PHYS 425 - w8l1

Last Time:

$$\sum_{\vec{k}} f(k) \Rightarrow \frac{V}{2\pi^2}\int_{k=0}^{\infty} f(k)\, k^2\, dk$$

sum over vector $\vec{k}$

fcn of mag. of $\vec{k}$

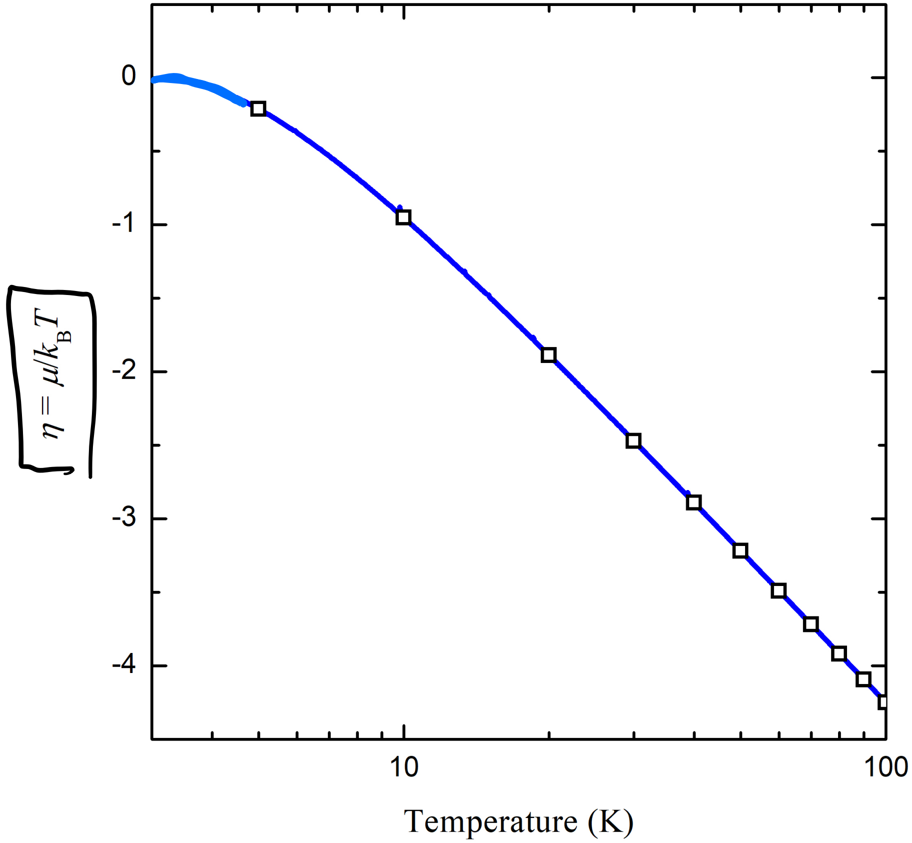
$$N = \sum_{\vec{k}} \bar{N}_{BE} = \underbrace{\frac{1}{e^{-\mu/k_BT}-1}}_{N_0} + \underbrace{\frac{1}{e^{(\varepsilon_1-\mu)/k_BT}-1}}_{N_1} + \underbrace{\frac{1}{e^{(\varepsilon_2-\mu)/k_BT}-1}}_{N_2} + \cdots$$

Using n & m for liquid ^4He

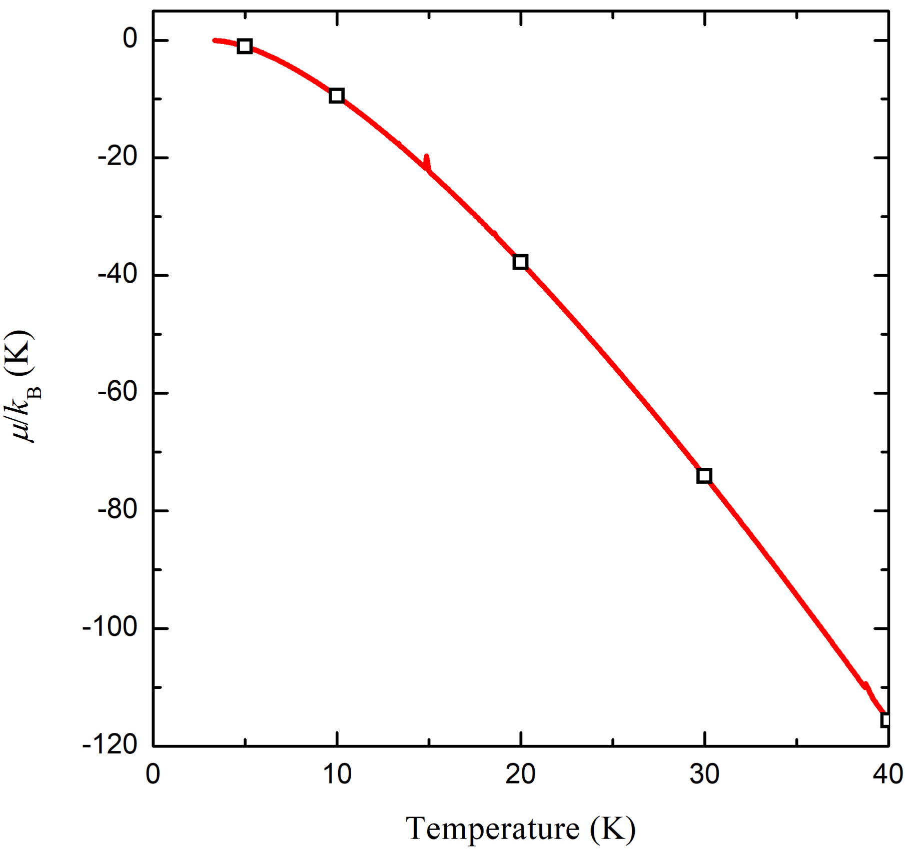
$$\left(\frac{3.42\,K}{T}\right)^{3/2} = \int_0^{\infty} \frac{x^2\, dx}{e^{x^2-\eta}-1}$$

$$\boxed{x^2 = \frac{\hbar^2 k^2}{2mk_BT} \qquad \eta = \frac{\mu}{k_BT}}$$

used to find T dependence of η & $\mu = \eta k_BT$



0
-1
-2
-3
-4
10
100
Temperature (K)
$\eta = \mu/k_B T$



0
-20
-40
-60
-80
-100
-120
0
10
20
30
40
μ/k_B (K)
Temperature (K)

Since we know temperature dependence of $\mu(T)$, can plot some of the terms in $N = \sum_{\vec{k}} \bar{N}_{BE}(k) = N_0 + N_1 + N_2 + \ldots + N_i + \ldots$

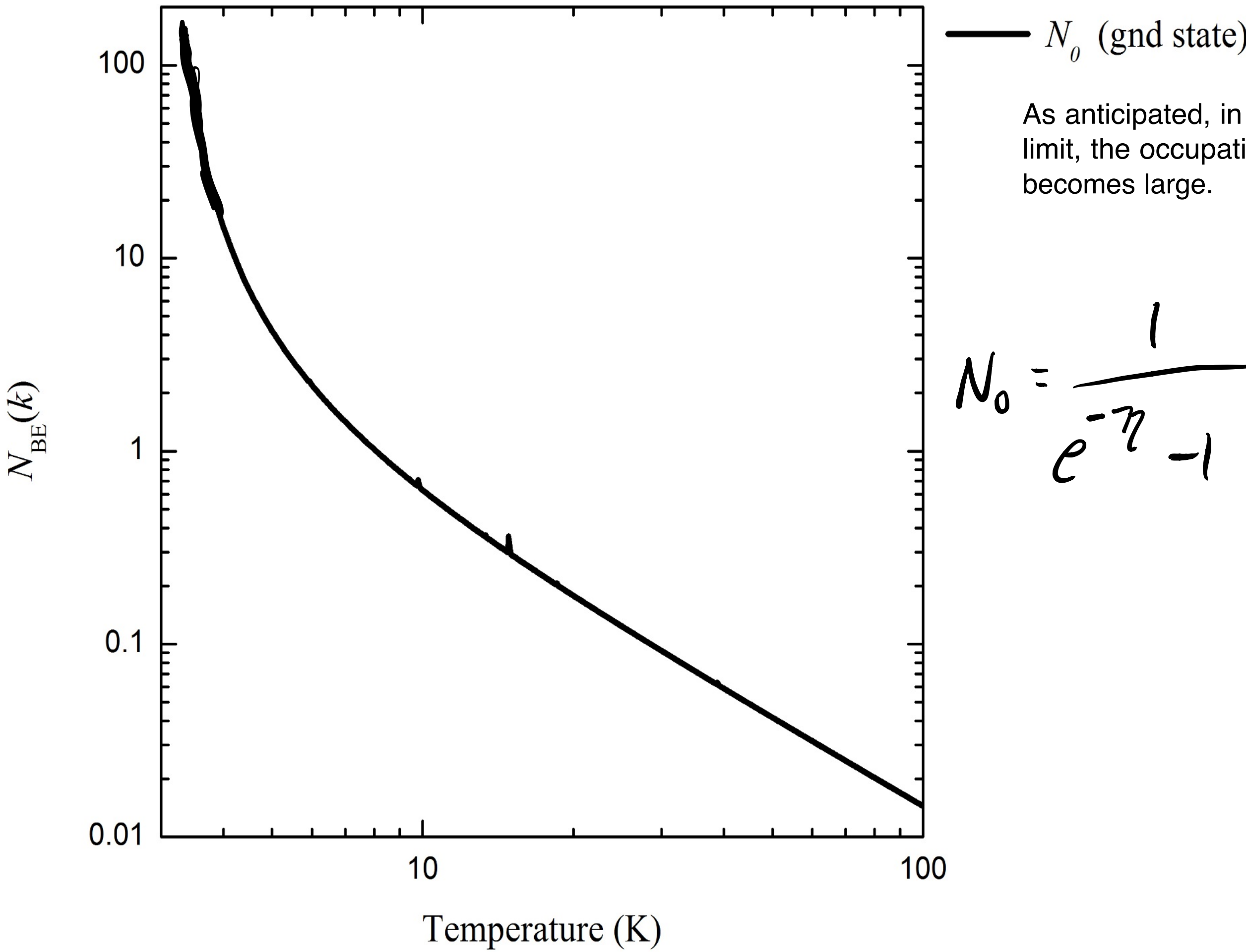
$$N_0 = \frac{1}{e^{-\mu/k_B T} - 1} = \frac{1}{e^{-\eta} - 1}$$

$$N_i = \frac{1}{e^{(\varepsilon_i - \mu)/k_B T} - 1} = \frac{1}{e^{\varepsilon_i/k_B T - \eta} - 1}$$

Define ε_i in terms of a temp.

$$T_i = \frac{\varepsilon_i}{k_B}$$

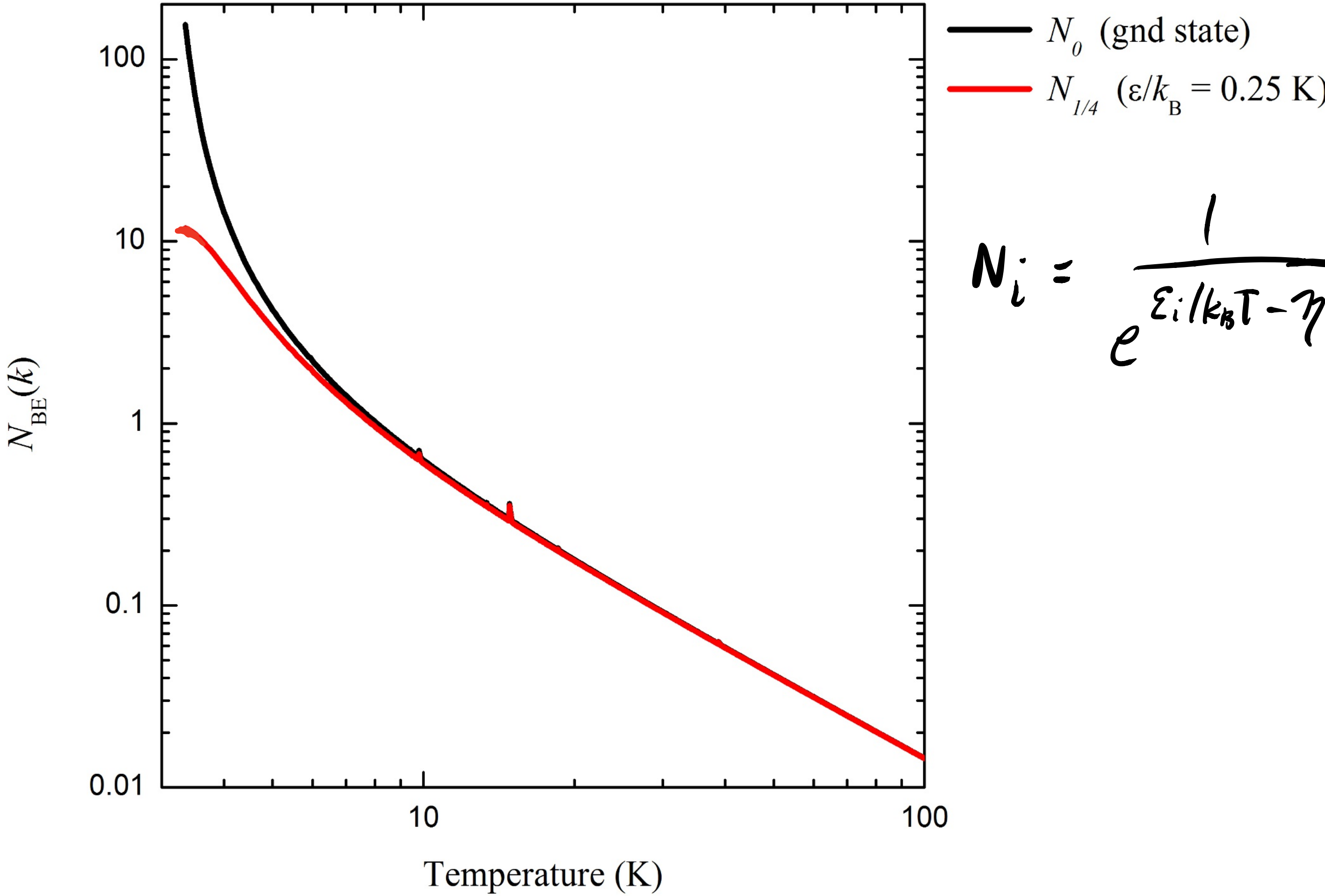
The Boson Dist'n Fcn ...



As anticipated, in the low-temperature limit, the occupation of the gnd state becomes large.

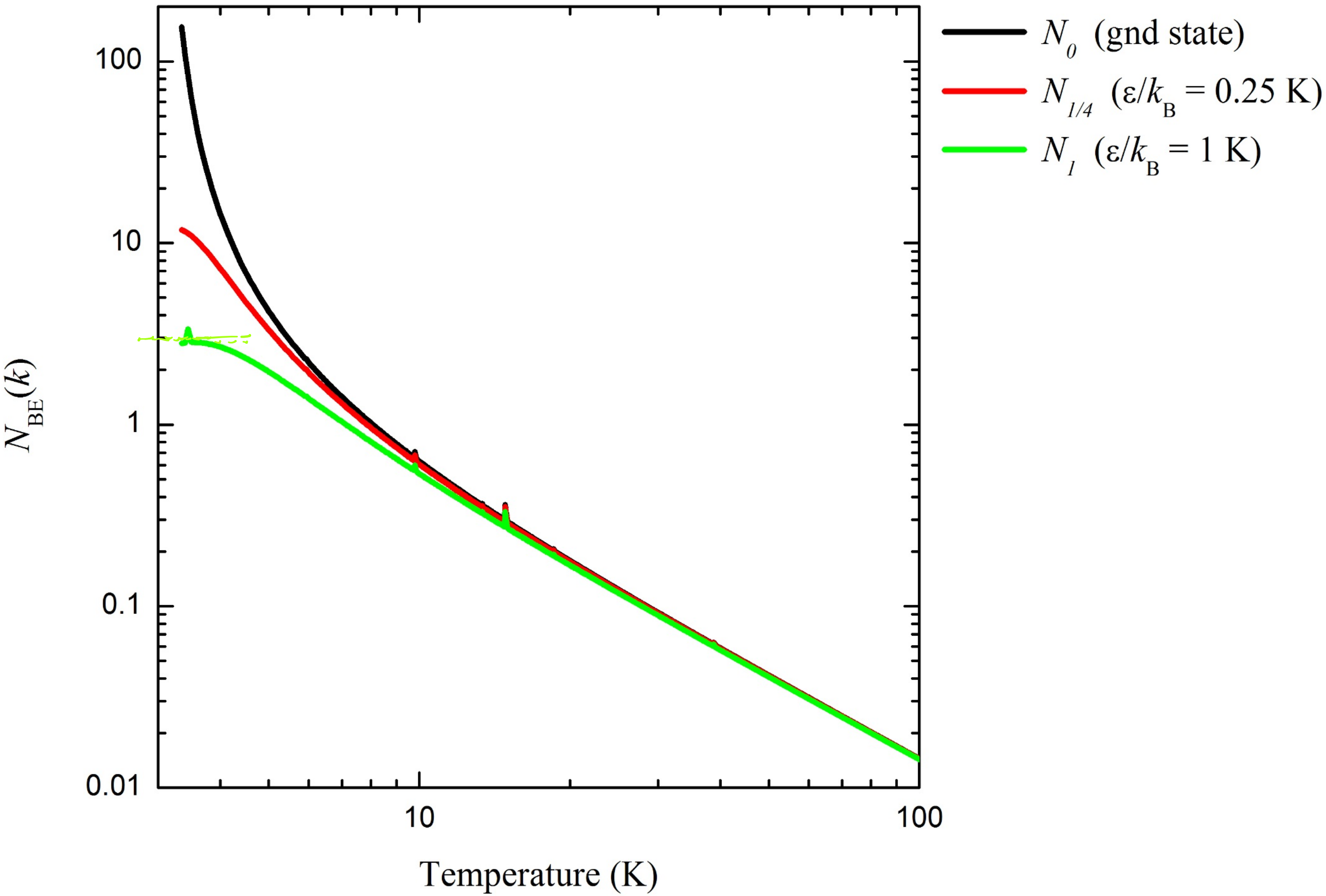
$$N_0 = \frac{1}{e^{-\eta} - 1}$$

The Boson Dist'n Fcn ...

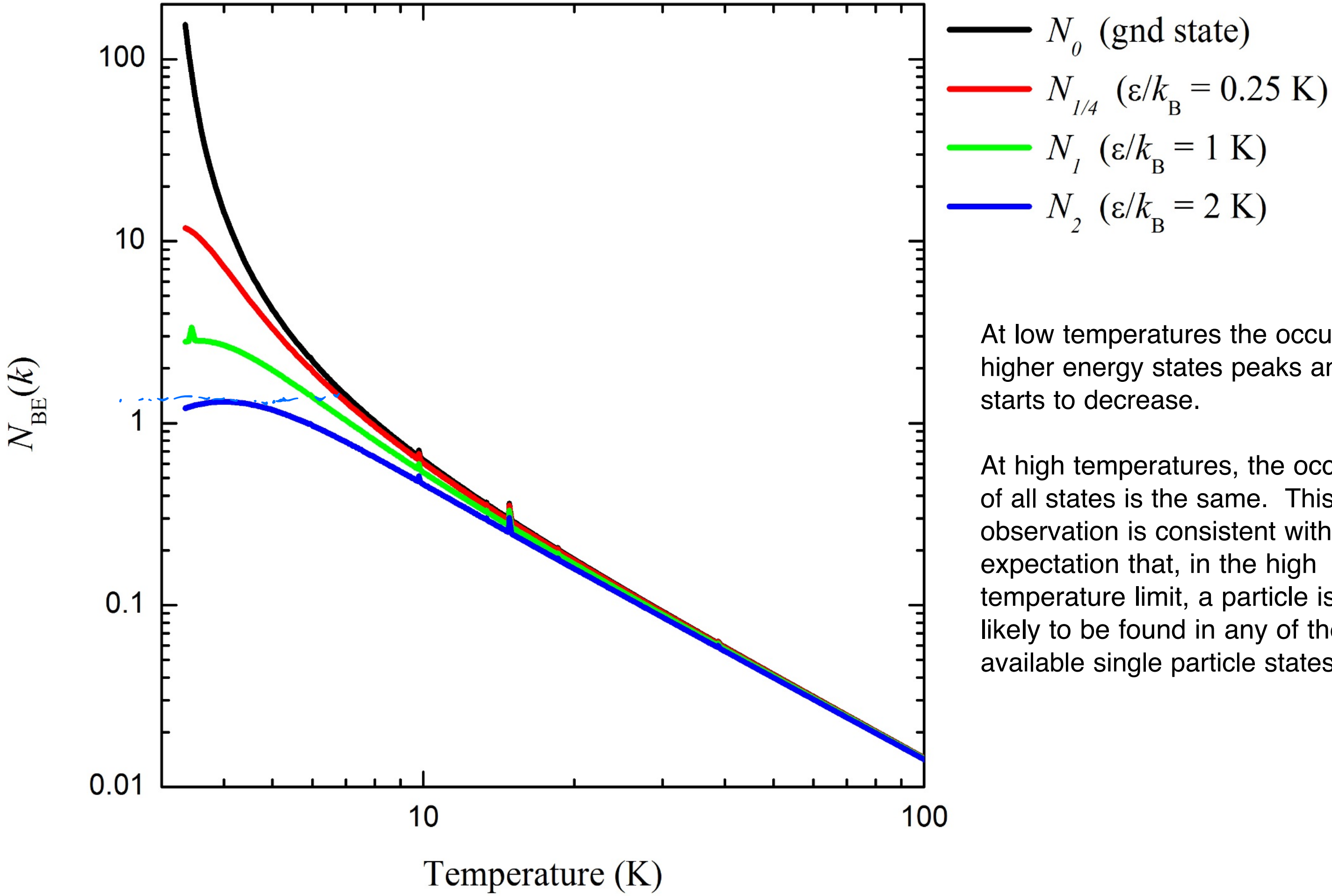


$$N_i = \frac{1}{e^{\varepsilon_i/k_BT - \eta} - 1}$$

The Boson Dist'n Fcn ...



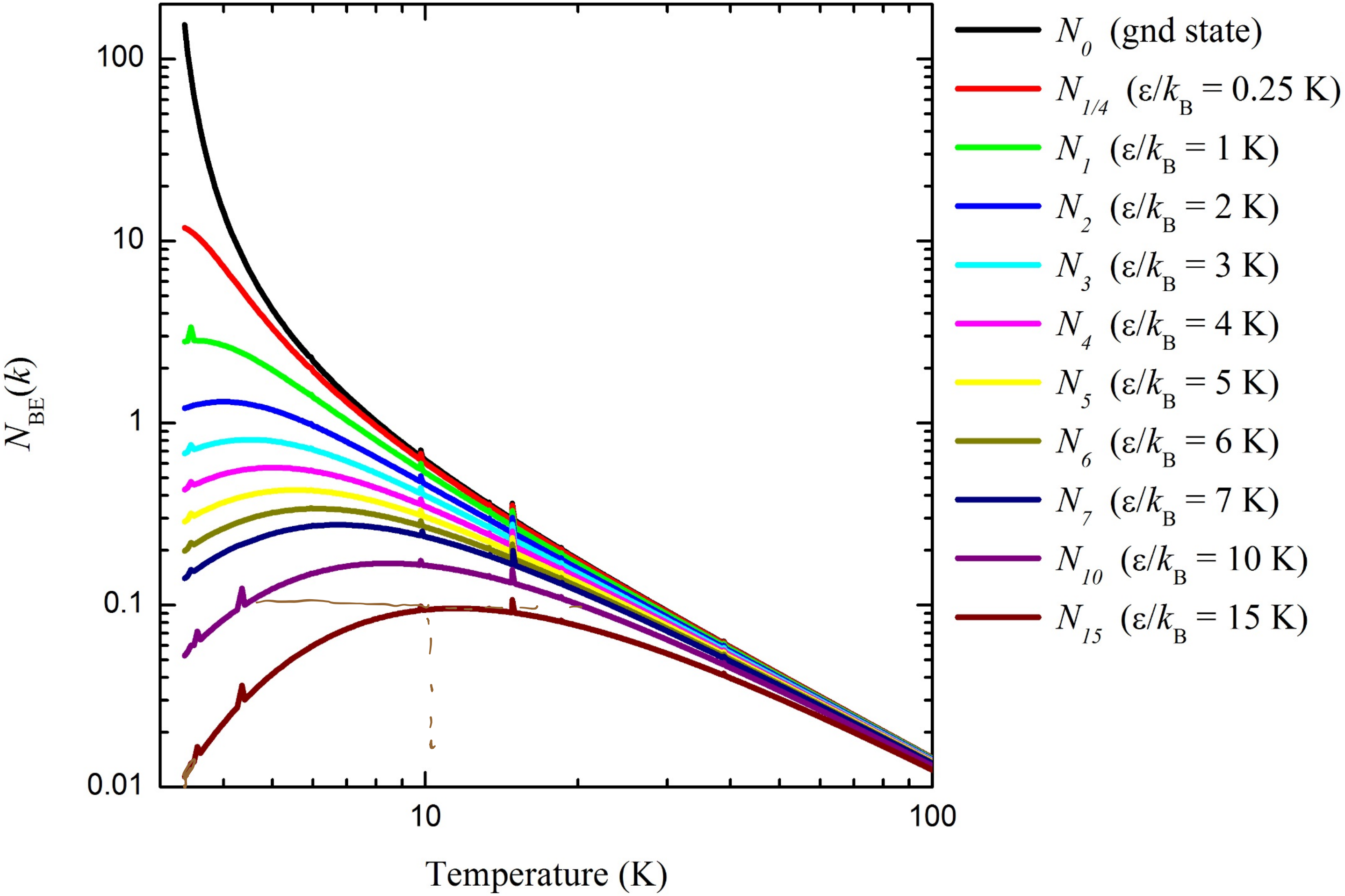
The Boson Dist'n Fcn ...



At low temperatures the occupation of higher energy states peaks and then starts to decrease.

At high temperatures, the occupation of all states is the same. This observation is consistent with the expectation that, in the high temperature limit, a particle is equally likely to be found in any of the available single particle states.

The Boson Dist'n Fcn ...



Return to $N = \sum_{\vec{k}} N_{BE} = \frac{1}{e^{-\eta}-1} + \frac{1}{e^{\varepsilon_1/k_BT-\eta}-1} + \frac{1}{e^{\varepsilon_2/k_BT-\eta}-1}$

$+ \dots$

Below so temp $\underbrace{T_c}_{\text{critical temp}}$, η will be close enough

to zero that $\frac{\varepsilon_1}{k_BT} - \eta \approx \frac{\varepsilon_1}{k_BT}$

$$\frac{\varepsilon_2}{k_BT} - \eta \approx \frac{\varepsilon_2}{k_BT}$$

For $T \lesssim T_c$

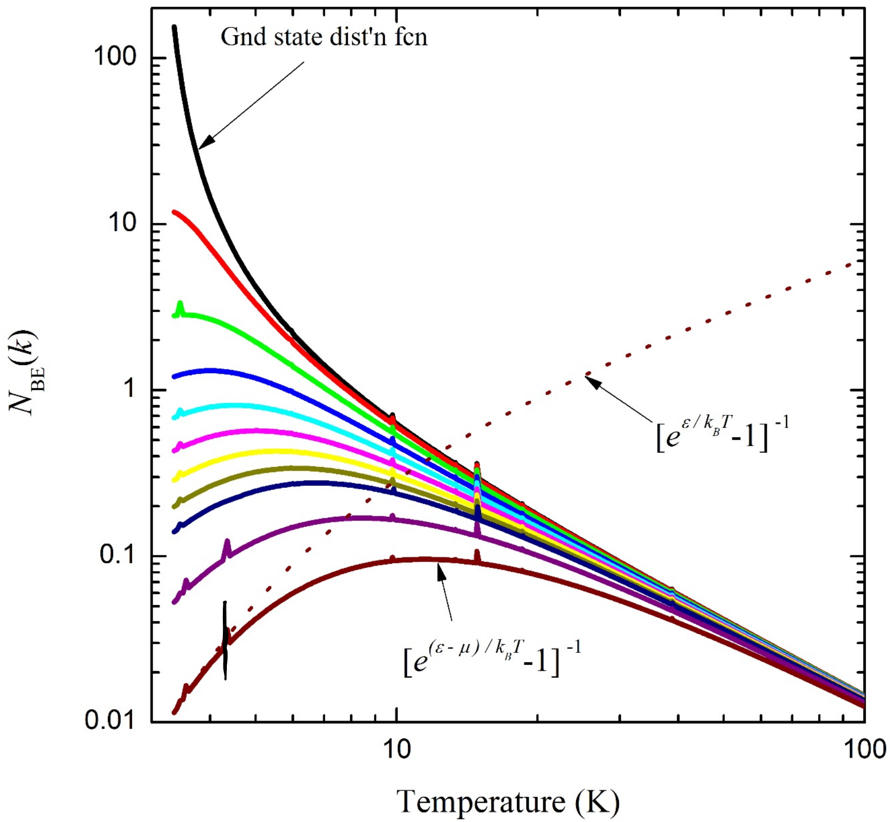
$$N \approx \underbrace{\frac{1}{e^{-\eta}-1}}_{\substack{\text{no. particles} \\ \text{in gnd state, } N_0}} + \underbrace{\frac{1}{e^{\varepsilon_1/k_BT}-1} + \frac{1}{e^{\varepsilon_2/k_BT}-1} + \dots}_{\substack{\text{no. of particles in} \\ \text{excited states, } N_{ex}}}$$

Plot $N_i = \frac{1}{e^{\varepsilon_i/k_BT-\eta}-1}$ & $N_j \approx \frac{1}{e^{\varepsilon_i/k_BT}-1}$ vs T

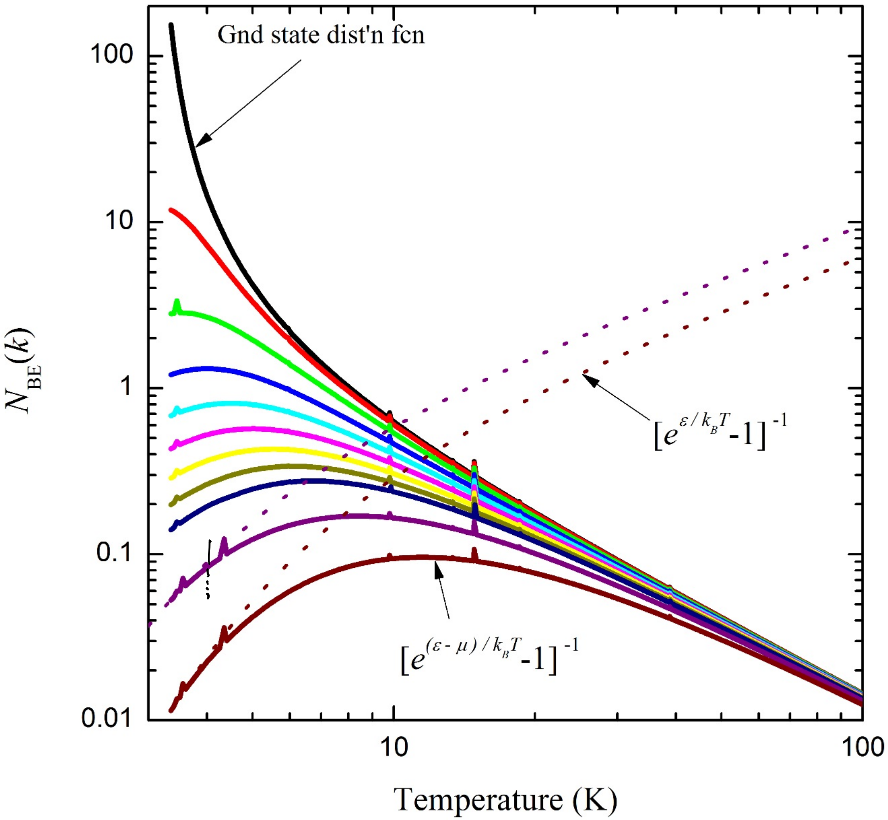
verify that approx works for $T \leq T_c$.

Below T_c: $N \approx N_0 + N_{\rm ex}$

$$= \frac{1}{e^{-\mu/k_{\rm B}T} - 1} + \left[\frac{1}{e^{\varepsilon_1/k_{\rm B}T} - 1} + \frac{1}{e^{\varepsilon_2/k_{\rm B}T} - 1} + \dots\right]$$



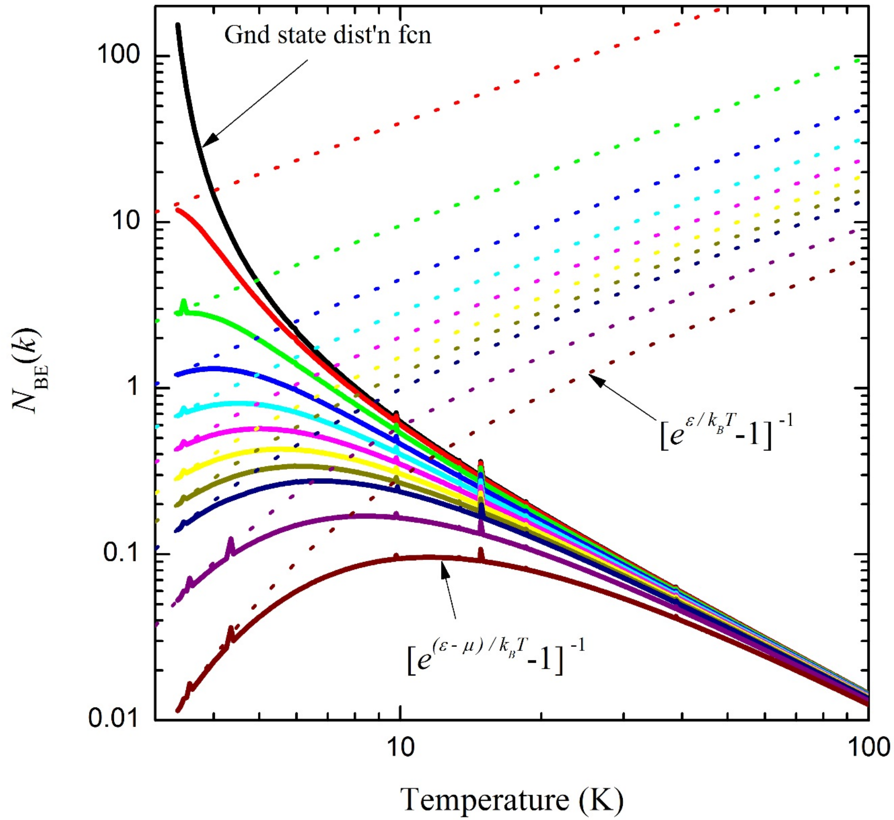
Below T_c: $$N \approx N_0 + N_{ex}$$
$$= \frac{1}{e^{-\mu/k_B T} - 1} + \left[\frac{1}{e^{\varepsilon_1/k_B T} - 1} + \frac{1}{e^{\varepsilon_2/k_B T} - 1} + \dots\right]$$



Below T_c: $N \approx N_0 + N_{ex}$

$$= \frac{1}{e^{-\mu/k_BT} - 1} + \left[\frac{1}{e^{\varepsilon_1/k_BT} - 1} + \frac{1}{e^{\varepsilon_2/k_BT} - 1} + \dots\right]$$

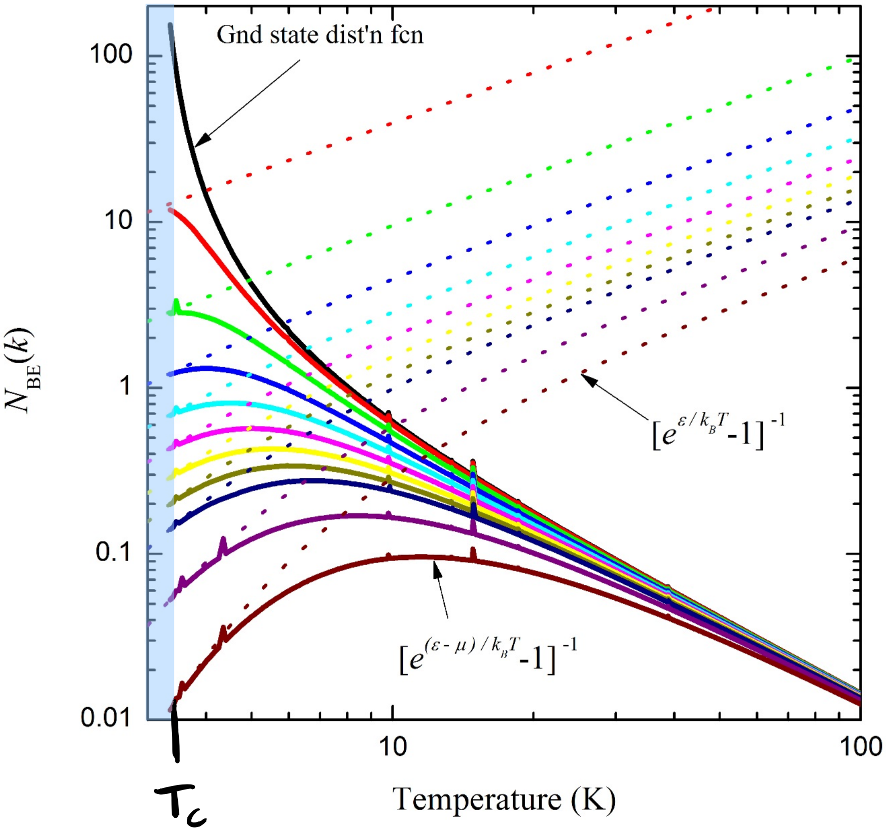
At sufficiently low temperatures, neglecting the chemical potential in the terms contributing to N_{ex} is a good approximation.



Below T_c: $$N \approx N_0 + N_{ex}$$

$$= \frac{1}{e^{-\mu/k_B T} - 1} + \left[\frac{1}{e^{\varepsilon_1/k_B T} - 1} + \frac{1}{e^{\varepsilon_2/k_B T} - 1} + \ldots \right]$$

There's a *sweet spot* where neglecting the chemical potential in N_{ex} is valid *and* N_0 is negligible compared to N_{ex}. This *sweet spot* defines the critical temperature T_C.



For $T < T_c$ $\quad N_{ex} \approx \sum_{\vec{k}} \frac{1}{e^{\varepsilon(k)/k_B T} - 1}$

$$= \frac{V}{2\pi^2} \int_{k=0}^{\infty} \frac{k^2 dk}{e^{\hbar^2 k^2/2mk_B T} - 1}$$

$$N_{ex} = \frac{V}{2\pi^2} \left(\frac{2mk_B T}{\hbar^2} \right)^{3/2} \int_0^{\infty} \frac{x^2 dx}{e^{x^2} - 1} \qquad a$$

At T_c $\quad N_{ex} \gg N_0$

Although N_0 has more particles than any individual excited state, there are many many excited states s.t. <u>total</u> no. of particles in excited states at $T = T_c$ still much greater than N_0.

$\therefore$ almost all particles in excited thats when $T = T_c$

$$N \approx \sum_{\vec{k}} \frac{1}{e^{\varepsilon(k)/k_B T_c} - 1} = \sum_{\vec{k}} \frac{1}{e^{\hbar^2 k^2/2mk_B T_c} - 1}$$

$$\approx \frac{V}{2\pi^2}\left(\frac{2mk_B T_c}{\hbar^2}\right)^{3/2} \underbrace{\int_0^\infty \frac{x^2\,dx}{e^{x^2}-1}}_{\text{evaluated numerically}} \quad (b)$$

Use this condition to define T_c.

evaluated numerically 1.15758

combine $\frac{(a)}{(b)}$ gives:

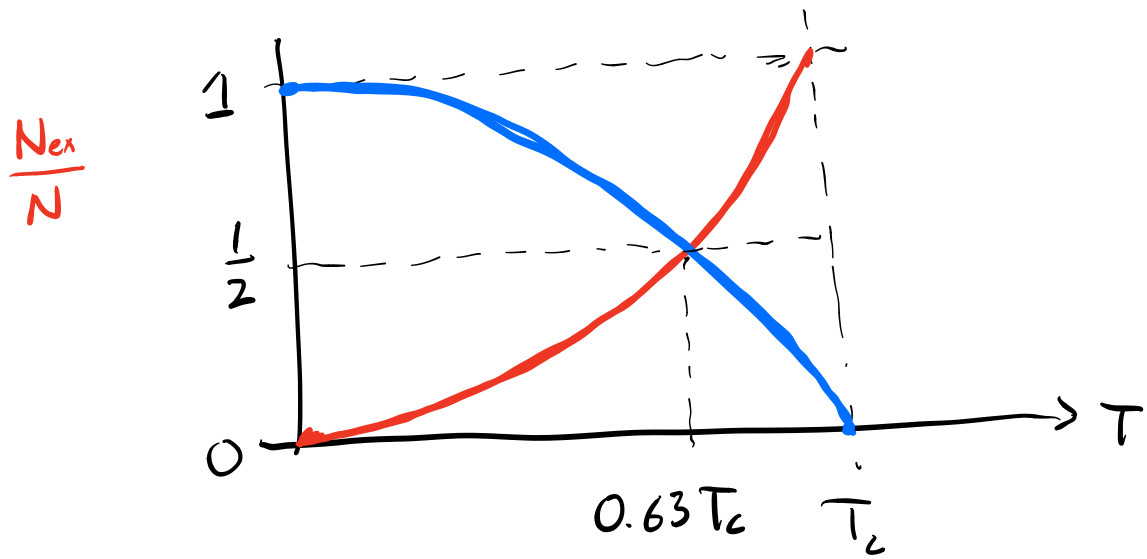
$$\boxed{\frac{N_{ex}}{N} = \left(\frac{T}{T_c}\right)^{3/2}}$$

find no. of particles in excited states for $T < T_c$

or using $N_0 = N - N_{ex}$ s.t. $\frac{N_0}{N} = 1 - \frac{N_{ex}}{N}$

$$\boxed{\frac{N_0}{N} = 1 - \left(\frac{T}{T_c}\right)^{3/2}}$$

no. of particles in gnd state $T < T_c$



Combining numerical factors in (b) gives

$$N \simeq 0.166\, V \left(\frac{m k_B T_c}{\hbar^2} \right)^{3/2} \qquad (*)$$

Note that at $T = T_c$, N is fixed since

V, $T = T_c$, m, k_B, $\hbar$ all const. or held fixed.

What if add particles to the system at $T = T_c$?

(*) suggests that N cannot change!

However, recall that to call ⊛ we assumed $N \approx N_{ex}$. It is really the no. of particles in excited states that is fixed. If add particles to system at $T = T_c$, they <u>must</u> go into gnd state.

From ⊛ $\boxed{k_B T_c = 3.31 \frac{\hbar^2}{m} n^{2/3}}$ critical temp for Bose-Einstein condensation.

Alternatively, if we keep N fixed, but reduce T below T_c, will find that a large fraction of particles in excited states move into the gnd state!

⇒ called Bose-Einstein Condensation (BEC). T_c is critical temp. below which BEC occurs.

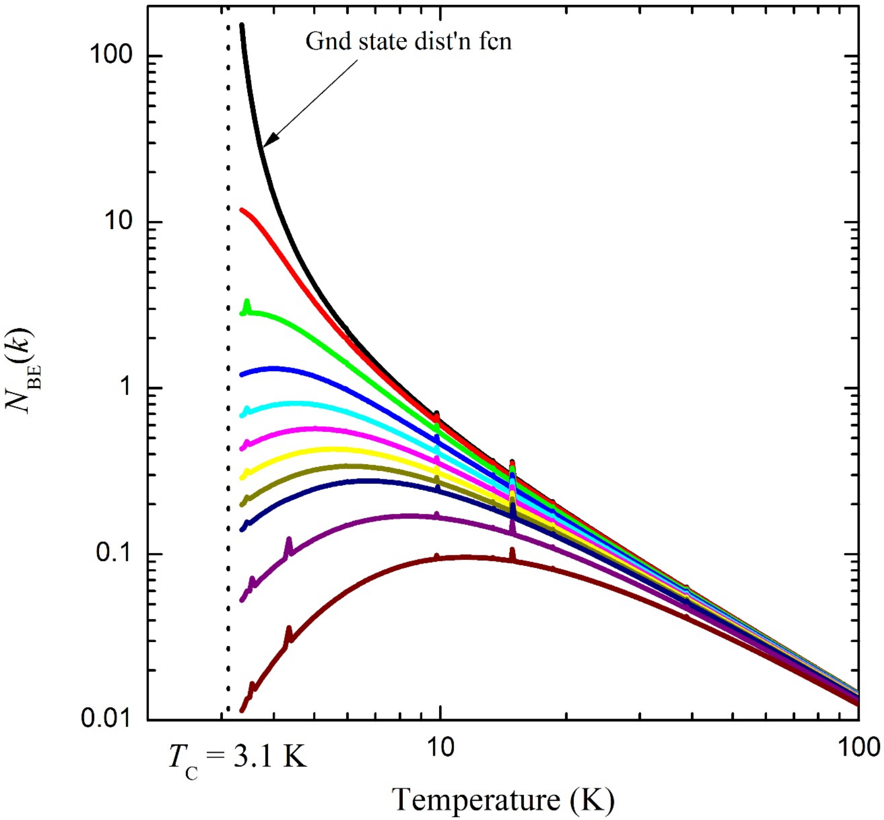
Boson-Einstein Condensation ...

for liquid ^{4}He : $n = 2.2 \times 10^{28}\ \mathrm{m}^{-3}$

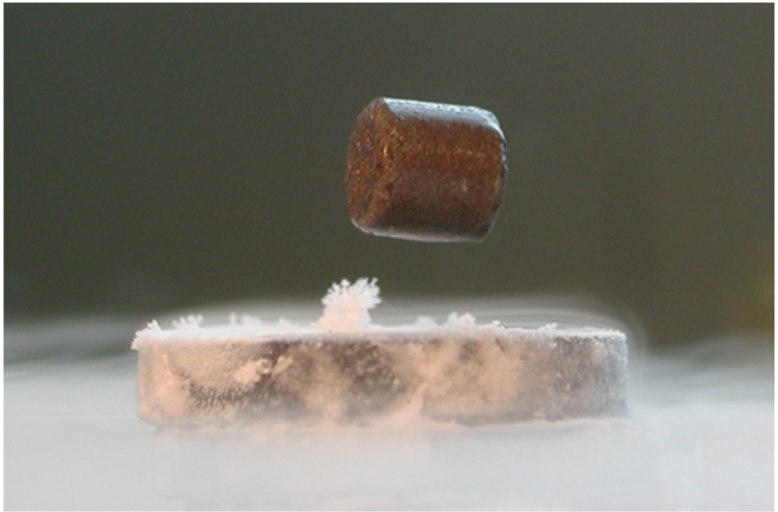
$$m = 6.68 \times 10^{-27}\ \mathrm{kg}$$

$$T_c = \frac{3.31}{k_B}\frac{\hbar^2}{m}n^{2/3} = \underline{3.1\ \mathrm{K}}$$

Experimentally, observe superfluid transition in L. ^{4}He at <u>2.17 K</u>



Macroscopic Quantum Physics ...



superconductivity

Superfluid ^{4}He