PHYS 425 - W10L1

Recall the Grand Partition fcn:

$$\Xi = \sum_{\substack{\text{all} \\ \text{states}}} e^{-(\varepsilon_i - \mu N_i)/k_B T}$$

For a single energy level that can be occupied by any number of Bosons:

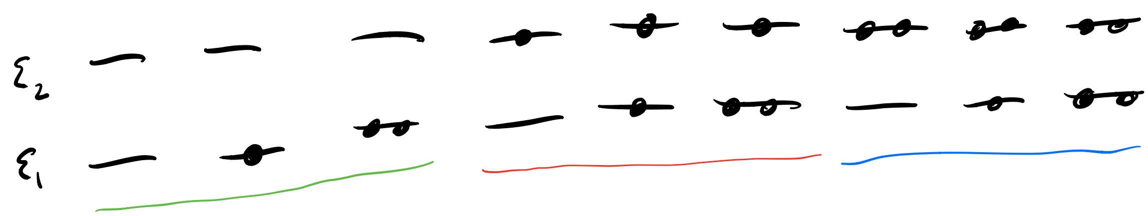
$$\Xi_{\vec{k}} = \frac{1}{1-e^{-(\varepsilon(k)-\mu)/k_B T}} = \left[1-e^{-(\varepsilon(k)-\mu)/k_B T}\right]^{-1}$$

single energy level

If consider Grand Canonical Ensemble of entire system (many levels of different $\vec{k}$ values):

$$\Xi = \prod_{\vec{k}} \Xi_{\vec{k}} = \prod_{\vec{k}} \left(1-e^{-(\varepsilon(k)-\mu)/k_B T}\right)^{-1}$$

Eg. Consider 2 levels each w/ 0, 1, or 2 particles



$$\Xi_{net} = 1 + e^{-(\varepsilon_1 - \mu)/k_BT} + e^{-(2\varepsilon_1 - 2\mu)/k_BT}$$
$$+ e^{-(\varepsilon_2 - \mu)/k_BT} + e^{-(\varepsilon_1 + \varepsilon_2 - 2\mu)/k_BT}$$
$$+ e^{-(2\varepsilon_1 + \varepsilon_2 - 3\mu)/k_BT}$$
$$+ e^{-(2\varepsilon_2 - 2\mu)/k_BT} + e^{-(\varepsilon_1 + 2\varepsilon_2 - 3\mu)/k_BT}$$
$$+ e^{-(2\varepsilon_1 + 2\varepsilon_2 - 4\mu)/k_BT}$$

c.t.
$$\left(1 + e^{-(\varepsilon_1 - \mu)/k_BT} + e^{-(2\varepsilon_1 - 2\mu)/k_BT}\right) \cdot \left(1 + e^{-(\varepsilon_2 - \mu)/k_BT} + e^{-(2\varepsilon_2 - 2\mu)/k_BT}\right)$$

$$\therefore \; \Xi_{net} = (\Xi_1)(\Xi_2)$$

In general,

$$\Xi = \prod_{\vec{k}} \left(1 - e^{-(\varepsilon(k)-\mu)/k_BT}\right)^{-1}$$

$\therefore$ grand Potential for many-level system is

$$\Phi_G = -k_BT \ln \Xi$$

$$= -k_BT \ln \left[\prod_{\vec{k}} \left(1 - e^{-(\varepsilon(k)-\mu)/k_BT}\right)^{-1} \right]$$

$$= -k_BT \sum_{\vec{k}} \ln \left[1 - e^{-(\varepsilon(k)-\mu)/k_BT}\right]^{-1}$$

$$= +k_BT \sum_{\vec{k}} \ln \underbrace{\left[1 - e^{-(\varepsilon(k)-\mu)/k_BT}\right]}_{\text{depends only on mag. of } \vec{k}}$$

Recall density of states

$$\sum_{\vec{k}} \underbrace{f(k)}_{\text{any fcn of mag. of } \vec{k}} = \frac{V}{2\pi^2} \int_0^\infty k^2 f(k)\, dk$$

↳ any fcn of mag. of $\vec{k}$

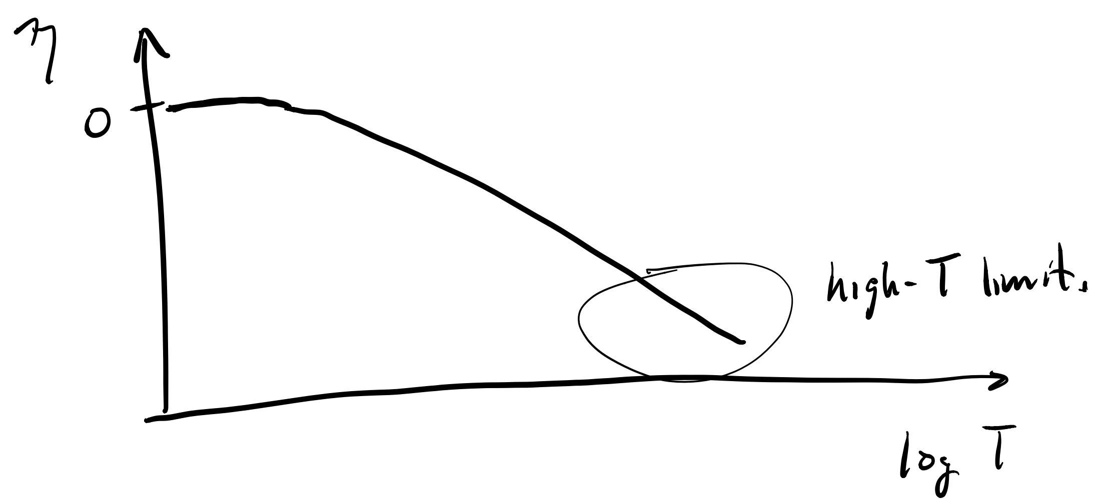
$$\boxed{\therefore\ \Phi_G = \frac{V}{2\pi^2} k_B T \int_0^\infty k^2 \ln\left[1 - e^{-(\varepsilon(k)-\mu)/k_B T}\right] dk}$$

General expression for grand potential of system of Bosons

- valid for any temp T
- valid for any form of $\varepsilon(k)$

Low-temp Limit:

Know $\eta = \frac{\mu}{k_B T}$ goes to zero as $T \to 0$



$\therefore$ in low-T limit

$$\boxed{\Phi_G \approx \frac{V}{2\pi^2} k_B T \int_0^\infty k^2 \ln\left[1 - e^{-\varepsilon(k)/k_B T}\right] dk \quad (1)}$$

only valid for Bosons for which $\frac{\mu}{k_B T} \to 0$ as $T \to 0$

high-temp limit:

For a gas of particles (Fermions or Bosons) in dilute high-temp limit

$$\eta = \frac{\mu}{k_B T} = \ln\left(\frac{n}{n_Q}\right) \text{ is large \& neg.}$$

$$n_Q = \left(\frac{m k_B T}{2\pi \hbar^2}\right)^{3/2} \qquad n = \frac{N}{V}$$

In this case

$$\bar{N}_{BE} = \frac{1}{e^{(\varepsilon(k)-\mu)/k_BT}-1} \approx e^{-(\varepsilon(k)-\mu)/k_BT}$$

$$\therefore \; e^{(\varepsilon(k)-\mu)/k_BT} \gg 1$$

$$\text{or} \quad 1 \gg \underbrace{e^{-(\varepsilon(k)-\mu)/k_BT}}_{\text{small.}}$$

Recall $\ln(1 \pm x) \approx \pm x$ for $|x| \ll 1$

$$\therefore \; \ln\Big[1 - \underbrace{e^{-(\varepsilon(k)-\mu)/k_BT}}_{\text{small}}\Big]$$

$$\approx -e^{-(\varepsilon(k)-\mu)/k_BT}$$

$$= -e^{-\varepsilon(k)/k_BT} e^{\eta}$$

$\therefore$ In high-T (dilute) limit

$$\boxed{\Phi_G \approx -\frac{V}{2\pi^2} k_BT e^{\eta} \int_0^{\infty} k^2 e^{-\varepsilon(k)/k_BT} dk \qquad (2)}$$

valid for Bosons & Fermions.

Recall $\Phi_G = U - TS - \mu N$

$$d\Phi_G = dU - TdS - SdT - \mu dN - Nd\mu$$

$$= \cancel{TdS} - PdV + \cancel{\mu dN} - \cancel{TdS} - SdT - \cancel{\mu dN} - Nd\mu$$

$$d\Phi_G = -PdV - SdT - Nd\mu$$

$$\therefore \quad P = -\left(\frac{\partial \Phi_G}{\partial V}\right)_{T,\mu}$$

$$S = -\left(\frac{\partial \Phi_G}{\partial T}\right)_{V,\mu} \rightsquigarrow C_V = T\frac{\partial S}{\partial T}$$

$$N = -\left(\frac{\partial \Phi_G}{\partial \mu}\right)_{V,T}$$

$\approx>$ can't use in low-T limit b/c set $\frac{\mu}{k_B T} \to 0$

For gas of free Bosons $\left(\varepsilon(k) = \frac{\hbar^2 k^2}{2m} \right)$

& use ② to find in high-T limit:

$$P = \frac{N k_B T}{V}$$ ideal gas law

$$S = N k_B \left[\ln\left(\frac{n_Q}{n}\right) + \frac{5}{2} \right]$$ Sackur-Tetrode eq'n.

$$C_V = \frac{3}{2} N k_B$$

$\boxed{\text{leave for homework}}$

Recovered classical results starting from Quantum theory of identical particles.

For gas of free Bosons use ① to find in low-T limit:

$$\left.\begin{aligned} P &\propto T^{5/2} \\ S &\propto T^{3/2} \\ C_V &\propto T^{3/2} \end{aligned}\right\} \text{homework.}$$