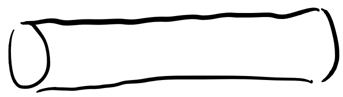
PHYS 425 - w4l2

Heat Transfer by Conduction through a Solid

"Derivation" of 1-D heat Eq'n.

(1) heat in a rod



heat capacity $C = \rho V c_v = \frac{dU}{dT}$

c_v: specific heat per unit mass

$\Delta U = \rho V c_v \Delta T$ (assuming that c_v temp indep. for a small ΔT)

$U_2 - U_1 = \rho V c_v (T_2 - T_1)$

or $U = \underbrace{\rho V c_v}_{C} T$ (heat in rod at temp T)

[similar to ideal gas where

$U = \underbrace{\frac{f}{2} N k_B}_{C} T \quad \therefore \quad U = CT$]

(2) Fourier's Law of Heat Transfer

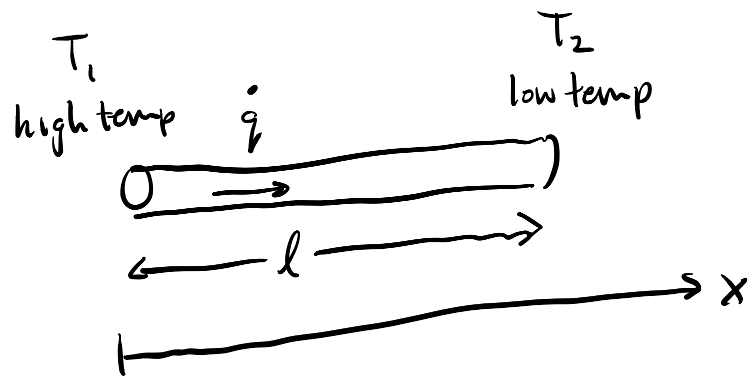
$$\dot{q} = \frac{\dot{Q}}{A} = -k \frac{\partial T}{\partial x}$$

heat flux

power per unit area

thermal conductivity

Defines thermal conductivity of a material



In the case drawn $\frac{\partial T}{\partial x} < 0$ $\therefore$ $\dot{q} > 0$ heat flows from left to right along $+x$.

In general $\kappa = \kappa(T)$ is temp dep.

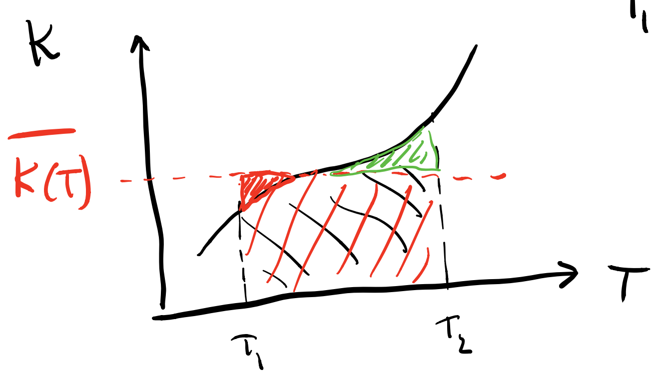
$$\frac{\dot{Q}}{A} = \frac{P}{A} = -\kappa(T)\frac{\partial T}{\partial x}$$

$$P\,dx = -A\,\kappa(T)\,dT$$

$$P\ell = -A\int_{T_1}^{T_2} \kappa(T)\,dT$$

$$= -A\,|T_2 - T_1|\,\underbrace{\frac{1}{|T_2-T_1|}\int_{T_1}^{T_2}\kappa(T)\,dT}_{\overline{\kappa(T)}} \quad *$$

avg. value of κ between $T_1 \leq T \leq T_2$.



$$\frac{\text{int. is area}}{\text{temp range}} = \text{avg. height of fcn}$$

$$\therefore \; P\ell = -A\,|T_2 - T_1|\,\overline{K(T)}$$

In our scenario, $T_1 > T_2$

or $T_2 - T_1 < 0$

$$\therefore \; -|T_2 - T_1| = T_2 - T_1 \equiv \Delta T$$

$$\therefore \; P\ell = A\,\Delta T\,\overline{K(T)}$$

or $$\Delta T = \frac{1}{\overline{K(T)}}\frac{\ell}{A} P$$

"Thermal circuit" equivalents

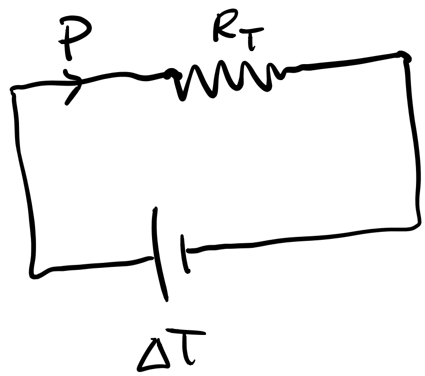
$\Delta V \to \Delta T$

$\sigma \to \overline{K(T)}$

$I = \dot{q} \to P = \dot{Q}$

$$\left(\text{reminiscent of } \Delta V = RI = \rho\frac{\ell}{A}I = \frac{1}{\sigma}\frac{\ell}{A}I \right)$$

Can often treat heat transfer by conduction using a equivalent circuit analogy

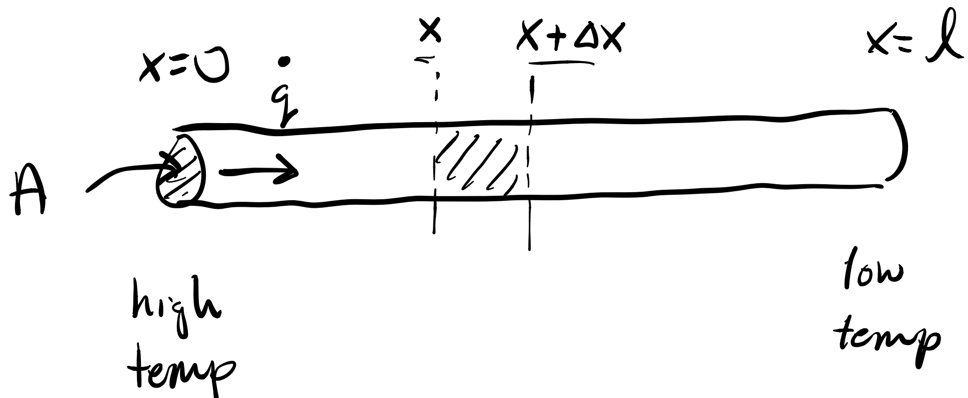


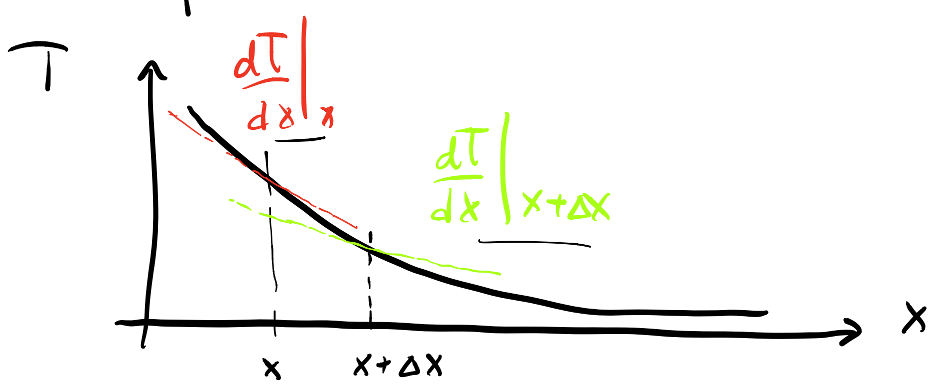
$$\Delta T - P R_T = 0$$

$$\Delta T = P R_T$$

$$R_T = \frac{1}{k}\frac{\ell}{A}$$ "thermal resistance"

(3) Consider a section of rod w/ heat flux $\dot{q}$ & apply conservation of energy





Heat within shaded segment $\rho V c_v T(x,t)$

$$\rho c_v A \Delta x \, T(x,t)$$

Conservation of Energy

change of energy in segment in time Δt = heat in from left − heat out to right

$$\cancel{A}\rho c_v \Delta x \, T(x, t+\Delta t) - \cancel{A}\rho c_v \Delta x \, T(x,t)$$

$$= \underbrace{\Delta t \cancel{A}\left(-k \left.\frac{\partial T}{\partial x}\right|_x\right)}_{\substack{\text{Fourier's Law} \\ \text{heat in from left}}} - \underbrace{\Delta t \cancel{A}\left(-k \left.\frac{\partial T}{\partial x}\right|_{x+\Delta x}\right)}_{\text{heat from right}}$$

$$\rho c_v \left[T(x,t+\Delta t) - T(x,t)\right]\Delta x$$

$$= k\left[\left.\frac{\partial T}{\partial x}\right|_{x+\Delta x} - \left.\frac{\partial T}{\partial x}\right|_x\right]\Delta t$$

or $$\frac{T(x, t+\Delta t) - T(x,t)}{\Delta t} = \frac{K}{\rho c_v}\left[\frac{\left.\frac{\partial T}{\partial x}\right|_{x+\Delta x} - \left.\frac{\partial T}{\partial x}\right|_{x}}{\Delta x}\right]$$

$\lim \Delta t, \Delta x \to 0$

$$\therefore \quad \frac{\partial T}{\partial t} = \frac{K}{\rho c_v}\frac{\partial^2 T}{\partial x^2}$$

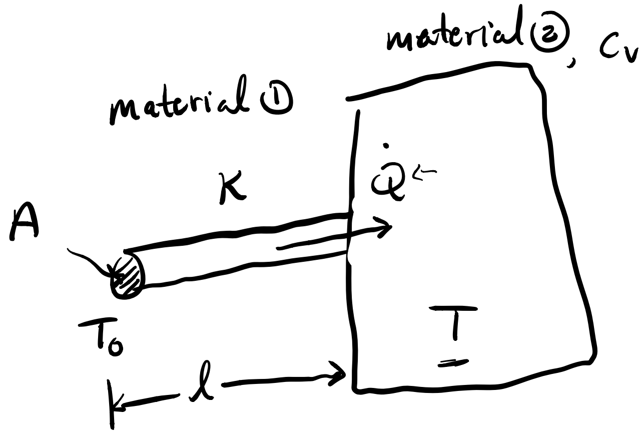
$$\boxed{\frac{\partial T(x,t)}{\partial t} = \alpha \frac{\partial^2 T(x,t)}{\partial x^2}}$$

1-D heat Eq'n

$\alpha \equiv \frac{K}{\rho c_v}$ is thermal diffusivity

$$[\alpha] = \frac{W}{\cancel{m}\cdot\cancel{K}}\ \frac{\cancel{m^3}^{\,2}}{\cancel{kJ}}\ \frac{\cancel{kJ}\,\cancel{K}}{J} = \frac{m^2}{s}$$

More on thermal circuits...



Assume that heat capacity of ① v. small c.t. that of ②.

→ no power absorbed by ①

→ ① just provides heat transfer

Assume that thermal conductivity of ② is very high

→ Temp of ② is always uniform

By Fourier's Law $\dot{q}$ in ① is:

(i) $$\frac{\dot{Q}}{A} = \frac{\overline{k(T)}}{l} \Delta T \qquad \left(\frac{dT}{dx} = \frac{\Delta T}{l} \right)$$

or $\dot{Q} = \frac{A}{\ell} \overline{K(T)} (T_0 - T)$

(ii) Material (2) absorbs incident heat

$$\frac{\Delta U}{\Delta T} = \rho V c_v \quad \therefore \quad \frac{\Delta U}{\Delta t} = \rho V c_v \frac{\Delta T}{\Delta t}$$

or $\dot{Q} = \rho V c_v \frac{dT}{dt}$

Equate $\dot{Q}$ expression

$$\frac{A}{\ell} \overline{K(T)} (T_0 - T) = \rho V c_v \frac{dT}{dt}$$

or $$-\frac{A}{\ell} \overline{K(T)} \frac{1}{\rho V c_v} = \frac{1}{T - T_0} \frac{dT}{dt}$$

$$\therefore \quad \underbrace{-\frac{A}{\ell} \frac{\overline{K(T)}}{\rho V c_v}}_{\equiv \frac{1}{\tau}} dt = \frac{dT}{T - T_0}$$

$$-\frac{t}{\tau} = \int \frac{dT}{T-T_0} = \ln(T-T_0) - B'$$

$\uparrow$ integration const.

$$\ln(T-T_0) = B' - \frac{t}{\tau}$$

$$T - T_0 = \underbrace{e^{B'}}_{\equiv B} e^{-t/\tau} \qquad \tau = \frac{\rho V c_v l}{A\,\overline{K(T)}}$$

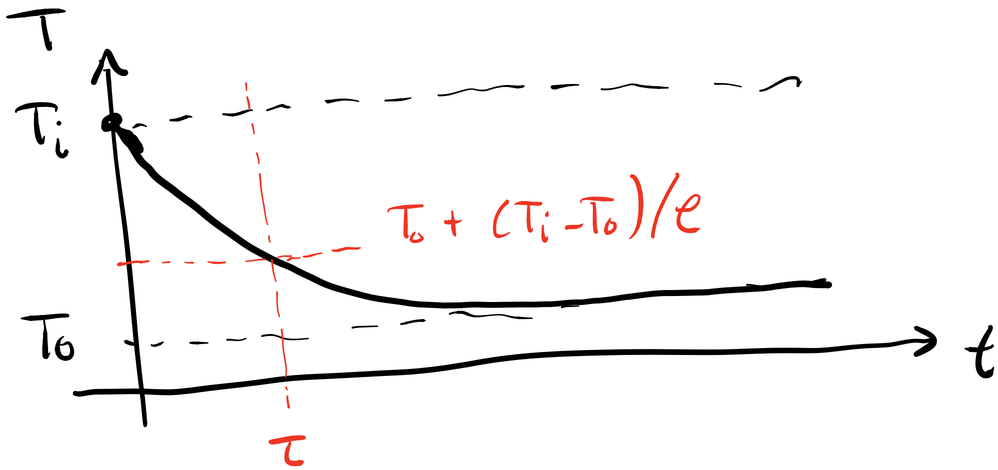
$$\therefore\ T = T_0 + B e^{-t/\tau}$$

If initially @ $t=0$, $T = T_i$

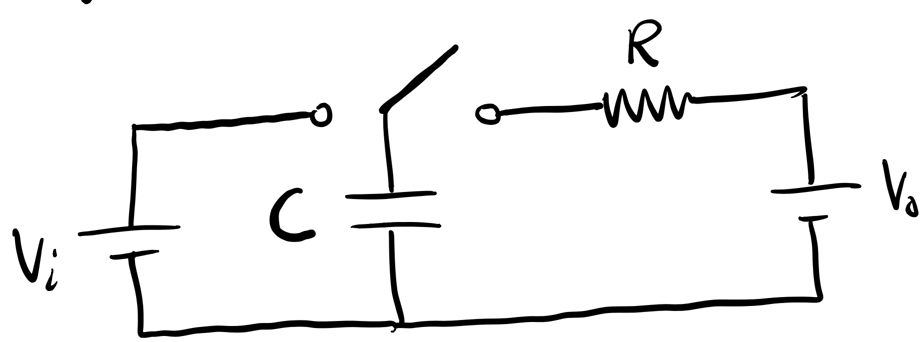
then

$$T_i = T_0 + B \quad \therefore \quad B = T_i - T_0$$

$$\boxed{\therefore\ T(t) = T_0 + (T_i - T_0) e^{-t/\tau}}$$

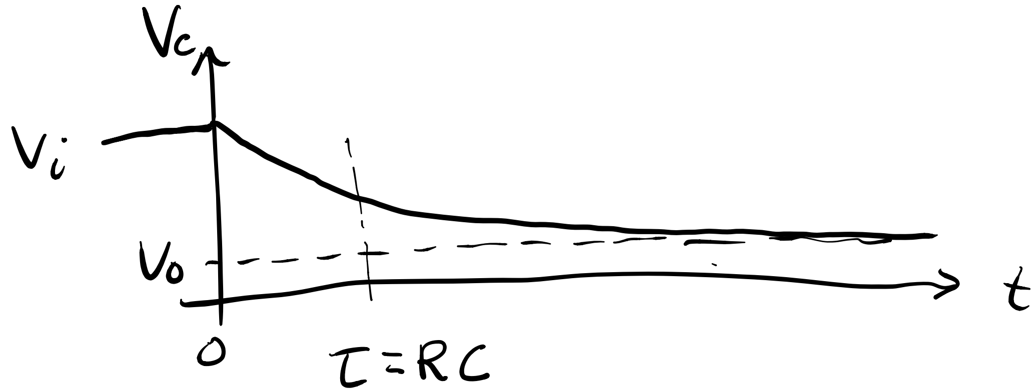


Exactly equiv. to a capacitor w/ initial volt. V_i discharging to a final volt. V_0 through a resistor R.



① Put switch to left to quickly charge cap to V_i $(t<0)$

② put switch to right to discharge cap to V_0 through R (assuming $V_0 < V_i$) @ $t=0$



Thermal time const.

$$\tau = \frac{\rho V c_v l}{A\,\overline{k(T)}} = \underbrace{\left(\frac{1}{\overline{k(T)}}\frac{l}{A}\right)}_{R_T}\underbrace{\left(\rho V c_v\right)}_{C_T}$$

R_T thermal resistance

C_T thermal capitance is heat capacity of large block.

Can set time to cool or heat material ② by adjusting thermal resistance of material ① and/or capacitance of material ②.