Contents

1 Topological Spaces and Continuous Maps

1.1 Elementary Topology

Given an inner product on an $\mathbb{R}$-vector space $\langle \cdot, \cdot \rangle$, one can define a norm $\|x\| = \sqrt{\langle x, x \rangle}$. Given a norm, one can define a metric $d(x, y) = \|x - y\|$. Given a metric d on a set X, one can define open sets in X:

given $a \in X$ and $r > 0$, $B(a, r) := \{x \in X \mid d(x, a) < r\}$. Then for $A \subseteq X$, we say A is open in X when $\forall a \in A \exists r > 0$ such that $B(a, r) \subseteq A$. Equivalently, for all $a \in A$, there is $b \in X$, $r > 0$ such that $a \in B(b, r) \subseteq A$.

Remark

The set of open sets on a metric space is called the *metric topology* on X.

Open sets in a metric space satisfy the following:

1. $\emptyset$ and X are open
2. arbitrary unions of open sets are open
3. finite intersections of open sets are open

Notation

For a set of sets S, the union of S is

$$\bigcup S := \{x \mid \exists A \in S, x \in A\} = \bigcup_{A \in S} A$$

In the case that $S \neq \emptyset$, the intersection of S is

$$\bigcap S := \{x \mid \forall A \in S, x \in A\} = \bigcap_{A \in S} A$$

Note

$\bigcap S$ would contain all elements as the condition $\forall A \in \emptyset$ would be vacuously satisfied. If we are given a universal set X, and S is known to be a set of subsets of X, then $\bigcap \emptyset = X$.

Definition 1.1

Let X be a set. $\mathcal{T} \subseteq \mathcal{P}(X)$ is called a *topology* on X if

1. $\emptyset, X \in \mathcal{T}$
2. If $S \subseteq \mathcal{T}$ is nonempty, then $\bigcup S \in \mathcal{T}$
3. If $S \subseteq \mathcal{T}$ is nonempty and finite, then $\bigcap S \in \mathcal{T}$

The elements of $\mathcal{T}$ are called the open sets of X. The closed sets are the compliments of the open sets.

Remark

To show 3 holds, it suffices to show the intersection of 2 open sets is open (by induction)

Definition 1.2

If X is a set, and $\mathcal{T}$ is a topology on X, then $(X, \mathcal{T})$ is called a *topological* space

Remark

When $f : X \to Y$ is a map between metric spaces, f is continuous iff $f^{-1}(V)$ is open in X for every open set $V \subseteq Y$.

Definition 1.3

For a map $f : X \to Y$ between topological spaces, we say that f is continuous when $f^{-1}(V)$ is open in X for every open set $V \subseteq Y$.

Example 1.1

if $f : A \subseteq \mathbb{R}^n \longrightarrow B \subseteq \mathbb{R}^m$ is an elementary function, then f is continuous.

Definition 1.4

When S, T are topologies on X with $S \subseteq T$, we say that S is coarser than T and T is finer than S. When $S \subsetneq T$, we use strictly coarser/finer.

Example 1.2

$\{\emptyset, X\}$ is a topology on X called the *trivial topology*

Example 1.3

$\mathcal{P}(X)$ is a topology on X called the *discrete topology*

Example 1.4

When $X = \emptyset, \mathcal{T} \subseteq \mathcal{P}(X) \Rightarrow \mathcal{T} \subseteq \{\emptyset\} \Rightarrow \mathcal{T} = \emptyset \vee \mathcal{T} = \{\emptyset\}$. Thus the only topology on $\emptyset$ is $\{\emptyset\}$.

Example 1.5

When $X = \{a\}$ the only topology is $\mathcal{T} = \{\emptyset, \{a\}\}$

Exercise 1.1.1

Find all topologies on the 2 and 3 element sets.

Definition 1.5

Let X be a topological space. Let $A \subseteq X$.

1. The *interior* of A (in X) denoted by $\operatorname{int}(A)$ is the union of all open sets in X which are contained in A.
2. The *closure* of A denoted $\overline{A}$ is the intersection of all closed sets in X which contain A.
3. The *boundary* of A, denoted by ∂A, given by $\partial A = \overline{A} \setminus \operatorname{int}(A)$

Note

The set of closed sets in a topological space is closed under arbitrary intersections and under finite unions. In particular $\emptyset, X$ are closed

Theorem 1.1

Let X be a topological space, $A \subseteq X$.

1. $\operatorname{int}(A)$ is open, and is the largest open set which is contained in A
2. $\overline{A}$ is closed, and is the smallest closed set which contains A
3. A is open iff $A = \operatorname{int}(A)$
4. A is closed iff $A = \overline{A}$
5. $\operatorname{int}(\operatorname{int}(A)) = \operatorname{int}(A)$
6. $\overline{\overline{A}} = \overline{A}$

Definition 1.6

Let X be a topological space, let $A \subseteq X$, let $a \in X$.

1. We say that a is an *interior point* of A when $a \in A$ and there is an open set U such that $a \in U \subseteq A$
2. We say that a is a *limit point* of A when for every open set $U \ni a$ we have $U \cap (A \setminus \{a\}) \neq \emptyset$. The set of limit points of A is denoted by A'
3. We say that a is a boundary point of A when every open set $U \ni a$, we have $U \cap A \neq \emptyset$ and $U \cap A^c \neq \emptyset$

Theorem 1.2

Let X be a topological space and let $A \subseteq X$.

1. $\text{int}(A)$ is equal to the set of all interior points
2. For $a \in X$,
$$a \in A' \Longleftrightarrow a \in \overline{A \setminus \{a\}}$$
3. A is closed iff $A' \subseteq A$
4. $\overline{A} = A \cup A'$
5. $\overline{A}$ is the disjoint union
$$\overline{A} = \text{int}(A) \sqcup \partial A$$
6. ∂A is equal to the set of boundary points of A

1.2 Topological Bases

Theorem 1.3

Let X be a set. Then the intersection of any set of topologies on X is also a topology on X.

Proof: Let $\{\mathcal{T}_\alpha\}_{\alpha \in I}$ be a collection of topologies on X. Let $\mathcal{T} = \bigcap_\alpha \mathcal{T}_\alpha$

1. Since $X, \emptyset \in \mathcal{T}_\alpha$ for all $\alpha \in I$. We have $X, \emptyset \in \mathcal{T}$
2. Let $\{U_i\} \subseteq \mathcal{T}$. For all $\alpha \in I$, we have each $U_i \in \mathcal{T}_\alpha$. Thus $\bigcup_i U_i \in \mathcal{T}_\alpha \Longrightarrow \bigcup_i U_i \in \mathcal{T}$ as desired.
3. Let $U_1, \ldots, U_n \in \mathcal{T}$. Then again for all $\alpha \in I$, we have each $U_i \in \mathcal{T}_\alpha$. Thus $\bigcap_{i=1}^n U_i \in \mathcal{T}_\alpha \Longrightarrow \bigcap_{i=1}^n U_i \in \mathcal{T}$

□

Corollary 1.4

When X is a set and $\mathcal{S}$ is any set of subsets of X (that is $S \subseteq \mathcal{P}(X)$), there is a unique smallest (coarsest) topology $\mathcal{T}$ on X which contains $\mathcal{S}$. Indeed $\mathcal{T}$ is the intersection of (the set of) all topologies on X containing $\mathcal{S}$.
This topology $\mathcal{T}$ is called the topology on X *generated by* $\mathcal{S}$

Definition 1.7

Let X be a set. A *basis of sets* on X is a set $\mathcal{B}$ of subsets of X (So $\mathcal{B} \subseteq \mathcal{P}(X)$) such that

1. $\mathcal{B}$ covers X, that is $\bigcup \mathcal{B} = X$
2. For every $C, D \in \mathcal{B}$ and $a \in C \cap D$. There is $B \in \mathcal{B}$ such that $a \in B \subseteq C \cap D$.

When $\mathcal{B}$ is a basis of sets in X and $\mathcal{T}$ is the topology on X generated by $\mathcal{B}$, we say that $\mathcal{B}$ is a *basis for* $\mathcal{T}$. The elements in $\mathcal{B}$ are called *basic open sets* in X.

Theorem 1.5 **Characterization of Open Sets in Terms of Basic Open Sets**

Let X be a topological space, Let $\mathcal{B}$ be a basis for the topology on X.

1. For $A \subseteq X$, A is open iff for every $a \in A$, there is $B \in \mathcal{B}$ such that $a \in B \subseteq A$ *
2. The open sets in X are the unions of (sets of) elements in $\mathcal{B}$

Equivalently,

1. $\mathcal{T} = \{A \subseteq X \mid \forall a \in A, \exists B \in \mathcal{B}\ a \in B \subseteq A\}$
2. $\mathcal{T} = \{\bigcup C \mid C \subseteq \mathcal{B}\}$

Proof: Let $\mathcal{T}$ be the topology on X (generated by $\mathcal{B}$). Let $\mathcal{S}$ be the set of all sets $A \subseteq X$ with property * ($\forall a \in A \exists B \in \mathcal{B} : a \in B \subseteq A$). And let $\mathcal{R}$ be the set of (arbitrary) unions of (sets of) elements in $\mathcal{B}$. Recall that $\mathcal{T}$ is the intersection of the set of all topologies on X which contain $\mathcal{B}$. Note that $\mathcal{S}$ contains $\mathcal{B}$ (obviously). Let us show that $\mathcal{S}$ is a topology on X. We have $\emptyset \in \mathcal{S}$ vacuously and $X \in \mathcal{S}$ because $\mathcal{B}$ covers X (given $a \in X$, we can choose $B \in \mathcal{B}$ with $a \in B$). When $U_k \in S$ for every $k \in K$ (where K is any index set). Let $a \in \bigcup_k U_k$. Choose $\ell \in K$ so that $a \in U_\ell$. Since $U_\ell \in \mathcal{S}$, we can choose $B \in \mathcal{B}$ so that $a \in B \subseteq U_\ell$. Since $U_\ell \subseteq \bigcup_k U_k$, we have $a \in B \subseteq \bigcup_k U_k$. Thus $\bigcup_k U_k$ satisfies *, hence $\bigcup_k U_k \in \mathcal{S}$ as required. Suppose $U, V \in \mathcal{S}$ Let $a \in U \cap V$. Since $U \in \mathcal{S}$ we can choose $C \in \mathcal{B}$ with $a \in C \subseteq U$. Since $V \in \mathcal{S}$, we can choose $D \in \mathcal{B}$ with $a \in D \subseteq V$. Since $\mathcal{B}$ is a basis, $C, D \in \mathcal{B}$ and $a \in C \cap D$, we can choose $B \in \mathcal{B}$ with $a \in B \subseteq C \cap D$. Then we have

$$a \in B \subseteq C \cap D \subseteq U \cap V$$

Thus $U \cap V$ satisfies * so that $U \cap V \in \mathcal{S}$ as required. Thus $\mathcal{S}$ is a topology on X containing $\mathcal{B}$, hence $\mathcal{T} \subseteq \mathcal{S}$. Let us show that $\mathcal{S} \subseteq \mathcal{R}$ let $U \in \mathcal{S}$. For each $a \in U$, choose $B_a \in \mathcal{B}$ with $a \in B_a \subseteq U$. Then we have

$$U = \bigcup_{a \in U} B_a \in \mathcal{R}$$

Thus $\mathcal{S} \subseteq \mathcal{R}$. Finally note that $\mathcal{R} \subseteq \mathcal{T}$ because if $U = \bigcup_k B_k$ with $B_k \in \mathcal{B}$, then each $B_k \in \mathcal{T}$, and $\mathcal{T}$ is a topology, so

$$U = \bigcup_{k \in K} B_k \in \mathcal{T}$$

$\square$

Theorem 1.6 **Characterization of a Basis in terms of the Open Sets**

Let X be a topological space with topology $\mathcal{T}$. Let $\mathcal{B} \subseteq \mathcal{T}$. Then $\mathcal{B}$ is a basis for $\mathcal{T}$ iff $\forall U \in \mathcal{T} \forall a \in U \exists B \in \mathcal{B} \quad a \in B \subseteq U$. *

Proof: If $\mathcal{B}$ is a basis for $\mathcal{T}$, then * holds by part 1 of the previous theorem. Suppose * holds. Let us show that $\mathcal{B}$ is a basis of sets in X. Note that $\mathcal{B}$ covers X since, taking $U = X$ in * we have $\forall a \in X \exists B \in \mathcal{B} \quad a \in B \subseteq X$. Also note that given $C, D \in \mathcal{B}$ and $a \in C \cap D$, then by taking $U = C \cap D$ in * (noting that $C, D \in \mathcal{B} \subseteq \mathcal{T}$ so that $U = C \cap D \in \mathcal{T}$) we can choose $B \in \mathcal{B}$ with $a \in B \subseteq C \cap D$. Thus $\mathcal{B}$ is a basis of sets in X. It remains to show that $\mathcal{T}$ is the topology generated by $\mathcal{B}$. Let $\mathcal{S}$ be the topology generated by $\mathcal{B}$. By part 1 of the previous theorem, S is the set of all unions of elements in $\mathcal{B}$. Also $\mathcal{S}$ is the smallest topology which contains $\mathcal{B}$. Since $\mathcal{B} \subseteq \mathcal{T}$ and $\mathcal{T}$ is a topology, we have $\mathcal{S} \subseteq \mathcal{T}$. Also we have $\mathcal{T} \subseteq \mathcal{S}$ because given $U \in \mathcal{T}$, by property *, for each $a \in U$, we can choose $B_a \in \mathcal{B}$ with $a \in B_a \subseteq U$, and then we have $U = \bigcup_{a \in U} B_a \in \mathcal{S}$ since it is a union of elements in $\mathcal{B}$ □

Example 1.6

When X is a metric space, the set $\mathcal{B}$ of all open balls in X is a basis for the metric topology on X.

Remark

We can use a basis for testing various topological properties:
When X is a topological space, and $\mathcal{B}$ is a basis for the topology on X, and $A \subseteq X$ and $a \in X$. Then

$$\begin{aligned} a \in \operatorname{int}(A) &\iff \exists B \in \mathcal{B} \text{ with } a \in B \subseteq A \\ a \in \overline{A} &\iff \forall B \in \mathcal{B} \text{ with } a \in B \quad B \cap A \neq \emptyset \\ a \in A' &\iff \forall B \in \mathcal{B} \text{ with } a \in B \quad (B \setminus \{a\}) \cap A \neq \emptyset \\ a \in \partial A &\iff \forall B \in \mathcal{B} \text{ with } a \in B \quad B \cap A \neq \emptyset \text{ and } B \cap (X \setminus A) \neq \emptyset \end{aligned}$$

Definition 1.8

A topological space X is called *Hausdorff* when for all $a, b \in X$ with $a \neq b$, there exist disjoint open sets U and V in X with $a \in U$ and $b \in V$.

Example 1.7

Metric spaces are Hausdorff

1.3 Subspaces

Definition 1.9 **Subspace Topology**

Let Y be a topological space with topology $\mathcal{S}$, and $X \subseteq Y$ be a subset. Let

$$\mathcal{T} := \{V \cap X \mid V \in \mathcal{S}\}$$

Then $\mathcal{T}$ is a topology on X:
Indeed $\emptyset \in \mathcal{S}$ so $\emptyset \cap X = \emptyset \in \mathcal{T}$ and $Y \in \mathcal{S}$ so $Y \cap X = X \in \mathcal{T}$. If K is any index set and $U_k \in \mathcal{T}$ for each $k \in K$, then for each $k \in K$ we can choose $V_k \in \mathcal{S}$ such that $U_k = V_k \cap X$ and then we have

$$\begin{aligned}\bigcup_{k \in K} U_k &= \bigcup_{k \in K} (V_k \cap X) \\ &= \left(\bigcup_{k \in K} V_k\right) \cap X \in \mathcal{T}\end{aligned}$$

since $\bigcup_{k \in K} V_k \in \mathcal{S}$. Similarly, when K is finite and $U_k \in \mathcal{T}$ for each $k \in K$ we have $\bigcap_{k \in K} U_k \in \mathcal{T}$ The topology $\mathcal{T}$ on X is called the *subspace topology* on X (inherited from the topology on Y).

Theorem 1.7

Let Y be a topological space, let $\mathcal{C}$ be a basis for the topology on Y. Let $X \subseteq Y$ be a subset. Then the set

$$\mathcal{B} = \{C \cap X \mid C \in \mathcal{C}\}$$

is a basis for the subspace topology on X.

Proof: Exercise □

Theorem 1.8

Let Z be a topological space, let $Y \subseteq Z$ be a subspace and $X \subseteq Y$ be a subset. Then the subspace topology on X inherited from Y is equal to the subspace topology on X inherited from Z.

Proof: Exercise □

Theorem 1.9

Let Y be a metric space, (using the metric topology) and let $X \subseteq Y$. Then the subspace topology on X (inherited from the topology on Y) is equal to the metric topology on X using the metric on X obtained by restricting the metric on Y.

Proof: Exercise □

1.4 Continuous Maps

Definition 1.10

Let X, Y be topological spaces.

1. For $f : X \to Y$ and $a \in X$, we say that f is *continuous at* a when for every open set $V \subseteq Y$ with $f(a) \in V$, there exists an open set $U \subseteq X$ with $a \in U \subseteq f^{-1}(V)$.
2. We say that f is *continuous* (in or on X) when for every open set $V \subseteq Y$, $f^{-1}(V)$ is open in X.
3. A *homeomorphism* from X to Y is is a bijective map $f : X \to Y$ such that both f and its inverse $f^{-1} : Y \to X$ are continuous. We say that X and Y are *homeomorphic*, and we write $X \cong Y$, when there exists a homeomorphism $f : X \to Y$. (and we remark that $f^{-1} : Y \to X$ is also a homeomorphism).

Theorem 1.10

Constant maps and inclusion maps are continuous.

Proof: For $f : X \to Y$ given by $f(x) = c \in Y$ for all $x \in X$. When V is open in Y,

$$f^{-1}(V) = \begin{cases} X \text{ if } c \in V \\ \emptyset \text{ if } c \notin V \end{cases}$$

When $X \subseteq Y$ is a subspace and $f : X \to Y$ is given by $f(x) = x$ for all $x \in X$, when V is open in Y.

$$\begin{aligned} f^{-1}(V) &= \{x \in X \mid f(x) \in V\} \\ &= \{x \in X \mid x \in V\} \\ &= V \cap X \end{aligned}$$

which is open in X. (when X uses the subspace topology) □

Remark

When Y is a topological space and $X \subseteq Y$ we shall assume, unless otherwise noted, that X uses the subspace topology.

Theorem 1.11 **Equivalent Definitions of Continuity**

Let $f : X \to Y$ be a map between topological spaces

1. f is continuous iff f is continuous at every $a \in X$
2. f is continuous iff for every closed set $K \subseteq Y$, $f^{-1}(K)$ is closed in X.
3. If $\mathcal{C}$ is a basis for the topology on Y then f is continuous iff for every $C \in \mathcal{C}$, $f^{-1}(C)$ is open in X.

Proof of 1: Suppose f is continuous on X. Let $a \in X$. Let V be an open set in Y with $f(a) \in V$. Let $U = f^{-1}(V)$, then $f^{-1}(V)$ is open, since f is continuous and $a \in U \subseteq f^{-1}(V)$. Suppose, conversely, that f is continuous at every $a \in X$. Let V be an open set in Y. For each $a \in f^{-1}(V)$ since f is continuous at a with $f(a) \in V$, we can choose an open set U_a in X with $a \in U_a \subseteq f^{-1}(V)$. Then

$$f^{-1}(V) = \bigcup_{a \in f^{-1}(V)} U_a$$

which is open in X, since it is a union in open sets in X. □

Theorem 1.12

Let $f : X \to Y$, $g : Y \to Z$ be continuous maps between topological spaces, then the composite map $h = g \circ f : X \to Z$ is continuous.

Proof: Show that $h^{-1}(W) = f^{-1}(g^{-1}(W))$ □

Remark

Homeomorphism of topological spaces behaves like an equivalence relation on the class of all topological spaces. For topological spaces X, Y, Z

1. $X \cong X$ (since id_X is a homeomorphism – a special case of the inclusion map)
2. If $X \cong Y$ then $Y \cong X$ (when $f : X \to Y$ is a homeomorphism, so is $f^{-1} : Y \to X$)
3. If $X \cong Y \cong Z$ then $X \cong Z$ (if $f : X \to Y, g : Y \to Z$ are homeomorphisms then so is $g \circ f$)

Theorem 1.13 — **Restriction of Domain and Restriction or Expansion of Codomain**

Let X, Y, Z be topological spaces. Suppose $f : X \to Y$ is continuous.

1. For any subspace $A \subseteq X$, the restriction $f|_A : A \to Y$ is continuous.
2. If $Y \subseteq Z$ is a subspace then $f : Y \to Z$ is continuous and if $B \subseteq Y$ with $f(X) \subseteq B$, then $f : X \to B$ is continuous.

Proof: Exercise □

Lemma 1.14 — **Glueing/Pasting Lemma**

Let $f : X \to Y$ be a map between topological spaces

1. If $X = \bigcup_{k \in K} U_k$ where each U_k is open in X and if each restriction map $f|_{U_k} : U_k \to Y$ is continuous (where U_k is using the subspace topology), then f is continuous.
2. If $X = C_1 \cup \cdots \cup C_n$ where each C_k is closed in X, and if each restriction $f|_{C_k} : C_k \to Y$ is continuous, then f is continuous.

Proof of 1: Suppose $X = \bigcup_{k \in K} U_k$ where each U_k is open in X and suppose each restriction $f|_{U_k}$ is continuous. Let $V \subseteq Y$ be open. Note that

$$\begin{aligned} f^{-1}(V) &= \{x \in X \mid f(x) \in V\} \\ &= \bigcup_{k \in K} \{x \in U_k \mid f(x) \in V\} \\ &= \bigcup_{k \in K} \left\{x \in U_k \;\middle|\; f|_{U_k}(x) \in V\right\} \\ &= \bigcup_{k \in K} f|_{U_k}^{-1}(V) \end{aligned}$$

For each $k \in K$, since $f|_{U_k}$ is continuous, we know that $f|_{U_k}^{-1}(V)$ is open in U_k. Since U_k is using the subspace topology, we can choose an open W_k in X such that $f|_{U_k}^{-1}(V) = W_k \cap U_k$. This is open in X since W_k and U_k are both open in X. Since $f^{-1}(V) = \bigcup_{k \in K} f|_{U_k}^{-1}(V)$ it is a union of open sets in X, so it is open in X. Thus f is continuous. □

Proof of 2: Exercise. First show that for $f : X \to Y$, f is continuous iff $f^{-1}(C)$ is closed in X for every closed set C in Y. And, show that when $A \subseteq X \subseteq Y$, A is closed in X (using the subspace topology from Y) iff $A = B \cap X$ for some closed set B in Y. □

Example 1.8

The map $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = \begin{cases} 2x & x \leq 0 \\ x^2 & x > 0 \end{cases}$ is continuous.

1.5 Examples of Homeomorphisms

Example 1.9

The circle

$$\{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1\}$$

in $\mathbb{R}^2$ is homeomorphic to the ellipse

$$\left\{(x, y) \in \mathbb{R}^2 \;\middle|\; \frac{(x-a)^2}{A^2} + \frac{(y-b)^2}{B^2} = 1\right\}$$

in $\mathbb{R}^2$

Example 1.10

$\mathbb{R} \cong (-1, 1) \subseteq \mathbb{R}$

Example 1.11

The standard unit n-sphere in $\mathbb{R}^{n+1}$ is the set

$$\mathbb{S}^n = \{x \in \mathbb{R}^{n+1} \mid \|x\| = 1\}$$

Where p is the north pole

$$p = e_{n+1} = (0, ..., 0, 1) \in \mathbb{S}^n$$

We have $\mathbb{S}^n \setminus \{p\} \cong \mathbb{R}^n$

2 Examples of Topological Spaces

Definition 2.1

Let X be a set. We sometimes write X_t to indicate that X is using the trivial topology $\mathcal{T}_t = \{\emptyset, X\}$. We sometimes write X_d to indicate X is using the discrete topology $\mathcal{T}_d = \mathcal{P}(X)$. We sometimes write X_c to indicate X is using the co-finite topology $\mathcal{T}_c = \{A \subseteq X \mid A = \emptyset \text{ or } X \setminus A \text{ is finite}\}$. Note the closed sets in X_c are exactly the finite ones and X.

Definition 2.2

When X is a metric space, we assume, unless otherwise indicated, that X uses the metric topology. Sometimes, we might write X_m to indicate that X is using the metric topology $\mathcal{T}_m$.

Definition 2.3

When Y is a topological space, and $X \subseteq Y$, we assume, unless otherwise indicated, that X uses the subspace topology. Sometimes, we might write X_s to indicate that X is using the subspace topology $\mathcal{T}_s$. When $X \subseteq \mathbb{R}^n$, we shall assume, unless otherwise indicated, that X is using $\mathcal{T}_m = \mathcal{T}_s$

Definition 2.4

Let X be a set. A (strict, linear or total) *order* on X is a binary relation $<$ on X such that

1. For all $x, y \in X$ exactly one of the following holds:
   a. $x < y$
   b. $x = y$
   c. $y < x$
2. For all $x, y, z \in X$, if $x < y$ and $y < z$ then $x < z$

An *ordered set* is a set X with an order $<$. When X is an ordered set, we also define $\leq, >, \geq$ by stipulating that for all $x, y \in X$

$$\begin{aligned} x \leq y &\Longleftrightarrow (x < y \vee x = y) \\ x > y &\Longleftrightarrow y < x \\ x \geq y &\Longleftrightarrow y \leq x \end{aligned}$$

Remark

In an ordered set X we can define an *upper bound*, a *lower bound*, the *supremum*, the *infimum*, the *maximum*, and the *minimum* of a subset $A \subseteq X$.

Example 2.1

Let X be an ordered set and $A \subseteq X$, $M = \max(A)$ when $M \in A$ with $M \geq x$ for all $x \in A$. Similarly, m for minimum.

Definition 2.5

When X is an ordered set, we have the following subsets which are called *intervals* in X. For $a, b \in X$ with $a < b$ we have

$$\begin{aligned} (a, b) &:= \{x \in X \mid a < x < b\} \\ (a, b] &:= \{x \in X \mid a < x \leq b\} \\ [a, b) &:= \{x \in X \mid a \leq x < b\} \\ [a, b] &:= \{x \in X \mid a \leq x \leq b\} \end{aligned}$$

Definition 2.6

Let X be an ordered set. The *order topology* on X is the topology $\mathcal{T}_o$ which is generated by the basis $\mathcal{B}_o$ of sets in X which consist of the following intervals:

- (a, b) where $a, b \in X$, $a < b$
- $(a, M]$ where $M = \max X$ and $a \in X$ with $a \neq M$ (in the case that X has a maximum)
- $[m, b)$ where $m = \min X$ and $b \in X$ with $b \neq m$ (in the case that X has a minimum)

We sometimes write X_o to indicate that X is using the order topology $\mathcal{T}_o$

Exercise 2.0.1

Verify $\mathcal{B}_o$ is a basis.

Example 2.2

$\mathbb{R} = \mathbb{R}_o = \mathbb{R}_m$

Definition 2.7

Let X be an ordered set the *lower limit topology* on X is the topology $\mathcal{T}_\ell$ generated by the basis $\mathcal{B}_\ell$ which consists of intervals of the form $[a, b)$ where $a, b \in X$ with $a < b$ we sometimes write X_ℓ to indicate that X is using the lower limit topology.

Note

on $\mathbb{R}$, $\mathcal{T}_\ell$ is not equal to $\mathcal{T}_m$. Note that when $a, b \in \mathbb{R}$ with $a < b$,

$$(a, b) = \bigcup_{n=m}^{\infty} \left[a + \frac{1}{n}, b\right) \text{ where } \frac{1}{m} < b - a$$

which is open in $\mathbb{R}_\ell$. So we have $\mathcal{T}_o \subseteq \mathcal{T}_\ell$

Example 2.3

Let $X = (0, 1) \cup \{2\} \subseteq \mathbb{R}$. Note that $\mathcal{T}_o \neq \mathcal{T}_m = \mathcal{T}_s$ on X. (Where X uses the standard order inherited from $\mathbb{R}$). For example $\{2\}$ is open in X_m. But is not open in X_o because any open set in X_o which contains 2, must contain a basic open set B with $2 \in B$. So it must contain a set of the form

$$B = (a, 2]_X = (a, 1) \cup \{2\} \text{ where } a \in (0, 1)$$

So they include elements other than 2

Example 2.4

When X is an ordered set, the *dictionary* (or *lexicographic*) order on X^2 is given by

$$(a, b) < (c, d) \Longleftrightarrow (a = c \text{ and } b < d) \text{ or } a < c$$

Note that on $\mathbb{R}^2$, the order topology $\mathcal{T}_o$ is not equal to the standard metric topology $\mathcal{T}_m$

2.1 Products of Topological Spaces

Definition 2.8

Let X, Y be sets, then the Cartesian product of X and Y is

$$X \times Y = \{(x, y) \mid x \in X, y \in Y\}$$

Definition 2.9

Let K be a non-empty index set and let X_k be a set for each $k \in K$. Then the Cartesian product of the (indexed set of) sets X_k, $k \in K$

$$\prod_{k \in K} X_k = \left\{ x : K \to \bigcup_{k \in K} X_k \,\middle|\, x(k) \in X_k \text{ for all } k \in K \right\}$$

and we write $x(k)$ as x_k. In the case that $K = \{1, ..., n\}$ we write

$$\prod_{k \in K} X_k = \prod_{k=1}^{n} X_k = X_1 \times \cdots \times X_n$$

In the case that $K = \mathbb{Z}^+$ we write

$$\prod_{k \in K} X_k = \prod_{k=1}^{\infty} X_k = X_1 \times X_2 \times \cdots$$

In the case that $K = \{1, ..., n\}$ and $X_k = X$ for all $k \in K$, we also write

$$\prod_{k \in K} X_k = \prod_{k \in K} X = \underbrace{X \times X \times \cdots \times X}_{n \text{ times}} = X^n$$

In the case that $K = \mathbb{Z}^+$, and $X_k = X$ for all $k \in K$, we also write

$$\prod_{k \in K} X_k = \prod_{k=1}^{\infty} = X \times X \times \cdots = X^\omega$$

In the case that X is a vector space, we write

$$X^\infty = \{x = (x_1, x_2 ...) \in X^\omega \mid x_k = 0 \text{ for all but finitely many } k \in \mathbb{Z}^+\}$$

In this case X^∞ and X^ω are both vector spaces.
When X_k is a set for each $k \in K$, for each $\ell \in K$ we have the projection map

$$p_\ell : \prod_{k \in K} X_k \to X_\ell$$

given by $p_\ell(x) = x_\ell = x(\ell)$. For any set Y, a function $f : Y \to \prod_{k \in K} X_k$ determines, and is determined by, its component functions

$$f_\ell : Y \to X_\ell$$

where $f_\ell = p_\ell \circ f$ so $f_\ell(y) = f(y)_\ell = f(y)(\ell)$

Definition 2.10

When X_k is a topological space for each $k \in K$, there are two commonly used topologies on $\prod_{k \in K} X_k$.

1. The *box topology* on $\prod_{k \in K} X_k$ is the topology generated by the basis of sets of the form
$$\prod_{k \in K} U_k \subseteq \prod_{k \in K} X_k$$
Where each U_k is open in X_k
2. The *product topology* on $\prod_{k \in K} X_k$ is the topology generated by the basis of sets consisting of the sets of the form $\prod_{k \in K} U_k$ where each U_k is open in X_k with $U_k = X_k$ for all but finitely many $k \in K$.

Note

The above two proposed bases are indeed bases of sets because
$$\left(\prod_{k \in K} U_k\right) \cap \left(\prod_{k \in K} V_k\right) = \prod_{k \in K} (U_k \cap V_k)$$
Also note that when K is finite, these two topologies are equal. When K is infinite, the box topology is finer than the product topology.

Theorem 2.1

Let $\mathcal{B}_k$ be a basis for X_k for each $k \in K$. Then the set of sets of the form
$$\prod_{k \in K} B_k \text{ where } B_k \in \mathcal{B}_k \text{ for all } k \in K$$
is a basis for the box topology on $\prod_{k \in K} X_k$, and the set of sets of the form
$$\prod_{k \in K} B_k \text{ where } B_k \in \mathcal{B}_k \cup \{X_k\} \text{ for all } k \in K$$
with $B_k = X_k$ for all but finitely many $k \in K$ is a basis for the product topology on $\prod_{k \in K} X_k$.

Proof: Exercise $\square$

Theorem 2.2

For each $k \in K$, let X_k be a subspace of Y_k (using the subspace topology). Then the box topology on $\prod_{k \in K} X_k$ is equal to the subspace topology on $\prod_{k \in K} X_k$ as a subspace of $\prod_{k \in K} Y_k$ using the box topology, and the product topology on $\prod_{k \in K} X_k$ is equal to the subspace topology on $\prod_{k \in K} X_k$ as a subspace of $\prod_{k \in K} Y_k$ using the product topology.

Theorem 2.3

Let Y be a topological space, and let X_k be a topological space for each $k \in K$, and let $f : Y \to \prod_{k \in K} X_k$. Then when $\prod_{k \in K} X_k$ uses the product topology, f is continuous if and only if each component map $f_\ell : Y \to X_\ell$ is continuous.

Proof: Suppose that f is continuous, then (using either the box or product topologies on $\prod_{k \in K} X_k$) each projection map $p_\ell : \prod_{k \in K} X_k \to X_\ell$ is continuous because when $U \subseteq X_\ell$ is open,

$$\begin{aligned} p_\ell^{-1}(U) &= \left\{ x \in \prod_{k \in K} X_k \,\middle|\, x_\ell = p_\ell(x) \in U \right\} \\ &= \prod_{k \in K} U_k \end{aligned}$$

where

$$U_k = \begin{cases} U & \text{if } k = \ell \\ X_k & \text{if } k \neq \ell \end{cases}$$

which is open in $\prod_{k \in K} X_k$ (using either the box or product topology) It follows that each component function f_ℓ is continuous because

$$f_\ell = p_\ell \circ f$$

Suppose, conversely, that each component map

$$f = p_\ell \circ f : Y \to \prod_{k \in K} X_k$$

is continuous, and that $\prod_{k \in K} X_k$ is using the product topology. To show that f is continuous, it suffices to show that $f^{-1}(B)$ is open in Y for every basic open set B in $\prod_{k \in K} X_k$. Let B be a basic open set (for the product topology) on $\prod_{k \in K} X_k$. Say $B = \prod_{k \in K} U_k$ where each U_k is open in X_k with $U_k = X_k$ for all but finitely many indices $k \in K$. Let $L \subseteq K$ be the finite set of all indices $k \in K$ for which $U_k \neq X_k$. We have

$$\begin{aligned} f^{-1}(B) &= \left\{ y \in Y \,\middle|\, f(y) \in \prod_{k \in K} U_k \right\} \\ &= \{ y \in Y \mid f_k(y) = f(y)_k \in U_k \text{ for all } k \in K \} \\ &= \{ y \in Y \mid f_k(y) \in U_k \text{ for all } k \in L \} \\ &= \bigcap_{k \in L} f_k^{-1}(U_k) \end{aligned}$$

Which is open in Y since it is a finite intersection of open sets in Y (with $f_k^{-1}(U_k)$) is open in Y because U_k is open in X_k and $f_k : Y \to X_k$ is continuous. $\square$

Remark

$$\mathbb{R}^\infty \subseteq \ell_1 \subseteq \ell_p \subseteq \ell_q \subseteq \ell_\infty \subseteq \mathbb{R}^\omega$$

for $1 \leq p \leq q \leq \infty$. Recall that these norms induce different topologies.

Question: do any of the p-norms induce the box or product topology on $\mathbb{R}^\infty \subseteq \mathbb{R}^\omega$?
Question: is there a norm or metric on $\mathbb{R}^\omega$ which induces the box or product topology?

Remark

Also, we have the p-norms on $\mathbb{R}^n$. They all give the same topology on $\mathbb{R}^n$. More generally, when X is a finite dimensional vector space, all norms on X induce the same topology on X. When $L : X \to Y$ is a linear map between normed linear spaces, L is continuous iff $\|L\|_{\text{op}} < \infty$ iff $L\left(\overline{B_X}(0,1)\right)$ is bounded in Y. And when X is finite dimensional, $\overline{B_X}(0,1)$ is compact and $L\left(\overline{B_X}(0,1)\right)$ is bounded, so L is continuous. In particular, when X is finite dimensional and $\|\cdot\|_1, \|\cdot\|_2$ are two norms on X,

$$\mathrm{id}_X : (X, \|\cdot\|_1) \longrightarrow (X, \|\cdot\|_2)$$

is continuous, and it is equal to its own inverse which is continuous, so id_X is a homeomorphism, so for a set $U \subseteq X$, U is open in $(X, \|\cdot\|_1)$ if and only if U is open in $(X, \|\cdot\|_2)$. Consequently, every finite dimensional vector space X has a *standard* topology. (Pick a basis $\{u_1, ..., u_n\}$, define

$$\left\langle \sum x_k u_k, \sum y_k u_k \right\rangle = \sum x_k y_k = x \cdot y$$

So the map $L : X \to \mathbb{R}^n$ given by

$$L\left(\sum x_k u_k\right) = \sum x_k e_k = x$$

is an inner product space isomorphism.) Then use the inner product to define a norm, a metric, and a topology. The resulting topology doesn't depend on the choice of basis.

2.2 Quotient Spaces

Definition 2.11

Let X be a set. Let $\sim$ be an equivalence relation on X. For $a \in X$, the *equivalence class* of a is

$$[a] = \{x \in X \mid a \sim x\}$$

Recall distinct equivalence classes are disjoint, and X is the disjoint union of distinct equivalence classes. The set of all equivalence classes is denoted by $X/\sim$, is called the quotient set of X by $\sim$.

$$X/\sim = \{[a] \mid a \in X\}$$

The map $q : X \to X/\sim$ given by $x \mapsto [x]$ is called the quotient map.

Definition 2.12

When X is a topological space, the *quotient topology* on $X/\sim$ is the topology obtained by stipulating that for $V \subseteq X/\sim$, V is open in $X/\sim$ if and only if $q^{-1}(V)$ is open in X.

Note

When $V \subseteq X/\sim$ so V is a set of equivalence classes.

$$\begin{aligned} q^{-1}(V) &= \{x \in X \mid q(x) \in V\} \\ &= \{x \in X \mid [x] \in V\} \\ &= \bigcup_{[x] \in V} [x] \\ &= \bigcup V \end{aligned}$$

Remark

For sets X and Y,

1. When Y is a topological space and $X \subseteq Y$ is a subset, the subspace topology is the coarsest topology on X for which the inclusion map $i : X \to Y$ is continuous
$$i^{-1}(V) = \{x \in X \mid i(x) \in V\} = \{x \in X \mid x \in V\} = V \cap X$$
2. When X and Y are both topological spaces, the product topology on $X \times Y$ is the coarsest topology for which the two projection maps $p_X : X \times Y \to X$, $p_Y : X \times Y \to Y$ are both continuous
$$p_X^{-1}(U) = U \times Y \quad p_Y^{-1}(V) = V \times X$$
3. When X is a topological space and $\sim$ an equivalence relation on X, the quotient topology on $X/\sim$ is the finest topology on $X/\sim$ for which the quotient map $q : X \to X/\sim$ is continuous

Note

Let X be a set and $\sim$ an equivalence relation on X. Note that any function $g : X/\sim \to Y$ (where Y is any set) determines and is determined by a function $f : X \to Y$ which is constant on equivalence classes (meaning that for $x_1, x_2 \in X$ if $x_1 \sim x_2$ then $f(x_1) = f(x_2)$) with g given by $g([x]) = f(x)$ and with f given by $f = g \circ q$. So $f(x) = g(q(x)) = g([x])$

Theorem 2.4

Let X, Y be topological spaces. Let $\sim$ be an equivalence relation on X. Let $f : X/\sim \to Y$. Let $g : X \to Y$ be the map given by $g(x) = f([x])$, that is $g = f \circ q$. Then f is continuous if and only if g is continuous.

Proof: If f is continuous, then g is continuous because $g = f \circ q$ which is the composite of two continuous maps. Suppose that g is continuous. Let $V \subseteq Y$, be open. We need to show that $f^{-1}(V)$ is open in $X/\sim$. By definition of the quotient topology

$$f^{-1}(V) \text{ is open in } X/\sim \iff q^{-1}(f^{-1}(V)) \text{ is open in } X$$

But

$$q^{-1}(f^{-1}(V)) = (f \circ q)^{-1}(V) = g^{-1}(V)$$

Which is open in X since g is continuous. □

Definition 2.13

For a group G and a set X, a *group action* of G on X is a function $* : G \times X \to X$, where we write $*\,(a, x)$ as $a * x$ or ax, such that

1. When $e \in G$ is the identity element we have $e * x = x$ for all $x \in X$.
2. For all $a, b \in G$ and all $x \in X$, we have

$$a * (b * x) = \underbrace{(ab)}_{\text{group op}} * x$$

We say that G *acts on* X (by using the group action).

Remark

A group action of G on X determines and is determined by a group homomorphism $\rho : G \to \mathrm{Perm}(X)$ where $\rho(a)(x) = a * x$ (the homomorphism ρ is called a *representation* of G)

Remark

Given an action of G on X, we can define an equivalence relation on X by

$$x \sim y \iff y = a * x \text{ for some } a \in G.$$

In this case, the equivalence class of x is called the *orbit of* x (we might write $[x]$ as $\mathrm{Orb}(x)$) and we write the quotient $X/\sim$ as X/G. So

$$\begin{aligned} X/G &= \{[x] \mid x \in X\} \\ &= \{\mathrm{Orb}(x) \mid x \in X\} \end{aligned}$$

Example 2.5

For $\mathbb{S}^1 = \{u \in \mathbb{R}^2 \mid \|u\| = 1\}$, we have $\mathbb{S}^1 \times \mathbb{R} \cong \mathbb{R}^2 \setminus \{0\}$. Define

$$\begin{aligned} f : \mathbb{S}^1 \times \mathbb{R} &\longrightarrow \mathbb{R}^2 \setminus \{0\} \\ (u, t) &\longmapsto e^t u \end{aligned}$$

and define

$$\begin{aligned} g : \mathbb{R}^2 \setminus \{0\} &\longrightarrow \mathbb{S}^1 \times \mathbb{R} \\ x &\longmapsto \left(\frac{x}{\|x\|}, \ln\|x\| \right) \end{aligned}$$

These maps are continuous (they are elementary functions) and they are inverses of each other.

Example 2.6

$\mathbb{S}^1$ acts on $\mathbb{R}^2 = \mathbb{C}$ by complex multiplication. For $a \in \mathbb{R}^2 = \mathbb{C}$,

$$\operatorname{Orb}(a) = [a] = \{ua \mid u \in \mathbb{S}^1\}$$

which is equal to the circle centered at 0 of radius $\|a\|$ (with $[0] = \{0\}$).
Show that $\mathbb{R}^2/\mathbb{S}^1 \cong [0, \infty) \subseteq \mathbb{R}$ we define

$$\begin{aligned} f : \mathbb{R}^2/\mathbb{S}^1 &\longrightarrow [0, \infty) \\ [x] &\longmapsto \|x\| \end{aligned}$$

and define

$$\begin{aligned} h : [0, \infty) &\longrightarrow \mathbb{R}^2/\mathbb{S}^1 \\ r &\longmapsto [r] = [(r, 0)] = \{re^{i\theta} \mid \theta \in \mathbb{R}\} \end{aligned}$$

Note that f is continuous because for the map $g : \mathbb{R}^2 \to [0, \infty) \subseteq \mathbb{R}$ given by $g(x) = \|x\|$. We have $g = f \circ q$. Since g is continuous, it follows that f is continuous. Also h is continuous because $h = q \circ i$ where $i : [0, \infty) \longrightarrow \mathbb{R}^2$ is the inclusion map $i(r) = (r, 0)$. Finally, note that f and h are inverses.

Example 2.7

$\mathbb{R}^+ = (0,\infty)$ acts on $\mathbb{R}^2$ be multiplication that is by $t * x = tx$. The orbits are for $o \neq x \in \mathbb{R}^2$, $[x] = \{tx \mid 0 < t \in \mathbb{R}\}$ which is the (open) ray from 0 through x and $[0] = \{0\}$. Each of the rays $[x]$ for $0 \neq x \in \mathbb{R}^2$ intersects a unique point on $\mathbb{S}^1$. Which gives a fairly natural bijective map

$$f : \mathbb{R}^2/\mathbb{R}^+ \longrightarrow \mathbb{S}^1 \cup \{0\}$$
$$[x] \longmapsto \begin{cases} \frac{x}{\|x\|} & \text{if } 0 \neq x \in \mathbb{R}^2 \\ 0 & \text{if } x = 0 \in \mathbb{R}^2 \end{cases}$$

The inverse $g : \mathbb{S}^1 \cup \{0\} \to \mathbb{R}^2/\mathbb{R}^+$ is given by $u \mapsto [u]$. Note that g is continuous ($g = q \circ i$ where i is the inclusion map $i : \mathbb{S}^1 \cup \{0\} \to \mathbb{R}^2$). But f is not continuous, for example the set $\{0\}$ is open in $\mathbb{S}^1 \cup \{0\}$ (it is an open ball) but $f^{-1}(\{0\}) = \{[0]\} \subseteq \mathbb{R}^2/\mathbb{R}^+$ and $q^{-1}(\{[0]\}) = \{0\}$ is not open in $\mathbb{R}^2$. In fact, $\mathbb{R}^2/\mathbb{R}^+ \ncong \mathbb{S}^1 \cup \{0\}$. One way to show this is to note that $\mathbb{S}^1 \cup \{0\}$ has a singleton which is open ($\{0\}$), but $\mathbb{R}^2/\mathbb{R}^+$ has no singleton which is open.

Remark

$\mathbb{R}^2/\mathbb{R}^+$ is not Hausdorff, so it is not metrizable (there is no metric we can define on $\mathbb{R}^2/\mathbb{R}^+$ for which that quotient topology is equal to the metric topology)

Example 2.8

$\mathbb{Z}$ acts by addition on $\mathbb{R}$ (by $n * x = x + n$). The orbits are the sets $[x] = \{x + n \mid n \in \mathbb{Z}\} = x + \mathbb{Z}$ Show that $\mathbb{R}/\mathbb{Z} \cong \mathbb{S}^1$. Define

$$\begin{aligned} f : \mathbb{R}/\mathbb{Z} &\longrightarrow \mathbb{S}^1 \\ [t] &\longmapsto e^{i2\pi t} \end{aligned}$$

(and note that when $[s] = [t]$ say $s = t + n$ where $n \in \mathbb{Z}$ we have

$$e^{i2\pi s} = e^{i2\pi(t+n)} = e^{i2\pi t}$$

) Note that f is continuous because the map $f : \mathbb{R} \to \mathbb{S}^1$ given by $g(t) = e^{i2\pi t}$ is continuous with $g = f \circ q$. The inverse map

$$\begin{aligned} h : \mathbb{S}^1 &\longrightarrow \mathbb{R}/\mathbb{Z} \\ e^{i\theta} &\longmapsto \left[\frac{\theta}{2\pi}\right] \end{aligned}$$

To see that h is continuous, we can express h in Cartesian coordinates. We remark that there is an angle map

$$\begin{aligned} \theta : \mathbb{R}^2 \setminus \{0\} &\longrightarrow [0, 2\pi) \\ (x, y) &\longmapsto \begin{cases} \arccos \frac{x}{\sqrt{x^2+y^2}} & \text{if } y > 0 \text{ or } (y = 0 \text{ and } x \neq 0) \\ 2\pi - \arccos \frac{x}{\sqrt{x^2+y^2}} & \text{if } y < 0 \text{ or } (y = 0 \text{ and } x < 0) \end{cases} \end{aligned}$$

This map is not continuous along the positive x-axis. In Cartesian coordinates, $h : \mathbb{S}^1 \to \mathbb{R}/\mathbb{Z}$ is given by

$$h(x, y) = \begin{cases} \left[\frac{1}{2\pi} \arccos(x)\right] & \text{if } y \geq 0 \\ \left[1 - \frac{1}{2\pi} \arccos(x)\right] & \text{if } y \leq 0 \end{cases}$$

that is by

$$h(x, y) = \begin{cases} h_1(x, y) \text{ if } (x, y) \in A \\ h_2(x, y) \text{ if } (x, y) \in B \end{cases}$$

Where

$$A = \left\{(x, y) \in \mathbb{S}^1 \;\middle|\; y \geq 0\right\}$$
$$B = \left\{(x, y) \in \mathbb{S}^1 \;\middle|\; y \leq 0\right\}$$

and

$$h_1(x, y) = \frac{1}{2\pi} \arccos x$$
$$h_2(x, y) = 1 - \frac{1}{2\pi} \arccos x$$

3 Connected, Path-Connected and Compact Spaces

Definition 3.1

Let X be a topological space. For subsets $A, B \subseteq X$, we say that A and B *separate* X when $A \neq \emptyset$, $B \neq \emptyset$, $A \cap B = \emptyset$ and $A \cup B = X$. We say that X is *disconnected* when there exist (nonempty disjoint) open sets $U, V \subseteq X$ which separate X. Otherwise, we say that X is *connected*.

Proposition 3.1

X is connected if and only if the only clopen sets are X and $\emptyset$.

Proof: If X is disconnected, we can find open sets $U, V \subseteq X$ which separate X then the sets $\emptyset, U, V, X$ are clopen. On the other hand, if $\emptyset \neq U \subsetneq X$ with both U both open and closed in X, then U and $V = X \setminus U$ are open sets in X which separate X. □

Exercise 3.0.1

When X is a metric space and $A \subseteq X$ is a subspace, then A is connected if and only if there do not exist open sets U, V in X such that $U \cap A \neq \emptyset$, $V \cap A \neq \emptyset$, $U \cap V = \emptyset$ and $A \subseteq U \cup V$.

Example 3.1

The connected sets in $\mathbb{R}$ are the intervals (including $\emptyset, \{a\}, \mathbb{R}$)

Example 3.2

The (non-empty) connected subsets of $\mathbb{Q}$ are the singletons (by using the density of the irrationals)

Theorem 3.2

If $f : X \to Y$ is a continuous map between topological spaces, and if X is connected, then $f(X)$ is connected.

Proof: Suppose X is connected and $f : X \to Y$ is continuous. By restricting the codomain, the map $f : X \to f(X)$ is also continuous. Suppose, for a contradiction that $f(X)$ is disconnected. Let U, V be open sets in $f(X)$ which separate $f(X)$. Then $f^{-1}(U)$ and $f^{-1}(V)$ are open sets in X which separate X, so that X is disconnected, giving the desired contradiction. □

Lemma 3.3

Let X be a subspace of Y. Suppose Y is disconnected. Let U, V be open sets in Y that separate Y. If X is connected, then $X \subseteq U$ or $X \subseteq V$.

Proof: Suppose $X \nsubseteq U$ and $X \nsubseteq V$. Since $U \cup V = Y$, it follows that $X \cap U \neq \emptyset$ and $X \cap V \neq \emptyset$. And these two sets are open sets in X which separate X. □

Theorem 3.4

Let $X = \bigcup_{k \in K} A_k$ where each subspace A_k is connected. With $\bigcap_k A_k \neq \emptyset$. Then X is connected.

Proof: Suppose, for a contradiction, that X is disconnected. Let U, V be open sets in X which separate X. Let $p \in \bigcap_{k \in K} A_k \subseteq X = U \cup V$. Either $p \in U$ or $p \in V$ (but not both) say $p \in U$. For each index k, since A_k is connected either $A_k \subseteq U$ or $A_k \subseteq V$ and since $p \in A_k$, $p \notin V$, we must have $A_k \subseteq U$. Since $A_k \subseteq U$ for every $k \in K$, we have $X = \bigcup_{k \in K} A_k \subseteq U$. This is not possible since U and V separate X. □

Theorem 3.5

The product of two connected spaces is connected.

Proof: Let X and Y be connected spaces. Suppose both X and Y are nonempty (since if either one was, $\emptyset$ is connected). Choose $a \in X$ and $b \in Y$ so $(a, b) \in X \times Y$. Since $X \times \{b\} \cong X$ and X is connected, it follows that $X \times \{b\}$ is connected. For each $x \in X$, since $\{x\} \times Y \cong Y$ and Y is connected, it follows that $\{x\} \times Y$ is connected. Since $X \times \{b\}$ and $\{x\} \times Y$ are connected and $(X \times \{b\}) \cap (\{x\} \times Y) \neq \emptyset$ (since (x, b) is in both), it follows from the previous theorem that the set $A_x = (X \times \{b\}) \cup (\{x\} \times Y)$ is connected. Since each A_x is connected and $\bigcap_{x \in X} A_x \neq \emptyset$ (indeed (a, b) is in the intersection) it follows that $\bigcup_{x \in X} A_x = X \times Y$ is connected. □

Lemma 3.6

Let X be a subspace of Y. Let U, V be subsets of X which separate X (not necessarily open). Then U is open in X if and only if $U \cap \overline{V} = \emptyset$. Symmetrically, V is open in X if and only if $V \cap \overline{U} = \emptyset$ where $\overline{U} = \mathrm{Cl}_Y(U), \overline{V} = \mathrm{Cl}_Y(V)$

Theorem 3.7

Let X be a topological space, let A, B be subspaces with $A \subseteq B \subseteq \overline{A}$. If A is connected, then so is B. In particular, if A is connected, then so is $\overline{A}$.

Proof: Suppose A is connected. Suppose for a contradiction that B is not connected. Let $U, V \subseteq B$ be open sets in B which separate B. Since A is connected and U, V are open sets in B, which separate B, by previous lemma, either $A \subseteq U$ or $A \subseteq V$. Say $A \subseteq U$. Since $A \subseteq U$ we have $\overline{A} \subseteq \overline{U}$ so that $B \subseteq \overline{A} \subseteq \overline{U}$. By the previous lemma, $V \cap \overline{U} = \emptyset$ hence $V \cap B = \emptyset$, but $V \subseteq B$ so $V = \emptyset$ which contradicts the fact that U and V separate B. □

Theorem 3.8

Let X_k be a connected topological space for each $k \in K$. Then $\prod X_k$ is connected using the product topology.

Definition 3.2

When X is a topological space, and $A \subseteq X$, we say that A is *dense* in X when $\overline{A} = X$. Note that

$$\begin{aligned}\overline{A} = X &\iff \text{the only closed set } K \subseteq X \text{ with } A \subseteq K \text{ is } K = X \\ &\iff \text{the only open set } U \subseteq X \text{ with } A \cap U = \emptyset \text{ is } U = \emptyset \\ &\iff \text{for every nonempty open set } U \subseteq X \text{ we have } A \cap U \neq \emptyset\end{aligned}$$

When $\mathcal{B}$ is a basis for the topology on X, verify that $\overline{A} = X$ if and only if for all $\emptyset \neq B \in \mathcal{B}$ we have $A \cap B \neq \emptyset$.

Example 3.3

$\mathbb{R}^\omega = \prod_{k=1}^\infty \mathbb{R}$ using the box topology is <u>not</u> connected. Indeed verify that the sets

$$\begin{aligned}U &= \{x \in \mathbb{R}^\omega \mid \|x\|_\infty < \infty\} \\ &= \text{the set of all bounded sequences in } \mathbb{R}\end{aligned}$$

and

$$\begin{aligned}V &= \{x \in \mathbb{R}^\omega \mid \|x\|_\infty = \infty\} \\ &= \text{the set of all unbounded sequences in } \mathbb{R}\end{aligned}$$

are open in $\mathbb{R}^\omega$ (with the box topology) and they cover $\mathbb{R}^\omega$.

3.1 Connected Components

Definition 3.3

Let X be a topological space. Define a binary relation $\sim$ on X by stipulating that for $a, b \in X$

$$a \sim b \iff \text{there exists a connected subspace } A \subseteq X \text{ with } a, b \in A$$

Note that $\sim$ is an equivalence relation. Indeed $a \sim a$ since $\{a\}$ is connected. If $a \sim b$ then obviously $b \sim a$. If $a \sim b$ and $b \sim c$ then we can choose connected subspaces $A, B \subseteq X$ with $a, b \in A$, $b, c \in B$, then by a previous lemma, since $b \in A \cap B$, we have $A \cup B$ is connected, and $a, c \in A \cup B$, so that $a \sim c$. The equivalence classes in X under $\sim$ are called the *connected components* of X. (Note that the connected components are disjoint and they cover X).

Theorem 3.9

Let X be a topological space. The connected components of X are the maximal connected subspaces of X. Indeed, each connected component of X is connected, and every non-empty connected subspace of X is contained inside exactly one of the connected components.

Proof: □

3.2 Path-Connectedness

Definition 3.4

Let X be a topological space. For $a, b \in X$, a (continuous) *path* from a to b in X is a continuous map $\alpha : [0, 1] \subseteq \mathbb{R} \to X$ with $\alpha(0) = a$ and $\alpha(1) = b$. We say that X is *path connected* when for every $a, b \in X$ there exists a path from a to b in X.

Theorem 3.10

Every path-connected space is connected.

Proof: Suppose X is path-connected. Suppose, for a contradiction, that X is not connected. Choose open sets $U, V \subseteq X$ which separate X. Choose $a \in U$ and $b \in V$. Since X is path-connected we can choose a path $\alpha : [0, 1] \subseteq \mathbb{R} \to X$ with $\alpha(0) = a$ $\alpha(1) = b$. Then the sets $\alpha^{-1}(U)$ and $\alpha^{-1}(V)$ are open and separate $[0, 1]$, contradiction. □

Theorem 3.11

The image of a path connected space under a continuous map is path connected. In particular, for topological spaces X and Y. If $X \cong Y$, then X is path connected if and only if Y is path connected.

Proof: Let $f : X \to Y$ be continuous and suppose X is path connected. Let $c, d \in f(X)$. Choose $a, b \in X$ with $f(a) = c$, $f(b) = d$. Since X is path connected, we can choose a path α in X from a to b. Then $\beta = f \circ \alpha$ is path in Y from c to d. □

Note

Convex sets are path connected (in normed linear spaces). More generally, the image of a convex set (in a normed linear spaces) under a continuous map is path connected, hence connected.

Example 3.4

$A = \{x \in \mathbb{R}^2 \mid 1 \leq \|x\| \leq 2\}$ is the image of $[1, 2] \times [0, 2\pi]$ under the polar coordinates map $p : \mathbb{R}^2 \to \mathbb{R}^2$ given by $p(r, \theta) = (r \cos \theta, r \sin \theta)$ and thus path connected. (Using the fact that rectangles (also balls) are convex and hence connected).

Proposition 3.12

Using the product topology, a product of path-connected spaces is path connected.

Proof: Let X_k be path connected for each $k \in K$. Let $a, b \in \prod X_k$. For each $k \in K$, choose a path α_k in X_k from a_k to b_k. Then the map $\alpha : [0, 1] \to \prod X_k$ given by

$$\alpha(t)(k) = \alpha(t)_k = \alpha_k(t)$$

is a (continuous) path in $\prod X_k$ from a to b. □

Remark

Using the box topology, this isn't true.

Definition 3.5

Let X be a topological space. Define a binary relation $\sim$ on X by stipulating that for $a, b \in X$

$$a \sim b \iff \text{there exists a path in } X \text{ from } a \text{ to } b$$

Note that this is an equivalence relation on X, indeed for $a, b, c \in X$:

1. $a \sim a$ since the constant path κ_a is a path from a to a in X.
2. If $a \sim b$ then there is a path α from a to b. Then $\beta(t) = \alpha(1-t)$
3. If $a \sim b$ and $b \sim c$ with paths α, β then $\gamma : [0,1] \to X$ given by

$$\gamma(t) = \begin{cases} \alpha(2t) & \text{for } 0 \leq t \leq \frac{1}{2} \\ \beta(2t-1) & \text{for } \frac{1}{2} \leq t \leq 1 \end{cases}$$

   is a (continuous) path in X from a to c (by the glueing lemma).

The equivalence classes in X under $\sim$ are called the *path components of* X

Theorem 3.13

Let X be a topological space. The path components of X are the maximal path connected subspaces of X. Indeed, each path component of X is path connected, and every path connected subspace of X is contained in exactly one of the path components of X.

Proof: path components are path connected by the definition of $\sim$. Let A be any path connected subspace of X. Let P, Q be any path components for which $A \cap P \neq \emptyset$ and $A \cap Q \neq \emptyset$. Choose $p \in A \cap P$ and $q \in A \cap Q$. Since $p, q \in A$ and A is path connected, we have $p \sim q$ and hence $P = [p] = [q] = Q$ since the path components cover X and A intersects with a unique path component P, we have $A \subseteq P$. $\square$

Note

In a topological space X, since each connected subspace of X is contained in a unique connected component of X, and since each path component of X is path connected, hence connected, it follows that each connected component of X is a (disjoint) union of some of the path components of X.

Example 3.5

Let $A = \left\{\left(x, \sin\frac{1}{x}\right) \mid o < x \leq \frac{1}{\pi}\right\}$. Let $B = \{(0, y) \mid -1 \leq y \leq 1\}$. Let $X = A \cup B$. We see that $\overline{A} = A \cup B = X$. Note that A is path connected because it is the image of the convex set $\left(0, \frac{1}{\pi}\right]$ under the continuous map $g : \left(0, \frac{1}{p}\right) \to \mathbb{R}^2$ given by $g(x) = \left(s, \sin\frac{1}{x}\right)$. Also B is convex hence path connected. Note that X is connected since it is the closure of a connected set A. We claim that X is not path connected, indeed there is no path in X from a point in A to a point in B. Since A and B are path connected with $\left(\frac{1}{\pi}, 0\right) \in A$ and $(0, 0) \in B$, it suffices to show that there is no path in $X = A \cup B$ from $\left(\frac{1}{\pi}, 0\right)$ to $(0, 0)$. Suppose for a contradiction that there is such a path $\alpha : [0, 1] \to A \cup B$ from $\left(\frac{1}{\pi}, 0\right)$ to $(0, 0)$ in $X = A \cup B$. Note that the map $\alpha : [0, 1] \to \mathbb{R}^2$ is continuous, say α is given by $\alpha(t) = (x(t), y(t))$ where $x, y : [0, 1] \to \mathbb{R}$ are both continuous with $(x(t), y(t)) \in X = A \cup B$ for all $t \in [0, 1]$ and with $x(0) = \frac{1}{\pi}, x(1) = 0, y(0) = y(1) = 0$. Also recall that when $(x, y) \in X = A \cup B$ with $x > 0$ we have $(x, y) \in A$ so that $y = \sin\frac{1}{x}$. Since $x : [0, 1] \to \mathbb{R}$ is continuous with $x(0) = \frac{1}{\pi}$ and $x(1) = 0$. By IVT, we can choose $0 < t_1 < t_2 < \dots < 1$ so that $x(t_n) = \frac{2}{(2n+1)\pi}$ and hence $y(t_n) = \sin\frac{1}{x(t_n)} = \sin\frac{(2n+1)\pi}{2} = (-1)^n$. Since $(t_n)_{n\geq 1}$ is increasing and bounded above (by 1) it converges with $\lim_{n\to\infty} t_n = s = \sup\{t_n \mid n \in \mathbb{N}\} \leq 1$ and we have $0 < t_n < s \leq 1$ for all $n \in \mathbb{N}$. Since $t_n \to s$ and since α is continuous at s, we have

$$(x(s), y(s)) = \alpha(s) = \lim_{n\to\infty} \alpha(t_n) = \left(\lim_{n\to\infty} x(t_n), \lim_{n\to\infty} y(t_n)\right)$$

so we have

$$\lim_{n\to\infty} x(t_n) = \lim_{n\to\infty} \frac{2}{(2n+1)\pi} = 0$$

but

$$\lim_{n\to\infty} y(t_n) = \lim_{n\to\infty} (-1)^n$$

which does not exist. In conclusion, $X = A \cup B$ is connected, but not path connected. Since X is connected, it only has one connected component, namely X. Since X is not path connected, it has at least 2 path components so, since A and B are path connected with $X = A \cup B$, A and B are the two path components of X.

3.3 Compactness

Definition 3.6

Let X be a topological space. For a set $\mathcal{S}$ of subsets of X, we say that $\mathcal{S}$ *covers* X or that $\mathcal{S}$ is *a cover* of X when $X = \bigcup \mathcal{S}$. When $\mathcal{S}$ is a cover of X, a *subcover* is a subset $\mathcal{R} \subseteq \mathcal{S}$ such that $X = \bigcup \mathcal{R}$. An *open cover* of X is a set of open sets which covers X. We say that X is *compact* when every open cover of X has a finite subcover.

Theorem 3.14

The image of a compact space under a continuous map is compact.

Proof: Let $f : X \to Y$ be a map between topological spaces. Suppose that X is compact and f is continuous. Note the map $f : X \to f(X)$ (by restricting codomain) is continuous. We claim that $f(X)$ is compact. Let $\mathcal{T}$ be an open cover of $f(X)$. Let $\mathcal{S} = \{f^{-1}(V) \mid V \in \mathcal{T}\}$. Then $\mathcal{S}$ is an open cover of X. Since X is compact, $\mathcal{S}$ has a finite subcover, $V_1, ..., V_n \in \mathcal{T}$ so that $X = \bigcup_{k=1}^{n} f^{-1}(V_k)$. Then $f(X) = \bigcup_{k=1}^{n} V_k$ so that $\{V_1, ..., V_n\}$ is a finite subcover of $\mathcal{T}$. Thus $f(X)$ is compact, as claimed. □

Theorem 3.15 **Heine-Borel**

For $A \subseteq \mathbb{R}^n$, A is compact iff A is closed and bounded.

Definition 3.7

Let X be a subspace of Y. For a set $\mathcal{T}$ of subsets of Y, we say $\mathcal{T}$ covers X in Y or $\mathcal{T}$ is a cover of X in Y, when $X \subseteq \bigcup \mathcal{T}$. When $\mathcal{T}$ is a cover of X in Y, a subcover of $\mathcal{T}$ (of X in Y) is a subset $\mathcal{R} \subseteq \mathcal{T}$ such that $X \subseteq \bigcup \mathcal{R}$. An open cover of X in Y is a set $\mathcal{T}$ of open sets in Y with $X \subseteq \bigcup \mathcal{T}$. We say that X is compact in Y when every open cover of X in Y has a finite subcover (of X in Y).

Theorem 3.16

Let X be a subspace of Y. Then X is compact (in itself) iff X is compact in Y.

Proof: Suppose X is compact (in X) let $\mathcal{T}$ be an open cover of X in Y. Let $\mathcal{S} = \{V \cap X \mid V \in \mathcal{T}\}$. Note that $\mathcal{S}$ is an open cover of X. Since X is compact in itself, we can choose $V_1, ..., V_n \in \mathcal{T}$ such that $X = \bigcup_{k=1}^{n} (V_k \cap X) = \bigcup_{k=1}^{n} V_k \cap X$. Then $X \subseteq \bigcup_{k=1}^{n} V_k$ so that $\{V_1, ..., V_n\}$ is a finite subcover of $\mathcal{T}$ (for X in Y). Suppose, conversely, that X is compact in Y. Let $\mathcal{S}$ be an open cover of X (in X). For each $U \in \mathcal{S}$ we can choose V_U open in Y such that $U = V_U \cap X$. Then $\mathcal{T} = \{V_U \mid U \in \mathcal{S}\}$ is an open cover of X in Y. Since X is compact in Y, we can choose $U_1, ..., U_n \in \mathcal{S}$ such that $X \subseteq \bigcup_{k=1}^{n} V_{U_k}$. Then

$$X = \bigcup_{k=1}^{n} V_{U_k} \cap X = \bigcup_{k=1}^{n} \left(V_{U_k} \cap X\right) = \bigcup_{k=1}^{n} U_k$$

so that $\{U_1, ..., U_n\}$ is a finite subcover of $\mathcal{S}$ (of X in X). □

Remark

When X is a subspace of a metric space Y (but not in general when X is a subspace of a topological space Y), we have an analogous result for the connectedness of X in Y: X is connected in Y when there do not exist open sets U, V in Y which separate X in Y, meaning that $U \cap X \neq \emptyset, V \cap X \neq \emptyset, U \cap V = \emptyset, X \subseteq U \cup V$. Verify that (when Y is a metric space) X is connected (in itself) iff X is connected in Y.

Theorem 3.17

Every closed subspace of a compact topological space is compact.

Proof: Let X be a subspace of Y. Suppose Y is compact (in Y) and that X is closed in Y. Let $\mathcal{S}$ be an open cover of X in Y. Since X is closed in Y, $X^c = Y \setminus X$ is open in Y. Note that $\mathcal{S} \cup \{X^c\}$ is an open cover of Y. Since Y is compact, we can choose a finite subcover of $\mathcal{S} \cup \{X^c\}$ so we can choose a finite subset $\mathcal{R} \subseteq \mathcal{S}$ such that $\mathcal{R} \cup \{X^c\}$ covers Y. Then $\mathcal{R}$ is a finite subcover of $\mathcal{S}$ (of X in Y). □

Theorem 3.18

Every compact subspace of a Hausdorff space is closed.

Proof: Let $X \subseteq Y$ be a subspace. Suppose that X is compact and Y is Hausdorff. We shall show $X^c = Y \setminus X$ is open in Y. Let $b \in X^c$. For each $a \in X$, since Y is Hausdorff we can choose disjoint open sets U_a and V_a in Y with $a \in U_a$ and $b \in V_a$. Note that $\mathcal{S} = \{U_a \mid a \in X\}$ is an open cover of X in Y. Since X is compact, we can choose $a_1, \ldots, a_n \in X$ such that $X \subseteq \bigcup_{k=1}^n U_{a_k}$. Let $U = \bigcup_{k=1}^n U_{a_k}$ and $V = \bigcap_{k=1}^n V_{a_k}$. Note that $X \subseteq U$, $b \in V$ and $U \cap V = \emptyset$. Since $X \subseteq U$ and $U \cap V = \emptyset$, we also have $X \cap V = \emptyset$ so that $V \subseteq X^c$. Hence X^c is open in Y, and X is closed. □

Theorem 3.19

Let $f : X \to Y$ be a map between topological spaces. Suppose f is continuous and bijective. Suppose is X is compact and Y is Hausdorff. Then $f^{-1} : Y \to X$ is continuous so that f is a homeomorphism.

Proof: Let $g = f^{-1} : Y \to X$. To show that g is continuous, we show that $g^{-1}(K)$ is closed in Y for every closed set K in X. Let K be a closed set in X. Note that since $g = f^{-1}$ we have $g^{-1}(K) = f(K)$. Since K is closed in X and X is compact, K is compact. Since f is continuous, $f(K)$ is compact. Since $f(K)$ is a closed subspace of the Hausdorff space Y, $f(K)$ is closed in Y. □

Example 3.6

1. Recall that $\mathbb{R} \cong (0,1), \mathbb{S}^1 \times \mathbb{R} \cong \mathbb{R}^2 \setminus \{0\}, \mathbb{R}/\mathbb{Z} \cong \mathbb{S}^1$,
2. $\mathbb{R} \ncong [0,1]$ since $[0,1]$ is compact but $\mathbb{R}$ is not.
3. $\mathbb{R} \ncong [0,1)$ since $[0,1) \setminus \{0\}$ is connected but one cannot remove any point from $\mathbb{R}$ and remain connected.
4. No two of $\mathbb{R}, \mathbb{R}^2, \mathbb{S}^1, \mathbb{S}^2$ are homeomorphic. Since $\mathbb{R}, \mathbb{R}^2$ are not compact, but $\mathbb{S}^1, \mathbb{S}^2$ are. $\mathbb{R}$ and $\mathbb{R}^2$ are not homeomorphic since one can remove a point from $\mathbb{R}^1$ and disconnect it but you cannot do that with $\mathbb{R}^2$. $\mathbb{S}^1 \ncong \mathbb{S}^2$ since $\mathbb{S}^1 \setminus \{p\} \cong \mathbb{R}$ and $\mathbb{S}^2 \setminus \{p\} \cong \mathbb{R}^2$.

Theorem 3.20

Let X be a topological space. Then X is compact if and only if X has the following property which we call the *finite intersection property on closed sets*: For every set $\mathcal{T}$ of closed subsets of X, if every finite subset of $\mathcal{T}$ has nonempty intersection, then $\bigcap \mathcal{T}$ is non empty.

Definition 3.8

A *partially ordered set* is a set X with a partial order $\leq$ such that $\forall x, y, z \in X$

1. $x \leq x$
2. If $x \leq y$ and $y \leq x$ then $x = y$
3. If $x \leq y$ and $y \leq z$ then $x \leq z$

Definition 3.9

A *chain* in X is a subset $C \subseteq X$ such that for all $x, y \in C$ we have $x \leq y$ or $y \leq x$

Lemma 3.21 **Zorn's Lemma**

Let X be a partially ordered set. If every chain in X has an upper bound in X, then X has a maximal element ($\exists a \in X \nexists x \in X \ a < x$)

Theorem 3.22 **Tychanoff's Theorem**

The product of a set of compact spaces is compact, using the product topology.

4 Countability and Separation Axioms

Definition 4.1

Let X be a topological space.

1. We say that X is *first countable* when for each $a \in X$, there exists a countable set $\mathcal{B}$ of open sets in X such that for every open $U \ni a$ there exists $B \in \mathcal{B}$ such that $a \in B \subseteq U$.
2. We say that X is *second countable* when there exists a countable basis $\mathcal{B}$ for the topology on X. We say that *Lindelöff* when every open cover of X has a countable subcover.
3. We say that X is *separable* when there exists a countable dense subset of X ($A \subseteq X$ such that $\overline{A} = X$).

Theorem 4.1

1. Every metric space is first-countable.
2. For every metric space X, X is second countable if and only if X is Lindelöff if and only if X is separable.

Theorem 4.2

Let X be a topological space. If X is second countable then

1. X is first countable
2. X is Lindelöff
3. X is separable

Proof: Suppose X is second countable and $\mathcal{B}$ a countable basis for the topology on X. It is clear that X is first countable (take $\mathcal{B}_a = \mathcal{B}$). To show X is Lindelöff, let $\mathcal{S}$ be an open cover of X. For each $a \in X$, choose $U_a \in \mathcal{S}$ with $a \in U_a$, then choose $B_a \in \mathcal{B}$ with $a \in B_a \subseteq U_a$. Then $\{B_a \mid a \in X\}$ is a countable open cover since $\{B_a \mid a \in X\} \subseteq \mathcal{B}$. Choose $a_1, \ldots$ in X such that $\{B_a \mid a \in X\} = \left\{B_{a_1}, B_{a_2}, \ldots\right\}$. Then $\left\{U_{a_1}, \ldots\right\}$ is a countable subcover of $\mathcal{S}$. To show X is separable, write $\mathcal{B} = \{B_1, B_2, \ldots\}$. For each $k \geq 1$, choose $a_k \in B_k$. Then $\{a_1, a_2, \ldots\}$ is dense in X since $a_k \in A \cap B_k$ for each $k \geq 1$ so that $A \cap B_k \neq \emptyset$. □

Example 4.1

Here are some examples to show that when X is not second countable, the other three properties do not imply one another

	first countable	Lindelöff	separable	second countable
$\mathbb{R}_\ell$	✓	✓	✓	×
I_o^2	✓	✓ (compact)	×	×
Γ	✓	×	✓	×
$\mathbb{R}_{\text{cf}}$	×	✓ (compact)	✓	×
$\mathbb{R}_{\text{cc}}$	×	✓	×	×
$\mathbb{R}_d$	✓	×	×	×

Note $\mathbb{R}_\ell$ is $\mathbb{R}$ with the lower limit topology. I_o^2 denotes the topological space with underlying set $I^2 = [0,1]^2$ and using the dictionary order topology. Γ is the Moore plane. $\mathbb{R}_{\text{cf}}$ is R with the co-finite topology. $\mathbb{R}_{\text{cc}}$ is $\mathbb{R}$ with the co-countable topology.

Theorem 4.3

1. Every subspace of a first countable space is first countable.
2. Every subspace of a second countable space is second countable.

Theorem 4.4

1. The product of any two first countable spaces is first countable.
2. The product of any two second countable spaces is second countable.
3. The product of any separable countable spaces is separable.

Note

A subspace of a Lindelöff space need not be Lindelöff. A subspace of a separable space need not be separable. The product of two Lindelöff spaces need not be Lindelöff.

Definition 4.2

Let X be a topological space

1. We say that X is T1 or that the 1-point subsets are closed in X when the 1-point subsets are closed in X.
2. We say that X is T2, or that X is Hausdorff, when for all $a, b \in X$ with $a \neq b$, there exists disjoint open sets $U, V \subseteq X$ with $a \in U$ and $b \in V$.
3. We say that X is T3, or that X is *regular*, when 1-point subsets of X are closed and for all $a \in X$ and for every closed set B in X with $a \notin B$, there exist disjoint open sets $U, V \subseteq X$ with $a \in U$ and $B \subseteq V$.
4. We say that X is T4 or *normal* when 1-point subsets of X are closed and for any two disjoint closed subsets $A, B \subseteq X$, there exist disjoint open sets $U, V \subseteq X$ such that $A \subseteq U$ and $B \subseteq V$.
5. We say that X is *metrizable* when there is a metric on X for which the metric topology is equal to the topology on X.

Theorem 4.5

Let X be a topological space.

1. If X is metrizable, then X is normal
2. If X is normal, then X is regular
3. If X is regular, then X is Hausdorff
4. If X is Hausdorff, then the 1-point subsets of X are closed.

Example 4.2

- Not every space X in which the 1-point subsets are closed is Hausdorff: for example, let X be an infinite set with the cofinite topology.
- Not every Hausdorff space is regular: for example, let $K = \left\{ \frac{1}{n} \mid n \in \mathbb{Z}^+ \right\}$ and let $\mathbb{R}_K$ be the topological space whose underlying set is $\mathbb{R}$ with basis consisting of sets of the form (a, b) with $a < b$ and $(a, b) \setminus K$ with $a < b$. This space is Hausdorff but not regular because the point 0 cannot be separated from the closed set K.
- Not every regular space is normal for example, verify that (1) $\mathbb{R}_\ell$ is normal (hence regular), (2) the product of two regular spaces is regular so that, in particular, $\mathbb{R}_\ell \times \mathbb{R}_\ell$ is regular, but (3) $\mathbb{R}_\ell \times \mathbb{R}_\ell$ is not normal.

Note

All of the countability and separation properties are invariant under homeomorphism.

Proposition 4.6

$\mathbb{R}_\ell$ is not second countable.

Proof: Let $\mathcal{B}$ be any basis for $\mathbb{R}_\ell$. For each $a \in \mathbb{R}_\ell$, choose a basic open set $B_a \in \mathcal{B}$ with $a \in B_a \subseteq [a, a+1)$ and note that $a = \min B_a$. Define $F : \mathbb{R}_\ell \to \mathcal{B}$ by $F(a)$ by $F(a) = B_a$. Then F is injective since if $F(a) = F(b)$ then $B_a = B_b$. So $a = \min B_a = \min B_b = b$. Since F is injective,

$$2^{\aleph_0} = |\mathbb{R}_\ell| \leq |\mathcal{B}|$$

so that $\mathcal{B}$ is uncountable. □

Theorem 4.7

1. Every subspace of a T1 space is T1
2. Every subspace of a Hausdorff space is Hausdorff
3. Every subspace of a regular space is regular
4. Every subspace of a metrizable space is metrizable

Proof: Let $X \subseteq Y$ be a subspace.

1. Suppose Y is T1. For $a \in X$, $\{a\}$ is closed in Y and $\{a\} = \{a\} \cap X$, so it is closed in X.
2. Suppose Y is Hausdorff. Let $a, b \in X$ with $a \neq b$. Since Y is Hausdorff, we can choose disjoint open sets U, V in Y with $a \in U$ and $b \in V$. Then $U \cap X$ and $V \cap X$ are disjoint open sets in X with $a \in U \cap X$ and $b \in V \cap X$.
3. Suppose Y is regular. Since Y is T1, so is X. Let $a \in X$ and let B be a closed set with $a \notin B$. Note that since B is closed in X, $B = \mathrm{Cl}_X B = \overline{B} \cap X$ where $\overline{B} = \mathrm{Cl}_Y B$. Since $a \in X$ and $a \notin B = \overline{B} \cap X$ we have $a \notin \overline{B}$. Since Y is regular, we can choose disjoint open sets U, V in Y with $a \in U$ and $\overline{B} \subseteq V$. Then $U \cap X$ and $V \cap X$ are disjoint open sets in X with $a \in U \cap X$ and $B = \overline{B} \cap X \subseteq V \cap X$
4. The proof of the last part is an exercise.

□

Remark

A subspace of a normal space need not be normal. We shall soon see that for any set K, $[0,1]^K$ is normal. It can be (maybe) shown that $(0,1)^K \subseteq [0,1]^K$ is not normal when K is uncountable

Theorem 4.8 **Alternate Definition of Regularity**

Let X be a T1 space. Then X is regular if and only if for every $a \in X$ and every open set W in X with $a \in W$, there exists an open set U in X with $a \in U \subseteq \overline{U} \subseteq W$.

Proof: Suppose X is regular. Let $W \ni a$ be open. Then W^c is closed and $a \notin W^c$ so, since X is regular, we can choose disjoint open U, V with $a \in U$, $W^c \subseteq V$. Since $U \cap V = \emptyset$ so $U \subseteq V^c$ which is closed, so $\overline{U} \subseteq V^c$ and since $W^c \subseteq V$ we have $V^c \subseteq W$ so we have $a \in U \subseteq \overline{U} \subseteq V^c \subseteq W$.
Suppose conversely, that for every open W in X and every $a \in W$ there exists an open set U in X with $a \in U \subseteq \overline{U} \subseteq W$. Let $a \in X$ and B be a closed set in X with $a \notin B$. Taking $W = B^c$, we choose an open set U in X with $a \in U \subseteq \overline{U} \subseteq B^c$. Let $V = \left(\overline{U}\right)^c$, which is open since $\overline{U} \subseteq B^c$ we have $B \subseteq \left(\overline{U}\right)^c = V$, and since $U \subseteq \overline{U}$ we have $U \cap \left(\overline{U}\right)^c = \emptyset$, that is $U \cap V = \emptyset$. □

Theorem 4.9 **The Closure of a Box**

For each $k \in K$, let X_k be a topological space, and let $A_k \subseteq X_k$. Then using either the product or the box topology,

$$\overline{\prod_{k \in K} A_k} = \prod_{k \in K} \overline{A_k}$$