PHYS 425 – w13l1

Last Time: Eq'n of motion of superfluid

$$\boxed{\underbrace{\frac{\partial \vec{v}_s}{\partial t} + (\vec{v}_s \cdot \vec{\nabla})\vec{v}_s}_{\text{acceleration of superfluid}} = \underbrace{-\frac{1}{m_4}\vec{\nabla}\mu}_{\text{gradient of P.E.} \Rightarrow \text{Force.}}}$$

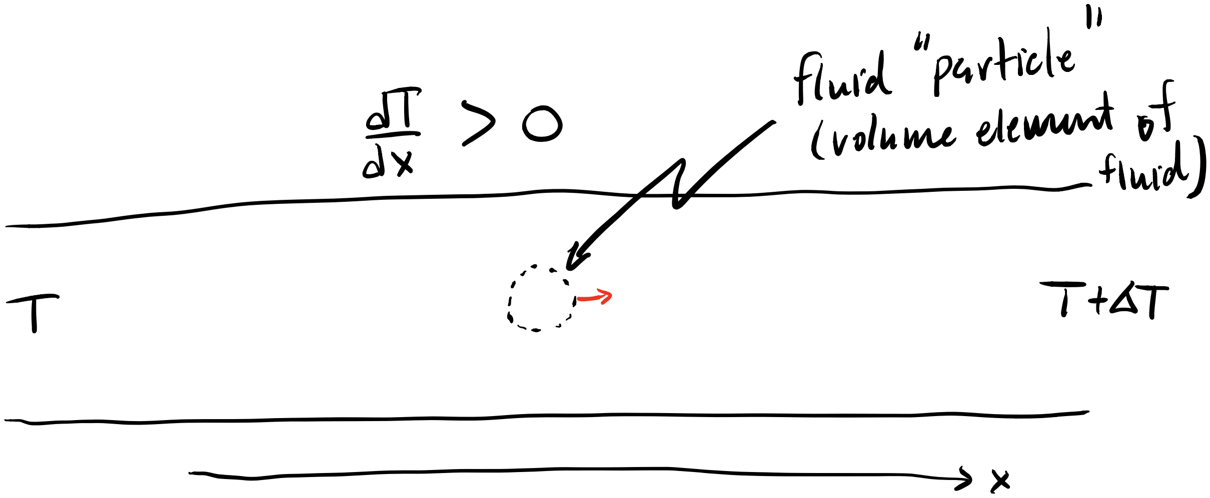
Today: ① Interpret $\frac{\partial f}{\partial t} + (\vec{v} \cdot \vec{\nabla})f$ Material derivative

② Evaluate $\vec{\nabla}\mu$

③ Understand the <u>thermomechanical effect</u> in He-II (liquid ^{4}He below T_λ).

① Material derivative of fcn f

is $$\frac{Df}{Dt} \equiv \frac{\partial f}{\partial t} + \vec{v} \cdot \vec{\nabla} f$$



Imagine a fluid in a pipe <u>at rest</u>

There's a temp gradient along length of pipe.

If temp at ends of pipe are const, then

$\frac{dT}{dt} = 0$ for our fluid particle.

① Now imagine that we can heat the right end of the pipe while fluid remains at rest.

Then $\frac{dT}{dt} \neq 0$ of fluid particle is nonzero due to instantaneous rate of change of temp at location of fluid particle.

② Now return to fixed temps T (left) & T+ΔT (right) but now fluid flows along pos. x dir'n. In this case, temp of fluid particle changes b/c it moves to a location in the pipe that has a diff. temp.

$$\Delta T = \Delta x \frac{dT}{dx}$$

$$\frac{\Delta T}{\Delta t} = \frac{\Delta x}{\Delta t}\frac{dT}{dt} = V_x \frac{dT}{dt}$$

In general $\frac{dT}{dt} = (\vec{v} \cdot \vec{\nabla}) T$

If $\frac{dT}{dt}$ at ends of pipes nonzero & fluid partilce was moving, then

$$\frac{\Delta T}{\Delta t} = \left.\frac{dT}{dt}\right|_{\text{pos. of particle}} + V_x \frac{dT}{dt}$$

In general $\dfrac{DT}{Dt} = \dfrac{\partial T}{\partial t} + (\vec{v}_s \cdot \vec{\nabla})T$

For superfluid eq'n of motion

$$\frac{D\vec{v}_s}{Dt} = \frac{\partial \vec{v}_s}{\partial t} + (\vec{v}_s \cdot \vec{\nabla})\vec{v}_s = -\frac{1}{m_4}\vec{\nabla}\mu$$

② Find an expression for $\vec{\nabla}\mu$

Gibb's Free Energy & Chemical Potential,

$$G \equiv U - T\Sigma + pV$$

Σ : entropy

S : phase in $\psi = \psi_0 e^{iS}$

$$dG = dU - Td\Sigma - \Sigma dT + pdV + Vdp$$

$$= \underbrace{\cancel{Td\Sigma} - \cancel{pdV} + \mu dN}_{dU} - \cancel{Td\Sigma} - \Sigma dT + \cancel{pdV} + VdP$$

$$\therefore\ dG = \mu dN - \Sigma dT + Vdp$$

$$\therefore \mu = \left(\frac{\partial G}{\partial N}\right)_{T,P} \quad \left[\text{also } \Sigma = -\left(\frac{\partial G}{\partial T}\right)_{N,P} \quad V = \left(\frac{\partial G}{\partial P}\right)_{N,T} \right]$$

Note that G is a system energy

- it must be an extensive quantity
- i.e. it must scale with system size

$$G \propto N$$

$$\Downarrow$$

$$G = N f(T, P, \ldots)$$

$$\therefore \mu = \left(\frac{\partial G}{\partial N}\right)_{T,P} = f(T, P, \ldots)$$

$$\Rightarrow \mu = f \Rightarrow \boxed{G = N\mu}$$

So for a system of fixed particle no. (N const)

$$dG = N\,d\mu = -\Sigma\, dT + V dp$$

$$\frac{\partial \mu}{\partial x} = -\frac{\Sigma}{N}\frac{\partial T}{\partial x} + \underbrace{\frac{V}{N}}_{\frac{1}{n}}\frac{\partial p}{\partial x}$$

$$\therefore \frac{\partial \mu}{\partial x} = -\frac{\Sigma}{N}\frac{\partial T}{\partial x} + \frac{1}{n}\frac{\partial p}{\partial x}$$

similar for y & z coords.

$$\boxed{\therefore \vec{\nabla}\mu = -\frac{\Sigma}{N}\vec{\nabla}T + \frac{1}{n}\vec{\nabla}p}$$

③ Thermomechanical Effect in He-II

sub $\vec{\nabla}\mu$ into superfluid Eq'n of motion.

$$\frac{\partial \vec{v}_s}{\partial t} + (\vec{v}_s \cdot \vec{\nabla})\vec{v}_s = -\frac{1}{m_4}\vec{\nabla}\mu$$

$$= \frac{\Sigma}{Nm_4}\vec{\nabla}T - \frac{1}{nm_4}\vec{\nabla}P$$

$Nm_4 = \rho V$ where $\underset{\substack{\uparrow \\ \text{total} \\ \text{density}}}{\rho} = \underset{\substack{\uparrow \\ \text{normal} \\ \text{fluid} \\ \text{density}}}{\rho_n} + \rho_s$ superfluid density

$$nm_4 = \frac{Nm_4}{V} = \rho$$

$$\boxed{\frac{\partial \vec{v}_s}{\partial t} + (\vec{v}_s \cdot \vec{\nabla})\vec{v}_s = \frac{\Sigma}{\rho V}\vec{\nabla}T - \frac{1}{\rho}\vec{\nabla}P}$$

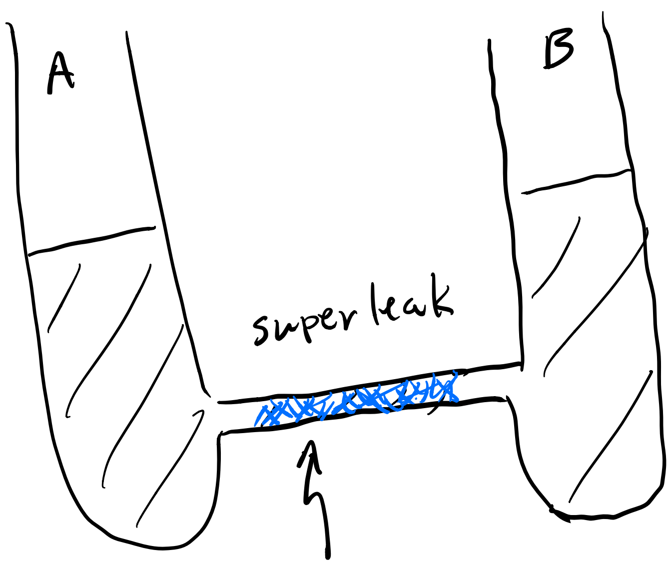
Thermomechanical Effect.

Temp. & press. gradients both cause acceleration of superfluid.

In equilibrium $\vec{v}_s$ is const &

$$0 = \frac{\Sigma}{\cancel{\rho} V}\vec{\nabla}T - \frac{1}{\cancel{\rho}}\vec{\nabla}P$$

or $$\boxed{\frac{\Delta P}{\Delta T} = \frac{\Sigma}{V}}$$ London Eq'n.

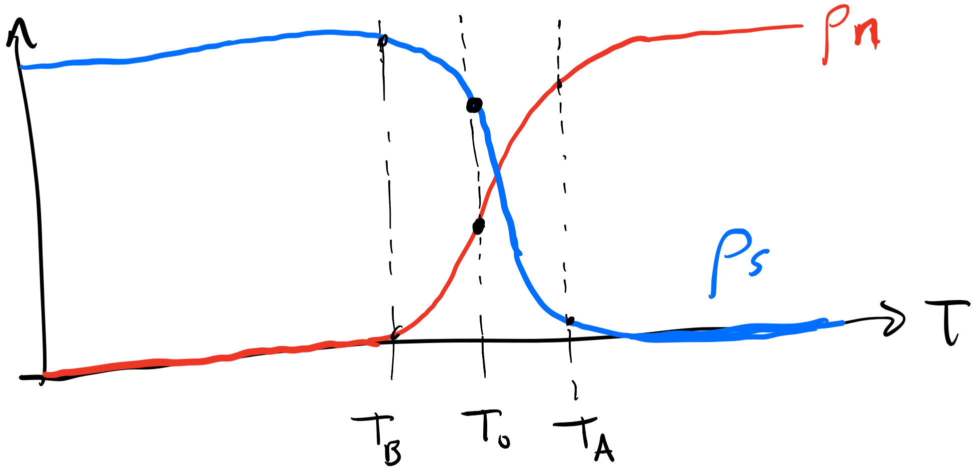


Normal fluid <u>cannot</u> pass through
Only superfluid can pass through

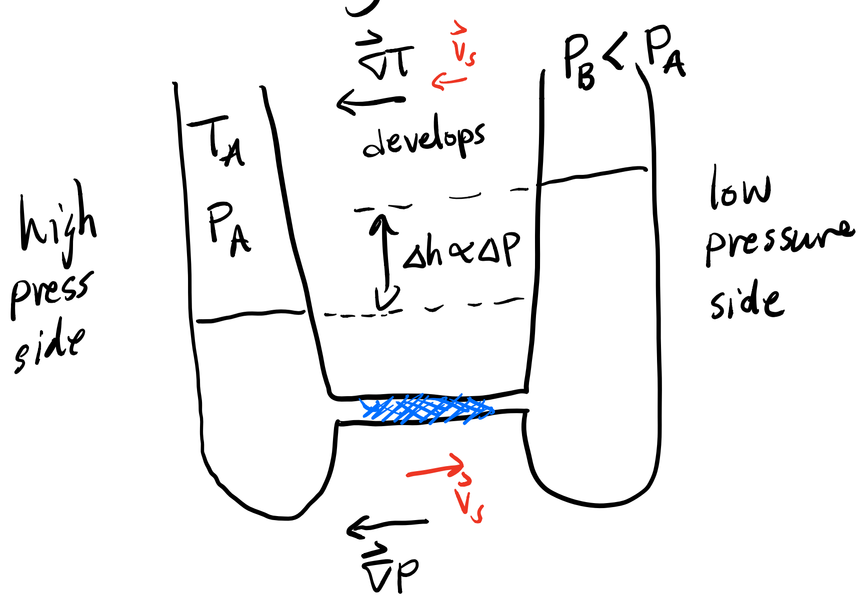
System is filled w/ Helium II

$\rho = \rho_n + \rho_s$

Start w/ $T_A = T_B$ s.t. have equal proportions of ρ_n & ρ_s on each side.



Now apply pressure to side A.



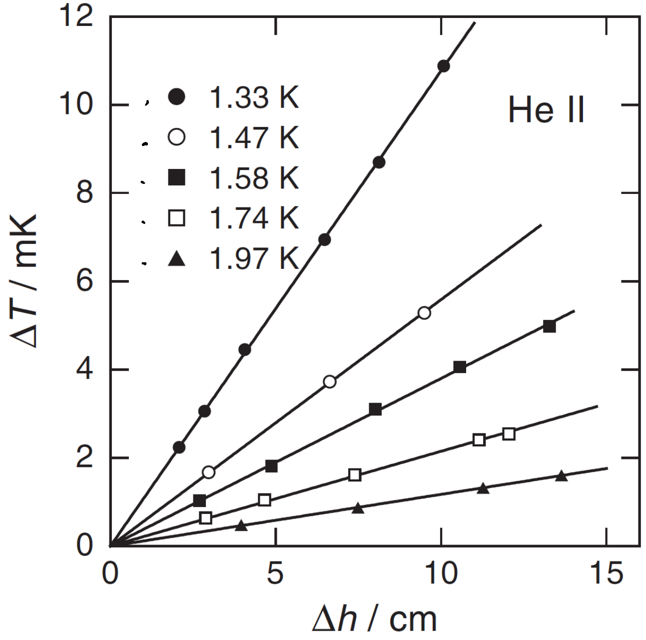
From Ⓐ if start w/ uniform temp & apply pressure gradient we get superfluid flow in opp. dir'n of $\vec{\nabla}p$

Flow continues until

$$\frac{\Sigma}{\rho V} \vec{\nabla} T \text{ balances } -\frac{1}{\rho} \vec{\nabla} p$$

or until $\boxed{\Delta T = \frac{V}{\Sigma} \Delta P}$ from

London eq'n.



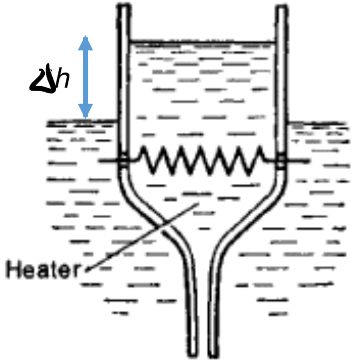
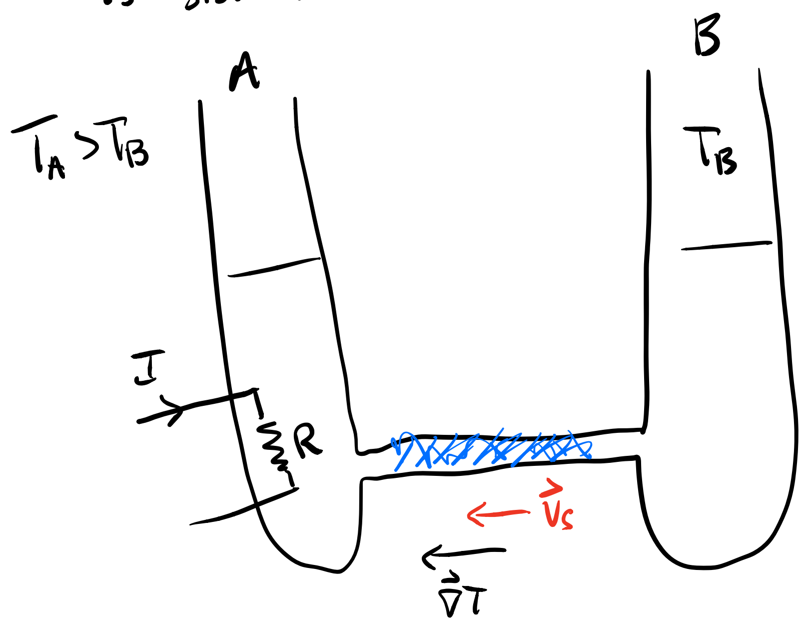


Fig. 2.16. Temperature difference as a function of the level difference in the two beakers that are connected via a superleak [59]

Now consider the inverse experiment.

Start w/ $\vec{\nabla} P = 0$ & heat the Helium II is side A.



w/ $\vec{\nabla} P = 0$

$$\frac{\partial \vec{v}_s}{\partial t} + (\vec{v}_s \cdot \vec{\nabla})\vec{v}_s = \frac{\Sigma}{\rho V} \vec{\nabla} T$$

superfluid accelerates in dir'n of $\vec{\nabla} T$ (towards high temp).

Flow continues until $\Delta P = \frac{\Sigma}{V} \Delta T$

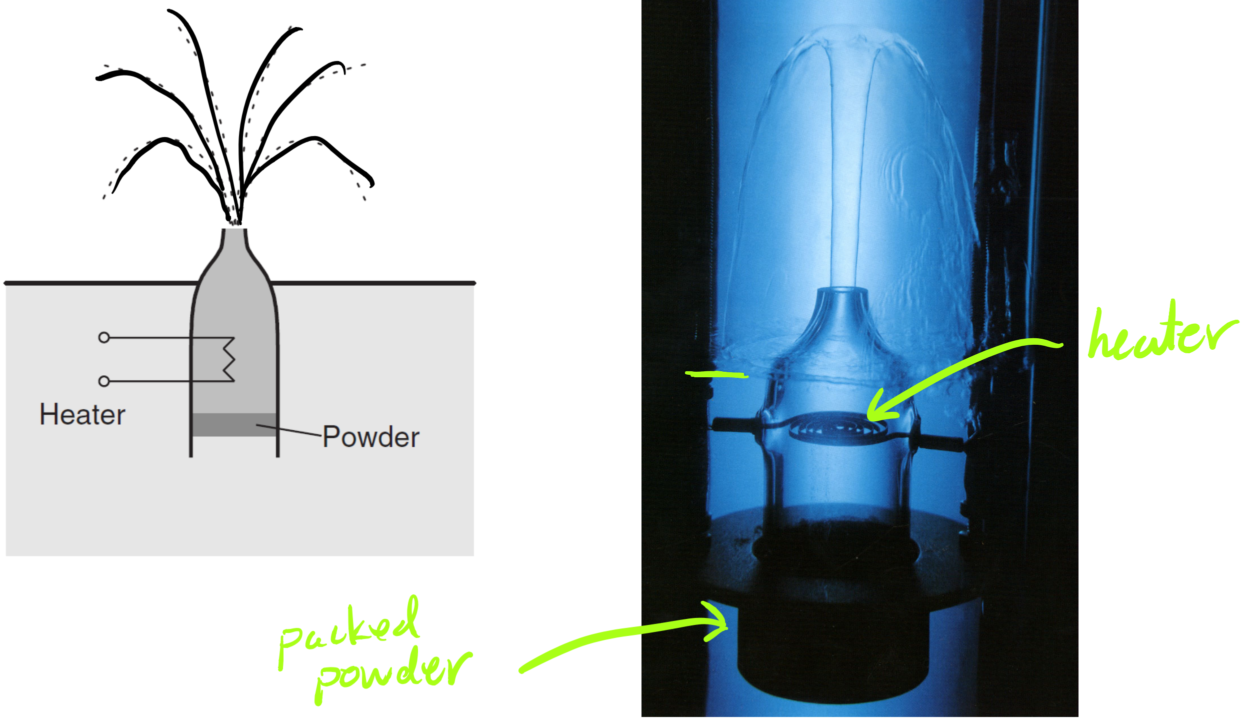


Fig. 2.7. (**a**) Schematic sketch of an experimental setup used to demonstrate the fountain effect. The helium inside and outside the flask has been drawn in a slightly different shade for clarity. (**b**) Photo of a fountain generated in helium II [47]