PHYS 425 - w3l3

Mean Free Path ℓ of Gas Particles

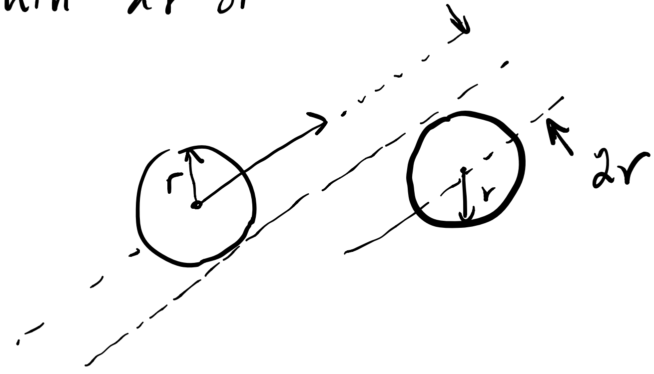
Quasi-ideal gases ... to understand thermal conduction in real gases, need to consider collisions between gas particles.

A gas for which thermodynamic properties determined by ideal gas approx ($PV = Nk_BT$) but in which molecules undergo collisions is called quasi-ideal.

Will consider elastic collision (K.E. conserved)

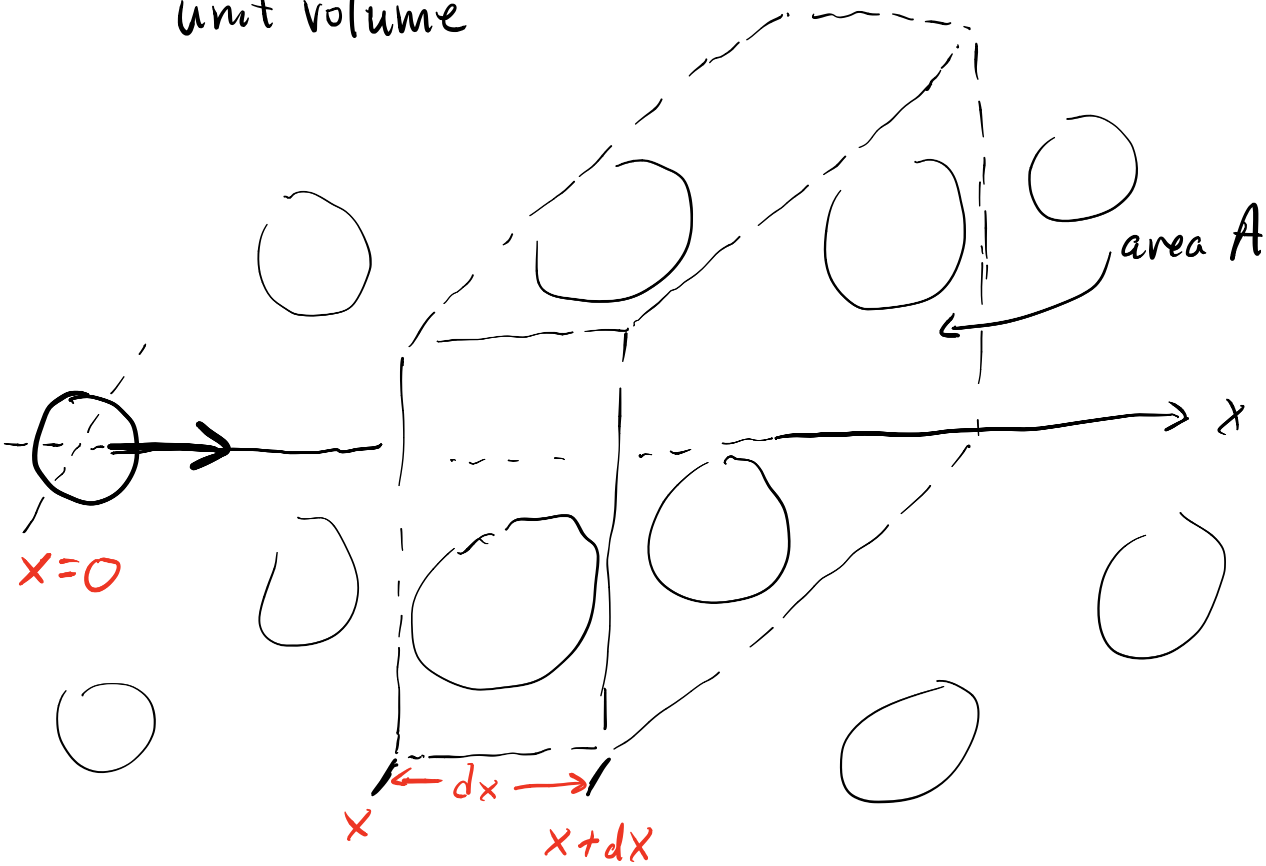
→ scattering

Particles will collide if their centres come within $2r$ of eachother.



Scattering cross-section $\sigma_c = \pi(2r)^2$

Suppose that there are n particles per unit volume



In a layer of thickness dx & area A perpendicular to dir'n of motion of a particle moving in x-dir'n there are:

$n\,A\,dx$ particles each w/ scattering cross-section σ_c.

Total scattering cross-section

$$\sigma_c \, n A \, dx.$$

Prob. that incoming particle is scattered in x to $x+dx$ is $\dfrac{\sigma_c n \cancel{A} dx}{\cancel{A}} = \sigma_c n \, dx$

or prob. that particle gets through region of thickness dx is $1 - \sigma_c n \, dx$

Assume particle originates from $x=0$.

Prob. that particle makes it to x w/o scattering is $\equiv P(x)$

Prob. that particle goes from x to $x+dx$ w/o scattering is $(1 - \sigma_c n \, dx)$

$\therefore$ Prob. that particle initially at $x=0$ goes to x unscattered & then to $x+dx$ w/o scattering is:

$$\underbrace{P(x+dx)}_{\substack{\text{prob. go from} \\ x=0 \text{ to } x+dx \\ \text{w/o scattering}}} = \underbrace{P(x)}_{\substack{\text{prob. go from} \\ x=0 \text{ to } x \\ \text{w/ scattering}}} \cdot \overbrace{\left(1-\sigma_c n \, dx\right)}^{\substack{\text{prob. go from } x \\ \text{to } x+dx \text{ w/o scattering}}}$$

$$P(x+dx) = P(x) - P(x)\,\sigma_c n \, dx$$

$$\therefore \underbrace{\frac{P(x+dx)-P(x)}{dx}}_{\frac{dP(x)}{dx}} = -P(x)\,\sigma_c n$$

$$\frac{dP(x)}{dx} = -P(x)\,\sigma_c n \Rightarrow \frac{dP(x)}{P(x)} = -\sigma_c n \, dx$$

Integrate:

$$\int_{P(0)}^{P(x)} \frac{dP(x)}{P(x)} = \int_0^x -\sigma_c n \, dx$$

$$\ln\left(\frac{P(x)}{P(0)}\right) = -\sigma_c n x$$

or $$P(x) = P(0) e^{-\sigma_c n x}$$

$P(0)$ is prob. of going from $x=0$ to $x=0$ w/o scattering! $\therefore P(0) = 1$

$$P(x) = e^{-\sigma_c n x} = e^{-x/\ell}$$

$$\ell \equiv \frac{1}{\sigma_c n}$$ & ℓ is called the mean free path of the particle.

ℓ is the avg. dist. that a gas particle travels between scattering events.

As expected $l \propto n^{-1}$ higher density

$\rightarrow$ higher collision rate

$\rightarrow$ lower mfp. l

The radius r of molecules not well defined.
Effective radius of N_2 or O_2 molecules

$$r \approx 1.5\,\text{Å} = 1.5 \times 10^{-10}\,\text{m}$$
$$= 0.15 \times 10^{-9}\,\text{m} = 0.15\,\text{nm}$$

For an ideal gas at room temp

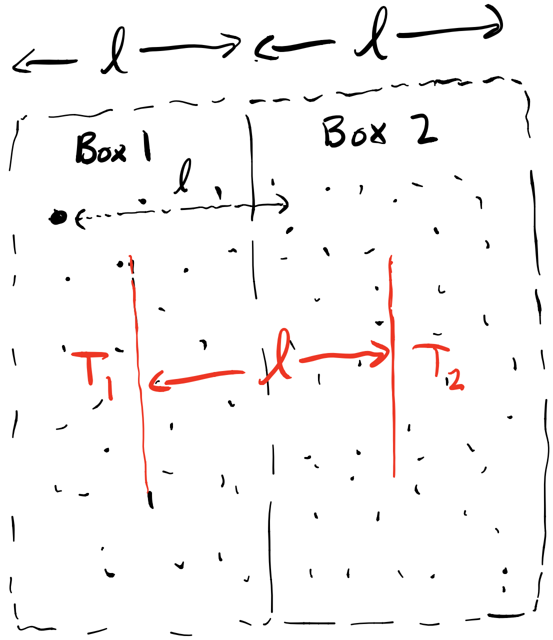
$$n = \frac{N}{V} = \frac{P}{k_B T} \simeq 2.4 \times 10^{25}\,\text{m}^{-3}$$

$$\boxed{l = \frac{1}{\pi (2r)^2 n} \simeq 140\,\text{nm}}$$

typical mfp for air at STP

Thermal Conductivity of Gas

Attempt to calc. heat flow that crosses Boundary btn box 1 & box 2.



$\Delta t = \frac{\ell}{\bar{v}}$ is avg. time between collisions

$\bar{v}$ is avg. speed of gas molecules.

Molecule that crosses plane from left to right → will have started somewhere within Box 1.

If total energy in Box 1 is U_1, then energy crossing plane left to right is

$$\approx \frac{U_1}{2}$$ (half molecules moving right & enter Box 2, half moving left & do not enter Box 2).

Likewise, have $\frac{U_2}{2}$ moving right to left across boundary.

Change in energy of box 2:

$$Q = \frac{1}{2}U_1 - \frac{1}{2}U_2 = -\frac{1}{2}(U_2 - U_1)$$

$$C_v = \frac{\Delta U}{\Delta T} \Rightarrow \Delta U = C_v \Delta T$$

heat capacity of particles in one box.

$$Q = -\frac{1}{2}C_v(T_2 - T_1)$$

T_2 is avg. temp in Box 2

T_1 is avg. temp in Box 1.

Temp gradient is $\frac{dT}{dx}$

$$\Delta T = \frac{dT}{dx}\ell = T_2 - T_1$$

$$Q = -\frac{1}{2} c_v \ell \frac{dT}{dx}$$

kappa: thermal conductivity of gas.

By definition heat flux $\frac{\dot{Q}}{A} = -K\frac{dT}{dt}$

$$\frac{Q}{\Delta t A} = -\underbrace{\frac{1}{2}\frac{c_v \ell}{A \Delta t}}_{\equiv K}\frac{dT}{dx}$$

The thermal conductivity of quasi-ideal gas is $K = \frac{c_v \ell}{2A\Delta t}$

For our scenario $\Delta t = \frac{\ell}{\bar{v}}$

$V = Al$ is the volume of box.

$$K = \frac{C_v\, l}{2\frac{V}{\cancel{l}}\frac{\cancel{l}}{\bar{v}}} = \frac{1}{2}\frac{C_v}{V}\bar{v}l$$

mass of each particle

write heat capacity $C_v = N m c_v$ ← specific heat or heat capacity per unit mass.

no. of particles

$$K = \frac{1}{2}\frac{N m c_v}{V}\bar{v}l = \boxed{\frac{1}{2} n m c_v \bar{v} l}$$

Thermal conducity of quasi-ideal gas.

units $[K] = \frac{W}{K \cdot m}$

Note a full (but difficult) calculation give

$$\boxed{K = \frac{1}{3} n m c_v \bar{v} l}$$