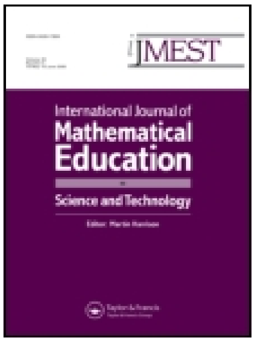

International Journal of Mathematical Education in Science and Technology

ISSN: 0020-739X (Print) 1464-5211 (Online) Journal homepage: https://www.tandfonline.com/loi/tmes20

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To cite this article: S. Velasco , F. L. Román , A. González & J. A. White (2006) Statistical estimation of some irrational numbers using an extension of Buffon's needle experiment, International Journal of Mathematical Education in Science and Technology, 37:6, 735-740, DOI: 10.1080/00207390500432675

To link to this article: https://doi.org/10.1080/00207390500432675

Published online: 20 Feb 2007.

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Statistical estimation of some irrational numbers using an extension of Buffon's needle experiment

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(*Received 4 February 2005*)

In the nineteenth century many people tried to seek a value for the most famous irrational number, π, by means of an experiment known as Buffon's needle, consisting of throwing randomly a needle onto a surface ruled with straight parallel lines. Here we propose to extend this experiment in order to evaluate other irrational numbers, such as $\sqrt{2}$, $\sqrt{3}$, and the golden ratio, $\Phi = (1+\sqrt{5})/2$, by simply replacing the needle by a suitable regular polygon and then calculating the probability that a line intersects two consecutive sides of the polygon. A computer simulation of the experiment by means of Monte Carlo methods is reported.

1. Introduction

Perhaps, the most famous experiment to calculate an irrational number is Buffon's needle. In 1777, Georges Louis Leclerc, Comte de Buffon [1], showed that the probability that a needle with length L, randomly dropped onto a horizontal plane ruled with equally spaced, straight parallel lines a distance d apart, will intersect one of the lines is

$$p = \frac{2L}{\pi d} \tag{1}$$

provided that $L \leqslant d$. Some years later, Laplace [2, 3] suggested Buffon's needle as a way of calculating the number π. Since then, many people have tried to estimate π from this method, sometimes with unbelievable luck [4, 5]. In any case, Buffon's needle has become a paradigmatic problem of Geometric Probability [6].

In 1937, Uspensky [7, 8] showed that replacing the needle by a plate shaped with any convex curve and with generalized diameter less than d, the probability that the plate will intersect one line is

$$p(P) = \frac{P}{\pi d} \tag{2}$$

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where P is the perimeter of the curve. Therefore, for a regular polygon of n sides of length L, such probability is

$$p(n) = \frac{nL}{\pi d} \tag{3}$$

which includes Buffon's result (1) as a particular case ($n=2$). In other words, by throwing, for example, an equilateral triangle or a square one can also evaluate π once the ratio L/d is known.

Interestingly, the above expansion opens new possibilities with respect to the original Buffon's needle. For example, because of the convexity of the polygon, one line can only intersect two sides of the polygon, then the following question is pertinent: what is the probability that a line intersects two consecutive sides of the polygon? The aim of this paper is to show that the answer to this question provides a simple way to calculate some irrational numbers. In particular, replacing the needle by a square, a regular pentagon and a regular hexagon in Buffon's experiment will give an estimate of $\sqrt{2}$, the golden ratio, $\Phi = (1+\sqrt{5})/2$, and $\sqrt{3}$, respectively.

2. Probability calculations

Consider a plane surface ruled with parallel lines a distance d apart and a regular polygon of n sides of length L. The dimensions of the polygon are small enough so that it cannot intersect simultaneously two parallel lines. After dropping randomly the polygon onto the surface consider the following two events: (1) event $A \equiv$ a line intersects the polygon; (2) event $B \equiv$ a line intersects two consecutive sides of the polygon. The probability $p(A)$ is given by equation (3). The probability $p(B)$ can be obtained by considering that there are n different mutually exclusive cases leading to event B and that each of these cases has the same probability, i.e.

$$p(B) = n\, p_{ab} \tag{4}$$

where p_{ab} is the probability that a line will intersect two given consecutive sides a and b of the polygon. On the other hand, as is shown in figure 1, the sides a and b together with the associated diagonal D form a triangle. The probability p_{ab} will be given by the probability that a line will intersect this triangle minus the probability that a line will intersect the diagonal D, i.e. taking into account expressions (1) and (2),

$$p_{ab} = \frac{2L+D}{\pi d} - \frac{2D}{\pi d} = \frac{2L-D}{\pi d} \tag{5}$$

Substitution of equation (5) into equation (4) yields

$$p(B) = \frac{n(2L-D)}{\pi d} \tag{6}$$

which, like probability $p(A)$, depends both on the geometry of the polygon and on the distance between parallel lines.

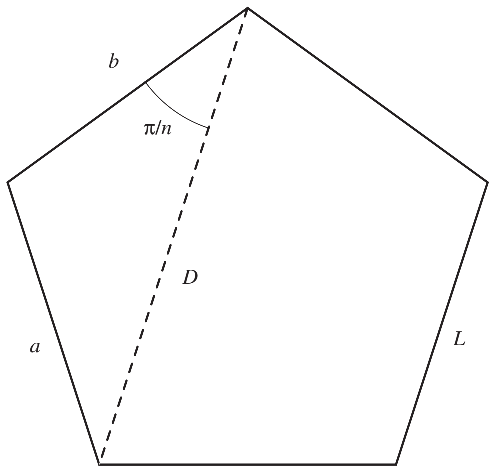

Figure 1. A regular pentagon ($n=5$) showing two consecutive sides, a and b, and the associated diagonal D.

The conditional probability $p(B|A)$ gives us the probability that event B will occur given that even A has occurred already, and it is given by

$$p(B|A) \equiv \frac{p(A \bigcap B)}{p(A)} \tag{7}$$

where $p(B \cap A)$ is the probability that event B and event A occur. But, in this case event B implies event A so that $B \subset A$ and, then, $p(A \cap B) = p(B)$. Therefore, substitution of equations (3) and (6) into equation (7) yields

$$p(B|A) = \frac{p(B)}{p(A)} = 2 - \frac{D}{L} = 2\left[1 - \cos\left(\frac{\pi}{n}\right)\right] \equiv p_n \tag{8}$$

which depends only on the number n of sides of the polygon and where the last equality is valid for $n \geqslant 3$. In particular, for $n = 3, 4, 5,$ and 6, expression (8) becomes, respectively,

$$p_3 = 1, \quad p_4 = 2 - \sqrt{2}, \quad p_5 = 2 - \Phi, \quad \text{and} \quad p_6 = 2 - \sqrt{3}, \tag{9}$$

where $\Phi = (1 + \sqrt{5})/2$ is the so–called golden ratio. Note that for $n \to \infty$ the polygon becomes a circumference and $p_\infty = 0$.

3. Computer simulations

Expression (8) is the desired result and leads, to the authors' knowledge, to a novel finding: Buffon's experiment allows one to estimate the irrational numbers $\sqrt{2}$, Φ, and $\sqrt{3}$, with the single change of the 'needle' shape and *without measuring any length*. Therefore, if N is the number of times a regular polygon of n sides is dropped, N_c is the number of times that a line intersects the polygon, and N_{cc} is the number

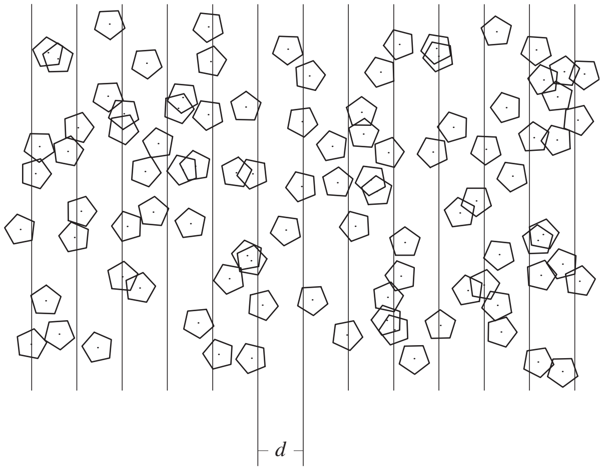

Figure 2. Buffon's experiment with pentagons. Sample snapshot with 100 pentagons.

of times that one finds a line intersects two consecutive sides of the polygon, one has that N_c/N is an experimental estimate of $p(A)$, N_{cc}/N is an experimental estimate of $p(B)$, and N_{cc}/N_c is an experimental estimate of p_n.

One can make an actual realization of the experiment by throwing different-shaped pieces onto a horizontal ruled plane, but this can be awfully tedious; it is much better, of course, to resort to very simple simulations in a personal computer where one can run millions of trials in a few seconds.

We have performed estimations for $\sqrt{2}$, Φ, and $\sqrt{3}$, from standard Monte Carlo simulations with $d = 2L$. In figure 2 we show a snapshot of the experiment with 100 pentagons. Figure 3 shows the evolution of the results of $r_n \equiv 2 - N_{cc}/N_c$ ($n = 4, 5,$ and 6) versus the number of trials N. The estimate of the probable error in each case is the standard deviation of the mean $\sigma_m = \sigma/\sqrt{N_c}$, where σ is the standard deviation of the individual estimates [9]. For example, for $N = 10^6$ we have obtained: $r_4 = 1.4141 \pm 0.0013$ ($\sqrt{2} = 1.414213\ldots$), $r_5 = 1.6181 \pm 0.0009$ ($\Phi = 1.618033\ldots$), and $r_6 = 1.7322 \pm 0.0004$ ($\sqrt{3} = 1.732050\ldots$).

To conclude, we have extended Buffon's needle experiment in order to evaluate the irrational numbers $\sqrt{2}$, Φ, and $\sqrt{3}$, by simply replacing the needle by a square, a regular pentagon, and a regular hexagon, respectively. These evaluations do not need the measurement of any length. The only requirement is that the corresponding polygon does not intersect simultaneously two parallel lines.

Acknowledgements

We acknowledge financial support by Comisión Interministerial de Ciencia y Tecnología (CICYT) of Spain under Grants BFM 2002-01225 FEDER and BFM 2003-07106 FEDER, and by Junta de CyL-FSE of Spain under Grants SA080/04 and SA092/04.

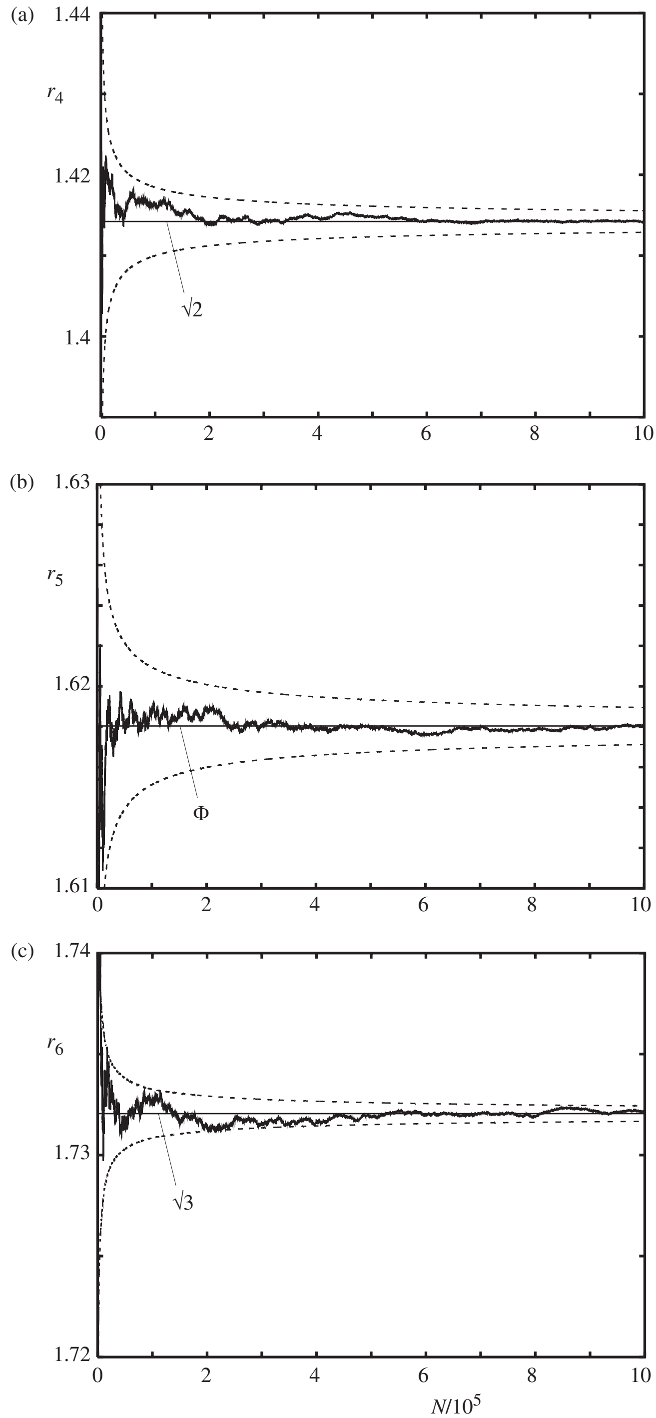

Figure 3. Results of the simulations for squares (*a*), pentagons (*b*), and hexagons (*c*). N is the number of trials and $r_n = 2 - N_{cc}/N_c$, where N_c is the number of times that a line intersects a polygon, and N_{cc} the number of times that a line intersects two consecutive sides of a polygon. The straight lines correspond to the values $2 - p_4 = \sqrt{2}$, $2 - p_5 = \Phi$, and $2 - p_6 = \sqrt{3}$. The dashed lines correspond to the standard deviation of the mean σ_m.

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On the local maxima of a constrained quadratic form

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(*Received 30 July 2004*)

This note presents a brief and partial review of the work of Broom, Cannings and Vickers [1]. It also presents some simple examples of an extension of the their formalism to non-symmetric matrices.

1. Introduction

Broom, Cannings and Vickers [1], in an interesting paper, considered the problem of the number of local maxima of a constrained quadratic form. This is essentially a problem of nonlinear programming, and can be stated as follows. Let $A = (a_{ij})$ be a real, symmetric $n \times n$ matrix, where n is any positive integer, $n \geqslant 2$. Let Δ_n be a region in R^n given as follows:

$$\Delta_n = \left\{ \underline{x} \in R^n; x_i \geqslant 0, \sum_i x_i = 1 \right\} \tag{1}$$

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