PHYS 425 - w11l2

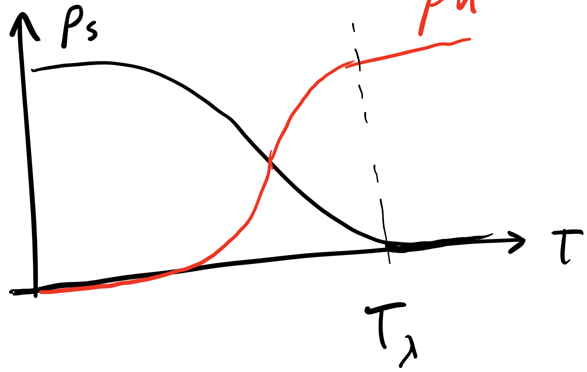
Today: ① Postulate & interpret a condensate wavefcn.

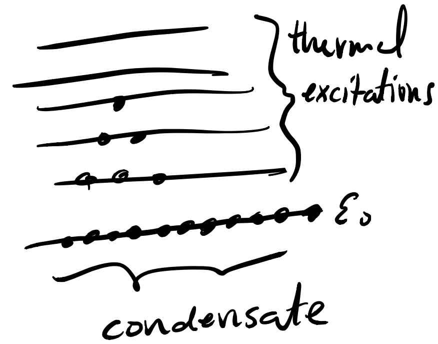
② Determine the critical velocity associated w/ superfluidity.

Have seen experimentally that below $T_\lambda = 2.17\,K$ liquid ^4He behaves as it composed of two interpenetrating fluids.

- the "normal fluid" fraction is viscous & has non-zero entropy.
- the "superfluid" fraction flow w/o dissipation & has zero entropy.

Identify superfluid fraction w/ condensate (particles in ground state) & normal fluid fraction w/ thermal excitations into higher energy states.





Today, we introduce a wavefcn that describes the entire condensate & write supercurrent (mass flow) in terms of the phase of wavefcn.

Apart from superfluid flow & charge flow in superconductors, atomic electron orbits only other known currents that flow w/o dissipation.

Each e^- in atom occupies a quantum state which is an eigenfcn of the Hamiltonian. In 1954, by analogy, London argued that superfluid currents are also quantum in nature & that a wavefcn extends throughout the superfuild volume.

Condensate wavefcn

$$\Psi(\vec{r}) = \Psi_0 e^{iS(\vec{r})}$$

Ψ_0 amplitude

$S(\vec{r})$ phase

(in general Ψ_0 can depend on position, but for now assume Ψ_0 const)

Condenstate momentum

$$\hat{p} = -i\hbar\vec{\nabla}$$

$$\hat{p}\,\Psi(\vec{r}) = -i\hbar\vec{\nabla}\left(\Psi_0 e^{iS(\vec{r})}\right)$$

$$= -i\hbar\,\underline{\Psi_0 e^{iS(\vec{r})}}\; i\vec{\nabla}S(\vec{r})$$

$$= \underbrace{\hbar\vec{\nabla}S(\vec{r})}_{\text{eigenvalue}}\,\Psi(\vec{r})$$

$$\boxed{\therefore\ \vec{p} = \hbar \vec{\nabla} S(\vec{r})}$$

Interpret this momentum as the momentum of one particle in the superfluid s.t. the superfluid velocity $\vec{v}_s$ is:

$$\vec{p} = m_4 \vec{v}_s \quad \text{or} \quad \vec{v}_s = \frac{\hbar}{m_4} \vec{\nabla} S(\vec{r})$$

When superfluid is at rest, $\vec{v}_s = 0$

$$\Rightarrow \vec{\nabla} S(\vec{r}) = 0 \Rightarrow S = \text{const.}$$

Entire condensate has the same phase.

When $\vec{v}_s$ is uniform & const., S varies smoothly in dir'n of $\vec{v}_s$

$\therefore$ can be no abrupt changes in S through superfluid.

↳ Qualitative understanding of how const. superfluid velocity can be maintained for long times.

→ sudden change in $\vec{V}_s$

$$\vec{V}_s \propto \vec{\nabla} S(\vec{r})$$

since $S(\vec{r})$ smooth

$\vec{\nabla} S(\vec{r}) \sim$ const.

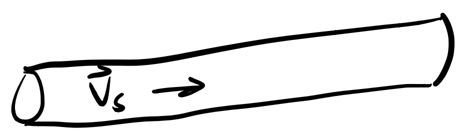
All particles in condensate have same momentum.

If suddenly try to change $\vec{V}_s$ & ∴ $S(\vec{r})$ of one particle, it must also change $S(\vec{r})$ of all particles in condensate & $\vec{V}$ of all condensate particles

→ unlike.

Landau Criterion for Superfluidity

Consider superfluid He II at $T=0$ moving in narrow tube.



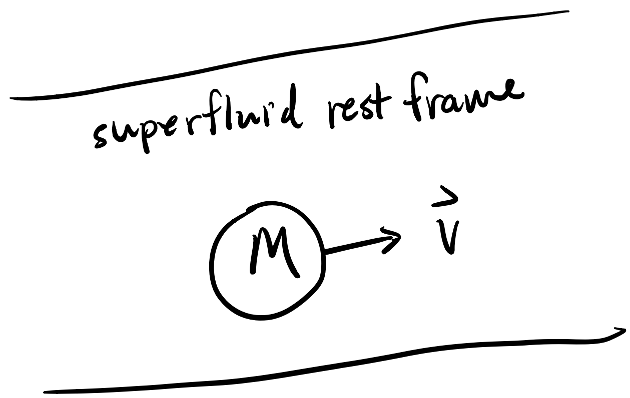
Claim: above a critical velocity v_{cr}, superfluidity destroyed b/c when condensate atoms collide w/ irregularities in pipe wall, they are removed from condensate & promoted to excited states.

↳ excitations created w/ equiv. loss of K.E. in superfluid.

↳ when v_{cr} reached, He II flow ceases to be frictionless.

Estimate of v_{cr}.

Imagine large mass M moving at const $\vec{v}$ through superfluid at rest



minimum value of $\vec{v}$ which creates an excitation is the critical velocity.

Suppose excitation in superfluid of energy $\varepsilon(p)$ & momentum $\vec{p} = \hbar \vec{k}$ occurs.

Conservation of energy:

$$\frac{1}{2}M v^2 = \frac{1}{2}M v_1^2 + \varepsilon(p)$$

Conservation of momentum:

$$M\vec{v} = M\vec{v}_1 + \vec{p}$$

$$\hookrightarrow \vec{v}_1 = \vec{v} - \frac{\vec{p}}{M}$$

$$v_1^2 = \vec{v}_1 \cdot \vec{v}_1 = \boxed{v^2 - \frac{2\vec{p}\cdot\vec{v}}{M} + \frac{p^2}{M^2}}$$

$$\therefore \cancel{\frac{1}{2}M v^2} = \cancel{\frac{1}{2}M v^2} - \vec{p}\cdot\vec{v} + \frac{p^2}{2M} + \varepsilon(p)$$

$$\rightarrow \therefore \varepsilon(p) - \vec{p}\cdot\vec{v} + \frac{p^2}{2M} = 0$$

Assume that mass M is very lagre s.t.

$\frac{p^2}{2M}$ negligible.

$\therefore \quad \varepsilon(p) = \vec{p} \cdot \vec{V} = pV\cos\theta$

since $\cos\theta \leq 1$, must have

$$\varepsilon(p) \leq pV \quad \text{or} \quad V \geq \frac{\varepsilon(p)}{p}$$ condition for excitations

i.e. any V greater than $\frac{\varepsilon(p)}{p}$ causes excitations & $\therefore$ flow is <u>not</u> frictionless.

To achieve frictionless flow require speeds

$$V < V_{cr} = \left[\frac{\varepsilon(p)}{p}\right]_{min}.$$

Values of p that minimize $\frac{\varepsilon(p)}{p}$ determined from

$$\frac{d}{dp}\left(\frac{\varepsilon(p)}{p}\right) = 0$$

$$\cancel{\frac{1}{p}}\frac{d\varepsilon}{dp} - \frac{\varepsilon}{p^{\cancel{2}}} = 0$$

$$\therefore \quad \frac{d\varepsilon}{dp} = \frac{\varepsilon(p)}{p}$$

Example 1. Critical velocity of a system of non-interacting Bosons.

$$\varepsilon(p) = \frac{p^2}{2m}$$

$$\frac{d\varepsilon}{dp} = \frac{p}{m} \qquad\qquad \frac{\varepsilon}{p} = \frac{p}{2m}$$

To find p that minimizes $\frac{\varepsilon(p)}{p}$, we require

$$\frac{d\varepsilon}{dp} = \frac{\varepsilon}{p} \implies \frac{p}{\cancel{m}} = \frac{p}{2\cancel{m}}$$

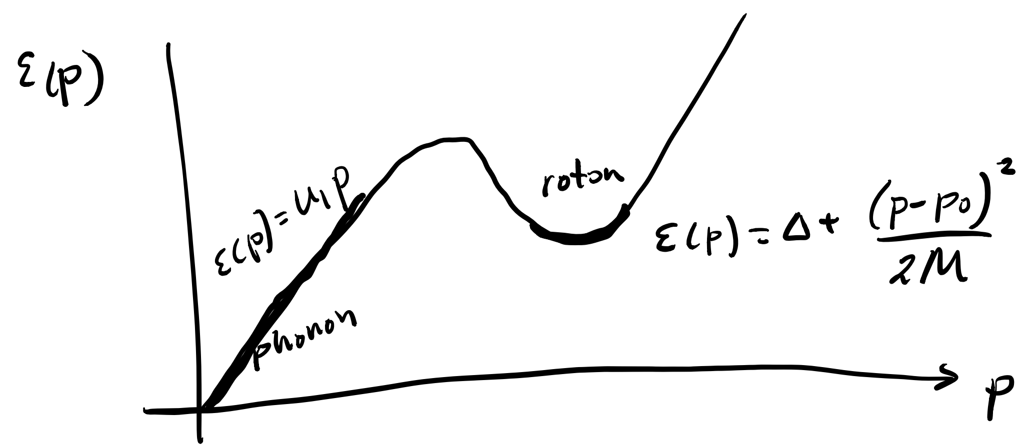
only sol'n is $p=0$!

$$V_{cr} = \left[\frac{\varepsilon(p)}{p}\right]\Bigg|_{min} = \frac{p}{2m}\Bigg|_{p=0} = 0.$$

Critical velocity of system of free Bosons is zero. Flow of any velocity sufficient to create excitations out of condensate.

→ no frictionless flow → no superfluidity.

Landau's spectrum



Phonon critical velocity:

test $\frac{\varepsilon(p)}{p} = \frac{d\varepsilon(p)}{dp}$

$\frac{\varepsilon(p)}{p} = u_1 \qquad \frac{d\varepsilon(p)}{dp} = u_1$

$\therefore \dfrac{\varepsilon(p)}{p} = \dfrac{d\varepsilon(p)}{dp}$ satisfied for any value of p along linear phonon curve.

$$V_{cr} = \left[\frac{\varepsilon(p)}{p}\right]_{min} = \frac{u_1 \cancel{p}}{\cancel{p}} = u_1$$

For excitations into phonon state, the critical velocity is $V_{cr,ph} = \underline{u_1 = 239\,m/s}$

Roton critical velocity.

$$\varepsilon(p) = \Delta + \frac{(p-p_0)^2}{2M}$$

$$\frac{d\varepsilon(p)}{dp} = \frac{p-p_0}{M} \qquad \frac{\varepsilon(p)}{p} = \frac{\Delta}{p} + \frac{(p-p_0)^2}{2Mp}$$

$\therefore$ p that minimizes $\dfrac{\varepsilon(p)}{p}$ determined

from

$$\frac{p-p_0}{m} = \frac{\Delta}{p} + \frac{(p-p_0)^2}{2mp}$$

$$2p^2 - \cancel{2p_0p} = 2M\Delta + p^2 - \cancel{2p_0p} + p_0^2$$

$$\therefore\ p^2 = p_0^2 + 2M\Delta$$

$$\text{or} \quad p = \sqrt{p_0^2 + 2M\Delta}$$

However, can note that

$$p_0^2 = 4.06\times10^{-48}\left(\frac{kg\,m}{s}\right)^2$$

$$2M\Delta = 2.6\times10^{-49}\left(\frac{kg\,m}{s}\right)^2$$

using

$$\frac{p_0}{\hbar} = 19.2\ \mathrm{nm}^{-1}$$

$$\frac{\Delta}{k_B} = 8.65\,K$$

$$M = 0.16\,m_4$$

$$p_0^2 + 2M\Delta \approx p_0^2 \quad \text{or} \quad p \approx p_0.$$

$$\left. \left(\frac{\varepsilon(p)}{p} \right) \right|_{p=p_0} = \frac{\Delta}{p_0} + \cancelto{0}{\frac{(p_0 - p_0)^2}{2M p_0}}$$

$$\therefore \quad V_{cr,r} \approx \frac{\Delta}{p_0} = \underline{59\,m/s}$$

critical velocity to excite atoms from condensate into roton states.