PHYS 425 - w5l2

Last Time:

symmetric 2-particle w.f.

$$\psi_S(x_1, x_2) = \phi_A(x_1)\phi_B(x_2) + \phi_A(x_2)\phi_B(x_1)$$

antisymmetric 2-particle w.f.

$$\psi_A(x_1, x_2) = \phi_A(x_1)\phi_B(x_2) - \phi_A(x_2)\phi_B(x_1)$$

What if two particles are in same state $A=B$?

Symmetric case

$$\psi_S(x_1, x_2) = 2\phi_A(x_1)\phi_A(x_2) \quad \text{no problem.}$$

$$\psi_A(x_1, x_2) = \phi_A(x_1)\phi_A(x_2) - \phi_A(x_2)\phi_A(x_1)$$
$$= 0!$$

In this case w.f. vanishes

Antisymmetric wavefcn does not allow two identical particles to occupy the same quantum state.

→ Pauli Exclusion Principle.

In Nature particles w/ integer spin ($S = 0, 1, 2, \ldots$) have symmetric wavefcns & are called Bosons

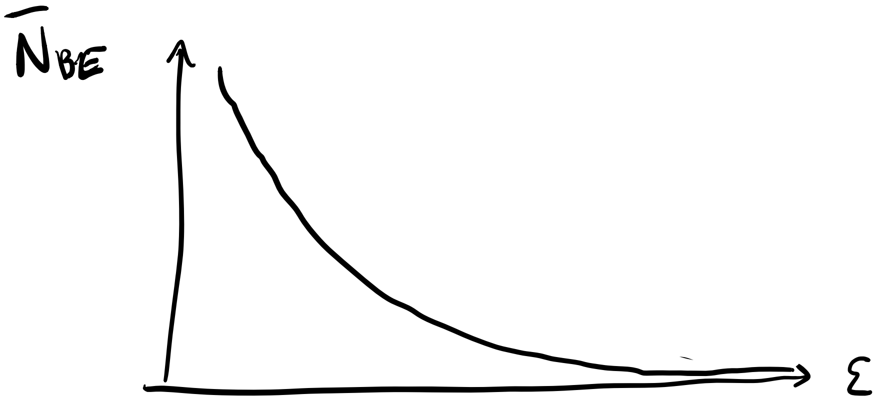
Eg. photons
phonons
^4He atom
Gluon
⋮

can have any no. of identical Bosons in the same quantum state.

Bose-Einstein dist'n gives avg. no. of Bosons in state w/ energy ε at temp T.

$$\bar{N}_{BE} = \frac{1}{e^{(\varepsilon-\mu)/k_B T} - 1}$$

large ε, $\bar{N}_{BE} \to 0$



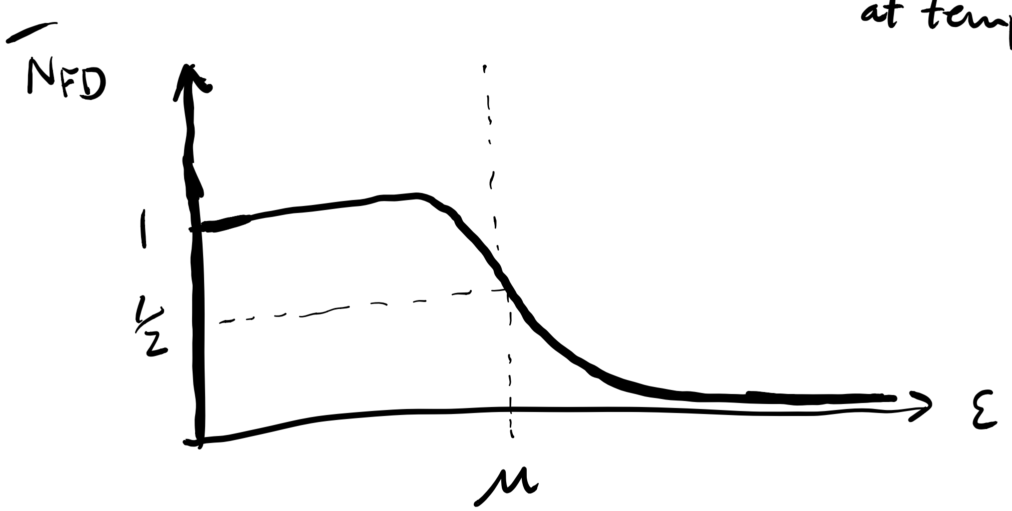
In Nature particles w/ half integer spin ($s = \frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \dots$) have antisymmetric w.f. & are called Fermions.

E.g. electrons
quarks
protons
neutrons
^3He atoms
⋮

can have only zero or one identical Fermion in same quantum state → obey Pauli Exclusion Principle.

$$\bar{N}_{FD} = \frac{1}{e^{(\varepsilon - \mu)/k_B T} \underline{\underline{+}}\ 1}$$

Fermi-Dirac dist'n gives avg. no. of Fermions in state w/ energy ε at temp. T



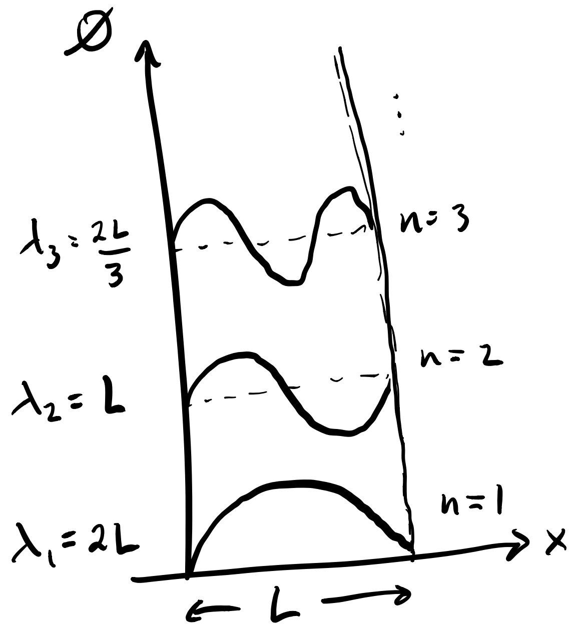
Proof that spin $\frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \ldots$ have antisymmetric w.f.s & spin $0, 1, 2, \ldots$ have symmetric w.f.s called spin-statistics theorem & requires relativistic Q.M.

Recall the wavefcn for a particle in a 1-D box.

$$\phi(x) = A \sin\left(\frac{\boxed{n\pi}\, x}{\boxed{L}}\right)$$

normalization const.

$$\left.\begin{aligned} \phi(0) &= 0 \\ \phi(L) &= 0 \end{aligned}\right\} \forall n$$



$$\rightarrow \lambda_n = \frac{2L}{n}$$

Engergy given by $\hat{H}\phi(x) = E\,\phi(x)$

K.E. of particle

where $\hat{H} = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}$ is the Hamiltonian.

Define wave number $k_n = \frac{2\pi}{\lambda_n} = \frac{n\pi}{L}$

s.t. $\phi(x) = A \sin kx$

$$\therefore \hat{H}(\phi(x)) = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}(A \sin kx)$$

$$= \underbrace{\frac{\hbar^2 k^2}{2m}}(A \sin kx)$$

$$E(k) = \frac{\hbar^2 k^2}{2m} = \frac{n^2 \pi^2 \hbar^2}{2m L^2}$$

$$n = 1, 2, 3, \ldots$$

By analogy, the w.f. of a particle in a 3-D box (cube of sides L) is

$$\phi(x, y, z) = A \sin\left(\frac{n_x \pi x}{L}\right) \sin\left(\frac{n_y \pi y}{L}\right) \sin\left(\frac{n_z \pi z}{L}\right)$$

$$= A \sin(k_x x) \sin(k_y y) \sin(k_z z)$$

$$k_i = \frac{n_i \pi}{L} \qquad i = x, y, z$$

Boundary conditions

$$\emptyset(x, y, z) = 0 \text{ when } \left.\begin{matrix} x \\ y \\ z \end{matrix}\right\} = 0$$

$$\text{or } \left.\begin{matrix} x \\ y \\ z \end{matrix}\right\} = L$$

$$\hat{\mathcal{H}} = -\frac{\hbar^2}{2m}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)$$

$$\hat{\mathcal{H}}\emptyset(x, y, z) = \frac{\hbar^2}{2m}\underbrace{\left(k_x^2 + k_y^2 + k_z^2\right)}_{\equiv k^2}$$

$$= \frac{\pi^2\hbar^2}{2mL^2}\left(n_x^2 + n_y^2 + n_z^2\right)$$

$n_x = 1, 2, 3, \ldots$

$n_y = 1, 2, 3, \ldots$

$n_z = 1, 2, 3, \ldots$

Consider a collection of non-interacting Bosons confined to 3-D box in $T=0$ grnd state.

→ all Bosons go into the lowest energy state $n_x = n_y = n_z = 1$

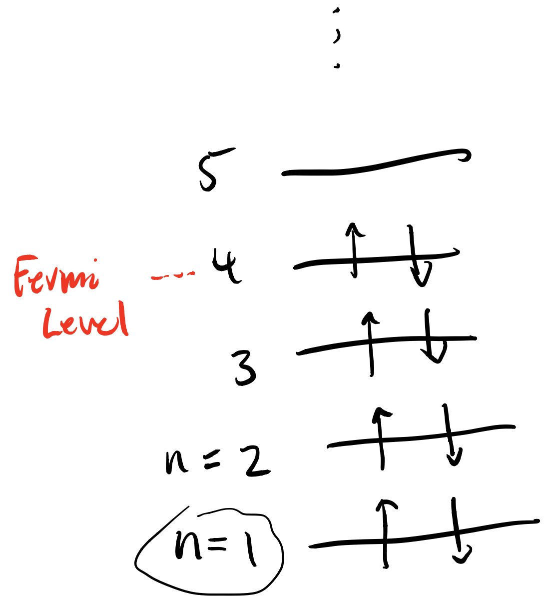
∴ for N identical non-interacting Bosons the system's grnd state energy is:

$$E_0 = N \frac{3\pi^2\hbar^2}{2mL^2}$$

What if we had N non-interacting Fermions like ^3He atoms. ^3He has spin $\frac{1}{2}$, so can put two ^3He atoms in each state

→ one spin up & the other spin down.

In 1-D well would have:



$E_3 = 9E_1$

$E_2 = 4E_1$

$E_1 = \dfrac{\pi^2 \hbar^2}{2mL^2}$

If have N particles, fill up to the $n = \frac{N}{2}$ state.

Highest occupied state is called the Fermi Level.

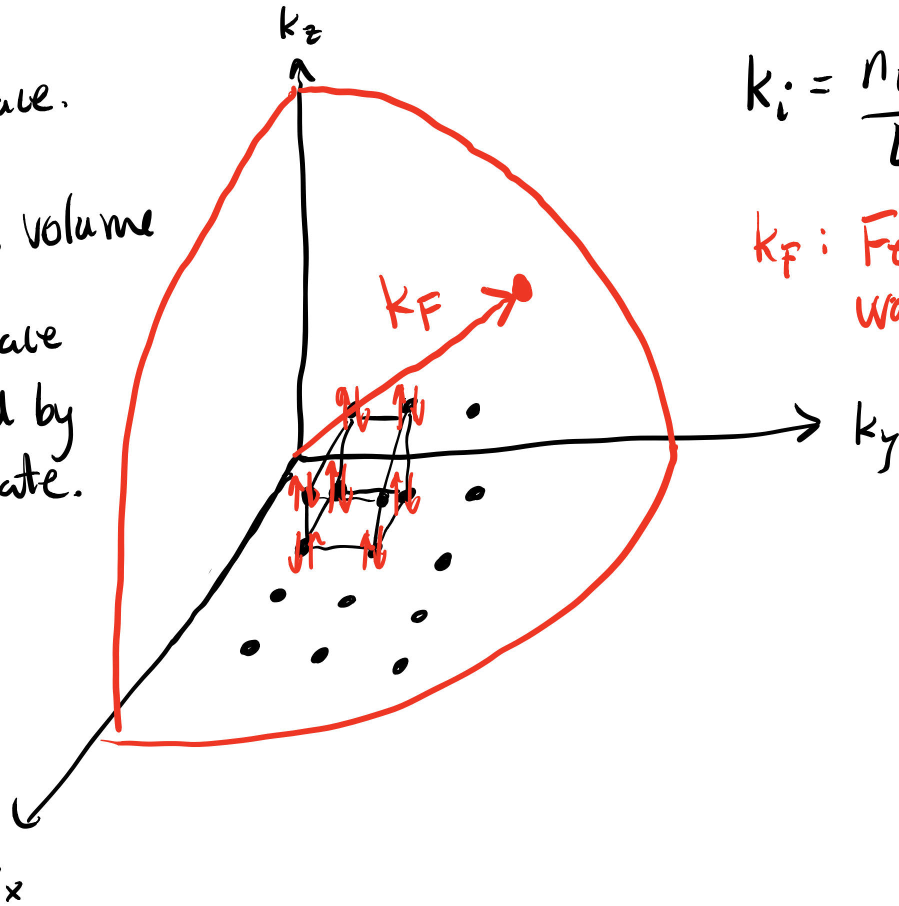
The Fermi energy is the energy of the highest occupied state

In 1-D $E_F = \left(\frac{N}{2}\right)^2 \dfrac{\pi^2 \hbar^2}{2mL^2}$

What would be the Fermi Energy if we had N identical Fermions in a 3-D box?
(like conduction electrons in a metal)

k-space.

$\left(\frac{\pi}{L}\right)^3$ is volume of k-space occupied by each state.



$$k_i = \frac{n_i \pi}{L}$$

k_F : Fermi wavevector

Fill lowest energy states in k-space one by one w/ two particles of opposite spin until used up all N particles. Will form one $\frac{1}{8}$ of a sphere called the Fermi sphere.

The volume of $\frac{1}{8}$ sphere is

$$\frac{1}{8}\,\frac{4}{3}\pi k_F^3 = \frac{1}{6}\pi k_F^3$$

No. of states inside the Fermi sphere is the volume of sphere divide by the volume occupied per state:

$$\frac{\frac{1}{6}\pi k_F^3}{\left(\frac{\pi}{L}\right)^3} = \frac{L^3 k_F^3}{6\pi^2}$$

Can put two particles per state (one $\uparrow$ one $\downarrow$)

$\therefore$ we require $\frac{N}{2} = \frac{L^3 k_F^3}{6\pi^2}$

$$\therefore \boxed{k_F^3 = \frac{3\pi^2 N}{L^3}}$$

Since $E = \frac{\hbar^2 k^2}{2m}$

we have $\boxed{E_F = \frac{\hbar^2 k_F^2}{2m} = \frac{\hbar^2}{2m}\left(\frac{3\pi^2 N}{L^3}\right)^{2/3}}$

Fermi energy of identical fermions in 3-D box.

Important point to remember when we discucs mixtures of ^3He & ^4He ...

$\rightarrow$ $E_F \propto N^{2/3}$