PHYS 425 - w12l1

Summary: Conditions for Superfluidity

Cool system of Bosons below the BEC temp.

- macroscopic no. of particles in gnd state.
- quantum volume λ_D^3 occupied by each particle exceeds the available volume per particle $\frac{V}{N} = \frac{1}{n}$
- particle wavefcns overlap
- describe entire condensate using a single macroscopic w.f.

$$\psi(\vec{r}) = \psi_0 e^{iS}$$

- $\vec{V}_s = \frac{\hbar}{m_4} \vec{\nabla} S$ $S(\vec{r})$ smoothly varying fcn.

- Sudden change in $\vec{V}_s$ would result in sudden change in velocities of macroscopic no. of particles $\rightarrow$ unlikely event.

- superfluid flow destroyed above critical velocity $\vec{V}_{cr}$. For a system of free particles w/ only K.E., $\vec{V}_{cr} = 0$
→ no superfluidity for non-interacting system of Bosons.

Two-Fluid Model (Tilley & Tilley Ch. 2)

Postulate that condensate governed by a single w.f.

$$\Psi(\vec{r},t) = \Psi_0(\vec{r},t)\, e^{iS(\vec{r},t)}$$

Assume that w.f. is sol'n to time-dependent Schrödinger Eq'n.

$$\underbrace{i\hbar \frac{\partial \Psi}{\partial t}}_{\text{Energy}} = \underbrace{-\frac{\hbar^2}{2m_4}\nabla^2 \Psi}_{\text{K.E.}} + \underbrace{V(\vec{r})\Psi}_{\text{P.E.}}$$

Interpretation of wavefcn:

Already saw that gradient of phase describes superfluid velocity

$$\vec{v}_s = \frac{\hbar}{m_4} \vec{\nabla} S$$

What about amplitude ψ_0

For a single-particle wavefcn (eg. e^- in H atom)

→ $\psi^* \psi$ describes prob. density of finding particle at position $\vec{r}$.

If have extremely large no. of particles w/ same wavefcn & look in any volume $dx\,dy\,dz$ then no. of particles expected in that volume is prop. to $\psi^* \psi \, dx\,dy\,dz$

$\therefore$ when have enormous no. of particles w/ same wavefcn, then $\Psi^*\Psi$ can be used to describe the density of particles.

Choose normalization of $\Psi(\vec{r},t)$ s.t.
$\Psi^*\Psi$ is equal to avg. no. of superfluid atoms per unit volume.

$$\Psi^*(\vec{r},t)\Psi(\vec{r},t) = \Psi_0^2(\vec{r},t) = \frac{\rho_s}{m_4}$$

ρ_s is mass density of superfluid particles

$\frac{\rho_s}{m_4}$ is no. density of " "

Consider a volume of superfluid at rest (no. kinetic energy). By first law of thermo., the change in internal energy of system is described by:

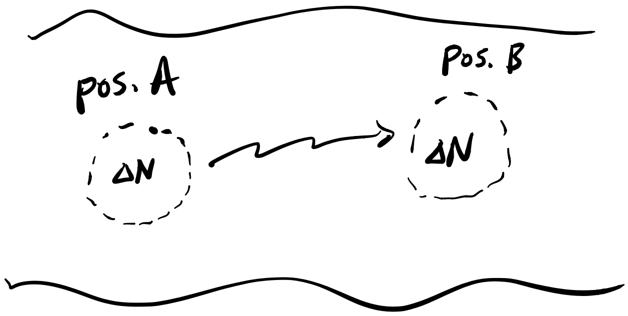
$$dU_0 = TdS - pdV + \mu\, dN$$

at rest

$$\Rightarrow \mu = \left(\frac{\partial U_0}{\partial N}\right)_{V,S}$$

For a moving superfluid, must add kinetic energy $U_{s,k}$ to get total system energy

$$\rightarrow U = U_{s,k} + U_0$$



imagine moving a volume of superfluid w/ ΔN particles from pos. A to pos. B.

- $dV = 0$
- Superfluid has no entropy $dS = 0$
- Flow of superfluid involves no dissipation

$$\therefore \Delta U = 0$$

When remove ΔN particles from A

$$\Delta U_{0A} = -\left(\frac{\partial U_0}{\partial N}\right)^{A}_{S,V} \Delta N$$

add ΔN particles to B

$$\Delta U_{0B} = +\left(\frac{\partial U_0}{\partial N}\right)^{B}_{S,V} \Delta N$$

$$\Delta U = 0 = \Delta U_{s,k} + \Big[\underbrace{\left(\frac{\partial U_0}{\partial N}\right)^B_{S,V}}_{\mu_B} - \underbrace{\left(\frac{\partial U_0}{\partial N}\right)^A_{S,V}}_{\mu_A} \Big] \Delta N$$

conservation of energy

$$\Delta U = 0 = \underbrace{\Delta U_{s,k}}_{\text{change in K.E.}} + \underbrace{(\mu_B - \mu_A)\Delta N}_{\text{change in P.E.}}$$

$(\mu_B - \mu_A)\Delta N$ is change in P.E. of ΔN ^4He atoms

$\Rightarrow$ μ is P.E. per particle of superfluid.

Time-dep. Schrödinger Eq'n becomes

$$\boxed{i\hbar \frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2m_4}\nabla^2\psi + \mu\psi}$$

Sub $\psi = \psi_0 e^{iS}$ into Schrodinger Eq'n.

$\psi_0(\vec{r},t)$ & $S(\vec{r},t)$

Note: $\nabla^2 = \vec{\nabla}\cdot\vec{\nabla}$

$$\vec{\nabla}\cdot(a\vec{A}) = \vec{A}\cdot\vec{\nabla}a + a\vec{\nabla}\cdot\vec{A}$$

$$i\hbar \frac{\partial}{\partial t}\left(\psi_0 e^{iS}\right) = \frac{-\hbar^2}{2m_4}\vec{\nabla}\cdot\left[\vec{\nabla}\left(\psi_0 e^{iS}\right)\right] + \mu\psi_0 e^{iS}$$

$$i\hbar\left(\frac{\partial\psi_0}{\partial t}\right)e^{iS} + i\hbar i\left(\frac{\partial S}{\partial t}\right)\psi_0 e^{iS}$$

$$= \frac{-\hbar^2}{2m_4}\vec{\nabla}\cdot\left[\left(\vec{\nabla}\psi_0\right)e^{iS} + i\left(\vec{\nabla}S\right)\psi_0 e^{iS}\right] + \mu\psi_0 e^{iS}$$

$$= \frac{-\hbar^2}{2m_4}\left[e^{iS}\nabla^2\psi_0 + i\left(\vec{\nabla}\psi_0\right)\cdot\left(\vec{\nabla}S\right)e^{iS}\right.$$

$$\left. + i\psi_0 e^{iS}\nabla^2 S + i\left(\vec{\nabla}S\right)\cdot\vec{\nabla}\left(\psi_0 e^{iS}\right)\right] + \mu\psi_0 e^{iS}$$

$$= -\frac{\hbar^2}{2m_4}\Big[e^{iS}\nabla^2\psi_0 + ie^{iS}\vec{\nabla}\psi_0\cdot\vec{\nabla}S + i\psi_0 e^{iS}\nabla^2 S$$

$$+ i\,\vec{\nabla}S\cdot\left(e^{iS}\vec{\nabla}\psi_0 + i\vec{\nabla}S\,\psi_0 e^{iS}\right)\Big] + \mu\psi_0 e^{iS}$$

$$\therefore\quad i\hbar\left(\frac{\partial\psi_0}{\partial t}\right)e^{iS} - \hbar\left(\frac{\partial S}{\partial t}\right)\psi_0 e^{iS}$$

$$= -\frac{\hbar^2}{2m_4}\Big[e^{iS}\nabla^2\psi_0 + ie^{iS}\vec{\nabla}\psi_0\cdot\vec{\nabla}S + i\psi_0 e^{iS}\nabla^2 S$$

$$+ ie^{iS}\vec{\nabla}S\cdot\vec{\nabla}\psi_0 - (\vec{\nabla}S)^2\psi_0 e^{iS}\Big] + \mu\psi_0 e^{iS}$$

divide by $\psi_0 e^{iS}$

$$\frac{i\hbar}{\psi_0}\left(\frac{\partial\psi_0}{\partial t}\right) - \hbar\left(\frac{\partial S}{\partial t}\right)$$

$$= -\frac{\hbar^2}{2m_4}\left[\frac{\nabla^2\psi_0}{\psi_0} - (\vec{\nabla}S)^2 + 2i\,\frac{\vec{\nabla}\psi_0\cdot\vec{\nabla}S}{\psi_0} + i\nabla^2 S\right]$$

$$+\mu$$

2 Eq'ns (real & imaginary parts)

Examine the real parts:

$$-\hbar\left(\frac{\partial S}{\partial t}\right) = -\frac{\hbar^2}{2m_4}\left[\frac{\nabla^2\psi_0}{\psi_0} - (\vec{\nabla}S)^2\right] + \mu$$

$$\hbar\left(\frac{\partial S}{\partial t}\right) = \frac{\hbar^2}{2m_4}\frac{\nabla^2\psi_0}{\psi_0} - \frac{\hbar^2}{2m_4}(\vec{\nabla}S)^2 - \mu$$

using $\vec{V}_s = \frac{\hbar}{m_4}\vec{\nabla}S \Rightarrow \vec{\nabla}S = \frac{m_4}{\hbar}\vec{V}_s$

$$\psi_0^2 = \frac{\rho_s}{m_4} \Rightarrow \psi_0 = \sqrt{\frac{\rho_s}{m_4}}$$

$$\hbar\frac{\partial S}{\partial t} = \frac{\hbar^2}{2m_4}\frac{\nabla^2\sqrt{\rho_s}}{\sqrt{\rho_s}} - \frac{1}{2}m_4 V_s^2 - \mu$$

In many circumstances, the superfluid density is uniform s.t. $\vec{\nabla}\sqrt{\rho_s}$ small.

$$\text{(\#)} \quad \hbar \frac{\partial s}{\partial t} \approx -\left(\mu + \frac{1}{2} m_4 v_s^2\right)$$

This will be the eq'n that describes the hydrodynamics / fluid dynamics) of the superfluid. This is an energy expression, the gradient will lead to a force eq'n (eq'n of motion)