PHYS 425 - W9L3

Last Time: Grand Canonical Ensemble

Grand partition fcn $\Xi = \sum_j e^{-(U_j - \mu N_j)/k_B T}$

Grand potential $\Phi_G = -k_B T \ln \Xi$

$$\bar{N} = -\left(\frac{\partial \Phi_G}{\partial \mu}\right)_{V,T}$$

Consider a single energy level of energy ε that can be occupied by either zero or one particles.

occupied

$N_1 = 1$

energy $U_1 = \varepsilon$

empty

$N_0 = 0$

energy $U_0 = 0$

$$\Xi = \sum_j e^{-(U_j - \mu N_j)/k_B T}$$

$$= 1 + e^{-(\varepsilon - \mu)/k_B T}$$

$$\therefore \Phi_G = -k_B T \ln\left(1 + e^{-(\varepsilon-\mu)/k_B T}\right)$$

$$\frac{\partial \Phi_G}{\partial \mu} = \frac{-\cancel{k_B T}}{1 + e^{-(\varepsilon-\mu)/k_B T}} \frac{1}{\cancel{k_B T}} e^{-(\varepsilon-\mu)/k_B T}$$

$$= -\frac{e^{-(\varepsilon-\mu)/k_B T}}{1 + e^{-(\varepsilon-\mu)/k_B T}} \cdot \frac{e^{(\varepsilon-\mu)/k_B T}}{e^{(\varepsilon-\mu)/k_B T}}$$

$$= \frac{-1}{e^{(\varepsilon-\mu)/k_B T} + 1}$$

$$\boxed{\bar{N}_{FD} = -\left(\frac{\partial \Phi_G}{\partial \mu}\right) = \frac{1}{1 + e^{(\varepsilon-\mu)/k_B T}}}$$

Fermi-Dirac dist'n. Avg. no. of identical Fermions in a state of energy ε at temp. T.

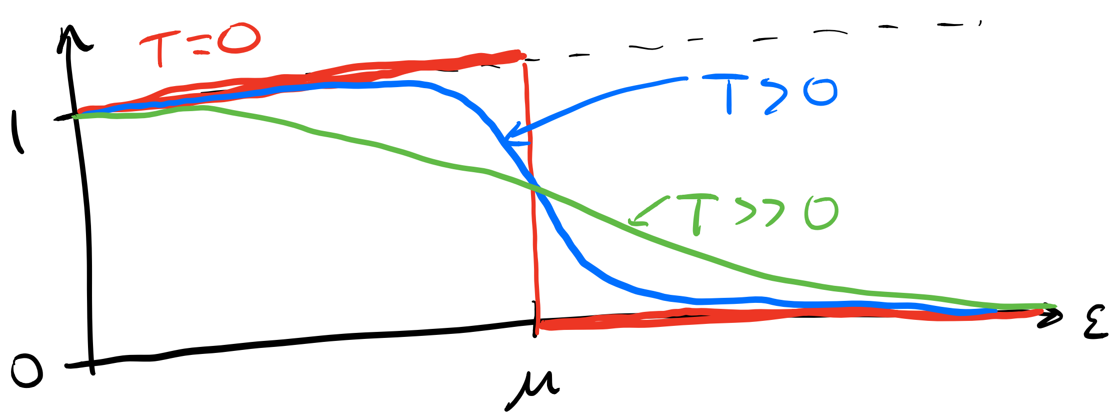
Plot $\bar{N}_{FD}$ vs ε for $T \to 0$.

$$\varepsilon - \mu > 0 \qquad e^{(\varepsilon-\mu)/k_BT} \to \infty$$

$$\bar{N}_{FD} = \frac{1}{e^{(\varepsilon-\mu)/k_BT}+1} \to 0$$

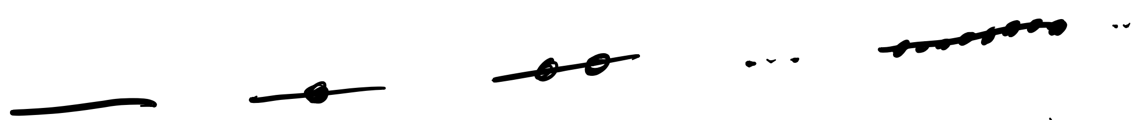
$$\varepsilon - \mu < 0 \qquad e^{(\varepsilon-\mu)/k_BT} \to 0$$

$$\bar{N}_{FD} = \frac{1}{e^{(\varepsilon-\mu)/k_BT}+1} \to 1$$



What if could have any no. of particles (identical) in a quantum state of energy ε?

$$\Xi = \sum_{j=0}^{\infty} e^{-(\varepsilon_j - \mu N_j)/k_B T}$$



empty $N=0$ energy $=0$ | $N=1$ energy $=\varepsilon$ | $N=2$ energy $=2\varepsilon$ | ... | N particles energy $=N\varepsilon$

$$\Xi = 1 + e^{-(\varepsilon-\mu)/k_B T} + e^{-2(\varepsilon-\mu)/k_B T} + \cdots$$

$$+ e^{-N(\varepsilon-\mu)/k_B T} + \cdots$$

$$= 1 + e^{-(\varepsilon-\mu)/k_B T} + \left(e^{-(\varepsilon-\mu)/k_B T}\right)^2$$

$$+ \cdots + \left(e^{-(\varepsilon-\mu)/k_B T}\right)^N + \cdots$$

geometric series $\displaystyle\sum_{q=0}^{\infty} r^q = \frac{1}{1-r}$ for $|r|<1$

Ξ is a geometric series w/ $r = e^{-(\varepsilon-\mu)/k_BT}$

If $\underline{\underline{\varepsilon - \mu > 0}}$, then

$$\Xi = \frac{1}{1 - e^{-(\varepsilon-\mu)/k_BT}}$$

$$\Phi_G = -k_BT \ln \Xi = k_BT \ln\left(\frac{1}{\Xi}\right)$$

$$= k_BT \ln\left(1 - e^{-(\varepsilon-\mu)/k_BT}\right)$$

$$\bar{N}_{BE} = -\left(\frac{\partial \Phi_G}{\partial \mu}\right) = \frac{+\cancel{k_BT}}{1 - e^{-(\varepsilon-\mu)/k_BT}}\left(\frac{+e^{-(\varepsilon-\mu)/k_BT}}{\cancel{k_BT}}\right)$$

$$\therefore \bar{N}_{BE} = \frac{e^{-(\varepsilon-\mu)/k_BT}}{1 - e^{-(\varepsilon-\mu)/k_BT}} \qquad \frac{e^{(\varepsilon-\mu)/k_BT}}{e^{(\varepsilon-\mu)/k_BT}}$$

$$\boxed{\bar{N}_{BE} = \frac{1}{e^{(\varepsilon-\mu)/k_BT} - 1}}$$

Avg. no. of Bosons in a state of energy ε at temp. T.

Bose-Einstein dist'n.

Ideal Monatomic gas entropy given by Sakur-Tetrode Eq'n.

$$S = N k_B \left\{ \ln\left(\frac{V}{N}\right) + \frac{3}{2} \ln \underbrace{\left(\frac{mU}{3\pi\hbar^2 N}\right)}_{\frac{m k_B T}{2\pi\hbar^2} \equiv \frac{1}{\lambda_D^2} \, *} + \frac{5}{2} \right\}$$

Determine chemical pot. of this classical ideal gas from $\mu = -T\left(\frac{\partial S}{\partial N}\right)_{U,V}$

$$\left(\frac{\partial S}{\partial N}\right)_{U,V} = k_B \left\{ \ln\left(\frac{V}{N}\right) + \frac{3}{2}\ln\left(\frac{mU}{3\pi\hbar^2 N}\right) + \cancel{\frac{5}{2}} \right\}$$

$$\cancel{+ \cancel{N} k_B \underbrace{\left\{ -\frac{1}{\cancel{N}} - \frac{3}{2}\frac{1}{\cancel{N}} \right\}}_{-\frac{5}{2} k_B}}$$

$$\mu = -T\left(\frac{\partial S}{\partial N}\right)_{U,V} = -k_BT\left\{\ln\left(\frac{V}{N}\right) + \frac{3}{2}\ln\underbrace{\left(\frac{mU}{3\pi\hbar^2 N}\right)}_{\frac{1}{\lambda_D^2}}\right\}$$

$$n = \frac{N}{V}$$

$$\mu = -k_BT\left\{-\ln n + \frac{3}{2}\ln\left(\frac{1}{\lambda_D^2}\right)\right\}$$

$$= k_BT\left\{\ln n - \ln\left(\frac{1}{\lambda_D^3}\right)\right\}$$

λ_D^3 quantum volume of particle (small in classical limit)

$n_Q = \frac{1}{\lambda_D^3}$ (large in classical limit)

$n = \frac{N}{V}$ small for a classical (non-interacting / dilute) gas.

$$\mu = k_BT\left(\ln n - \ln n_Q\right)$$

$$\boxed{\therefore\ \mu = k_BT\ln\left(\frac{n}{n_Q}\right)}$$

chemical pot. of monatomic ideal gas.

$$\therefore \frac{\mu}{k_B T} = \ln\left(\frac{n}{n_Q}\right)$$

For classical gas

$n << n_Q$

$$\therefore -\frac{\mu}{k_B T} = -\ln\left(\frac{n}{n_Q}\right) = \ln\left(\frac{n_Q}{n}\right) >> 1$$

$$\bar{N}_{FD} = \frac{1}{e^{(\varepsilon-\mu)/k_B T}+1} \qquad \bar{N}_{BE} = \frac{1}{e^{(\varepsilon-\mu)/k_B T}-1}$$

In classical limit

$$e^{(\varepsilon-\mu)/k_B T} \pm 1 = e^{\varepsilon/k_B T}\underbrace{e^{-\mu/k_B T}}_{\frac{n_Q}{n} >> 1} \pm 1$$

$$= e^{\varepsilon/k_B T}\left(\frac{n_Q}{n}\right) \pm 1 \approx e^{\varepsilon/k_B T}\frac{n_Q}{n}$$

$\therefore$ in classical limit

$$\bar{N}_{FD} \approx \bar{N}_{BE} \approx \frac{n}{n_Q} e^{-\varepsilon/k_B T}$$

In the high-temp, dilute limit the Fermi-Dirac and Bose-Einstein dist'ns become identical.

$\therefore$ Behaviour of classical gas does not depend on whether gas of particles is identical Fermions or Bosons.

Why? $$\bar{N}_{classical} \simeq \frac{n}{n_Q} e^{-\varepsilon/k_B T} \ll 1$$

avg. no. of particles in state w/ energy ε at temp. T.

$\therefore$ Regardless of if one has Bosons or Fermions, average occupation of quantum states always much less than one. $\rightsquigarrow$ Whether or not particles need to obey Pauli Exclusion principle becomes irrelevant.