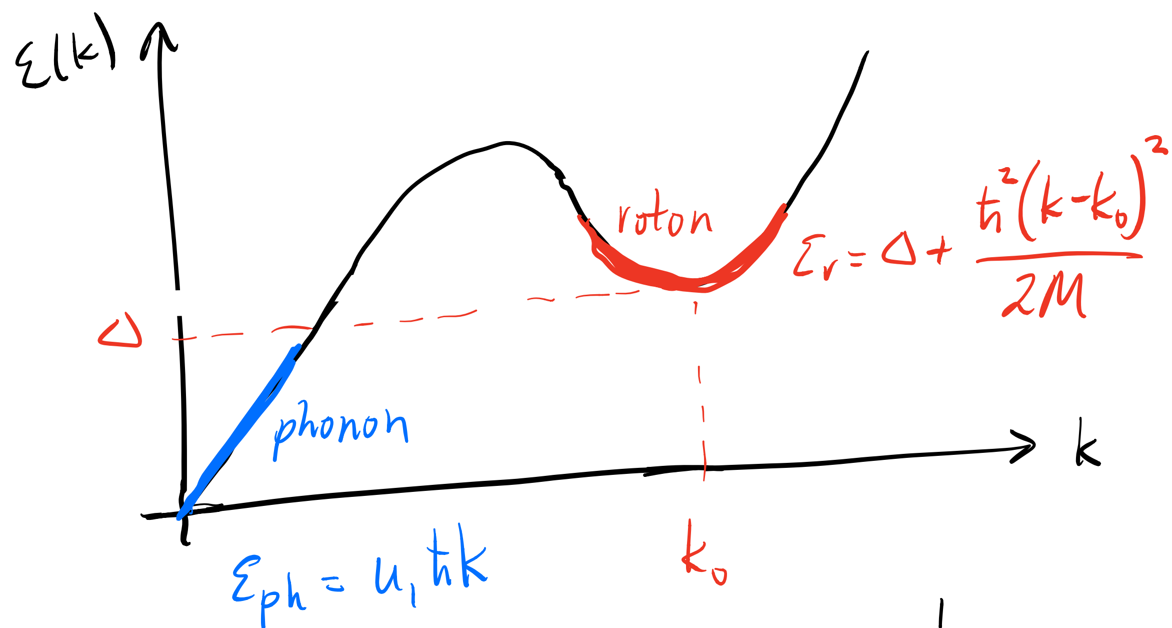
PHYS 425 - w11l1

Last Time: Landau's dispersion relation for liquid ^{4}He excitations



Using $\Phi_G = \frac{V}{2\pi^2} k_B T \int_0^\infty k^2 \ln\left[1 - e^{-\varepsilon(k)/k_BT}\right] dk$

$S = -\left(\frac{\partial \Phi_G}{\partial T}\right)_{V,\mu}$ & $C_V = T\left(\frac{\partial S}{\partial T}\right)$

found $C_{V,ph} \propto T^3$ ←

Sound $\quad v = f\lambda$

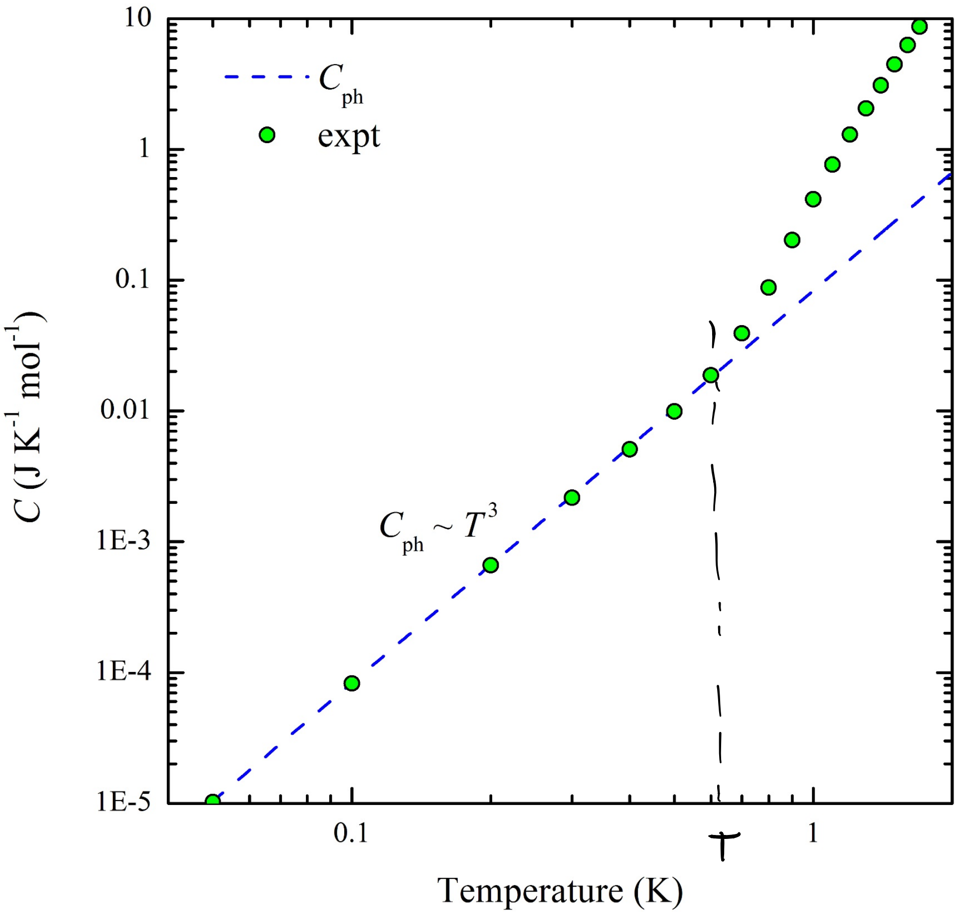
$$f = v\,\frac{1}{\lambda}$$

$$\omega = v k \qquad \text{phonon dispersion}$$

$$E = v\hbar k \leftrightarrow \varepsilon_{ph} = u_1 \hbar k$$

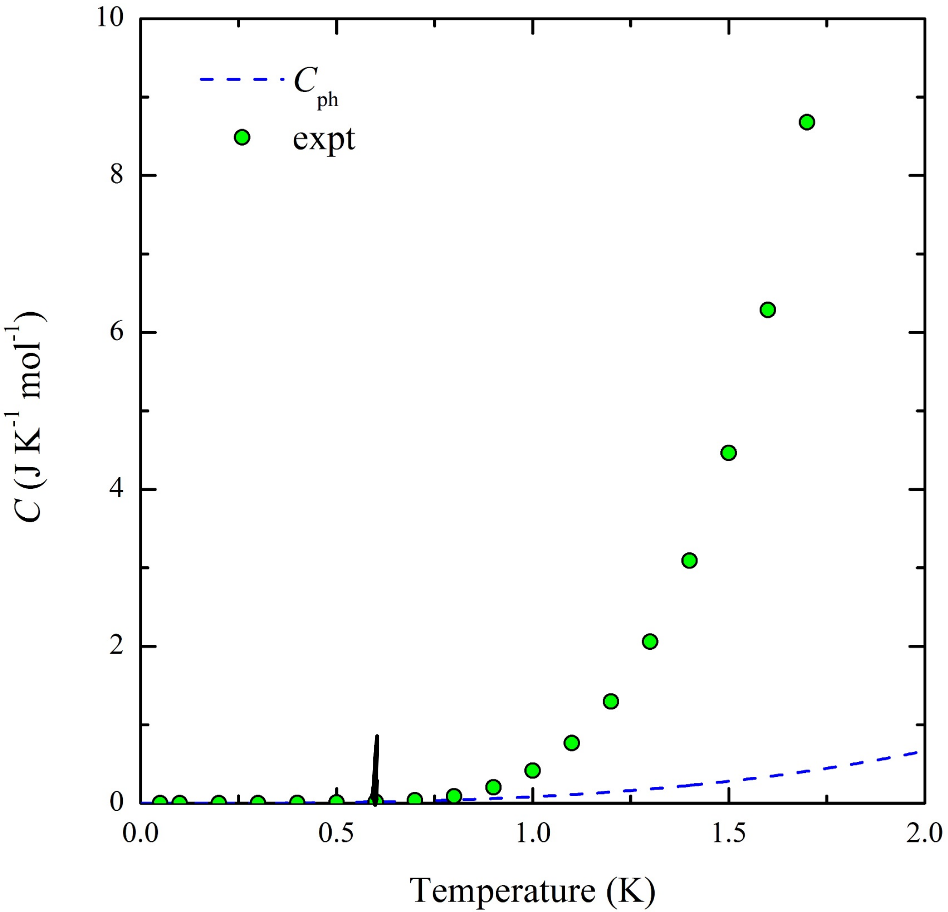
He II Specific Heat

The low-temp phonon contribution to the specific heat of He II.

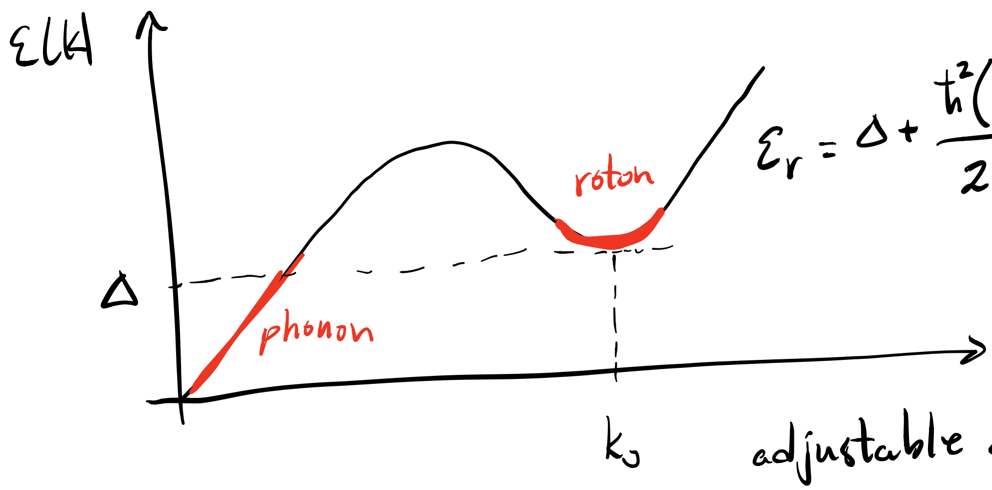


He II Specific Heat

We still need to describe the "activation" of the specific heat at higher temperature (but still below).



Superfluid ^{4}He dispersion relation



$$\mathcal{E}_r = \Delta + \frac{\hbar^2(k-k_0)^2}{2M}$$

adjustable parameters

Δ, k_0, M

At lowest temps, thermal energy $k_B T < \Delta$ inadequate to excite ^{4}He atoms into roton states. Once cross a threshold temp, start to occupy roton states & get another contribution to C_V.

Rotons: $$\mathcal{E}_r = \Delta + \frac{\hbar^2(k-k_0)^2}{2M}$$

at low-T

$$\Phi_{G,r} = \frac{V}{2\pi^2} k_B T \int_0^\infty k^2 \ln\left[1 - e^{-\Delta/k_BT} e^{-\hbar^2(k-k_0)^2/2Mk_BT}\right] dk$$

rotons only present at high energy

$$e^{-\Delta/k_BT} \text{ small}$$

$$\ln\left[1 - e^{-\Delta/k_BT} e^{-\hbar^2(k-k_0)^2/2Mk_BT}\right]$$

$$\approx -e^{-\Delta/k_BT} e^{-\hbar^2(k-k_0)^2/2Mk_BT}$$

$$\therefore \Phi_{G,r} \approx -\frac{V}{2\pi^2} k_BT e^{-\Delta/k_BT} \int_0^\infty k^2 e^{-\hbar^2(k-k_0)^2/2Mk_BT} dk$$

$$x^2 = \frac{\hbar^2(k-k_0)^2}{2Mk_BT} \qquad k - k_0 = \underbrace{\sqrt{\frac{2Mk_BT}{\hbar^2}}}_{\equiv \frac{1}{\lambda}} x$$

$$k = \frac{x}{\lambda} + k_0 \qquad k=0 \quad x = -\lambda k_0$$

$$dk = \frac{dx}{\lambda} \qquad k = \infty \quad x = \infty$$

$$\Phi_{G,r} \simeq -\frac{V}{2\pi^2} k_B T \frac{e^{-\Delta/k_B T}}{\lambda} \int_{-\lambda k_0}^{\infty} \left(\frac{x}{\lambda} + k_0\right)^2 e^{-x^2} dk$$

$$= -\frac{V}{2\pi^2} k_B T \frac{e^{-\Delta/k_B T}}{\lambda^3} \int_{-\lambda k_0}^{\infty} (x + \lambda k_0)^2 e^{-x^2} dx$$

can already see exponential T-dependence emerging

In this expression if $\lambda k_0 \gg 1$ (which it is)

then e^{-x^2} very small when $x \lesssim -\lambda k_0$.

i.e. very little area under $(x+\lambda k_0)^2 e^{-x^2}$

for $-\infty < x < -\lambda k_0$

$\therefore$ can safely replace lower limit of integral w/ $-\infty$.

$$\therefore \Phi_{a,r} \approx -\frac{V}{2\pi^2} k_B T \frac{e^{-\Delta/k_B T}}{\lambda^3} \underbrace{\int_{-\infty}^{\infty} (x+\lambda k_0)^2 e^{-x^2} dx}_{\text{Maple } \frac{\sqrt{\pi}}{2}\left(1+2(\lambda k_0)^2\right)}$$

$$\approx \sqrt{\pi}\,(\lambda k_0)^2$$

$$\therefore \Phi_{a,r} \simeq -\frac{V}{2\pi^{3/2}} k_B T \frac{e^{-\Delta/k_B T}}{\lambda \hbar^2} (\hbar k_0)^2$$

$$\frac{1}{\lambda} = \sqrt{\frac{2M k_B T}{\hbar^2}}$$

$$\boxed{\Phi_{a,r} \simeq -2V(\hbar k_0)^2 \sqrt{M} \left(\frac{k_B T}{2\pi \hbar^2}\right)^{3/2} e^{-\Delta/k_B T}}$$

$$S_r = -\left(\frac{\partial \Phi_G}{\partial T}\right)_{V,\mu}$$

$$= \frac{2V(\hbar k_0)^2 \sqrt{M k_B T}\, k_B\, e^{-\Delta/k_B T}}{(2\pi)^{3/2}\hbar^3}\left(\frac{\Delta}{k_B T} + \frac{3}{2}\right)$$

$$\boxed{C_{V,r} = T\frac{\partial S}{\partial T} = \frac{2V(\hbar k_0)^2 \sqrt{M k_B T}\, k_B\, e^{-\Delta/k_B T}}{(2\pi)^{3/2}\hbar^3}\left\{\left(\frac{\Delta}{k_B T}\right)^2 + \frac{\Delta}{k_B T} + \frac{3}{4}\right\}}$$

For roton state in superfluid ^4He

$$k_0 = 19.2 \times 10^9\ m^{-1}$$

$$M = 0.16\, m_4 = 1.07 \times 10^{-27}\ kg$$

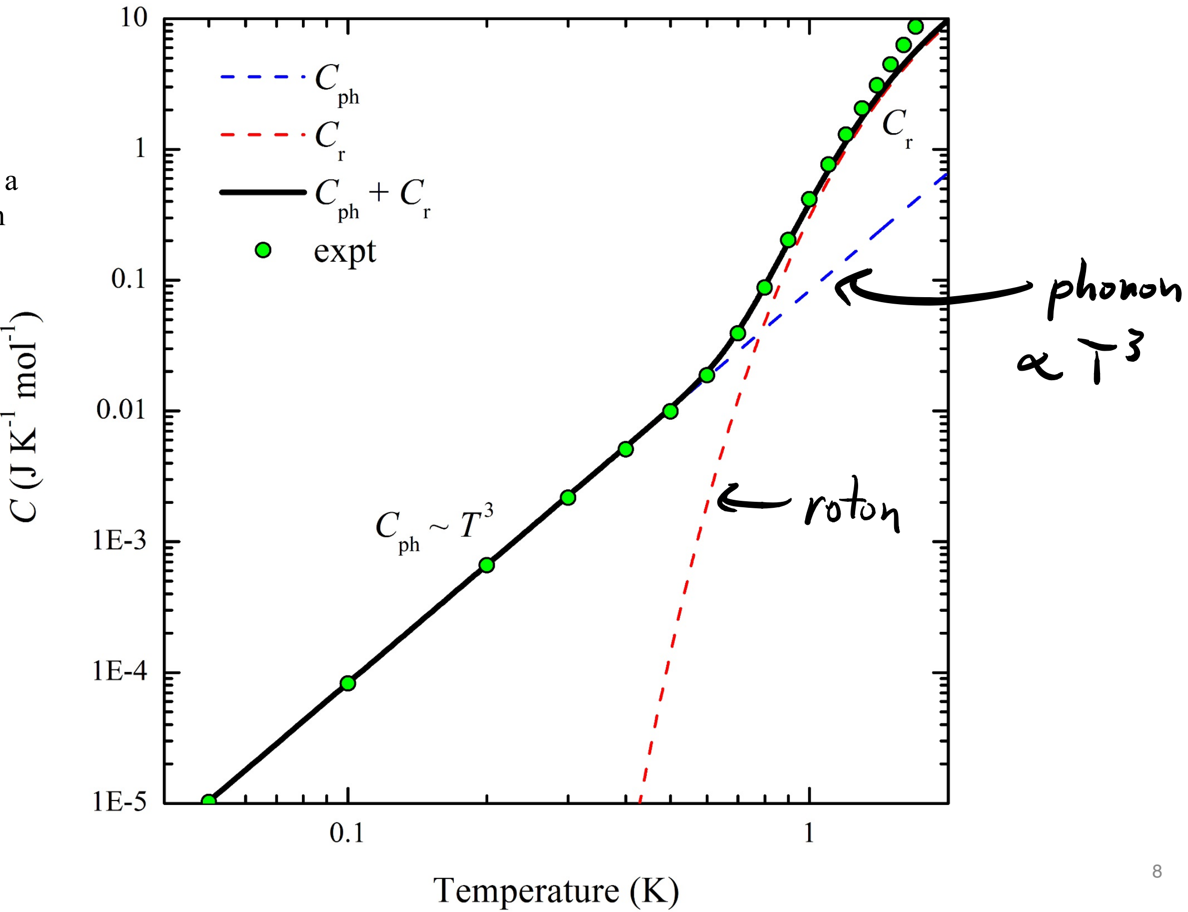
$$\frac{\Delta}{k_B} = 8.65K$$

$$\frac{1}{\lambda} = \sqrt{\frac{2Mk_BT}{\hbar^2}}$$

$$k_0\lambda \simeq 11.7 \gg 1$$

He II Specific Heat

The roton states result in a specific heat contribution that “activates” at sufficiently high temperatures.



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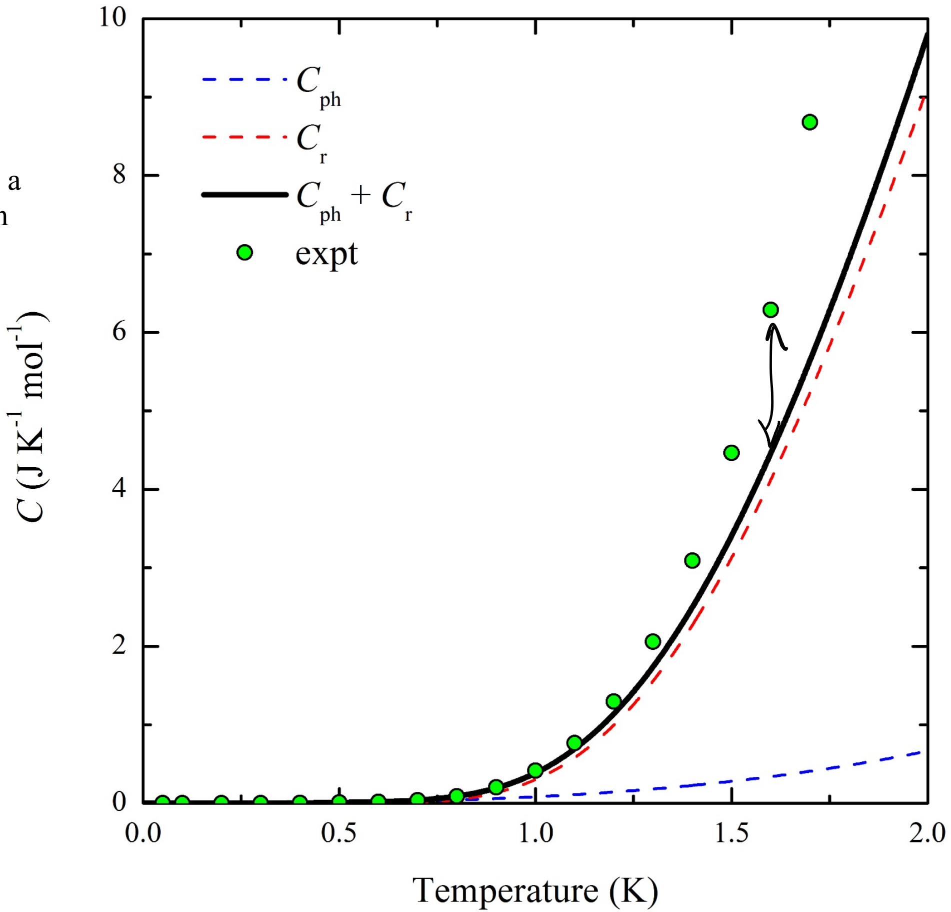
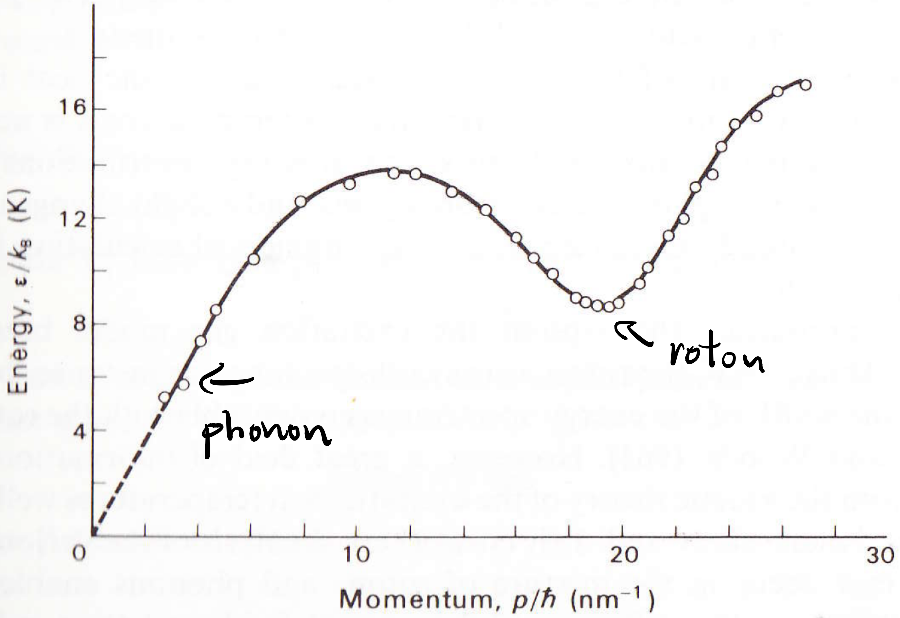


$u_1 = 239\,\mathrm{m\,s^{-1}}$ $\quad p_0/\hbar = 19{\cdot}2\,\mathrm{nm^{-1}}$ ← k_0

$\Delta/k_B = 8{\cdot}65\,\mathrm{K}$ $\quad \mu_r = 0{\cdot}16\,m_4.$ ← M



He II excitation spectrum obtained from neutron-scattering experiments (Henshaw and Woods 1961).