

## Contents

1	Topological Spaces and Continuous Maps .....	2
1.1	Elementary Topology .....	2
1.2	Topological Bases .....	5
1.3	Subspaces .....	8
1.4	Continuous Maps .....	9
1.5	Examples of Homeomorphisms .....	11
2	Examples of Topological Spaces .....	12
2.1	Products of Topological Spaces .....	14
2.2	Quotient Spaces .....	18
3	Connected, Path-Connected and Compact Spaces .....	24

# 1 Topological Spaces and Continuous Maps

## 1.1 Elementary Topology

Given an inner product on an  $\mathbb{R}$ -vector space  $\langle \cdot, \cdot \rangle$ , one can define a norm  $\|x\| = \sqrt{\langle x, x \rangle}$ . Given a norm, one can define a metric  $d(x, y) = \|x - y\|$ . Given a metric  $d$  on a set  $X$ , one can define open sets in  $X$ :

given  $a \in X$  and  $r > 0$ ,  $B(a, r) := \{x \in X \mid d(x, a) < r\}$ . Then for  $A \subseteq X$ , we say  $A$  is open in  $X$  when  $\forall a \in A \exists r > 0$  such that  $B(a, r) \subseteq A$ . Equivalently, for all  $a \in A$ , there is  $b \in X$ ,  $r > 0$  such that  $a \in B(b, r) \subseteq A$ .

### Remark

The set of open sets on a metric space is called the *metric topology* on  $X$ .

Open sets in a metric space satisfy the following:

1.  $\emptyset$  and  $X$  are open
2. arbitrary unions of open sets are open
3. finite intersections of open sets are open

### Notation

For a set of sets  $S$ , the union of  $S$  is

$$\bigcup S := \{x \mid \exists A \in S, x \in A\} = \bigcup_{A \in S} A$$

In the case that  $S \neq \emptyset$ , the intersection of  $S$  is

$$\bigcap S := \{x \mid \forall A \in S, x \in A\} = \bigcap_{A \in S} A$$

### Note

$\bigcap S$  would contain all elements as the condition  $\forall A \in \emptyset$  would be vacuously satisfied. If we are given a universal set  $X$ , and  $S$  is known to be a set of subsets of  $X$ , then  $\bigcap \emptyset = X$ .

### Definition 1.1.1

Let  $X$  be a set.  $\mathcal{T} \subseteq \mathcal{P}(X)$  is called a *topology* on  $X$  if

1.  $\emptyset, X \in \mathcal{T}$
2. If  $S \subseteq \mathcal{T}$  is nonempty, then  $\bigcup S \in \mathcal{T}$
3. If  $S \subseteq \mathcal{T}$  is nonempty and finite, then  $\bigcap S \in \mathcal{T}$

The elements of  $\mathcal{T}$  are called the open sets of  $X$ . The closed sets are the compliments of the open sets.

**Remark**

To show 3 holds, it suffices to show the intersection of 2 open sets is open (by induction)

**Definition 1.1.2**

If  $X$  is a set, and  $\mathcal{T}$  is a topology on  $X$ , then  $(X, \mathcal{T})$  is called a *topological space*

**Remark**

When  $f : X \rightarrow Y$  is a map between metric spaces,  $f$  is continuous iff  $f^{-1}(V)$  is open in  $X$  for every open set  $V \subseteq Y$ .

**Definition 1.1.3**

For a map  $f : X \rightarrow Y$  between topological spaces, we say that  $f$  is continuous when  $f^{-1}(V)$  is open in  $X$  for every open set  $V \subseteq Y$ .

**Example 1.1.1**

if  $f : A \subseteq \mathbb{R}^n \rightarrow B \subseteq \mathbb{R}^m$  is an elementary function, then  $f$  is continuous.

**Definition 1.1.4**

When  $S, T$  are topologies on  $X$  with  $S \subseteq T$ , we say that  $S$  is coarser than  $T$  and  $T$  is finer than  $S$ . When  $S \subsetneq T$ , we use strictly coarser/finer.

**Example 1.1.2**

$\{\emptyset, X\}$  is a topology on  $X$  called the *trivial topology*

**Example 1.1.3**

$\mathcal{P}(X)$  is a topology on  $X$  called the *discrete topology*

**Example 1.1.4**

When  $X = \emptyset$ ,  $\mathcal{T} \subseteq \mathcal{P}(X) \Rightarrow \mathcal{T} \subseteq \{\emptyset\} \Rightarrow \mathcal{T} = \emptyset \vee \mathcal{T} = \{\emptyset\}$ . Thus the only topology on  $\emptyset$  is  $\{\emptyset\}$ .

**Example 1.1.5**

When  $X = \{a\}$  the only topology is  $\mathcal{T} = \{\emptyset, \{a\}\}$

**Exercise 1.1.1**

Find all topologies on the 2 and 3 element sets.

**Definition 1.1.5**

Let  $X$  be a topological space. Let  $A \subseteq X$ .

1. The *interior* of  $A$  (in  $X$ ) denoted by  $\text{int}(A)$  is the union of all open sets in  $X$  which are contained in  $A$ .
2. The *closure* of  $A$  denoted  $\overline{A}$  is the intersection of all closed sets in  $X$  which contain  $A$ .
3. The *boundary* of  $A$ , denoted by  $\partial A$ , given by  $\partial A = \overline{A} \setminus \text{int}(A)$

**Note**

The set of closed sets in a topological space is closed under arbitrary intersections and under finite unions. In particular  $\emptyset, X$  are closed

**Theorem 1.1.1**

Let  $X$  be a topological space,  $A \subseteq X$ .

1.  $\text{int}(A)$  is open, and is the largest open set which is contained in  $A$
2.  $\overline{A}$  is closed, and is the smallest closed set which contains  $A$
3.  $A$  is open iff  $A = \text{int}(A)$
4.  $A$  is closed iff  $A = \overline{A}$
5.  $\text{int}(\text{int}(A)) = \text{int}(A)$
6.  $\overline{\overline{A}} = \overline{A}$

**Definition 1.1.6**

Let  $X$  be a topological space, let  $A \subseteq X$ , let  $a \in X$ .

1. We say that  $a$  is an *interior point* of  $A$  when  $a \in A$  and there is an open set  $U$  such that  $a \in U \subseteq A$
2. We say that  $a$  is a *limit point* of  $A$  when for every open set  $U \ni a$  we have  $U \cap (A \setminus \{a\}) \neq \emptyset$ . The set of limit points of  $A$  is denoted by  $A'$
3. We say that  $a$  is a *boundary point* of  $A$  when every open set  $U \ni a$ , we have  $U \cap A \neq \emptyset$  and  $U \cap A^c \neq \emptyset$

**Theorem 1.1.2**

Let  $X$  be a topological space and let  $A \subseteq X$ .

1.  $\text{int}(A)$  is equal to the set of all interior points
2. For  $a \in X$ ,

$$a \in A' \iff a \in \overline{A \setminus \{a\}}$$

3.  $A$  is closed iff  $A' \subseteq A$
4.  $\overline{A} = A \cup A'$
5.  $\overline{A}$  is the disjoint union

$$\overline{A} = \text{int}(A) \sqcup \partial A$$

6.  $\partial A$  is equal to the set of boundary points of  $A$

**1.2 Topological Bases****Theorem 1.2.1**

Let  $X$  be a set. Then the intersection of any set of topologies on  $X$  is also a topology on  $X$ .

**Proof:** Let  $\{\mathcal{T}_\alpha\}_{\alpha \in I}$  be a collection of topologies on  $X$ . Let  $\mathcal{T} = \bigcap_{\alpha} \mathcal{T}_\alpha$

1. Since  $X, \emptyset \in \mathcal{T}_\alpha$  for all  $\alpha \in I$ . We have  $X, \emptyset \in \mathcal{T}$
2. Let  $\{U_i\} \subseteq \mathcal{T}$ . For all  $\alpha \in I$ , we have each  $U_i \in \mathcal{T}_\alpha$ . Thus  $\bigcup_i U_i \in \mathcal{T}_\alpha \implies \bigcup_i U_i \in \mathcal{T}$  as desired.
3. Let  $U_1, \dots, U_n \in \mathcal{T}$ . Then again for all  $\alpha \in I$ , we have each  $U_i \in \mathcal{T}_\alpha$ . Thus  $\bigcap_{i=1}^n U_i \in \mathcal{T}_\alpha \implies \bigcap_{i=1}^n U_i \in \mathcal{T}$

□

**Corollary 1.2.2**

When  $X$  is a set and  $\mathcal{S}$  is any set of subsets of  $X$  (that is  $S \subseteq \mathcal{P}(X)$ ), there is a unique smallest (coarsest) topology  $\mathcal{T}$  on  $X$  which contains  $\mathcal{S}$ . Indeed  $\mathcal{T}$  is the intersection of (the set of) all topologies on  $X$  containing  $\mathcal{S}$ .

This topology  $\mathcal{T}$  is called the topology on  $X$  *generated by*  $\mathcal{S}$

**Definition 1.2.1**

Let  $X$  be a set. A *basis of sets* on  $X$  is a set  $\mathcal{B}$  of subsets of  $X$  (So  $\mathcal{B} \subseteq \mathcal{P}(X)$ ) such that

1.  $\mathcal{B}$  covers  $X$ , that is  $\bigcup \mathcal{B} = X$
2. For every  $C, D \in \mathcal{B}$  and  $a \in C \cap D$ . There is  $B \in \mathcal{B}$  such that  $a \in B \subseteq C \cap D$ .

When  $\mathcal{B}$  is a basis of sets in  $X$  and  $\mathcal{T}$  is the topology on  $X$  generated by  $\mathcal{B}$ , we say that  $\mathcal{B}$  is a *basis for*  $\mathcal{T}$ . The elements in  $\mathcal{B}$  are called *basic open sets* in  $X$ .

**Theorem 1.2.3****Characterization of Open Sets in Terms of Basic Open Sets**

Let  $X$  be a topological space, Let  $\mathcal{B}$  be a basis for the topology on  $X$ .

1. For  $A \subseteq X$ ,  $A$  is open iff for every  $a \in A$ , there is  $B \in \mathcal{B}$  such that  $a \in B \subseteq A$  \*
2. The open sets in  $X$  are the unions of (sets of) elements in  $\mathcal{B}$

Equivalently,

1.  $\mathcal{T} = \{A \subseteq X \mid \forall a \in A, \exists B \in \mathcal{B} \ a \in B \subseteq A\}$
2.  $\mathcal{T} = \{\bigcup C \mid C \subseteq \mathcal{B}\}$

**Proof:** Let  $\mathcal{T}$  be the topology on  $X$  (generated by  $\mathcal{B}$ ). Let  $\mathcal{S}$  be the set of all sets  $A \subseteq X$  with property \* ( $\forall a \in A \exists B \in \mathcal{B} : a \in B \subseteq A$ ). And let  $\mathcal{R}$  be the set of (arbitrary) unions of (sets of) elements in  $\mathcal{B}$ . Recall that  $\mathcal{T}$  is the intersection of the set of all topologies on  $X$  which contain  $\mathcal{B}$ . Note that  $\mathcal{S}$  contains  $\mathcal{B}$  (obviously). Let us show that  $\mathcal{S}$  is a topology on  $X$ . We have  $\emptyset \in \mathcal{S}$  vacuously and  $X \in \mathcal{S}$  because  $\mathcal{B}$  covers  $X$  (given  $a \in X$ , we can choose  $B \in \mathcal{B}$  with  $a \in B$ ). When  $U_k \in \mathcal{S}$  for every  $k \in K$  (where  $K$  is any index set). Let  $a \in \bigcup_k U_k$ . Choose  $\ell \in K$  so that  $a \in U_\ell$ . Since  $U_\ell \in \mathcal{S}$ , we can choose  $B \in \mathcal{B}$  so that  $a \in B \subseteq U_\ell$ . Since  $U_\ell \subseteq \bigcup_k U_k$ , we have  $a \in B \subseteq \bigcup_k U_k$ . Thus  $\bigcup_k U_k$  satisfies \*, hence  $\bigcup_k U_k \in \mathcal{S}$  as required. Suppose  $U, V \in \mathcal{S}$ . Let  $a \in U \cap V$ . Since  $U \in \mathcal{S}$  we can choose  $C \in \mathcal{B}$  with  $a \in C \subseteq U$ . Since  $V \in \mathcal{S}$ , we can choose  $D \in \mathcal{B}$  with  $a \in D \subseteq V$ . Since  $\mathcal{B}$  is a basis,  $C, D \in \mathcal{B}$  and  $a \in C \cap D$ , we can choose  $B \in \mathcal{B}$  with  $a \in B \subseteq C \cap D$ . Then we have

$$a \in B \subseteq C \cap D \subseteq U \cap V$$

Thus  $U \cap V$  satisfies \* so that  $U \cap V \in \mathcal{S}$  as required. Thus  $\mathcal{S}$  is a topology on  $X$  containing  $\mathcal{B}$ , hence  $\mathcal{T} \subseteq \mathcal{S}$ . Let us show that  $\mathcal{S} \subseteq \mathcal{R}$  let  $U \in \mathcal{S}$ . For each  $a \in U$ , choose  $B_a \in \mathcal{B}$  with  $a \in B_a \subseteq U$ . Then we have

$$U = \bigcup_{a \in U} B_a \in \mathcal{R}$$

Thus  $\mathcal{S} \subseteq \mathcal{R}$ . Finally note that  $\mathcal{R} \subseteq \mathcal{T}$  because if  $U = \bigcup_k B_k$  with  $B_k \in \mathcal{B}$ , then each  $B_k \in \mathcal{T}$ , and  $\mathcal{T}$  is a topology, so

$$U = \bigcup_{k \in K} B_k \in \mathcal{T}$$

□

**Theorem 1.2.4****Characterization of a Basis in terms of the Open Sets**

Let  $X$  be a topological space with topology  $\mathcal{T}$ . Let  $\mathcal{B} \subseteq \mathcal{T}$ . Then  $\mathcal{B}$  is a basis for  $\mathcal{T}$  iff  $\forall U \in \mathcal{T} \forall a \in U \exists B \in \mathcal{B} \ a \in B \subseteq U$ . \*

**Proof:** If  $\mathcal{B}$  is a basis for  $\mathcal{T}$ , then \* holds by part 1 of the previous theorem. Suppose \* holds. Let us show that  $\mathcal{B}$  is a basis of sets in  $X$ . Note that  $\mathcal{B}$  covers  $X$  since, taking  $U = X$  in \* we have  $\forall a \in X \exists B \in \mathcal{B} \ a \in B \subseteq X$ . Also note that given  $C, D \in \mathcal{B}$  and  $a \in C \cap D$ , then by taking  $U = C \cap D$  in \* (noting that  $C, D \in \mathcal{B} \subseteq \mathcal{T}$  so that  $U = C \cap D \in \mathcal{T}$ ) we can choose  $B \in \mathcal{B}$  with  $a \in B \subseteq C \cap D$ . Thus  $\mathcal{B}$  is a basis of sets in  $X$ . It remains to show that  $\mathcal{T}$  is the topology generated by  $\mathcal{B}$ . Let  $\mathcal{S}$  be the topology generated by  $\mathcal{B}$ . By part 1 of the previous theorem,  $\mathcal{S}$  is the set of all unions of

elements in  $\mathcal{B}$ . Also  $\mathcal{S}$  is the smallest topology which contains  $\mathcal{B}$ . Since  $\mathcal{B} \subseteq \mathcal{T}$  and  $\mathcal{T}$  is a topology, we have  $\mathcal{S} \subseteq \mathcal{T}$ . Also we have  $\mathcal{T} \subseteq \mathcal{S}$  because given  $U \in \mathcal{T}$ , by property \*, for each  $a \in U$ , we can choose  $B_a \in \mathcal{B}$  with  $a \in B_a \subseteq U$ , and then we have  $U = \bigcup_{a \in U} B_a \in \mathcal{S}$  since it is a union of elements in  $\mathcal{B}$   $\square$

### Example 1.2.1

When  $X$  is a metric space, the set  $\mathcal{B}$  of all open balls in  $X$  is a basis for the metric topology on  $X$ .

### Remark

We can use a basis for testing various topological properties:

When  $X$  is a topological space, and  $\mathcal{B}$  is a basis for the topology on  $X$ , and  $A \subseteq X$  and  $a \in X$ . Then

$$a \in \text{int}(A) \iff \exists B \in \mathcal{B} \text{ with } a \in B \subseteq A$$

$$a \in \overline{A} \iff \forall B \in \mathcal{B} \text{ with } a \in B \quad B \cap A \neq \emptyset$$

$$a \in A' \iff \forall B \in \mathcal{B} \text{ with } a \in B \quad (B \setminus \{a\}) \cap A \neq \emptyset$$

$$a \in \partial A \iff \forall B \in \mathcal{B} \text{ with } a \in B \quad B \cap A \neq \emptyset \text{ and } B \cap (X \setminus A) \neq \emptyset$$

### Definition 1.2.2

A topological space  $X$  is called *Hausdorff* when for all  $a, b \in X$  with  $a \neq b$ , there exist disjoint open sets  $U$  and  $V$  in  $X$  with  $a \in U$  and  $b \in V$ .

### Example 1.2.2

Metric spaces are Hausdorff

### 1.3 Subspaces

#### Definition 1.3.1

#### Subspace Topology

Let  $Y$  be a topological space with topology  $\mathcal{S}$ , and  $X \subseteq Y$  be a subset. Let

$$\mathcal{T} := \{V \cap X \mid V \in \mathcal{S}\}$$

Then  $\mathcal{T}$  is a topology on  $X$ :

Indeed  $\emptyset \in \mathcal{S}$  so  $\emptyset \cap X = \emptyset \in \mathcal{T}$  and  $Y \in \mathcal{S}$  so  $Y \cap X = X \in \mathcal{T}$ . If  $K$  is any index set and  $U_k \in \mathcal{T}$  for each  $k \in K$ , then for each  $k \in K$  we can choose  $V_k \in \mathcal{S}$  such that  $U_k = V_k \cap X$  and then we have

$$\begin{aligned} \bigcup_{k \in K} U_k &= \bigcup_{k \in K} (V_k \cap X) \\ &= \left( \bigcup_{k \in K} V_k \right) \cap X \in \mathcal{T} \end{aligned}$$

since  $\bigcup_{k \in K} V_k \in \mathcal{S}$ . Similarly, when  $K$  is finite and  $U_k \in \mathcal{T}$  for each  $k \in K$  we have  $\bigcap_{k \in K} U_k \in \mathcal{T}$ . The topology  $\mathcal{T}$  on  $X$  is called the *subspace topology* on  $X$  (inherited from the topology on  $Y$ ).

#### Theorem 1.3.1

Let  $Y$  be a topological space, let  $\mathcal{C}$  be a basis for the topology on  $Y$ . Let  $X \subseteq Y$  be a subset. Then the set

$$\mathcal{B} = \{C \cap X \mid C \in \mathcal{C}\}$$

is a basis for the subspace topology on  $X$ .

**Proof:** Exercise □

#### Theorem 1.3.2

Let  $Z$  be a topological space, let  $Y \subseteq Z$  be a subspace and  $X \subseteq Y$  be a subset. Then the subspace topology on  $X$  inherited from  $Y$  is equal to the subspace topology on  $X$  inherited from  $Z$ .

**Proof:** Exercise □

#### Theorem 1.3.3

Let  $Y$  be a metric space, (using the metric topology) and let  $X \subseteq Y$ . Then the subspace topology on  $X$  (inherited from the topology on  $Y$ ) is equal to the metric topology on  $X$  using the metric on  $X$  obtained by restricting the metric on  $Y$ .

**Proof:** Exercise □



## 1.4 Continuous Maps

### Definition 1.4.1

Let  $X, Y$  be topological spaces.

1. For  $f : X \rightarrow Y$  and  $a \in X$ , we say that  $f$  is *continuous at  $a$*  when for every open set  $V \subseteq Y$  with  $f(a) \in V$ , there exists an open set  $U \subseteq X$  with  $a \in U \subseteq f^{-1}(V)$ .
2. We say that  $f$  is *continuous* (in or on  $X$ ) when for every open set  $V \subseteq Y$ ,  $f^{-1}(V)$  is open in  $X$ .
3. A *homeomorphism* from  $X$  to  $Y$  is a bijective map  $f : X \rightarrow Y$  such that both  $f$  and its inverse  $f^{-1} : Y \rightarrow X$  are continuous. We say that  $X$  and  $Y$  are *homeomorphic*, and we write  $X \cong Y$ , when there exists a homeomorphism  $f : X \rightarrow Y$ . (and we remark that  $f^{-1} : Y \rightarrow X$  is also a homeomorphism).

### Theorem 1.4.1

Constant maps and inclusion maps are continuous.

**Proof:** For  $f : X \rightarrow Y$  given by  $f(x) = c \in Y$  for all  $x \in X$ . When  $V$  is open in  $Y$ ,

$$f^{-1}(V) = \begin{cases} X & \text{if } c \in V \\ \emptyset & \text{if } c \notin V \end{cases}$$

When  $X \subseteq Y$  is a subspace and  $f : X \rightarrow Y$  is given by  $f(x) = x$  for all  $x \in X$ , when  $V$  is open in  $Y$ .

$$\begin{aligned} f^{-1}(V) &= \{x \in X \mid f(x) \in V\} \\ &= \{x \in X \mid x \in V\} \\ &= V \cap X \end{aligned}$$

which is open in  $X$ . (when  $X$  uses the subspace topology) □

### Remark

When  $Y$  is a topological space and  $X \subseteq Y$  we shall assume, unless otherwise noted, that  $X$  uses the subspace topology.

### Theorem 1.4.2

### Equivalent Definitions of Continuity

Let  $f : X \rightarrow Y$  be a map between topological spaces

1.  $f$  is continuous iff  $f$  is continuous at every  $a \in X$
2.  $f$  is continuous iff for every closed set  $K \subseteq Y$ ,  $f^{-1}(K)$  is closed in  $X$ .
3. If  $\mathcal{C}$  is a basis for the topology on  $Y$  then  $f$  is continuous iff for every  $C \in \mathcal{C}$ ,  $f^{-1}(C)$  is open in  $X$ .

**Proof of 1:** Suppose  $f$  is continuous on  $X$ . Let  $a \in X$ . Let  $V$  be an open set in  $Y$  with  $f(a) \in V$ . Let  $U = f^{-1}(V)$ , then  $f^{-1}(V)$  is open, since  $f$  is continuous and  $a \in U \subseteq f^{-1}(V)$ . Suppose, conversely, that  $f$  is continuous at every  $a \in X$ . Let  $V$  be an open set in  $Y$ . For each  $a \in f^{-1}(V)$  since  $f$  is continuous at  $a$  with  $f(a) \in V$ , we can choose an open set  $U_a$  in  $X$  with  $a \in U_a \subseteq f^{-1}(V)$ . Then

$$f^{-1}(V) = \bigcup_{a \in f^{-1}(V)} U_a$$

which is open in  $X$ , since it is a union in open sets in  $X$ . □

### Theorem 1.4.3

Let  $f : X \rightarrow Y$ ,  $g : Y \rightarrow Z$  be continuous maps between topological spaces, then the composite map  $h = g \circ f : X \rightarrow Z$  is continuous.

**Proof:** Show that  $h^{-1}(W) = f^{-1}(g^{-1}(W))$  □

### Remark

Homeomorphism of topological spaces behaves like an equivalence relation on the class of all topological spaces. For topological spaces  $X, Y, Z$

1.  $X \cong X$  (since  $\text{id}_X$  is a homeomorphism – a special case of the inclusion map)
2. If  $X \cong Y$  then  $Y \cong X$  (when  $f : X \rightarrow Y$  is a homeomorphism, so is  $f^{-1} : Y \rightarrow X$ )
3. If  $X \cong Y \cong Z$  then  $X \cong Z$  (if  $f : X \rightarrow Y, g : Y \rightarrow Z$  are homeomorphisms then so is  $g \circ f$ )

### Theorem 1.4.4

### Restriction of Domain and Restriction or Expansion of Codomain

Let  $X, Y, Z$  be topological spaces. Suppose  $f : X \rightarrow Y$  is continuous.

1. For any subspace  $A \subseteq X$ , the restriction  $f|_A : A \rightarrow Y$  is continuous.
2. If  $Y \subseteq Z$  is a subspace then  $f : Y \rightarrow Z$  is continuous and if  $B \subseteq Y$  with  $f(X) \subseteq B$ , then  $f : X \rightarrow B$  is continuous.

**Proof:** Exercise □

### Lemma 1.4.5

### Glueing/Pasting Lemma

Let  $f : X \rightarrow Y$  be a map between topological spaces

1. If  $X = \bigcup_{k \in K} U_k$  where each  $U_k$  is open in  $X$  and if each restriction map  $f|_{U_k} : U_k \rightarrow Y$  is continuous (where  $U_k$  is using the subspace topology), then  $f$  is continuous.
2. If  $X = C_1 \cup \dots \cup C_n$  where each  $C_k$  is closed in  $X$ , and if each restriction  $f|_{C_k} : C_k \rightarrow Y$  is continuous, then  $f$  is continuous.

**Proof of 1:** Suppose  $X = \bigcup_{k \in K} U_k$  where each  $U_k$  is open in  $X$  and suppose each restriction  $f|_{U_k}$  is continuous. Let  $V \subseteq Y$  be open. Note that

$$\begin{aligned}
f^{-1}(V) &= \{x \in X \mid f(x) \in V\} \\
&= \bigcup_{k \in K} \{x \in U_k \mid f(x) \in V\} \\
&= \bigcup_{k \in K} \{x \in U_k \mid f|_{U_k}(x) \in V\} \\
&= \bigcup_{k \in K} f|_{U_k}^{-1}(V)
\end{aligned}$$

For each  $k \in K$ , since  $f|_{U_k}$  is continuous, we know that  $f|_{U_k}^{-1}(V)$  is open in  $U_k$ . Since  $U_k$  is using the subspace topology, we can choose an open  $W_k$  in  $X$  such that  $f|_{U_k}^{-1}(V) = W_k \cap U_k$ . This is open in  $X$  since  $W_k$  and  $U_k$  are both open in  $X$ . Since  $f^{-1}(V) = \bigcup_{k \in K} f|_{U_k}^{-1}(V)$  it is a union of open sets in  $X$ , so it is open in  $X$ . Thus  $f$  is continuous.  $\square$

**Proof of 2:** Exercise. First show that for  $f : X \rightarrow Y$ ,  $f$  is continuous iff  $f^{-1}(C)$  is closed in  $X$  for every closed set  $C$  in  $Y$ . And, show that when  $A \subseteq X \subseteq Y$ ,  $A$  is closed in  $X$  (using the subspace topology from  $Y$ ) iff  $A = B \cap X$  for some closed set  $B$  in  $Y$ .  $\square$

### Example 1.4.1

The map  $f : \mathbb{R} \rightarrow \mathbb{R}$  given by  $f(x) = \begin{cases} 2x & x \leq 0 \\ x^2 & x > 0 \end{cases}$  is continuous.

## 1.5 Examples of Homeomorphisms

### Example 1.5.1

The circle

$$\{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1\}$$

in  $\mathbb{R}^2$  is homeomorphic to the ellipse

$$\left\{ (x, y) \in \mathbb{R}^2 \mid \frac{(x-a)^2}{A^2} + \frac{(y-b)^2}{B^2} = 1 \right\}$$

in  $\mathbb{R}^2$

### Example 1.5.2

$\mathbb{R} \cong (-1, 1) \subseteq \mathbb{R}$

**Example 1.5.3**

The standard unit  $n$ -sphere in  $\mathbb{R}^{n+1}$  is the set

$$\mathbb{S}^n = \{x \in \mathbb{R}^{n+1} \mid \|x\| = 1\}$$

Where  $p$  is the north pole

$$p = e_{n+1} = (0, \dots, 0, 1) \in \mathbb{S}^n$$

We have  $\mathbb{S}^n \setminus \{p\} \cong \mathbb{R}^n$

## 2 Examples of Topological Spaces

**Definition 2.0.1**

Let  $X$  be a set. We sometimes write  $X_t$  to indicate that  $X$  is using the trivial topology  $\mathcal{T}_t = \{\emptyset, X\}$ . We sometimes write  $X_d$  to indicate  $X$  is using the discrete topology  $\mathcal{T}_d = \mathcal{P}(X)$ . We sometimes write  $X_c$  to indicate  $X$  is using the co-finite topology  $\mathcal{T}_c = \{A \subseteq X \mid A = \emptyset \text{ or } X \setminus A \text{ is finite}\}$ . Note the closed sets in  $X_c$  are exactly the finite ones and  $X$ .

**Definition 2.0.2**

When  $X$  is a metric space, we assume, unless otherwise indicated, that  $X$  uses the metric topology. Sometimes, we might write  $X_m$  to indicate that  $X$  is using the metric topology  $\mathcal{T}_m$ .

**Definition 2.0.3**

When  $Y$  is a topological space, and  $X \subseteq Y$ , we assume, unless otherwise indicated, that  $X$  uses the subspace topology. Sometimes, we might write  $X_s$  to indicate that  $X$  is using the subspace topology  $\mathcal{T}_s$ . When  $X \subseteq \mathbb{R}^n$ , we shall assume, unless otherwise indicated, that  $X$  is using  $\mathcal{T}_m = \mathcal{T}_s$ .

**Definition 2.0.4**

Let  $X$  be a set. A (strict, linear or total) *order* on  $X$  is a binary relation  $<$  on  $X$  such that

1. For all  $x, y \in X$  exactly one of the following holds:
  - a.  $x < y$
  - b.  $x = y$
  - c.  $y < x$
2. For all  $x, y, z \in X$ , if  $x < y$  and  $y < z$  then  $x < z$

An *ordered set* is a set  $X$  with an order  $<$ . When  $X$  is an ordered set, we also define  $\leq, >, \geq$  by stipulating that for all  $x, y \in X$

$$x \leq y \iff (x < y \vee x = y)$$

$$x > y \iff y < x$$

$$x \geq y \iff y \leq x$$

**Remark**

In an ordered set  $X$  we can define an *upper bound*, a *lower bound*, the *supremum*, the *infimum*, the *maximum*, and the *minimum* of a subset  $A \subseteq X$ .

**Example 2.0.1**

Let  $X$  be an ordered set and  $A \subseteq X$ ,  $M = \max(A)$  when  $M \in A$  with  $M \geq x$  for all  $x \in A$ . Similarly,  $m$  for minimum.

**Definition 2.0.5**

When  $X$  is an ordered set, we have the following subsets which are called *intervals* in  $X$ . For  $a, b \in X$  with  $a < b$  we have

$$(a, b) := \{x \in X \mid a < x < b\}$$

$$(a, b] := \{x \in X \mid a < x \leq b\}$$

$$[a, b) := \{x \in X \mid a \leq x < b\}$$

$$[a, b] := \{x \in X \mid a \leq x \leq b\}$$

**Definition 2.0.6**

Let  $X$  be an ordered set. The *order topology* on  $X$  is the topology  $\mathcal{T}_o$  which is generated by the basis  $\mathcal{B}_o$  of sets in  $X$  which consist of the following intervals:

- $(a, b)$  where  $a, b \in X$ ,  $a < b$
- $(a, M]$  where  $M = \max X$  and  $a \in X$  with  $a \neq M$  (in the case that  $X$  has a maximum)
- $[m, b)$  where  $m = \min X$  and  $b \in X$  with  $b \neq m$  (in the case that  $X$  has a minimum)

We sometimes write  $X_o$  to indicate that  $X$  is using the order topology  $\mathcal{T}_o$

**Exercise 2.0.1**

Verify  $\mathcal{B}_o$  is a basis.

**Example 2.0.2**

$$\mathbb{R} = \mathbb{R}_o = \mathbb{R}_m$$

**Definition 2.0.7**

Let  $X$  be an ordered set the *lower limit topology* on  $X$  is the topology  $\mathcal{T}_\ell$  generated by the basis  $\mathcal{B}_\ell$  which consists of intervals of the form  $[a, b)$  where  $a, b \in X$  with  $a < b$  we sometimes write  $X_\ell$  to indicate that  $X$  is using the lower limit topology.

**Note**

on  $\mathbb{R}$ ,  $\mathcal{T}_\ell$  is not equal to  $\mathcal{T}_m$ . Note that when  $a, b \in \mathbb{R}$  with  $a < b$ ,

$$(a, b) = \bigcup_{n=m}^{\infty} \left[ a + \frac{1}{n}, b \right) \text{ where } \frac{1}{m} < b - a$$

which is open in  $\mathbb{R}_\ell$ . So we have  $\mathcal{T}_o \subseteq \mathcal{T}_\ell$

**Example 2.0.3**

Let  $X = (0, 1) \cup \{2\} \subseteq \mathbb{R}$ . Note that  $\mathcal{T}_o \neq \mathcal{T}_m = \mathcal{T}_s$  on  $X$ . (Where  $X$  uses the standard order inherited from  $\mathbb{R}$ ). For example  $\{2\}$  is open in  $X_m$ . But is not open in  $X_o$  because any open set in  $X_o$  which contains 2, must contain a basic open set  $B$  with  $2 \in B$ . So it must contain a set of the form

$$B = (a, 2]_X = (a, 1) \cup \{2\} \text{ where } a \in (0, 1)$$

So they include elements other than 2

**Example 2.0.4**

When  $X$  is an ordered set, the *dictionary* (or *lexicographic*) order on  $X^2$  is given by

$$(a, b) < (c, d) \iff (a = c \text{ and } b < d) \text{ or } a < c$$

Note that on  $\mathbb{R}^2$ , the order topology  $\mathcal{T}_o$  is not equal to the standard metric topology  $\mathcal{T}_m$

**2.1 Products of Topological Spaces****Definition 2.1.1**

Let  $X, Y$  be sets, then the Cartesian product of  $X$  and  $Y$  is

$$X \times Y = \{(x, y) \mid x \in X, y \in Y\}$$

**Definition 2.1.2**

Let  $K$  be a non-empty index set and let  $X_k$  be a set for each  $k \in K$ . Then the Cartesian product of the (indexed set of) sets  $X_k$ ,  $k \in K$

$$\prod_{k \in K} X_k = \left\{ x : K \rightarrow \bigcup_{k \in K} X_k \mid x(k) \in X_k \text{ for all } k \in K \right\}$$

and we write  $x(k)$  as  $x_k$ . In the case that  $K = \{1, \dots, n\}$  we write

$$\prod_{k \in K} X_k = \prod_{k=1}^n X_k = X_1 \times \dots \times X_n$$

In the case that  $K = \mathbb{Z}^+$  we write

$$\prod_{k \in K} X_k = \prod_{k=1}^{\infty} X_k = X_1 \times X_2 \times \dots$$

In the case that  $K = \{1, \dots, n\}$  and  $X_k = X$  for all  $k \in K$ , we also write

$$\prod_{k \in K} X_k = \prod_{k \in K} X = \underbrace{X \times X \times \dots \times X}_{n \text{ times}} = X^n$$

In the case that  $K = \mathbb{Z}^+$ , and  $X_k = X$  for all  $k \in K$ , we also write

$$\prod_{k \in K} X_k = \prod_{k=1}^{\infty} X = X \times X \times \dots = X^{\omega}$$

In the case that  $X$  is a vector space, we write

$$X^{\infty} = \{x = (x_1, x_2, \dots) \in X^{\omega} \mid x_k = 0 \text{ for all but finitely many } k \in \mathbb{Z}^+\}$$

In this case  $X^{\infty}$  and  $X^{\omega}$  are both vector spaces.

When  $X_k$  is a set for each  $k \in K$ , for each  $\ell \in K$  we have the projection map

$$p_{\ell} : \prod_{k \in K} X_k \rightarrow X_{\ell}$$

given by  $p_{\ell}(x) = x_{\ell} = x(\ell)$ . For any set  $Y$ , a function  $f : Y \rightarrow \prod_{k \in K} X_k$  determines, and is determined by, its component functions

$$f_{\ell} : Y \rightarrow X_{\ell}$$

where  $f_{\ell} = p_{\ell} \circ f$  so  $f_{\ell}(y) = f(y)_{\ell} = f(y)(\ell)$

**Definition 2.1.3**

When  $X_k$  is a topological space for each  $k \in K$ , there are two commonly used topologies on  $\prod_{k \in K} X_k$ .

1. The *box topology* on  $\prod_{k \in K} X_k$  is the topology generated by the basis of sets of the form

$$\prod_{k \in K} U_k \subseteq \prod_{k \in K} X_k$$

Where each  $U_k$  is open in  $X_k$

2. The *product topology* on  $\prod_{k \in K} X_k$  is the topology generated by the basis of sets consisting of the sets of the form  $\prod_{k \in K} U_k$  where each  $U_k$  is open in  $X_k$  with  $U_k = X_k$  for all but finitely many  $k \in K$ .

**Note**

The above two proposed bases are indeed bases of sets because

$$\left( \prod_{k \in K} U_k \right) \cap \left( \prod_{k \in K} V_k \right) = \prod_{k \in K} (U_k \cap V_k)$$

Also note that when  $K$  is finite, these two topologies are equal. When  $K$  is infinite, the box topology is finer than the product topology.

**Theorem 2.1.1**

Let  $\mathcal{B}_k$  be a basis for  $X_k$  for each  $k \in K$ . Then the set of sets of the form

$$\prod_{k \in K} B_k \text{ where } B_k \in \mathcal{B}_k \text{ for all } k \in K$$

is a basis for the box topology on  $\prod_{k \in K} X_k$ , and the set of sets of the form

$$\prod_{k \in K} B_k \text{ where } B_k \in \mathcal{B}_k \cup \{X_k\} \text{ for all } k \in K$$

with  $B_k = X_k$  for all but finitely many  $k \in K$  is a basis for the product topology on  $\prod_{k \in K} X_k$ .

**Proof:** Exercise □

**Theorem 2.1.2**

For each  $k \in K$ , let  $X_k$  be a subspace of  $Y_k$  (using the subspace topology). Then the box topology on  $\prod_{k \in K} X_k$  is equal to the subspace topology on  $\prod_{k \in K} X_k$  as a subspace of  $\prod_{k \in K} Y_k$  using the box topology, and the product topology on  $\prod_{k \in K} X_k$  is equal to the subspace topology on  $\prod_{k \in K} X_k$  as a subspace of  $\prod_{k \in K} Y_k$  using the product topology.



**Theorem 2.1.3**

Let  $Y$  be a topological space, and let  $X_k$  be a topological space for each  $k \in K$ , and let  $f : Y \rightarrow \prod_{k \in K} X_k$ . Then when  $\prod_{k \in K} X_k$  uses the product topology,  $f$  is continuous if and only if each component map  $f_\ell : Y \rightarrow X_\ell$  is continuous.

**Proof:** Suppose that  $f$  is continuous, then (using either the box or product topologies on  $\prod_{k \in K} X_k$ ) each projection map  $p_\ell : \prod_{k \in K} X_k \rightarrow X_\ell$  is continuous because when  $U \subseteq X_\ell$  is open,

$$\begin{aligned} p_\ell^{-1}(U) &= \left\{ x \in \prod_{k \in K} X_k \mid x_\ell = p_\ell(x) \in U \right\} \\ &= \prod_{k \in K} U_k \end{aligned}$$

where

$$U_k = \begin{cases} U & \text{if } k = \ell \\ X_k & \text{if } k \neq \ell \end{cases}$$

which is open in  $\prod_{k \in K} X_k$  (using either the box or product topology) It follows that each component function  $f_\ell$  is continuous because

$$f_\ell = p_\ell \circ f$$

Suppose, conversely, that each component map

$$f = p_\ell \circ f : Y \rightarrow \prod_{k \in K} X_k$$

is continuous, and that  $\prod_{k \in K} X_k$  is using the product topology. To show that  $f$  is continuous, it suffices to show that  $f^{-1}(B)$  is open in  $Y$  for every basic open set  $B$  in  $\prod_{k \in K} X_k$ . Let  $B$  be a basic open set (for the product topology) on  $\prod_{k \in K} X_k$ . Say  $B = \prod_{k \in K} U_k$  where each  $U_k$  is open in  $X_k$  with  $U_k = X_k$  for all but finitely many indices  $k \in K$ . Let  $L \subseteq K$  be the finite set of all indices  $k \in K$  for which  $U_k \neq X_k$ . We have

$$\begin{aligned} f^{-1}(B) &= \left\{ y \in Y \mid f(y) \in \prod_{k \in K} U_k \right\} \\ &= \{y \in Y \mid f_k(y) = f(y)_k \in U_k \text{ for all } k \in K\} \\ &= \{y \in Y \mid f_k(y) \in U_k \text{ for all } k \in L\} \\ &= \bigcap_{k \in L} f_k^{-1}(U_k) \end{aligned}$$

Which is open in  $Y$  since it is a finite intersection of open sets in  $Y$  (with  $f_k^{-1}(U_k)$ ) is open in  $Y$  because  $U_k$  is open in  $X_k$  and  $f_k : Y \rightarrow X_k$  is continuous. □

**Remark**

$$\mathbb{R}^\infty \subseteq \ell_1 \subseteq \ell_p \subseteq \ell_q \subseteq \ell_\infty \subseteq \mathbb{R}^\omega$$

for  $1 \leq p \leq q \leq \infty$ . Recall that these norms induce different topologies.

Question: do any of the  $p$ -norms induce the box or product topology on  $\mathbb{R}^\infty \subseteq \mathbb{R}^\omega$ ?

Question: is there a norm or metric on  $\mathbb{R}^\omega$  which induces the box or product topology?

**Remark**

Also, we have the  $p$ -norms on  $\mathbb{R}^n$ . They all give the same topology on  $\mathbb{R}^n$ . More generally, when  $X$  is a finite dimensional vector space, all norms on  $X$  induce the same topology on  $X$ . When  $L : X \rightarrow Y$  is a linear map between normed linear spaces,  $L$  is continuous iff  $\|L\|_{\text{op}} < \infty$  iff  $L(\overline{B_X}(0, 1))$  is bounded in  $Y$ . And when  $X$  is finite dimensional,  $\overline{B_X}(0, 1)$  is compact and  $L(\overline{B_X}(0, 1))$  is bounded, so  $L$  is continuous. In particular, when  $X$  is finite dimensional and  $\|\cdot\|_1, \|\cdot\|_2$  are two norms on  $X$ ,

$$\text{id}_X : (X, \|\cdot\|_1) \longrightarrow (X, \|\cdot\|_2)$$

is continuous, and it is equal to its own inverse which is continuous, so  $\text{id}_X$  is a homeomorphism, so for a set  $U \subseteq X$ ,  $U$  is open in  $(X, \|\cdot\|_1)$  if and only if  $U$  is open in  $(X, \|\cdot\|_2)$ . Consequently, every finite dimensional vector space  $X$  has a *standard* topology. (Pick a basis  $\{u_1, \dots, u_n\}$ , define

$$\left\langle \sum x_k u_k, \sum y_k u_k \right\rangle = \sum x_k y_k = x \cdot y$$

So the map  $L : X \rightarrow \mathbb{R}^n$  given by

$$L\left(\sum x_k u_k\right) = \sum x_k e_k = x$$

is an inner product space isomorphism.) Then use the inner product to define a norm, a metric, and a topology. The resulting topology doesn't depend on the choice of basis.

## 2.2 Quotient Spaces

**Definition 2.2.1**

Let  $X$  be a set. Let  $\sim$  be an equivalence relation on  $X$ . For  $a \in X$ , the *equivalence class* of  $a$  is

$$[a] = \{x \in X \mid a \sim x\}$$

Recall distinct equivalence classes are disjoint, and  $X$  is the disjoint union of distinct equivalence classes. The set of all equivalence classes is denoted by  $X/\sim$ , is called the quotient set of  $X$  by  $\sim$ .

$$X/\sim = \{[a] \mid a \in X\}$$

The map  $q : X \rightarrow X/\sim$  given by  $x \mapsto [x]$  is called the quotient map.

**Definition 2.2.2**

When  $X$  is a topological space, the *quotient topology* on  $X/\sim$  is the topology obtained by stipulating that for  $V \subseteq X/\sim$ ,  $V$  is open in  $X/\sim$  if and only if  $q^{-1}(V)$  is open in  $X$ .

**Note**

When  $V \subseteq X/\sim$  so  $V$  is a set of equivalence classes.

$$\begin{aligned} q^{-1}(V) &= \{x \in X \mid q(x) \in V\} \\ &= \{x \in X \mid [x] \in V\} \\ &= \bigcup_{[x] \in V} [x] \\ &= \bigcup V \end{aligned}$$

**Remark**

For sets  $X$  and  $Y$ ,

1. When  $Y$  is a topological space and  $X \subseteq Y$  is a subset, the subspace topology is the coarsest topology on  $X$  for which the inclusion map  $i : X \rightarrow Y$  is continuous

$$i^{-1}(V) = \{x \in X \mid i(x) \in V\} = \{x \in X \mid x \in V\} = V \cap X$$

2. When  $X$  and  $Y$  are both topological spaces, the product topology on  $X \times Y$  is the coarsest topology for which the two projection maps  $p_X : X \times Y \rightarrow X$ ,  $p_Y : X \times Y \rightarrow Y$  are both continuous

$$p_X^{-1}(U) = U \times Y \quad p_Y^{-1}(V) = V \times X$$

3. When  $X$  is a topological space and  $\sim$  an equivalence relation on  $X$ , the quotient topology on  $X/\sim$  is the finest topology on  $X/\sim$  for which the quotient map  $q : X \rightarrow X/\sim$  is continuous

**Note**

Let  $X$  be a set and  $\sim$  an equivalence relation on  $X$ . Note that any function  $g : X/\sim \rightarrow Y$  (where  $Y$  is any set) determines and is determined by a function  $f : X \rightarrow Y$  which is constant on equivalence classes (meaning that for  $x_1, x_2 \in X$  if  $x_1 \sim x_2$  then  $f(x_1) = f(x_2)$ ) with  $g$  given by  $g([x]) = f(x)$  and with  $f$  given by  $f = g \circ q$ . So  $f(x) = g(q(x)) = g([x])$

**Theorem 2.2.1**

Let  $X, Y$  be topological spaces. Let  $\sim$  be an equivalence relation on  $X$ . Let  $f : X/\sim \rightarrow Y$ . Let  $g : X \rightarrow Y$  be the map given by  $g(x) = f([x])$ , that is  $g = f \circ q$ . Then  $f$  is continuous if and only if  $g$  is continuous.

**Proof:** If  $f$  is continuous, then  $g$  is continuous because  $g = f \circ q$  which is the composite of two continuous maps. Suppose that  $g$  is continuous. Let  $V \subseteq Y$ , be open. We need to show that  $f^{-1}(V)$  is open in  $X/\sim$ . By definition of the quotient topology

$$f^{-1}(V) \text{ is open in } X/\sim \iff q^{-1}(f^{-1}(V)) \text{ is open in } X$$

But

$$q^{-1}(f^{-1}(V)) = (f \circ q)^{-1}(V) = g^{-1}(V)$$

Which is open in  $X$  since  $g$  is continuous. □

### Definition 2.2.3

For a group  $G$  and a set  $X$ , a *group action* of  $G$  on  $X$  is a function  $*$  :  $G \times X \rightarrow X$ , where we write  $*(a, x)$  as  $a * x$  or  $ax$ , such that

1. When  $e \in G$  is the identity element we have  $e * x = x$  for all  $x \in X$ .
2. For all  $a, b \in G$  and all  $x \in X$ , we have

$$a * (b * x) = \underbrace{(ab)}_{\text{group op}} * x$$

We say that  $G$  *acts on*  $X$  (by using the group action).

### Remark

A group action of  $G$  on  $X$  determines and is determined by a group homomorphism  $\rho : G \rightarrow \text{Perm}(X)$  where  $\rho(a)(x) = a * x$  (the homomorphism  $\rho$  is called a *representation* of  $G$ )

### Remark

Given an action of  $G$  on  $X$ , we can define an equivalence relation on  $X$  by

$$x \sim y \iff y = a * x \text{ for some } a \in G.$$

In this case, the equivalence class of  $x$  is called the *orbit of*  $x$  (we might write  $[x]$  as  $\text{Orb}(x)$ ) and we write the quotient  $X/\sim$  as  $X/G$ . So

$$\begin{aligned} X/G &= \{[x] \mid x \in X\} \\ &= \{\text{Orb}(x) \mid x \in X\} \end{aligned}$$

**Example 2.2.1**

For  $\mathbb{S}^1 = \{u \in \mathbb{R}^2 \mid \|u\| = 1\}$ , we have  $\mathbb{S}^1 \times \mathbb{R} \cong \mathbb{R}^2 \setminus \{0\}$ . Define

$$\begin{aligned} f : \mathbb{S}^1 \times \mathbb{R} &\longrightarrow \mathbb{R}^2 \setminus \{0\} \\ (u, t) &\longmapsto e^t u \end{aligned}$$

and define

$$\begin{aligned} g : \mathbb{R}^2 \setminus \{0\} &\longrightarrow \mathbb{S}^1 \times \mathbb{R} \\ x &\longmapsto \left( \frac{x}{\|x\|}, \ln\|x\| \right) \end{aligned}$$

These maps are continuous (they are elementary functions) and they are inverses of each other.

**Example 2.2.2**

$\mathbb{S}^1$  acts on  $\mathbb{R}^2 = \mathbb{C}$  by complex multiplication. For  $a \in \mathbb{R}^2 = \mathbb{C}$ ,

$$\text{Orb}(a) = [a] = \{ua \mid u \in \mathbb{S}^1\}$$

which is equal to the circle centered at 0 of radius  $\|a\|$  (with  $[0] = \{0\}$ ).

Show that  $\mathbb{R}^2/\mathbb{S}^1 \cong [0, \infty) \subseteq \mathbb{R}$  we define

$$\begin{aligned} f : \mathbb{R}^2/\mathbb{S}^1 &\longrightarrow [0, \infty) \\ [x] &\longmapsto \|x\| \end{aligned}$$

and define

$$\begin{aligned} h : [0, \infty) &\longrightarrow \mathbb{R}^2/\mathbb{S}^1 \\ r &\longmapsto [r] = [(r, 0)] = \{re^{i\theta} \mid \theta \in \mathbb{R}\} \end{aligned}$$

Note that  $f$  is continuous because for the map  $g : \mathbb{R}^2 \rightarrow [0, \infty) \subseteq \mathbb{R}$  given by  $g(x) = \|x\|$ . We have  $g = f \circ q$ . Since  $g$  is continuous, it follows that  $f$  is continuous. Also  $h$  is continuous because  $h = q \circ i$  where  $i : [0, \infty) \rightarrow \mathbb{R}^2$  is the inclusion map  $i(r) = (r, 0)$ . Finally, note that  $f$  and  $h$  are inverses.

**Example 2.2.3**

$\mathbb{R}^+ = (0, \infty)$  acts on  $\mathbb{R}^2$  by multiplication that is by  $t * x = tx$ . The orbits are for  $0 \neq x \in \mathbb{R}^2$ ,  $[x] = \{tx \mid 0 < t \in \mathbb{R}\}$  which is the (open) ray from 0 through  $x$  and  $[0] = \{0\}$ . Each of the rays  $[x]$  for  $0 \neq x \in \mathbb{R}^2$  intersects a unique point on  $\mathbb{S}^1$ . Which gives a fairly natural bijective map

$$f : \mathbb{R}^2 / \mathbb{R}^+ \longrightarrow \mathbb{S}^1 \cup \{0\}$$

$$[x] \mapsto \begin{cases} \frac{x}{\|x\|} & \text{if } 0 \neq x \in \mathbb{R}^2 \\ 0 & \text{if } x = 0 \in \mathbb{R}^2 \end{cases}$$

The inverse  $g : \mathbb{S}^1 \cup \{0\} \rightarrow \mathbb{R}^2 / \mathbb{R}^+$  is given by  $u \mapsto [u]$ . Note that  $g$  is continuous ( $g = q \circ i$  where  $i$  is the inclusion map  $i : \mathbb{S}^1 \cup \{0\} \rightarrow \mathbb{R}^2$ ). But  $f$  is not continuous, for example the set  $\{0\}$  is open in  $\mathbb{S}^1 \cup \{0\}$  (it is an open ball) but  $f^{-1}(\{0\}) = \{[0]\} \subseteq \mathbb{R}^2 / \mathbb{R}^+$  and  $q^{-1}(\{[0]\}) = \{0\}$  is not open in  $\mathbb{R}^2$ . In fact,  $\mathbb{R}^2 / \mathbb{R}^+ \not\cong \mathbb{S}^1 \cup \{0\}$ . One way to show this is to note that  $\mathbb{S}^1 \cup \{0\}$  has a singleton which is open ( $\{0\}$ ), but  $\mathbb{R}^2 / \mathbb{R}^+$  has no singleton which is open.

**Remark**

$\mathbb{R}^2 / \mathbb{R}^+$  is not Hausdorff, so it is not metrizable (there is no metric we can define on  $\mathbb{R}^2 / \mathbb{R}^+$  for which that quotient topology is equal to the metric topology)

**Example 2.2.4**

$\mathbb{Z}$  acts by addition on  $\mathbb{R}$  (by  $n * x = x + n$ ). The orbits are the sets  $[x] = \{x + n \mid n \in \mathbb{Z}\} = x + \mathbb{Z}$ . Show that  $\mathbb{R}/\mathbb{Z} \cong \mathbb{S}^1$ . Define

$$\begin{aligned} f : \mathbb{R}/\mathbb{Z} &\longrightarrow \mathbb{S}^1 \\ [t] &\longmapsto e^{i2\pi t} \end{aligned}$$

(and note that when  $[s] = [t]$  say  $s = t + n$  where  $n \in \mathbb{Z}$  we have

$$e^{i2\pi s} = e^{i2\pi(t+n)} = e^{i2\pi t}$$

) Note that  $f$  is continuous because the map  $f : \mathbb{R} \rightarrow \mathbb{S}^1$  given by  $g(t) = e^{i2\pi t}$  is continuous with  $g = f \circ q$ . The inverse map

$$\begin{aligned} h : \mathbb{S}^1 &\longrightarrow \mathbb{R}/\mathbb{Z} \\ e^{i\theta} &\longmapsto \left[ \frac{\theta}{2\pi} \right] \end{aligned}$$

To see that  $h$  is continuous, we can express  $h$  in Cartesian coordinates. We remark that there is an angle map

$$\begin{aligned} \theta : \mathbb{R}^2 \setminus \{0\} &\longrightarrow [0, 2\pi) \\ (x, y) &\longmapsto \begin{cases} \arccos \frac{x}{\sqrt{x^2+y^2}} & \text{if } y > 0 \text{ or } (y = 0 \text{ and } x \neq 0) \\ 2\pi - \arccos \frac{x}{\sqrt{x^2+y^2}} & \text{if } y < 0 \text{ or } (y = 0 \text{ and } x < 0) \end{cases} \end{aligned}$$

This map is not continuous along the positive  $x$ -axis. In Cartesian coordinates,  $h : \mathbb{S}^1 \rightarrow \mathbb{R}/\mathbb{Z}$  is given by

$$h(x, y) = \begin{cases} \left[ \frac{1}{2\pi} \arccos(x) \right] & \text{if } y \geq 0 \\ \left[ 1 - \frac{1}{2\pi} \arccos(x) \right] & \text{if } y \leq 0 \end{cases}$$

that is by

$$h(x, y) = \begin{cases} h_1(x, y) & \text{if } (x, y) \in A \\ h_2(x, y) & \text{if } (x, y) \in B \end{cases}$$

Where

$$\begin{aligned} A &= \{(x, y) \in \mathbb{S}^1 \mid y \geq 0\} \\ B &= \{(x, y) \in \mathbb{S}^1 \mid y \leq 0\} \end{aligned}$$

and

$$\begin{aligned} h_1(x, y) &= \frac{1}{2\pi} \arccos x \\ h_2(x, y) &= 1 - \frac{1}{2\pi} \arccos x \end{aligned}$$

### 3 Connected, Path-Connected and Compact Spaces

#### Definition 3.0.1

Let  $X$  be a topological space. For subsets  $A, B \subseteq X$ , we say that  $A$  and  $B$  *separate*  $X$  when  $A \neq \emptyset$ ,  $B \neq \emptyset$ ,  $A \cap B = \emptyset$  and  $A \cup B = X$ . We say that  $X$  is *disconnected* when there exist (nonempty disjoint) open sets  $U, V \subseteq X$  which separate  $X$ . Otherwise, we say that  $X$  is *connected*.

#### Proposition 3.0.1

$X$  is connected if and only if the only clopen sets are  $X$  and  $\emptyset$ .

**Proof:** If  $X$  is disconnected, we can find open sets  $U, V \subseteq X$  which separate  $X$  then the sets  $\emptyset, U, V, X$  are clopen. On the other hand, if  $\emptyset \neq U \subsetneq X$  with both  $U$  both open and closed in  $X$ , then  $U$  and  $V = X \setminus U$  are open sets in  $X$  which separate  $X$ .  $\square$

#### Exercise 3.0.1

When  $X$  is a metric space and  $A \subseteq X$  is a subspace, then  $A$  is connected if and only if there do not exist open sets  $U, V$  in  $X$  such that  $U \cap A \neq \emptyset$ ,  $V \cap A \neq \emptyset$ ,  $U \cap V = \emptyset$  and  $A \subseteq U \cup V$ .

#### Example 3.0.1

The connected sets in  $\mathbb{R}$  are the intervals (including  $\emptyset, \{a\}, \mathbb{R}$ )

#### Example 3.0.2

The (non-empty) connected subsets of  $\mathbb{Q}$  are the singletons (by using the density of the irrationals)

#### Theorem 3.0.2

If  $f : X \rightarrow Y$  is a continuous map between topological spaces, and if  $X$  is connected, then  $f(X)$  is connected.

**Proof:** Suppose  $X$  is connected and  $f : X \rightarrow Y$  is continuous. By restricting the codomain, the map  $f : X \rightarrow f(X)$  is also continuous. Suppose, for a contradiction that  $f(X)$  is disconnected. Let  $U, V$  be open sets in  $f(X)$  which separate  $f(X)$ . Then  $f^{-1}(U)$  and  $f^{-1}(V)$  are open sets in  $X$  which separate  $X$ , so that  $X$  is disconnected, giving the desired contradiction.  $\square$

#### Lemma 3.0.3

Let  $X$  be a subspace of  $Y$ . Suppose  $Y$  is disconnected. Let  $U, V$  be open sets in  $Y$  that separate  $Y$ . If  $X$  is connected, then  $X \subseteq U$  or  $X \subseteq V$ .

**Proof:** Suppose  $X \not\subseteq U$  and  $X \not\subseteq V$ . Since  $U \cup V = Y$ , it follows that  $X \cap U \neq \emptyset$  and  $X \cap V \neq \emptyset$ . And these two sets are open sets in  $X$  which separate  $X$ .  $\square$



**Theorem 3.0.4**

Let  $X = \bigcup_{k \in K} A_k$  where each subspace  $A_k$  is connected. With  $\bigcap_k A_k \neq \emptyset$ . Then  $X$  is connected.

**Proof:** Suppose, for a contradiction, that  $X$  is disconnected. Let  $U, V$  be open sets in  $X$  which separate  $X$ . Let  $p \in \bigcap_{k \in K} A_k \subseteq X = U \cup V$ . Either  $p \in U$  or  $p \in V$  (but not both) say  $p \in U$ . For each index  $k$ , since  $A_k$  is connected either  $A_k \subseteq U$  or  $A_k \subseteq V$  and since  $p \in A_k$ ,  $p \notin V$ , we must have  $A_k \subseteq U$ . Since  $A_k \subseteq U$  for every  $k \in K$ , we have  $X = \bigcup_{k \in K} A_k \subseteq U$ . This is not possible since  $U$  and  $V$  separate  $X$ . □

**Theorem 3.0.5**

The product of two connected spaces is connected.

**Proof:** Let  $X$  and  $Y$  be connected spaces. Suppose both  $X$  and  $Y$  are nonempty (since if either one was,  $\emptyset$  is connected). □