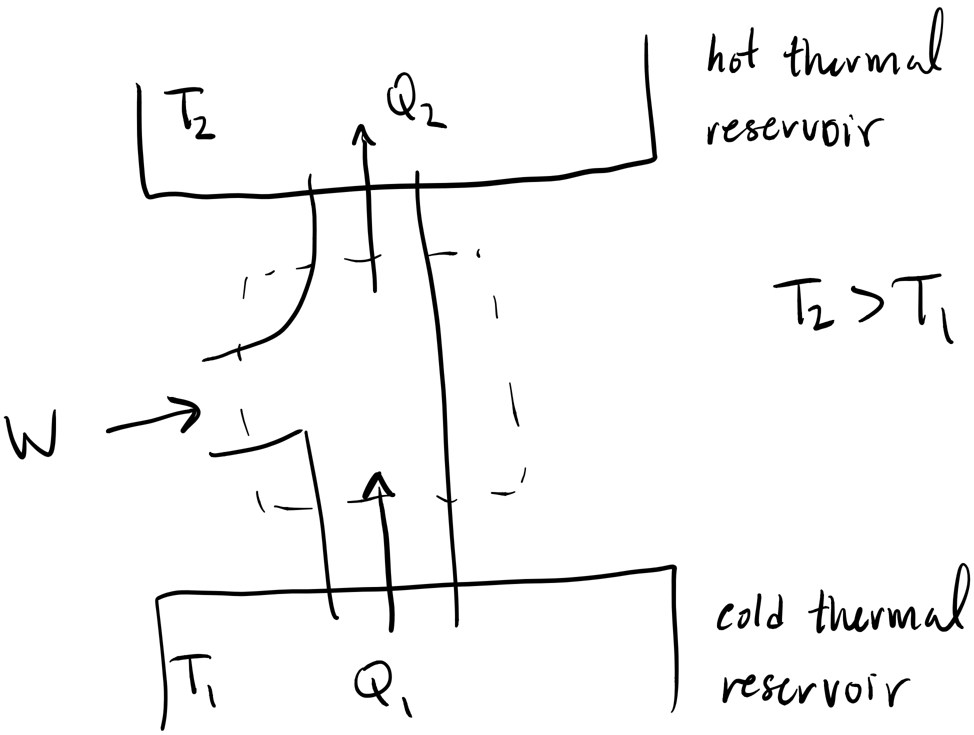
PHYS 425: Low-Temperature Physics

week 1, lecture 1 → w1l1

Course website: https://people.ok.ubc.ca/jbobowsk/phys425.html

Attaining Low-Temperatures:

Abstract diagram



Goal: ▨ do work W on system

▨ extract heat Q_1 from T_1

▨ deposit heat Q_2 into T_2

As an example, suppose system is an ideal gas:

Eq'n of state (ideal gas Law)

$$PV = Nk_BT$$

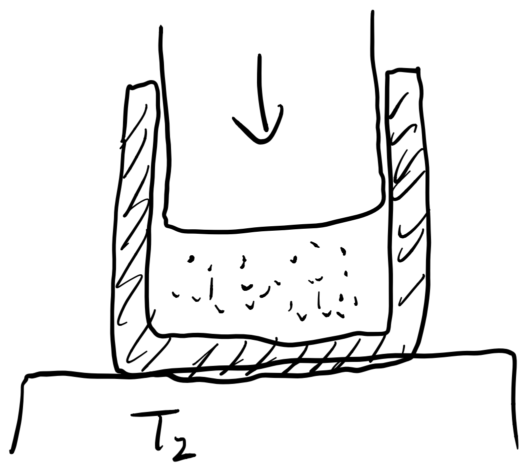
$$U = \frac{f}{2} Nk_BT$$

f: no. of degrees of freedom (dof)

monatomic $f = 3$

How can we extract Q_1 from cold reservoir & deposit Q_2 into hot reservoir?

(i) One way to do work on ideal gas is to compress it w/ a piston. Do compession isothermally by placing system (gas) in contact w/ thermal reservoir @ temp. T_2



If $T = T_2 = \text{const.}$, then $PV = \text{const}$

or $P \propto \frac{1}{V}$

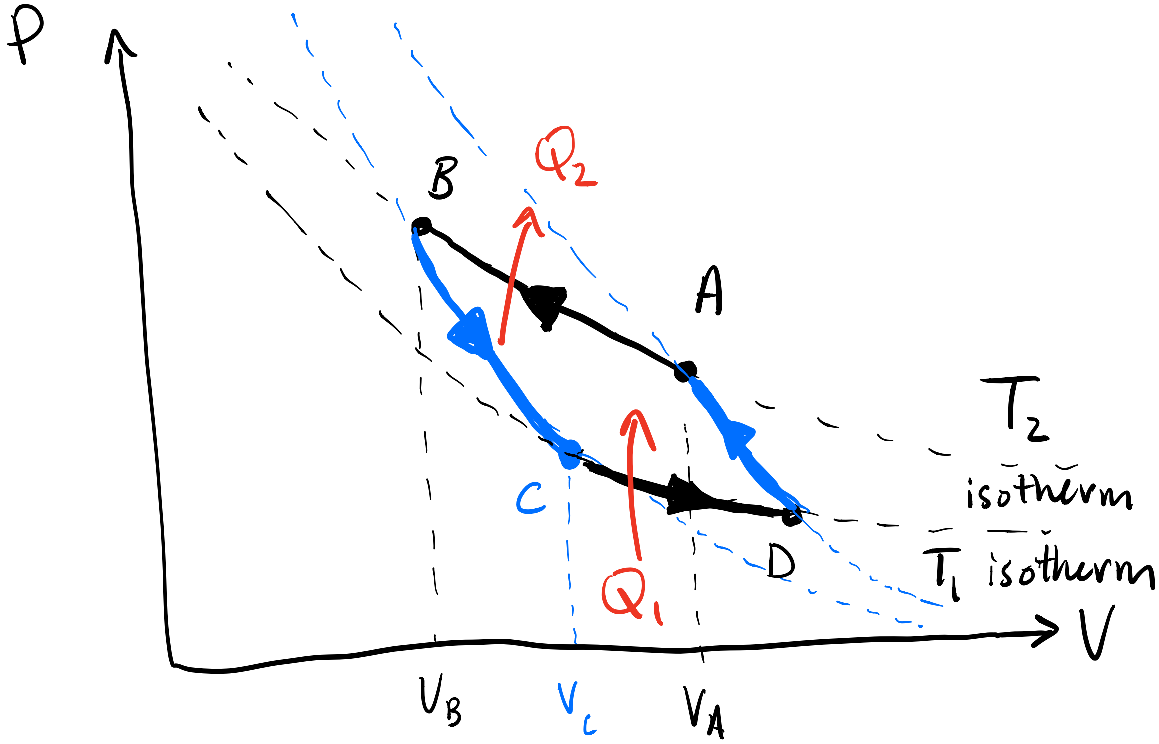
Process $A \to B$ isothermal compression.

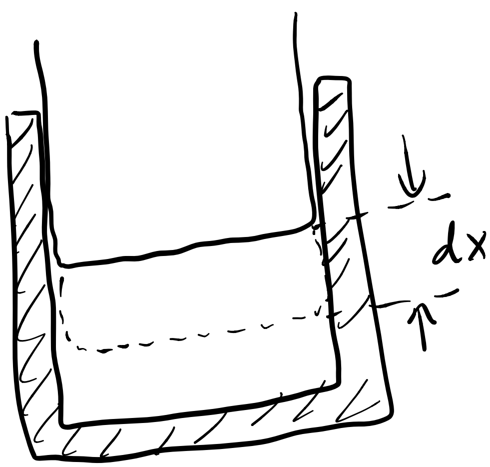
Change in sys. internal energy

$$dU = dQ - dW$$

work by sys.

→ heat added to sys.





$$dW = F\,dx$$

$$= \underbrace{\frac{F}{A}}_{P}\ \underbrace{A\,dx}_{dV}$$

A: cross-sectional area of piston

$$\therefore\ dW = P\,dV$$

For $A \to B$ process

$$\left.\begin{aligned} U_A &= \frac{f}{2} N k_B T_2 \\ U_B &= \frac{f}{2} N k_B T_2 \end{aligned}\right\} \quad dU = U_B - U_A = 0$$

$$\therefore \ dQ_2 = dW = PdV$$

$$Q_2 = \int PdV = Nk_B T_2 \int_{V_A}^{V_B} \frac{dV}{V}$$

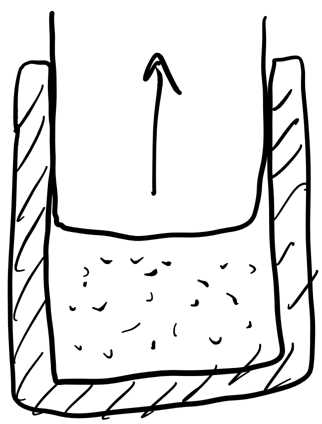
$$= Nk_B T_2 \ln \underbrace{\left(\frac{V_B}{V_A}\right)}_{\text{less than one}} < 0$$

$\therefore Q_2 < 0$ $\therefore$ heat removed from the sys & added to reservoir.

$$W_{AB} = Nk_B T_2 \ln\left(\frac{V_B}{V_A}\right) \quad (\text{negative})$$

work by the gas

(ii) Next, thermally isolate system from the surrounding (i.e. isolate gas from both thermal reservoirs). $\therefore$ System cannot exchange heat w/ anything $dQ = 0 \Rightarrow$ adiabatic process



adiabatic expansion.

Recall that for adiabatic process on ideal gas $PV^{\gamma} = \text{const.}$

$$\text{where } \gamma = \frac{f+2}{2} \quad \text{eg. } f = 3, \quad \gamma = \frac{5}{2} > 1$$

$$P_B V_B^{\gamma} = P_C V_C^{\gamma}$$

$$\underbrace{P_B V_B}_{Nk_B T_B} V_B^{\gamma-1} = \underbrace{P_C V_C}_{Nk_B T_C} V_C^{\gamma-1} \Rightarrow T_B V_B^{\gamma-1} = T_C V_C^{\gamma-1}$$

$T_B = T_2$ & Let's call $T_C = T_1$

$$T_2 V_B^{\gamma-1} = T_1 V_C^{\gamma-1}$$

$$\boxed{\frac{T_2}{T_1} = \left(\frac{V_C}{V_B}\right)^{\gamma-1}}$$

$V_C > V_B$ & b/c $\gamma > 1$

$\Rightarrow T_2 > T_1$

$$dU = \cancel{dQ}^{\,0 \text{ adiabatic}} - dW$$

$$\therefore \Delta U = -W$$

$$W = -\Delta U = -\frac{f}{2} N k_B (T_1 - T_2)$$

$$\therefore W_{BC} = \underbrace{-\frac{f}{2} N k_B (T_1 - T_2)}_{\text{positive}}$$ work by gas.

(iii) Now put sys in contact w/ reservoir @ T_1 & isothermally expand gas.

$$đQ_1 = đW$$

$$Q_1 = N k_B T_1 \ln\left(\frac{V_D}{V_C}\right)$$

positive ∴ heat is absorbed by system (removed for T_1 reservoir).

Extracted heat from cold reservoir.
⇒ refrigeration.

$$\boxed{W_{CD} = N k_B T_1 \ln\left(\frac{V_D}{V_C}\right)}$$

pos. work by gas.

(iv) Can return to A to complete cycle by adiabatically compressing gas from $V_D \to V_A$ ($đQ = 0$)

In this case $W_{DA} = -\Delta U = \boxed{-\frac{f}{2} N k_B (T_2 - T_1)}$

work by gas (negative)

like we add w/ other adiabatic process

$$T_1 V_D^{\gamma-1} = T_2 V_A^{\gamma-1}$$

$$\boxed{\frac{T_2}{T_1} = \left(\frac{V_D}{V_A}\right)^{\gamma-1}}$$

Net work on gas during one complete cycle.

$$W_{on\ gas} = -W_{AB} - W_{BC} - W_{CD} - W_{DA}$$

$$= -Nk_B T_2 \ln\left(\frac{V_B}{V_A}\right) + \frac{f}{2} N k_B (\cancel{T_1} - \cancel{T_2})$$

$$- N k_B T_1 \ln\left(\frac{V_D}{V_C}\right) + \frac{f}{2} N k_B (\cancel{T_2} - \cancel{T_1})$$

$$= -Nk_B \left[T_2 \ln\left(\frac{V_B}{V_A}\right) + T_1 \ln\left(\frac{V_D}{V_C}\right) \right]$$

However we found that:

$$\frac{T_2}{T_1} = \underbrace{\left(\frac{V_C}{V_B}\right)^{\gamma-1} = \left(\frac{V_D}{V_A}\right)^{\gamma-1}}$$

$$\frac{V_C}{V_B} = \frac{V_D}{V_A} \quad \text{or} \quad \frac{V_D}{V_C} = \frac{V_A}{V_B}$$

$$\therefore \quad W_{on\ gas} = -Nk_B\left[\underbrace{T_2 \ln\left(\frac{V_B}{V_A}\right)}_{\text{neg.}} + \underbrace{T_1 \ln\left(\frac{V_A}{V_B}\right)}_{\text{positive}}\right]$$

$$= -Nk_B\left[-T_2 \ln\left(\frac{V_A}{V_B}\right) + T_1 \ln\left(\frac{V_A}{V_B}\right)\right]$$

$$\boxed{W_{on\ gas} = Nk_B \underbrace{(T_2 - T_1)}_{\text{pos.}} \underbrace{\ln\left(\frac{V_A}{V_B}\right)}_{\text{pos.}}} \quad \boxed{\text{pos.}}$$

We have at least now designed a system that does work on gas & removes heat from cold reservoir.

→ refrigeration.

Define a coefficient of performance (COP) ξ (zeta)

$$\xi \equiv \frac{Q_1}{W_{\text{on gas}}}$$

want ξ to be as large as possible.

Recall $Q_1 = N k_B T_1 \ln\left(\frac{V_D}{V_C}\right) = N k_B T_1 \ln\left(\frac{V_A}{V_B}\right)$

$$\boxed{\therefore\ \xi = \frac{\cancel{N k_B} T_1 \cancel{\ln\left(\frac{V_A}{V_B}\right)}}{\cancel{N k_B}\left(T_2 - T_1\right)\cancel{\ln\left(\frac{V_A}{V_B}\right)}} = \frac{T_1}{T_2 - T_1}}$$