PHYS 425 - w5l3

Goal: Calculate heat capacity of a system of identical Fermions at low temp.

Low temp means $T \ll T_F = \frac{E_F}{k_B}$

Fermi temp.

Beautiful calc that connect microscopic quantum theory to measurable macroscopic thermodynamic quantity.

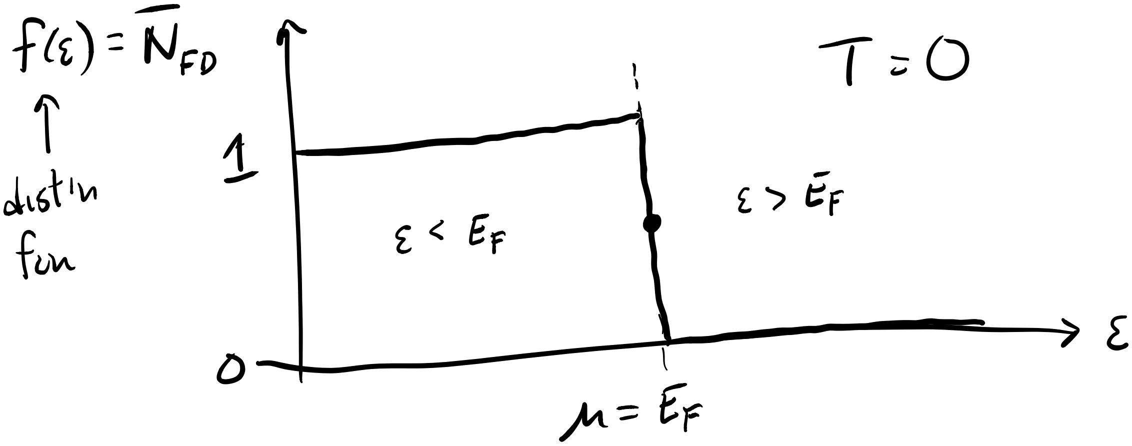
Results apply to:
- conduction electrons in a metal. ($T_F \approx 10^4 - 10^5$ K)
- neutrons in a neutron star
- ^{3}He atoms in L. ^{4}He

$\boxed{\text{Will show that } C_V \propto T \text{ when } T \ll T_F}$

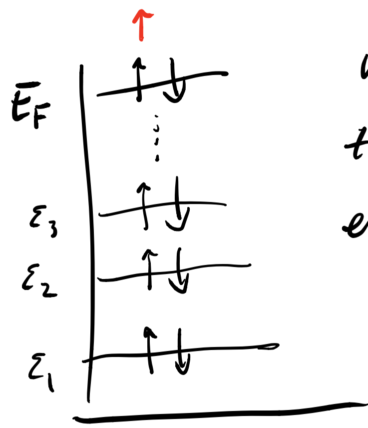
Density of States (DOS)

$D(\varepsilon)\,\underline{d\varepsilon}$ tell us how many quantum states there are between engeries of ε and $\varepsilon + d\varepsilon$

If we have a system of N Fermions at $T = 0$, then all states below E_F are filled and all states above E_F empty.



Recall chemical pot. μ tells us change in system energy when no. of particles is changed.



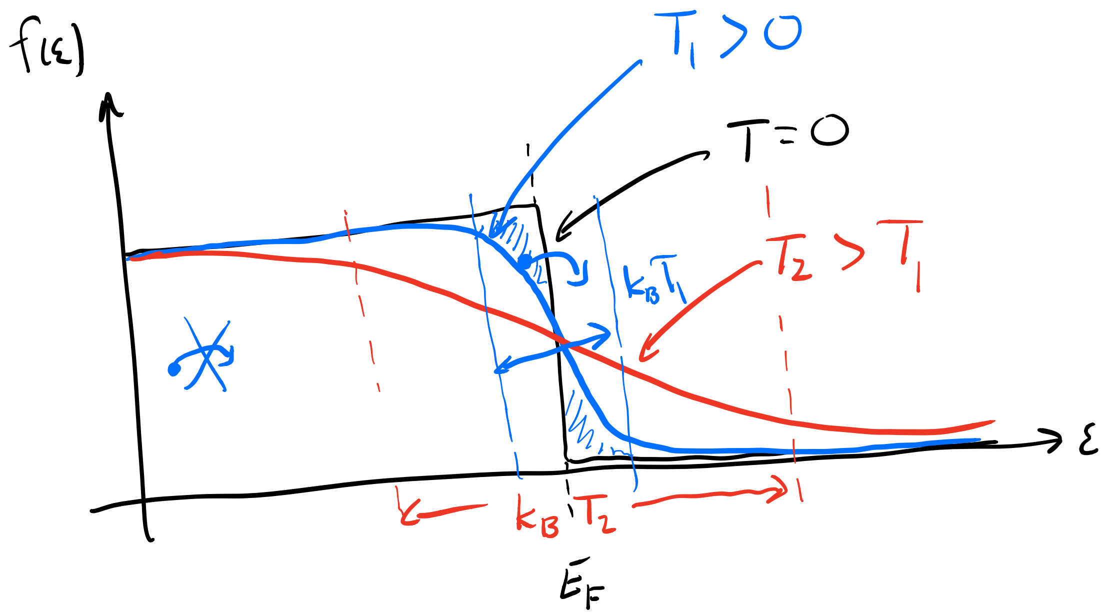
when add single particle to system at $T=0$, system energy increases by E_F

$\Rightarrow \therefore \mu = E_F$ @ $T=0$.

$$\therefore N = \underbrace{\int_0^{E_F} D(\varepsilon)\,d\varepsilon}_{\text{no. of states from } \varepsilon=0 \text{ to } \varepsilon=E_F.}$$

The Fermi-Dirac distr'n fcn $f(\varepsilon)$ gives prob. that, at temp T, a state w/ energy ε is occupied.

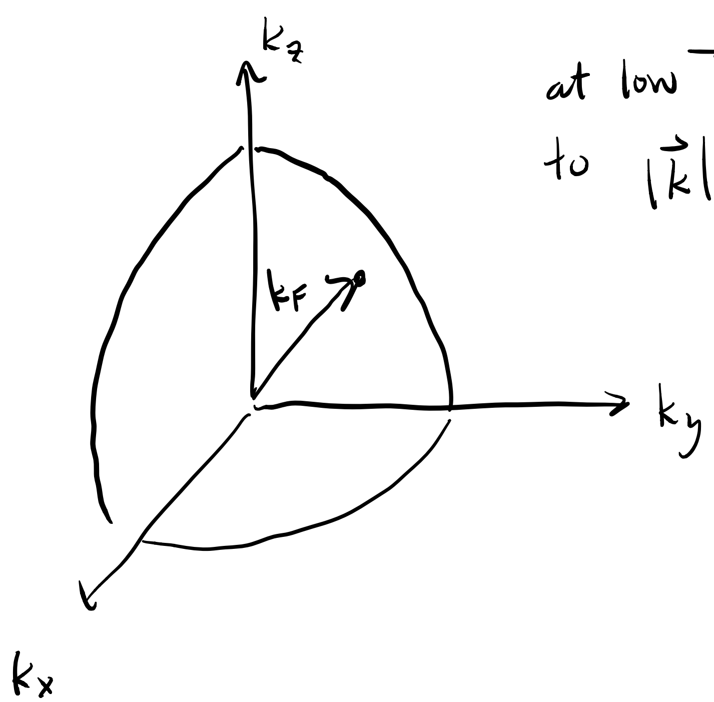
$$N = \int_0^\infty f(\varepsilon)\, D(\varepsilon)\, d\varepsilon$$



For $0 < T \ll T_F$, only particles in states near E_F can be excited into higher energy states. Particles in low energy states don't have enough thermal energy ($k_B T$) to reach unoccupied states above E_F.

The functional form of $f(\varepsilon)$ is

$$f(\varepsilon) = \frac{1}{\exp\left(\frac{\varepsilon-\mu}{k_B T}\right)+1}$$



at low T, states up to $|\vec{k}| < k_F$ filled

So another way to express N is:

$$N = 2 \int_{|\vec{k}| \le k_F} \frac{d\vec{k}}{\left(\frac{\pi}{L}\right)^3}$$

$\left(\frac{\pi}{L}\right)^3 = \frac{\pi^3}{V}$

two particles per k-state.

$\left(\frac{\pi}{L}\right)^3$ ← volume occupied per k-state

$$N = \cancel{2} \int_0^{k_F} \frac{V}{\pi^3} \; \frac{1}{\cancel{8}} \cancel{4}\pi k^2 dk$$

$\uparrow$ pos. oct.

$$= \frac{V}{\pi^2} \int_0^{k_F} k^2 dk \quad \leftarrow$$

write $\quad dk = \frac{dk}{d\varepsilon} d\varepsilon \qquad\qquad \frac{d\varepsilon}{dk} = \frac{d}{dk}\left(\frac{\hbar^2 k^2}{2m}\right)$

$$= \frac{\hbar^2 k}{m}$$

$$\therefore \frac{dk}{d\varepsilon} = \frac{m}{\hbar^2 k}$$

$$\therefore N = \frac{V}{\pi^2} \int_0^{\varepsilon_F} k^{\cancel{2}} \frac{m}{\hbar^2 \cancel{k}} d\varepsilon \qquad\qquad \varepsilon = \frac{\hbar^2 k^2}{2m}$$

$$\rightarrow \quad k = \frac{\sqrt{2m\varepsilon}}{\hbar}$$

$$N = \frac{V}{\pi^2} \int_0^{\varepsilon_F} \frac{\sqrt{2m\varepsilon}}{\hbar} \; \frac{m}{\hbar^2} \, d\varepsilon$$

$$= \int_0^{\varepsilon_F} \frac{mV}{\pi^2 \hbar^3} \sqrt{2m\varepsilon} \; d\varepsilon \left(\frac{2}{2}\right)$$

$$= \int_0^{\varepsilon_F} \boxed{\frac{V}{2\pi^2}\left(\frac{2m}{\hbar^2}\right)^{3/2}\sqrt{\varepsilon}}\, d\varepsilon$$

Compare to (#) $\quad N = \int_0^{\varepsilon_F} \boxed{D(\varepsilon)}\, d\varepsilon$

$\therefore$ DOS is

$$\boxed{D(\varepsilon) = \frac{V}{2\pi^2}\left(\frac{2m}{\hbar^2}\right)^{3/2}\sqrt{\varepsilon}}$$

Next goal is to use DOS to calc. heat capacity.

Internal Energy

prob. state is occupied

$$U = \int_0^{\infty} \underset{\text{energy}}{\varepsilon}\, f(\varepsilon)\, \underbrace{D(\varepsilon)}_{\text{no. of states w/ energy } \varepsilon}\, d\varepsilon$$

$$\rightarrow \quad C_v = \left(\frac{\partial U}{\partial T}\right)_V = \int_0^\infty \varepsilon \frac{df(\varepsilon)}{dT} D(\varepsilon)\, d\varepsilon$$

First, note that

$$\rightarrow \int_0^\infty \frac{df(\varepsilon)}{dT} D(\varepsilon)\, d\varepsilon = \frac{d}{dT} \underbrace{\int_0^\infty f(\varepsilon) D(\varepsilon)\, d\varepsilon}_{N}$$

$$= \frac{dN}{dT} = 0 \quad \left(\begin{matrix}\text{fixed} \\ \text{no. of particles}\end{matrix}\right)$$

$$C_v = \frac{\partial U}{\partial T} = \int_0^\infty \varepsilon \frac{df(\varepsilon)}{dT} D(\varepsilon)\, d\varepsilon - \mu \underbrace{\int_0^\infty \frac{df(\varepsilon)}{dT} D(\varepsilon)\, d\varepsilon}_{0}$$

$$= \int_0^\infty (\varepsilon - \mu) \underbrace{\frac{df(\varepsilon)}{dT}}_{\text{replace}} D(\varepsilon)\, d\varepsilon$$

#

Evaluated df/dT

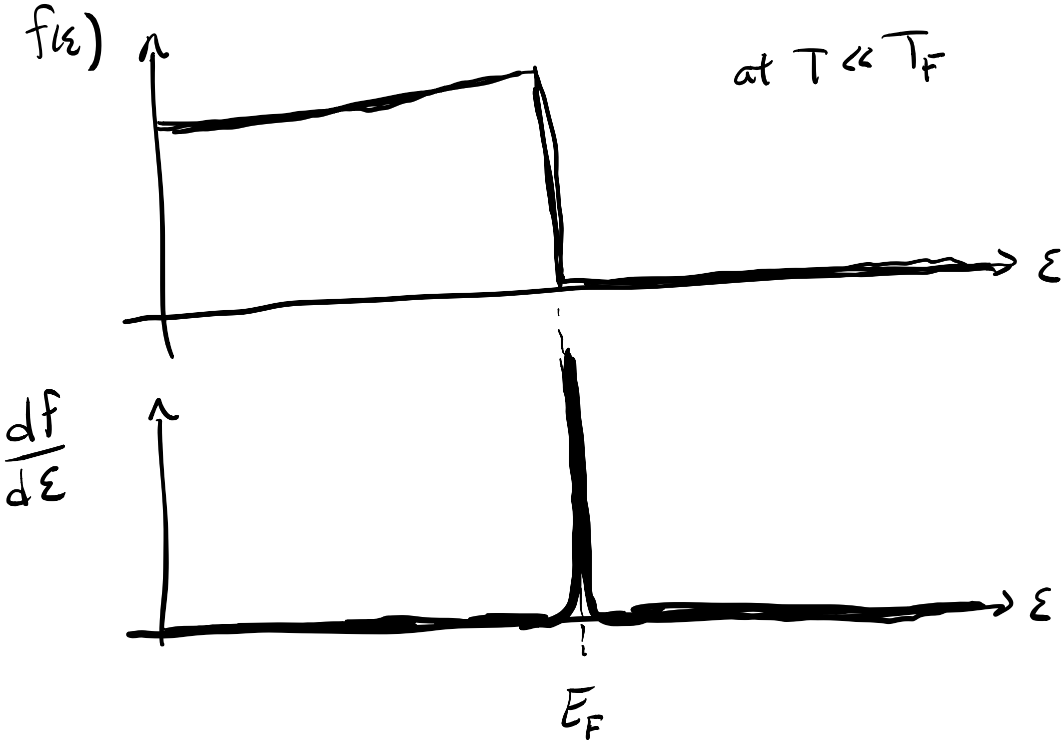
$$\frac{df(\varepsilon)}{dT} = \frac{d}{dT}\frac{1}{\exp\left(\frac{\varepsilon-\mu}{k_B T}\right)+1} = \frac{+1}{\left(\exp\left(\frac{\varepsilon-\mu}{k_B T}\right)+1\right)^2} \cdot \left(+\frac{(\varepsilon-\mu)}{k_B T^2}\exp\left(\frac{\varepsilon-\mu}{k_B T}\right)\right)$$

$$\frac{df(\varepsilon)}{dT} = \frac{\frac{\varepsilon-\mu}{k_B T^2}\exp\left(\frac{\varepsilon-\mu}{k_B T}\right)}{\left(\exp\left(\frac{\varepsilon-\mu}{k_B T}\right)+1\right)^2}$$

$$\frac{df(\varepsilon)}{d\varepsilon} = \frac{d}{d\varepsilon}\left(\frac{1}{\exp\left(\frac{\varepsilon-\mu}{k_B T}\right)+1}\right) = \frac{-\frac{1}{k_B T}\exp\left(\frac{\varepsilon-\mu}{k_B T}\right)}{\left(\exp\left(\frac{\varepsilon-\mu}{k_B T}\right)+1\right)^2}$$

Notice that $\boxed{\frac{df(\varepsilon)}{dT} = -\left(\frac{\varepsilon-\mu}{T}\right)\frac{df}{d\varepsilon}}$ #

$$\therefore C_v = -\int_0^\infty \frac{(\varepsilon-\mu)^2}{T} \frac{df(\varepsilon)}{d\varepsilon} D(\varepsilon)\, d\varepsilon$$



At low temps, $\frac{df(\varepsilon)}{d\varepsilon}$ is strongly peaked at $\varepsilon = E_F$. $\therefore$ in integral, can replace $D(\varepsilon)$ w/ $D(E_F)$.

$$C_v \approx -D(E_F) \int_{0}^{\infty} \frac{(\varepsilon-\mu)^2}{T} \underbrace{\frac{df}{d\varepsilon} d\varepsilon}_{df}$$

Note that when $\varepsilon = \infty, \quad f = 0$

$\varepsilon = 0, \quad f = 1$

$$C_v \approx -\frac{D(E_F)}{T} \int_{1}^{0} (\varepsilon-\mu)^2 \, df$$

$$= \frac{D(E_F)}{T} \int_{0}^{1} (\varepsilon-\mu)^2 \, df$$

If $f(\varepsilon) = \left[\exp\left(\frac{\varepsilon-\mu}{k_B T}\right) + 1\right]^{-1}$

$$\frac{1}{f} - 1 = \exp\left(\frac{\varepsilon-\mu}{k_B T}\right)$$

or $\boxed{k_B T \ln\left(\frac{1}{f} - 1\right)} = \varepsilon - \mu$

$$\therefore C_V = \frac{D(E_F)}{T}\int_0^1 (k_B T)^2 \left[\ln\left(\frac{1}{f}-1\right)\right]^2 df$$

$$= \underline{k_B^2}\,\underline{T}\, D(E_F) \underbrace{\int_0^1 \left[\ln\left(\frac{1}{f}-1\right)\right]^2 df}_{\frac{\pi^2}{3}\ (\text{exact sol'n})}$$

Recall $D(E_F) = \dfrac{V}{2\pi^2}\left(\dfrac{2m}{\hbar^2}\right)^{3/2}\sqrt{E_F}$

also recall $E_F = \dfrac{\hbar^2}{2m}\left(\dfrac{3\pi^2 N}{V}\right)^{2/3}$

$$\therefore E_F^{3/2} = \left(\frac{\hbar^2}{2m}\right)^{3/2} \frac{3\pi^2 N}{V}$$

$$C_V = k_B^2 T \frac{N}{N} \frac{V}{2\pi^2}\left(\frac{2m}{\hbar^2}\right)^{3/2} \frac{\pi^2}{3}\sqrt{E_F}$$

$$C_V = \frac{k_B^2 T}{2} N \pi^2 \underbrace{\frac{V}{3\pi^2 N}\left(\frac{2m}{\hbar^2}\right)^{3/2}}_{\frac{1}{E_F^{3/2}}} \sqrt{E_F}$$

$$C_V = k_B^2 T N \frac{\pi^2}{2} \frac{1}{E_F} = \frac{\pi^2}{2} k_B^{\cancel{2}} T N \frac{1}{\cancel{k_B} T_F}$$

low-temp heat capacity of identical Fermions is:

$$\boxed{C_V = \frac{\pi^2}{2} N k_B \frac{T}{T_F}} \qquad C_V \propto T$$