *space of broken trajectories* is

$$\overline{\mathcal{L}}(p,q) = \bigcup_{\{c_1,...,c_i\} \subseteq \text{Crit}(f)} \mathcal{L}(p,c_1) \times \mathcal{L}(c_1,c_2) \times ... \times \mathcal{L}(c_i,q)$$

We can define a topology on this space as follows:
For this, let $\lambda = (\lambda_1, ..., \lambda_l) \in \overline{\mathcal{L}}(p,q)$. Then λ connects a certain number of critical points via the trajectories λ_i, where λ_i exits a critical point c_i and enters c_{i+1}. Now let U_i^- be a neghboehood of the point at which λ_i exits the chosen Morse neighborhood around c_{i-1} and U_i^+ be a neighborhood of the point at which λ_i enters the Morse-neighborhood of c_i. Then U^- is the collection of the U_i- and U_+ the collection of the U_i^+. Then we say that a trajectory $\mu = (\mu_1, ..., \mu_k) \in \mathcal{W}(\lambda, U^-, U^+)$, if there exist integers

$$0 < i_0 < ... < i_k = l$$

such that:

- $\mu_j \in \mathcal{L}(c_{i_j}, c_{i_{j+1}})$ for every $j \leq k$
- μ_j exits the chart $\Omega(c_{j+1})$ the interior of the corresponding element in U^- and enters the chart $\Omega(c_j)$ through interior of the corresponding element in U^+.

The $\mathcal{W}(\lambda, U^-, U^+)$ form a fundamental system of open neighborhoods for a topology on $\overline{\mathcal{L}}(p,q)$.

But do they?

It is clear that the resulting topology coincides with the topology of $\mathcal{L}(p,q)$.

Is it really?

Remark. We have seen earlier that $\mathcal{L}(c_i, c_{i+1})$ is only well defined if $c_i \neq c_{i+1}$, and we know that the index is decreasing along trajectory lines, so $\mathcal{L}(c_i, c_{i+1}) = \varnothing$ if $\text{ind}\, c_i \leq \text{ind}\, c_{i+1}$. Then $\overline{\mathcal{L}}(p,q)$ is a tuple that gives a "direction" along trajectories from p to q.

As suggested by the notation, $\overline{\mathcal{L}}(p,q)$ can be endowed with a topology, such that it is the closure of $\mathcal{L}(p,q)$, indeed its compactification. To show this is the aim of the next couple sections.

Theorem 3.3. *The space $\overline{\mathcal{L}}(p,q)$ is compact.*

Corollary 3.4. *If* $\operatorname{ind}(p) = \operatorname{ind}(q) + 1$*, then* $\mathcal{L}(p,q)$ *is finite.*

Remark. Hence $n_X(p,q)$ is well defined.

Proof. The corollary follows immedeatly from the theorem, because in this case we have

$$\mathcal{L}(p,q) = \overline{\mathcal{L}}(p,q)$$

□

Proof of the theorem. Let $(l_n)_n$ be a sequence in $\overline{\mathcal{L}}(p,q)$. We begin by assuming, that $l_n \in \mathcal{L}(p,q)$.The trajectory l_n exits $\Omega(p)$ through a point l_n^- and enters $\Omega(q)$ at a point l_n^+. The point l_n^- is in the intersection of the unstable manifold and the boundary $\partial\Omega(p)$. This is a sphere and therefore compact. After extracting a subsequence, we may, and do, therefore assume that

$$\lim l_n^- = a^- \quad \text{and} \quad \lim l_n^+ = b^+$$

Let $\gamma(t) = \varphi_t(a^-)$ be the trajectory of a^-, and let $c_1 = \lim_{t\to\infty} \gamma(t)$. Then c_1 is a critical point and $\gamma \in \mathcal{L}(p, c_1)$. Let d^+ be the entrypoint of γ into $\Omega(c_1)$. By the theorem of the dependence of differential equation on the initial condition, for a large enough n, l_n must also enter $\Omega(c_1)$ through a point d_n^+. Then by the following lemma we get $\lim d_n^+ = d_n$. □

§4 Morse Homology

We set $C_*(M, f; R)$ as the R-Module generated by the critical points of f.