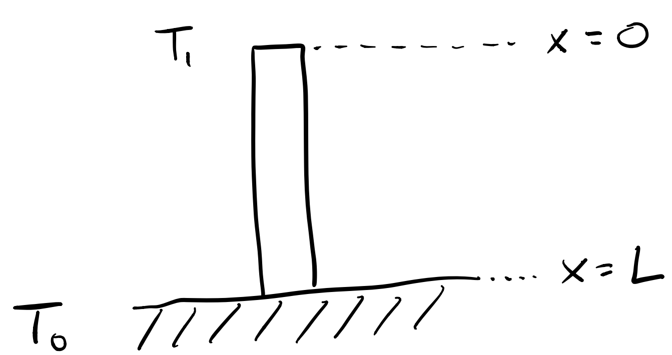
PHYS 425 – w4l3

Solving the 1-D Heat Equation

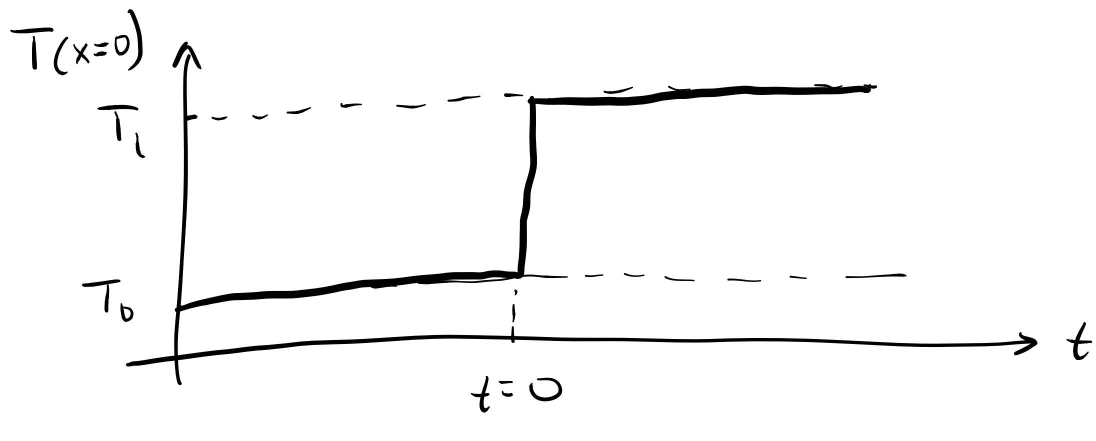
$$\frac{\partial T(x,t)}{\partial t} = \alpha \frac{\partial^2 T(x,t)}{\partial x^2} \quad \text{where } \alpha = \frac{K}{c_v \rho}$$

Assume initially have a bar at uniform temp.



Then temp is changed suddenly to T_1 at $x=0$

@ $x=L$, temp remains fixed at T_0.



Make a change of variables

$$u = T - T_0 \Rightarrow T = u + T_0$$

$$\frac{\partial T}{\partial t} = \frac{\partial u}{\partial t} \qquad \frac{\partial^2 T}{\partial x^2} = \frac{\partial^2 u}{\partial x^2}$$

$$\alpha \equiv c^2$$

$$\Rightarrow \quad \frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}$$

Initial Conditions & Boundary conditions

$$T(x,0) = T_0 \Rightarrow u(x,0) = 0 \quad \forall x \quad t < 0$$

$$T(L,t) = T_0 \Rightarrow u(L,0) = 0 \quad \forall t$$

$$T(0,t) = T_1 \Rightarrow u(0,t) = T_1 - T_0 \quad t > 0$$

Separation of variables

$$u(x,t) = F(x)\, G(t)$$

$$\frac{\partial u}{\partial t} = F \dot{G} \qquad \frac{\partial^2 u}{\partial x^2} = F'' G$$

$\therefore \; F\dot{G} = c^2 F'' G$ divide by $c^2 F G$

$$\underbrace{\frac{\dot{G}}{c^2 G}}_{\text{fcn of } t} = \underbrace{\frac{F''}{F}}_{\text{fcn of } x} = h \quad (\text{const})$$

Two homogeneous ordinary diff. eq'ns.

① $F'' - hF = 0$

② $\dot{G} - hc^2 G = 0$

Start w/ ①.

case (i) $h = \mu^2 > 0$

$\rightarrow$ $F'' - \mu^2 F = 0 \Rightarrow$ sol'n $\underline{F(x) = Ae^{\mu x} + Be^{-\mu x}}$

Initial condition @ $t = 0$

$$u(x,0) = 0 \; \begin{cases} u(0,0) = 0 \\ u(L,0) = 0 \end{cases}$$

$$u(0,0) = F(0)\,G(0)$$
$$= (A+B)\,G(0) = 0$$
require $A+B=0 \Rightarrow B=-A.$

$$u(L,0) = F(L)\,G(L) \cdot$$

$$u(L,0) = F(L)\,G(0)$$
$$= \Big(Ae^{\mu L} + \underbrace{B}_{-A}e^{-\mu L}\Big)\,G(0) = 0$$

$$= A\left(e^{\mu L} - e^{-\mu L}\right)G(0) = 0$$

$\mu \neq 0$

$\Rightarrow \boxed{A=0 \;\therefore\; B=0}$ $\quad h>0$ case.
$F=0.$

case ii $\quad h=0$

$$F'' - \cancel{h}^{\,0}F = 0, \quad F''=0.$$

sol'n $\quad F = ax+b \Leftarrow$
$F' = a$
$F'' = 0$ ✓

Boundary conditions

$$u(0,t) = T_1 - T_0 \qquad t > 0$$

$$= F(0)\,G(t) = T_1 - T_0$$

$$\therefore \; bG(t) = T_1 - T_0$$

$$u(L,t) = F(L)\,G(t) = 0$$

$$\therefore \; (aL + b)\,G(t) = 0$$

$$aL\,G(t) + \underbrace{b\,G(t)}_{T_1 - T_0} = 0$$

$$\therefore \quad a = \frac{-(T_1 - T_0)}{L\,G(t)}$$

Note that, since $\dot{G} - \overset{0}{h}c^2 G = 0$,

if $h = 0$, $\dot{G} = 0$ or $G(t) = \text{const}$.

can arbitrarilly set $G(t) = 1$

$$\boxed{\begin{aligned} b &= T_1 - T_0 \\ a &= \frac{-(T_1 - T_0)}{L} \end{aligned}}$$

$$\therefore F(x) = (T_1 - T_0) - \frac{(T_1 - T_0)}{L} x$$

$$\boxed{F(x) = (T_1 - T_0)\left(1 - \frac{x}{L}\right)} \quad h = 0 \text{ case}$$

case (iii) $h = -p^2 < 0$

$$\therefore F'' + p^2 F = 0$$

sol'n $F(x) = A\cos px + B\sin px$

Boundary condition

$u(x, 0) = 0$

$\therefore F(x) G(0) = 0$

$x = 0$

$F(0) G(0) = 0$

$\therefore A G(0) = 0$

$\therefore A = 0.$

$F(L) G(0) = 0$

$B\sin(pL) G(0) = 0$

$\therefore \sin(pL) = 0 \qquad p_n L = n\pi$

$$p_n = \frac{n\pi}{L}$$

$\therefore \; F_n(x) = B\sin\left(\frac{n\pi}{L}x\right)$

arbitrarily set $B = 1$

$$\boxed{F_n(x) = \sin\left(\frac{n\pi}{L}x\right)} \quad h = -p^2 < 0 \text{ case}$$

Now work on 2nd ODE

$$\dot{G} - hc^2 G = 0$$

consider only the $h = -p^2$ case

b/c $h = \mu^2 > 0$ case resulted in $F(x) = 0$

so $G(t)$ doesn't matter.

We also already saw that $h = 0$ case gave

$G = \text{const.}$

If $h = -p_n^2$

then $\dot{G} + (p_n c)^2 G = 0$

Know $p_n = \frac{n\pi}{L} \quad \therefore \; p_n c = \frac{n\pi c}{L} \equiv \lambda_n$

$$\dot{G} + \lambda_n^2 G = 0 \quad \rightsquigarrow \quad \frac{dG}{G} = -\lambda_n^2\, dt$$

sol'n is $G_n(t) = B_n e^{-\lambda_n^2 t}$

$$* \quad u(x,t) = \underbrace{(T_1 - T_0)\left(1 - \frac{x}{L}\right)}_{h=0} + \underbrace{\sum_{n=1}^{\infty} B_n e^{-\lambda_n^2 t} \sin\left(\frac{n\pi x}{L}\right)}_{h=-p^2<0 \text{ case}}$$

Still need to determine value of B_n.
Use initial condition

$$u(x,0) = 0 = (T_1 - T_0)\left(1 - \frac{x}{L}\right) + \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{L}\right)$$

<u>Note</u>: $\displaystyle\int_0^L \sin\left(\frac{m\pi x}{L}\right)\sin\left(\frac{n\pi x}{L}\right)dx = \frac{L}{2}\delta_{nm} \quad (m, n \neq 0)$

mult. $u(x,0)$ by $\sin\left(\frac{m\pi x}{L}\right)$ & integrate.

$$0 = \frac{2}{L}\int_0^L (T_1 - T_0)\left(1 - \frac{x}{L}\right)\sin\left(\frac{m\pi x}{L}\right)dx$$
$$+ \frac{2}{L}\sum_{n=1}^{\infty} B_n \int_0^L \sin\left(\frac{m\pi x}{L}\right)\sin\left(\frac{n\pi x}{L}\right)dx$$

$$\sum_{n=1}^{\infty} B_n \delta_{nm} = B_m$$

can solve $$\int_0^L \left(1-\frac{x}{L}\right)\sin\left(\frac{m\pi x}{L}\right)dx$$

by integration by parts. $$\int_a^b u\,dv = uv\Big|_a^b - \int_a^b v\,du$$

Result is $\frac{L}{m\pi}$

$$0 = \frac{2}{\cancel{L}}\left(T_1 - T_0\right)\frac{\cancel{L}}{m\pi} + B_m$$

$$\therefore \boxed{B_m = -\frac{2(T_1-T_0)}{m\pi}}$$

$$\therefore u(x,t) = (T_1-T_0)\left(1-\frac{x}{L}\right) - \frac{2(T_1-T_0)}{\pi}\sum_{n=1}^{\infty}\frac{e^{-\lambda_n^2 t}}{n}\sin\left(\frac{n\pi x}{L}\right)$$

$$= (T_1-T_0)\left[\left(1-\frac{x}{L}\right) - \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{e^{-\lambda_n^2 t}}{n}\sin\left(\frac{n\pi x}{L}\right)\right]$$

Recall that $u(x,t) = T(x,t) - T_0$

$$\boxed{\therefore T(x,t) = T_0 + (T_1 - T_0)\left[\left(1-\frac{x}{L}\right) - \frac{2}{\pi}\sum_{n=1}^{\infty} \frac{e^{-\lambda_n^2 t}}{n} \sin\left(\frac{n\pi x}{L}\right)\right]}$$

$$\boxed{\begin{aligned} \lambda_n &= \frac{n\pi c}{L} = \frac{n\pi}{L}\sqrt{\alpha} \\ \alpha &= \frac{K}{c_v \rho} \end{aligned}}$$

after a long time, the $\sum e^{-\lambda_n^2 t}$ term becomes negligible.

t large:

* $T(x) \approx T_0 + (T_1 - T_0)\left(1 - \frac{x}{L}\right)$

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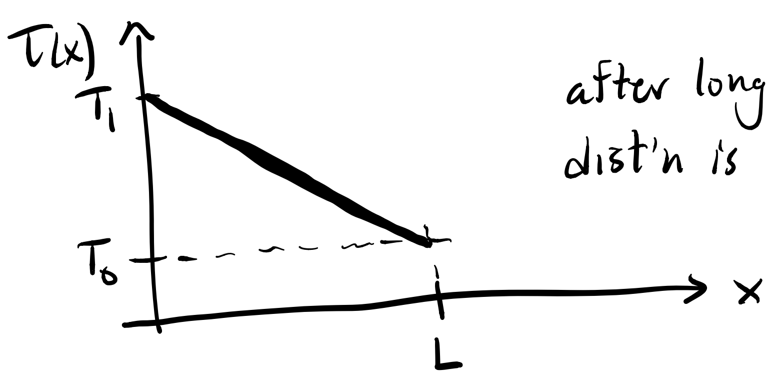


after long time, temp dist'n is linear in x.

Can now determine temp at any position along the rod at any time!

Eg. For Cu rod at 10 K $\alpha \approx 0.1 \frac{m^2}{s}$

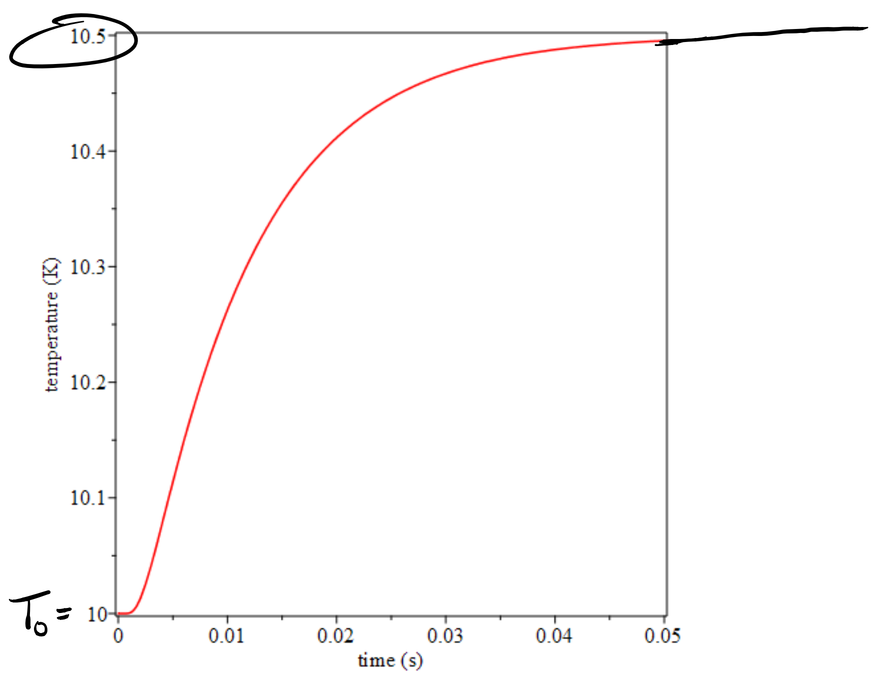
$T_0 = 10K$

$T_1 = 11K \quad (x=0)$

$L = 10cm$

Keeping terms up to $n = 500$

Plot $x = 0.05m$ (5cm) $\quad 0 \leq t \leq 0.05s$



Plot 2: $t = 0.015$ $0 \le x \le 0.1\,m$