PHYS 425 – W12L2

Last Time: Sub $\psi(\vec{r},t) = \psi_0 e^{iS}$

into $$i\hbar \frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2m_4} \nabla^2 \psi + \mu \psi$$

Lead to a complex eq'n (real & imaginary parts)

Real Part:

$$-\hbar \frac{\partial S}{\partial t} = -\frac{\hbar^2}{2m_4}\left[\frac{\nabla^2 \psi_0}{\psi_0} - (\vec{\nabla} S)^2\right] + \mu$$

$$\Downarrow$$

$$\circledast \quad \boxed{\hbar \frac{\partial S}{\partial t} \approx -\left(\mu + \frac{1}{2} m_4 v_s^2\right)}$$

Today ⇒ Imaginary Part

$$\frac{\hbar}{\psi_0} \frac{\partial \psi_0}{\partial t} = -\frac{\hbar^2}{2m_4}\left[2 \frac{\vec{\nabla}\psi_0 \cdot \vec{\nabla} S}{\psi_0} + \nabla^2 S\right]$$

This will lead continuity eq'n for superfluid

Recall: $\psi_0^2 = \frac{\rho_s}{m_4} \Rightarrow m_4 \psi_0^2 = \rho_s$

$$\frac{\hbar}{m_4} \vec{\nabla} S = \vec{V}_s$$

$$\frac{1}{\psi_0}\frac{\partial \psi_0}{\partial t} = -\frac{\hbar}{2m_4}\left[2\,\frac{\vec{\nabla}\psi_0 \cdot \vec{\nabla} S}{\psi_0} + \nabla^2 S\right]$$

mult. by $2m_4\psi_0^2$

$$\underbrace{2m_4\psi_0\frac{\partial \psi_0}{\partial t}}_{\frac{\partial}{\partial t}\left(m_4\psi_0^2\right) = \frac{\partial \rho_s}{\partial t}} = -\frac{\hbar}{m_4}\Big[\underbrace{2m_4\psi_0\vec{\nabla}\psi_0}_{\vec{\nabla}\left(m_4\psi_0^2\right) = \vec{\nabla}\rho_s}\cdot\vec{\nabla}S + \underbrace{m_4\psi_0^2}_{\rho_s}\nabla^2 S\Big]$$

$$2\psi_0\frac{\partial \psi_0}{\partial t} = \frac{\partial}{\partial t}\left(\psi_0^2\right)$$

$$\frac{1}{2}\frac{\partial}{\partial t}\left(\psi_0^2\right)$$

$$\therefore \frac{\partial \rho_s}{\partial t} = -\vec{\nabla}\rho_s\cdot\underbrace{\left(\frac{\hbar}{m_4}\vec{\nabla}S\right)}_{\vec{V}_s} - \underbrace{\frac{\hbar}{m_4}\rho_s\overbrace{\vec{\nabla}\cdot\vec{\nabla}S}^{\nabla^2 S}}_{\rho_s\vec{\nabla}\cdot\left(\frac{\hbar}{m_4}\vec{\nabla}S\right)}$$

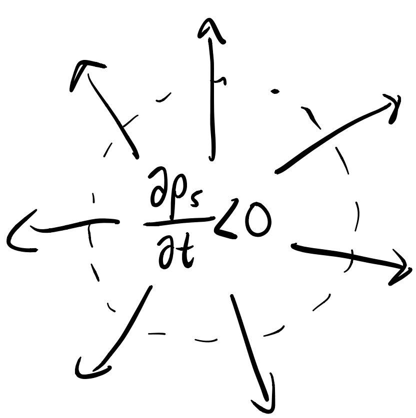
$$= \rho_s \vec{\nabla}\cdot\vec{v}_s$$

$$\therefore \quad \frac{\partial \rho_s}{\partial t} = \underbrace{-\vec{\nabla}\rho_s \cdot \vec{v}_s - \rho_s \vec{\nabla}\cdot\vec{v}_s}_{-\vec{\nabla}\cdot(\underbrace{\rho_s \vec{v}_s}_{\equiv \vec{j}_s})}$$

(mass flow rate per unit area)

$$\boxed{\frac{\partial \rho_s}{\partial t} = -\vec{\nabla}\cdot\vec{j}_s}$$

continuity eq'n for the superfluid



positive divergence

$\vec{\nabla}\cdot\vec{j}_s > 0$

Back to ④

$$\hbar \frac{\partial S}{\partial t} = -\left(\mu + \frac{1}{2} m_4 v_s^2\right)$$

Note: ① $\vec{\nabla}(\vec{A}\cdot\vec{B}) = (\vec{A}\cdot\vec{\nabla})\vec{B} + (\vec{B}\cdot\vec{\nabla})\vec{A}$

$$+ \vec{A}\times(\vec{\nabla}\times\vec{B}) + \vec{B}\times(\vec{\nabla}\times\vec{A})$$

② $\vec{\nabla}\times(\vec{\nabla} a) = 0$

Take gradient of ④

$$\hbar \frac{\partial}{\partial t}(\vec{\nabla} S) = -\vec{\nabla}\left(\mu + \frac{1}{2} m_4 \vec{v}_s\cdot\vec{v}_s\right)$$

$$\vec{\nabla} S = \frac{m_4}{\hbar}\vec{v}_s$$

$$\therefore \quad m_4 \frac{\partial \vec{v}_s}{\partial t} = -\vec{\nabla}\mu - \frac{1}{2} m_4 \underbrace{\vec{\nabla}(\vec{v}_s\cdot\vec{v}_s)}_{2(\vec{v}_s\cdot\vec{\nabla})\vec{v}_s + 2\vec{v}_s\times(\vec{\nabla}\times\vec{v}_s)}$$

but $\vec{\nabla}\times\vec{v}_s = \underbrace{\vec{\nabla}\times\left(\frac{\hbar}{m_4}\vec{\nabla} S\right)}_{=0 \text{ by } ②}$

$$\therefore \; m_4 \frac{\partial \vec{v}_s}{\partial t} = -\vec{\nabla}\mu - m_4 (\vec{v}_s \cdot \vec{\nabla})\vec{v}_s$$

or

$$\boxed{\underbrace{\frac{\partial \vec{v}_s}{\partial t} + (\vec{v}_s \cdot \vec{\nabla})\vec{v}_s}_{\text{acceleration}} = \underbrace{-\frac{1}{m_4}\vec{\nabla}\mu}_{\text{minus gradient of potential energy divided by mass}}} \quad \#$$

Euler Eq'n for an ideal fluid → zero viscosity

Eq'n of motion for the Fluid

$$\left(\vec{a} = \frac{\vec{F}}{m} \right)$$

$$\vec{v}_s = v_{sx}\hat{i} + v_{sy}\hat{j} + v_{sz}\hat{k}$$

$$\vec{v}_s = \vec{v}_s(x, y, z; t)$$

$$\frac{dv_{sx}}{dt} = \underbrace{\frac{\partial v_{sx}}{\partial x}\underbrace{\frac{\partial x}{\partial t}}_{v_{sx}} + \frac{\partial v_{sx}}{\partial y}\underbrace{\frac{\partial y}{\partial t}}_{v_{sy}} + \frac{\partial v_{sx}}{\partial z}\underbrace{\frac{\partial z}{\partial t}}_{v_{sz}}}_{[(\vec{v}_s \cdot \vec{\nabla})\vec{v}_s]_x} + \frac{\partial v_{sx}}{\partial t}$$

material derivative

compare to:

$$(\vec{v}_s \cdot \vec{\nabla})\vec{v}_s = \left(v_{sx}\frac{\partial}{\partial x} + v_{sy}\frac{\partial}{\partial y} + v_{sz}\frac{\partial}{\partial z}\right)\left(v_{sx}\hat{\imath} + v_{sy}\hat{\jmath} + v_{sz}\hat{k}\right)$$

$$\left(v_{sx}\frac{\partial v_{sx}}{\partial x} + v_{sy}\frac{\partial v_{sx}}{\partial y} + v_{sz}\frac{\partial v_{sx}}{\partial z}\right)\hat{\imath} + (\quad)\hat{\jmath} + (\quad)\hat{k}$$

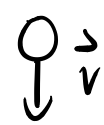
$\therefore$ $\frac{\partial \vec{v}_s}{\partial t} + (\vec{v}_s \cdot \vec{\nabla})\vec{v}_s$ just describes how $\vec{v}_s$ evolves w/ time $\rightsquigarrow$ acceleration

$$\boxed{\frac{\partial \vec{v}_s}{\partial t} + (\vec{v}_s \cdot \vec{\nabla})\vec{v}_s = -\frac{1}{m_4}\vec{\nabla}\mu}$$ #

No dissipation in #

→ like a mass falling in gravitational field

$\frac{1}{2}mv^2 + mgy = \text{const}$



take a time derivative

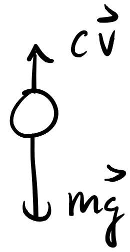
$$\cancel{m}\cancel{v}\frac{dv}{dt} + \cancel{m}g\cancel{v} = 0$$

$$\therefore \underbrace{-g}_{\substack{\text{minus gradient} \\ \text{of P.E. per unit mass}}} = \frac{dv}{dt} \leftarrow \text{acceleration}$$

$$\text{P.E.} = mgy$$

$$-\frac{\partial}{\partial y}(\text{P.E.}) = -mg$$

If include air resistance, Newton's Law becomes



$$ma = -mg + cv$$

$$\therefore -g = \frac{dv}{dt} - \frac{c}{m} v$$

↗ − gradient of P.E. ↑ acceleration ← dissipation

For comparison, the Naiver-Stokes eq'n for fluid flow for an incompressible fluid is:

$$\underbrace{\frac{\partial \vec{v}}{\partial t} + (\vec{v}\cdot\vec{\nabla})\vec{v}}_{\text{acceleration}} - \underbrace{\eta \vec{\nabla}^2 \vec{v}}_{\text{dissipation}} = \underbrace{\frac{-\vec{\nabla}\mu}{m}}_{\substack{-\text{ gradient} \\ \text{of P.E.}}}$$

η ← viscosity