PHYS 425 - w5l1

Road Map to 10 mK .... Understanding Dilution Refrigerators.

Mixtures of ^3He - ^4He

1. Understand the difference between the behaviour of systems of identical particles at low temps. $\Rightarrow$ Fermions & Bosons.

   - [x] Some QM review
   - [x] Develop w.f. for a pair of identical particles
   - [x] Deduce Pauli Exclusion principle
   - [x] Write down dist'n fcns for Fermions & Bosons (hope to derive these later in the course)

2. Systems of Fermions at Low temp.

- [x] Fermi Energy
- [x] Density of States
- [x] Low-temp heat capacity
- [x] Low-temp Entropy

3. Mixtures of ^{3}He-^{4}He

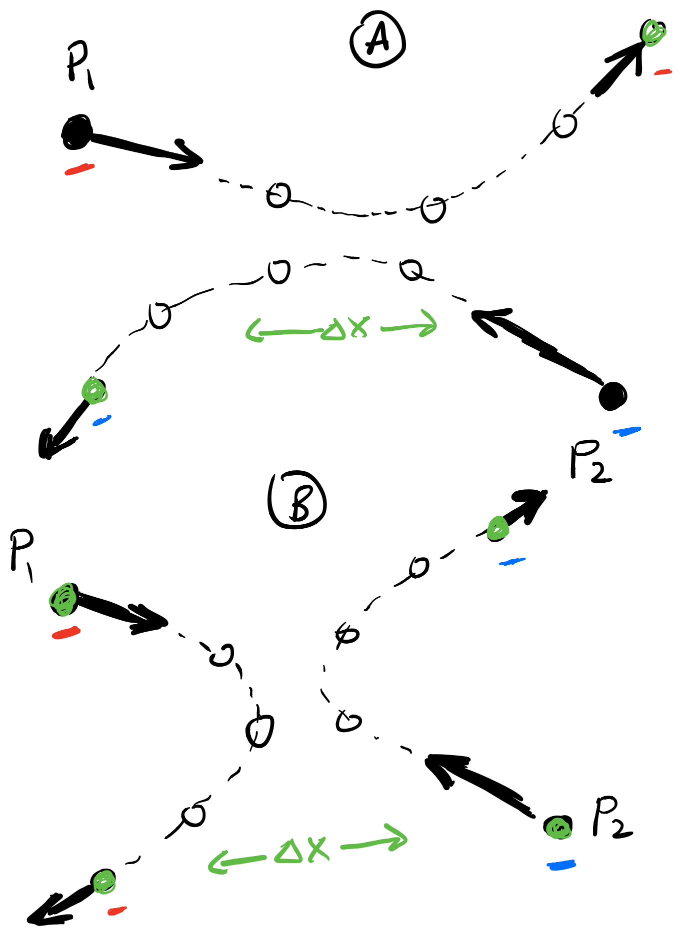
- [x] Intrepretation of Chemical Potential
- [x] Energetics of ^{3}He-^{4}He mixtures
- [x] Gibbs Free Energy & Chemical Pot.

4. Dilution Refrigerator

- [x] Analogy w/ Evapourative Cooling
- [x] Cooling Power
- [x] Base temp.
- [x] Operation of a dilution Refrigerator

Identical Particles

Classically, can always distinguish between particles. However, this is not always possible in case of Quantum Mechanics.



Classically, can track position & momentum of P_1 & P_2 w/ arbitrary precision & can always distinguish between scenarion (A) & (B) even if P_1 & P_2 are identical.

Howere, b/c of Heisenberg uncertainting principle (HUP) $\Delta x \Delta p \geq \frac{\hbar}{2}$ in Q.M. cannot always tell which particle is which after scattering event. If particles come within Δx of one another loose ability to distinguish between the two & $\therefore$ between scenarios Ⓐ & Ⓑ.

This implies that, for identical particles, exchanging positions of p_1 & p_2 leaves system unchanged.

Physically measurable quantities in Q.M. determined from abs. square of wavefcn:

Eg. $\underset{\sim}{|\psi(x)|^2 dx}$ gives prob. of finding particle between x & $x+dx$

$$\langle \psi | \mathcal{H} | \psi \rangle = E \langle \psi | \psi \rangle$$
$$= E \int \psi^*(x)\, \psi(x)\, dx$$
$$= E \int |\psi(x)|^2\, dx$$

is expectation value of system energy.

For a system of two identical particles require

$$\Rightarrow \underset{P_1 \text{ at } x_1,\ P_2 \text{ at } x_2}{|\psi(x_1, x_2)|^2}\, dx_1\, dx_2 = \underset{P_1 \text{ at } x_2,\ P_2 \text{ at } x_1}{|\psi(x_2, x_1)|^2}\, dx_1\, dx_2$$

i.e. since particles are identical, prob. of finding P_1 at x_1, P_2 at x_2 same as prob. of finding P_2 at x_1, P_1 at x_2.

This condition is satisfied if

$$\boxed{\psi(x_2, x_1) = e^{i\delta}\, \psi(x_1, x_2)}$$

where $e^{i\delta}$ is a phase factor.

$$|\psi(x_2, x_1)|^2 = e^{-i\delta}\psi^*(x_1, x_2)\, e^{+i\delta}\psi(x_1, x_2)$$

$$= |\psi(x_1, x_2)|^2$$

Define exchange operator

$$P_{12}\psi(x_1, x_2) = \psi(x_2, x_1) = e^{i\delta}\psi(x_1, x_2)$$

$$\therefore P_{12}\big(\underbrace{P_{12}\psi(x_1, x_2)}_{\psi(x_2, x_1)}\big) = P_{12}\underbrace{\psi(x_2, x_1)}_{e^{i\delta}\psi(x_1, x_2)}$$

$$= e^{i\delta}\big(\underbrace{P_{12}\psi(x_1, x_2)}_{\psi(x_2, x_1)}\big) = e^{i\delta}\underbrace{\psi(x_2, x_1)}_{e^{i\delta}\psi(x_1, x_2)}$$

$$= \boxed{e^{2i\delta}\psi(x_1, x_2)}$$

Must also be the case that

$$P_{12}\big(P_{12}\psi(x_1, x_2)\big) = \boxed{\psi(x_1, x_2)}$$

$$\therefore \psi(x_1, x_2) = e^{2i\delta} \psi(x_1, x_2)$$

Holds only if $e^{2i\delta} = 1$

or $\boxed{e^{i\delta} = \sqrt{1} = \pm 1}$

$$e^{i\delta} = \cos\delta + i\sin\delta = \pm 1$$

true if $\delta = 0, \pi$

Returning to ⊛, we find

$$\psi(x_2, x_1) = \pm \psi(x_1, x_2)$$

If $\psi(x_1, x_2) = +\psi(x_2, x_1)$ w.f. is said to be symmetric

If $\psi(x_1, x_2) = -\psi(x_2, x_1)$ w.f. is said to antisymmetric

Suppose that our system is two identical non-interacting particles.

Eg. Particles in a box.

Hamiltonian for P_1 is $\mathcal{H}_1$ s.t.

$$\mathcal{H}_1 \phi_A(x_1) = E_A \phi_A(x_1)$$

P_1 is state A w/ energy E_A

Hamiltonian for P_2 is $\mathcal{H}_2$ s.t.

$$\mathcal{H}_2 \phi_B(x_2) = E_B \phi_B(x_2)$$

P_2 in state B w/ energy E_B.

For example for free particles $\mathcal{H} = -\frac{\hbar^2}{2m}\frac{d^2}{dx^2}$

ϕ_A & ϕ_B are single-particle wavefcns of P_1 & P_2

Eg. particles in 1-D box

$$\phi_n(x) = \sqrt{\frac{2}{L}} \sin\left(\frac{n\pi x}{L}\right)$$

$\therefore$ Requirements for two-particle w.f. for net system are:

① $\psi(x_1, x_2) = \pm \psi(x_2, x_1)$ exchange property.

② $$\underbrace{(\mathcal{H}_1 + \mathcal{H}_2)}_{\text{net hamiltonian of non-interacting system.}} \psi(x_1, x_2) = (E_1 + E_2)\psi(x_1, x_2)$$

w.f. must be an eigenstate of net Hamiltonian.

<u>Try a guess</u>

$$\psi(x_1, x_2) = \phi_A(x_1) + \phi_B(x_2)$$

P_1 in state A $\quad$ P_2 in state B

test the exchange property

$$\psi(x_2, x_1) = \phi_A(x_2) + \phi_B(x_1) \neq \pm \psi(x_1, x_2)$$

P_2 in state A $\quad$ P_1 in state B

Fails exchange test.

Try another guess

$$\psi(x_1, x_2) = \phi_A(x_1)\,\phi_B(x_2)$$

exchange $\quad \psi(x_2, x_1) = \phi_A(x_2)\,\phi_B(x_1)$ fails exchange requirement.

Finally try

$$\psi(x_1, x_2) = \underbrace{\phi_A(x_1)\,\phi_B(x_2) \pm \phi_A(x_2)\,\phi_B(x_1)}_{\text{superposition of } P_1 \text{ in } A,\ P_2 \text{ in } B \text{ \& } P_1 \text{ in } B,\ P_2 \text{ in } A}$$

Makes some sense b/c w/ two identical particles one in state & other in state B, can't tell which is in A & which is in B.

Try exchange:

$$\psi(x_2, x_1) = \phi_A(x_2)\,\phi_B(x_1) \pm \phi_A(x_1)\,\phi_B(x_2)$$

$$= \pm\big(\underbrace{\phi_A(x_1)\,\phi_B(x_2) \pm \phi_A(x_2)\,\phi_B(x_1)}\big)$$

$\psi(x_1, x_2)$

$$\therefore \psi(x_2, x_1) = \pm \psi(x_1, x_2)$$ Exchange requirement satisfied!

Check that is two-particle w.f. is an eigenstate of $\mathcal{H}_1 + \mathcal{H}_2$.

$$(\mathcal{H}_1 + \mathcal{H}_2)\psi(x_1, x_2) = (\mathcal{H}_1 + \mathcal{H}_2)\left[\phi_A(x_1)\phi_B(x_2) \pm \phi_A(x_2)\phi_B(x_1)\right]$$

$$= \Big(\overbrace{\mathcal{H}_1 \phi_A(x_1)}^{E_A \phi_A(x_1)}\Big)\phi_B(x_2) \pm \phi_A(x_2)\Big(\overbrace{\mathcal{H}_1 \phi_B(x_1)}^{E_B \phi_B(x_1)}\Big)$$

$$+ \phi_A(x_1)\Big(\underbrace{\mathcal{H}_2 \phi_B(x_2)}_{E_B \phi_B(x_2)}\Big) \pm \Big(\underbrace{\mathcal{H}_2 \phi_A(x_2)}_{E_A \phi_A(x_2)}\Big)\phi_B(x_1)$$

$$= E_A \phi_A(x_1)\phi_B(x_2) \pm E_B \phi_A(x_2)\phi_B(x_1)$$

$$+ E_B \phi_A(x_1)\phi_B(x_2) \pm E_A \phi_A(x_2)\phi_B(x_1)$$

$$= E_A \Big(\overbrace{\phi_A(x_1)\phi_B(x_2) \pm \phi_A(x_2)\phi_B(x_1)}^{\psi(x_1, x_2)}\Big)$$

$$+ E_B \Big(\underbrace{\phi_A(x_1)\phi_B(x_2) \pm \phi_A(x_2)\phi_B(x_1)}_{\psi(x_1, x_2)}\Big)$$

$\therefore \ (\mathcal{H}_1 + \mathcal{H}_2)\psi(x_1, x_2) = (E_A + E_B)\psi(x_1, x_2)$

$\therefore$ our two-particle w.f. is an eigenstate of the total Hamiltonian.

Symmetric 2-particle w.f.

$$\psi_S(x_1, x_2) = \phi_A(x_1)\phi_B(x_2) + \phi_A(x_2)\phi_B(x_1)$$

Antisymmetric 2-particle w.f.

$$\psi_A(x_1, x_2) = \phi_A(x_1)\phi_B(x_2) - \phi_A(x_2)\phi_B(x_1)$$