Contents

1 Groups

1.1 Notation

1. $\mathbb{N} = \{1, 2, \ldots\}$
2. $\mathbb{Z} = \{\ldots, -1, 0, 1, \ldots\}$
3. $\mathbb{Q} = \left\{\frac{a}{b} : a \in \mathbb{Z}, b \in \mathbb{N}\right\}$
4. $\mathbb{R} =$ real numbers
5. $\mathbb{C} = \{a + bi : a, b \in \mathbb{R}, i^2 = -1\}$

For $n \in \mathbb{N}$, $\mathbb{Z}_n =$ integers modulo $n = \{[0], \ldots, [n-1]\}$ where $[r] = \{z \in \mathbb{Z} : Z \equiv r \bmod n\}$
We note that the set $S = \mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{R}, \mathbb{C}, \mathbb{Z}_n$ has 2 operations $+, \cdot$.
For $n \in \mathbb{N}$, an $n \times n$ matrix over $\mathbb{R}$ (or $\mathbb{Q}$ or $\mathbb{C}$) is an $n \times n$ array

$$A = [a_{ij}] = \begin{bmatrix} a_{11} & \ldots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \ldots & a_{nn} \end{bmatrix}$$

with $a_{ij} \in \mathbb{R}$.
Note we can also do $+, \cdot$. For $A, B \in M_n(\mathbb{R})$

$$A + B := [a_{ij} + b_{ij}] \quad A \cdot B := \left[\sum_{k=1}^{n} a_{ik} b_{kj}\right]$$

1.2 Groups

Definition 1.2.1

Let G be a set and $* : G \times G \to G$. We say G is a *group* if the following are satisfied:

1. Associativity: if $a, b, c \in G$, then $a * (b * c) = (a * b) * c$
2. Identity: there is $e \in G$ such that $a * e = e * a = a$ for all $a \in G$
3. Inverses: for all $a \in G$, there is $a^{-1} \in G$ such that $a * a^{-1} = a^{-1} * a = e$

Definition 1.2.2

A group is called *abelian* if $a * b = b * a$ for all $a, b \in G$

Exercise 1.2.1

Prove in the definition of a group, 1-sided identity and inverses are enough to have 2-sided identity and inverses

Proposition 1.2.1 **previous exercise**

Suppose G is a set, $* : G \times G \to G$ is associative. Suppose there is $e \in G$ such that $e * a = a$ for all $a \in G$. Further suppose that for every $a \in G$, there is $a^{-1} \in G$ such that $a^{-1} * a = e$. Then for all $a \in G$,

1. $a * e = a$
2. $a * a^{-1} = e$

Proof of 1: Let $a \in G$, then

$$a^{-1} * a * e = e * e = e = a^{-1} * a$$

Multiplying on the left by ${a^{-1}}^{-1}$ gives

$$\begin{aligned} & {a^{-1}}^{-1} * a^{-1} * a * e = {a^{-1}}^{-1} * a^{-1} * a \\ \implies & e * a * e = e * a \\ \implies & a * e = a \end{aligned}$$

□

Proof of 2: Let $a \in G$, then

$$a^{-1} * a * a^{-1} = e * a^{-1} = a^{-1}$$

Again multiplying on the left by ${a^{-1}}^{-1}$ gives

$$a * a^{-1} = e$$

□

Proposition 1.2.2

Let G be a group, let $a \in G$. Then

1. The group identity is unique
2. The inverse of a is unique

Proof of 1: Suppose e_1, e_2 are both identities. Then

$$e_1 = e_1 * e_2 = e_2$$

□

Proof of 2: Suppose b_1, b_2 are inverses of a. Then

$$b_1 = b_1 * e = b_1 * (a * b_2) = (b_1 * a) * b_2 = e * b_2 = b_2$$

□

Example 1.2.1

$(\mathbb{Z}, +), (\mathbb{Q}, +), (\mathbb{R}, +), (\mathbb{C}, +)$ are all abelian groups

Example 1.2.2

$(\mathbb{Z},\cdot), (\mathbb{Q},\cdot), (\mathbb{R},\cdot), (\mathbb{C},\cdot)$ are not groups as 0 has no inverse

Example 1.2.3

but $(\mathbb{Q}\setminus\{0\},\cdot), (\mathbb{R}\setminus\{0\},\cdot), (\mathbb{C}\setminus\{0\},\cdot)$ are abelian groups

Definition 1.2.3

For a set $(S,\cdot)$ let $S^* \subseteq S$ denote the set of all elements with inverses.

Exercise 1.2.2

what is $\mathbb{Z}_n^*$?

Example 1.2.4

$(M_n(\mathbb{R}),+)$ is an abelian group.

Example 1.2.5

Consider $\left(M_{n(\mathbb{R})},\cdot\right)$ The identity matrix is $\begin{bmatrix} 1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 1 \end{bmatrix} \in M_n(\mathbb{R})$ However, since not all $M \in M_n(\mathbb{R})$ have multiplicative inverses, $(M_n(\mathbb{R}),\cdot)$ is not a group.

Notation

$\mathrm{GL}_n(\mathbb{R}) = \{M \in M_n(\mathbb{R}) : \det(M) \neq 0\}$

Note

If $A, B \in \mathrm{GL}_n(\mathbb{R})$, then $\det(AB) = \det(A)\det(B) \neq 0$ Thus $AB \in \mathrm{GL}_n(\mathbb{R})$. The associativity of $\mathrm{GL}_n(\mathbb{R})$ inherits from $M_n(\mathbb{R})$. Also the identity matrix satisfies $\det(I) = 1 \neq 0$ and thus $I \in \mathrm{GL}_n(\mathbb{R})$. Finally, for $M \in \mathrm{GL}_n(\mathbb{R})$, there exists $M^{-1} \in M_n(\mathbb{R})$ such that $MM^{-1} = I = M^{-1}M$ since $\det(M^{-1}) = \frac{1}{\det(M)} \neq 0$, we have $M^{-1} \in \mathrm{GL}_n(\mathbb{R})$. Thus $(\mathrm{GL}_n(\mathbb{R}),\cdot)$ is a group, called the *general linear group of degree n over $\mathbb{R}$*

Note

if $n \geq 2$, then $\mathrm{GL}_n(\mathbb{R})$ is not abelian.

Exercise 1.2.3

What is $(\mathrm{GL}_1(\mathbb{R}),\cdot)$?

Example 1.2.6

Let G, H be groups. The *direct product* is the set $G \times H$ with the component wise operation defined by

$$(g_1, h_1) * (g_2, h_2) = (g_1 *_G g_2, h_1 *_H h_2)$$

One can check that $G \times H$ is a group with identity (e_G, e_H) and the inverse of (g, h) is (g^{-1}, h^{-1})

Note

One can show by induction that if $G_1, ..., G_n$ are groups, then $G_1 \times \cdots \times G_n$ is also a group.

Notation

Given a group G and $g_1, g_2 \in G$, we often denote $g_1 * g_2$ by $g_1 g_2$ and its identity by 1. Also the unique inverse of an element $g \in G$ is denoted by g^{-1}. Also for $n \in \mathbb{N}$, we define $g^n = g * g * \cdots * g$ (n-times) and $g^{-n} = (g^{-1})^n$. Finally, we denote $g^0 = 1$.

Proposition 1.2.3

Let G be a group and $g, h \in G$ we have

1. $g^{-1^{-1}} = g$
2. $(gh)^{-1} = h^{-1}g^{-1}$
3. $g^n g^m = g^{n+m}$ for all $n, m \in \mathbb{Z}$
4. $(g^n)^m = g^{nm}$ for all $n, m \in \mathbb{Z}$

Proof of 1: Since

$$g^{-1}g = 1 = gg^{-1}$$

so $g^{-1^{-1}} = g$ □

Proof of 2:

$$(gh)(h^{-1}g^{-1}) = g(hh^{-1})g^{-1} = g1g^{-1} = 1$$

Similarly,

$$(h^{-1}g^{-1})(gh) = 1$$

Thus $(gh)^{-1} = h^{-1}g^{-1}$ □

Proof of 3: We proceed by considering cases:

1. if $n = 0$ then

$$g^n g^m = g^0 g^m = 1g^m = g^m = g^{0+m} = g^{n+m}$$

2. if $n > 0$, we will proceed by induction on n. Case 1 establishes the base case. Let $m \in \mathbb{Z}$, $n \in \mathbb{Z}_{\geq 0}$. Suppose that $g^n g^m = g^{n+m}$ Then

$$g^{n+1}g^m = gg^ng^m = gg^{n+m} = g^{n+m+1}$$

3. if $n < 0$, then $n = -k$ for some $k \in \mathbb{N}$. We have

$$g^kg^ng^m = g^{k+n}g^m = g^0g^m = g^m$$

also

$$g^kg^{n+m} = g^{k+m+n} = g^m$$

Thus

$$g^kg^ng^m = g^kg^{n+m}$$

So

$$g^ng^m = g^{n+m}$$

as desired.

□

Proof of 4: We proceed by considering cases:

1. if $m = 0$, then $(g^n)^m = (g^n)^0 = 1 = g^0 = g^{n0} = g^{nm}$
2. if $m > 0$, then

$$(g^n)^m = \underbrace{g^ng^n\cdots g^n}_{m \text{ times}} = g^{nm}$$

3. if $m < 0$, then $m = -k$ for some $k \in \mathbb{N}$. We will induct on k. For $k = 1$ we see that $(g^n)^{-1} = g^{-n}$ since

$$g^ng^{-n} = g^{n-n} = g^0 = 1$$

Suppose $(g^n)^{-\ell} = g^{-n\ell}$ for all $1 \leq \ell \leq k$ Then

$$(g^n)^{-k-1} = (g^n)^{-k}(g^n)^{-1} = g^{-nk}g^{-n} = g^{-nk-n} = g^{-n(k+1)}$$

□

Exercise 1.2.4

prove 3,4

Warning

In general, it is not the case that if $g, h \in G$ then $(gh)^n = g^nh^n$, this is not true unless G is abelian

Proposition 1.2.4

Let G be a group and $g, h, f \in G$ Then

1. They satisfy the left and right cancellation. More precisely,
   a. if $gh = gf$ then $h = f$
   b. if $hg = fg$ then $h = f$
2. Given $a, b \in G$ the equations $ax = b$ and $ya = b$ have unique solutions for $x, y \in G$

Proof of 1-a: By left-multiplying by g^{-1}, we have

$$gh = gf \Longleftrightarrow g^{-1}gh = g^{-1}gf \Longleftrightarrow h = f$$

□

Proof of 1-b: similar to 1-a □

Proof of 2: Let $x = a^{-1}b$ then

$$ax = aa^{-1}b = b$$

If u is another solution, then $au = b = ax$. By 1-a, $u = x$. Similarly, $y = ba^{-1}$ is the unique solution of $ya = b$ □

1.3 Symmetric Groups

Definition 1.3.1

Given a non-empty set L, *a permutation* of L is a bijection from L to L. The set of all permutations of L is denoted by S_L

Example 1.3.1

Consider the set $L = \{1, 2, 3\}$ which has the following different permutations

$$\begin{pmatrix} 123 \\ 123 \end{pmatrix}, \begin{pmatrix} 123 \\ 132 \end{pmatrix}, \begin{pmatrix} 123 \\ 213 \end{pmatrix}, \begin{pmatrix} 123 \\ 231 \end{pmatrix}, \begin{pmatrix} 123 \\ 312 \end{pmatrix}, \begin{pmatrix} 123 \\ 321 \end{pmatrix}$$

Where $\left(\begin{smallmatrix} 123 \\ 123 \end{smallmatrix}\right)$ denotes the bijection

$$\sigma : \{1, 2, 3\} \longrightarrow \{1, 2, 3\}$$

$$\sigma(1) = 1, \sigma(2) = 2, \sigma(3) = 3$$

Notation

For $n \in \mathbb{N}$ we denote by $S_n = S_{\{1,2,\dots,n\}}$ the set of all permutations of $\{1, 2, \dots, n\}$. We have seen that the order of $S_3 = 3! = 6$. To consider the general S_n, we note that for a permutation $\sigma \in S_n$, there are n choices for $\sigma(1)$, $n - 1$ choices for $\sigma(2)$,..., 1 choice for $\sigma(n)$ Thus

Proposition 1.3.1

$|S_n| = n!$

Note

For Möbius quizzes, use “9 dots” for permutations.

Remark

Given $\sigma, \tau \in S_n$ we can compose them to get a new element $\sigma\tau$, where $\sigma\tau = \{1, 2, ..., n\} \to \{1, 2, ..., n\}$ given by $x \mapsto \sigma(\tau(x))$ Since both σ, τ are bijections, $\sigma\tau \in S_n$

Example 1.3.2

Compute $\sigma\tau$ and $\tau\sigma$ if

$$\sigma = \begin{pmatrix} 1234 \\ 3412 \end{pmatrix}, \quad \tau = \begin{pmatrix} 1234 \\ 2431 \end{pmatrix}$$

Then $\sigma\tau(1) = \sigma(2) = 4$,... Then $\sigma\tau = \left(\begin{smallmatrix} 1234 \\ 4213 \end{smallmatrix}\right)$, and $\tau\sigma = \left(\begin{smallmatrix} 1234 \\ 3124 \end{smallmatrix}\right)$
We note that $\sigma\tau \neq \tau\sigma$

Note

For any $\sigma, \tau \in S_n$ we have that $\tau\sigma, \sigma\tau \in S_n$ but $\sigma\tau \neq \tau\sigma$ in general on the other hand, for any σ, τ, μ we have $\sigma(\tau\mu) = (\sigma\tau)\mu$. Also note the *identity permutation* $\varepsilon \in S_n$ is defined as

$$\varepsilon = \begin{pmatrix} 12 \cdots n \\ 12 \cdots n \end{pmatrix}$$

Thus for any $\sigma \in S_n$, we have $\sigma\varepsilon = \varepsilon\sigma = \sigma$
Finally, for $\sigma \in S_n$, since it is a bijection, there is a unique bijection $\sigma^{-1} \in S_n$ called the *inverse permutation* of σ such that for all $x, y \in \{1, 2, ..., n\}$

$$\sigma^{-1}(x) = y \iff \sigma(y) = x$$

It follows that

$$\sigma(\sigma^{-1}(x)) = \sigma(y) = x$$

and

$$\sigma^{-1}(\sigma(y)) = y$$

i.e we have

$$\sigma\sigma^{-1} = \sigma^{-1}\sigma = \varepsilon$$

Example 1.3.3

$$\sigma = \begin{pmatrix} 12345 \\ 45123 \end{pmatrix}$$

Then

$$\sigma^{-1} = \begin{pmatrix} 12345 \\ 34512 \end{pmatrix}$$

From the above we have

Proposition 1.3.2

$(S_n, \circ)$ is a group, called the *symmetric group of degree* n

Exercise 1.3.1

Write down all rotations and reflections that fix an equilateral triangle. Then check why it is the "same" as S_3

Example 1.3.4

Consider

$$\sigma = \begin{pmatrix} 123456789(10) \\ 317694258(10) \end{pmatrix} \in S_{10}$$

We note that $1 \to 3 \to 7 \to 2 \to 1$ and $4 \to 6 \to 4$ and $5 \to 9 \to 8$ and $10 \to 10$ Thus σ can be *decomposed* into one 4-cycle (1372), one 2-cycle (46), and one 3-cycle (598) and one 1-cycle (10) (we usually do not write 1-cycles) Note that these cycles are *pairwise disjoint* and we have

$$\sigma = (1372)(46)(598)$$

We can also write $\sigma = (46)(598)(1372)$, or $\sigma = (64)(985)(7213)$

Theorem 1.3.3 Cycle Decomposition

If Given $\sigma \in S_n$ with $\sigma \neq \varepsilon$, then σ is a product of (one or more) disjoint cycles of length at least 2. This factorization is unique up to the order of the factors.

Proof: See bonus 1. □

Convention

Every permutation of S_n can be regarded as a permutation in S_{n+1} by fixing the number $n + 1$, thus

$$S_1 \subseteq S_2 \subseteq \cdots \subseteq S_n \subseteq S_{n+1}$$

1.4 Cayley Tables

Definition 1.4.1

For a finite group G, defining its operation by means of a table is sometimes convenient. Given $x, y \in G$, the product xy is the entry of the table in the row corresponding to x and the column corresponding to y, such a table is a *Cayley table*.

Remark

By cancellation, the entries in each row or column of a Cayley table are all distinct

Example 1.4.1

Consider $(\mathbb{Z}_2, +)$ its Cayley table is

$\mathbb{Z}_2$	$[0]$	$[1]$
$[0]$	$[0]$	$[1]$
$[1]$	$[1]$	$[0]$

Example 1.4.2

Consider the group $\mathbb{Z}^* = \{1, -1\}$. Its Cayley table is

$\mathbb{Z}^*$	1	-1
1	1	-1
-1	-1	1

Note

If we replace 1 by $[0]$ and -1 by $[1]$ the Cayley tables of $\mathbb{Z}^*$ and $\mathbb{Z}_2$ become the same. In this case, we say $\mathbb{Z}^*$ and $\mathbb{Z}_2$ are *isomorphic* denoted by

$$\mathbb{Z}^* \cong \mathbb{Z}_2$$

Example 1.4.3

For $n \in \mathbb{N}$, the *cyclic group of order* n is defined by

$$C_n = \{1, a, a^2, ..., a^{n-1}\} \text{ with } a^n = 1 \text{ and } 1, a, ..., a^{n-1} \text{ are distinct}$$

The Cayley table of C_n is as follows

C_n	1	a	a^2	$\cdots$	a^{n-2}	a^{n-1}
1	1	a	a^2	$\cdots$	a^{n-2}	a^{n-1}
a	a	a^2	a^3	$\cdots$	a^{n-1}	1
a^2	a^2	a^3	a^4	$\cdots$	1	a
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$
a^{n-2}	a^{n-2}	a^{n-1}	1	$\cdots$	a^{n-4}	a^{n-3}
a^{n-1}	a^{n-1}	1	a	$\cdots$	a^{n-3}	a^{n-2}

Proposition 1.4.1

Let G be a group. Up to isomorphism, we have

1. If $|G| = 1$, then $G \cong \{1\}$
2. If $|G| = 2$, then $G \cong C_2$
3. If $|G| = 3$, then $G \cong C_3$
4. If $|G| = 4$, then $G \cong C_4$ or $G \cong K_4 \cong C_2 \times C_2$

Proof of 1: obviously □

Proof of 2: If $|G| = 2$ then $G = \{1, g\}$ with $g \neq 1$ Then $g^2 = g$ or $g^2 = 1$. We note that if $g^2 = g$, then $g = 1$ contradiction.thus $g^2 = 1$. Thus the Cayley table is as follows

G	1	g
1	1	g
g	g	1

which is the same as C_2 □

Proof of 3: If $|G| = 3$, then $G = \{1, g, h\}$ with $g \neq 1, h \neq 1, g \neq h$ By cancellation, we have $gh \neq g, gh \neq h$, thus $gh = 1$. Similarly, we have $hg = 1$. Also, on the row for g, we have $g1 = g$, $gh = 1$. Since all entries in this row are distinct, we have $g^2 = h$. Similarly, we have $h^2 = g$. Thus we obtain the following Cayley table

G	1	g	h
1	1	g	h
g	g	h	1
h	h	1	g

Which is the same as C_3. □

Proof of 4: See assignment 1 □

Exercise 1.4.1

Consider the symmetry group of a non-square rectangle. How is it related to K_4?

2 Subgroups

2.1 Subgroups

Definition 2.1.1

Let G be a group and $H \subseteq G$. IF H itself is a group, then we say H is a *subgroup* of G.

Note

We note that since G is a group, for $h_1, h_2, h_3 \in H \subseteq G$, we have

$$h_1(h_2 h_3) = (h_1 h_2) h_3$$

Thus

Proposition 2.1.1 **Subgroup Test**

Let G be a group, $H \subseteq G$. Then H is a subgroup of G if

1. If $h_1, h_2 \in H$, then $h_1 h_2 \in H$
2. $1_H \in H$
3. If $h \in H$, then $h^{-1} \in H$

Exercise 2.1.1

Prove that $1_H = 1_G$

Example 2.1.1

Given a group G, then $\{1\}, G$ are subgroups of G

Example 2.1.2

We have a chain of groups

$$(\mathbb{Z}, +) \subseteq (\mathbb{Q}, +) \subseteq (\mathbb{R}, +) \subseteq (\mathbb{C}, +)$$

Example 2.1.3

Define

$$\mathrm{SL}_n(\mathbb{R}) = (\mathrm{SL}_n(\mathbb{R}), \cdot) := \{M \in M_n(\mathbb{R}), \det(M) = 1\} \subseteq \mathrm{GL}_n(\mathbb{R})$$

Note that the identity matrix $I \in \mathrm{SL}_n(\mathbb{R})$. Let $A, B \in \mathrm{SL}_n(\mathbb{R})$, then

$$\det(AB) = \det(A)\det(B) = 1 \cdot 1 = 1$$

and

$$\det(A^{-1}) = \frac{1}{\det(A)} = \frac{1}{1} = 1$$

i.e. $AB, A^{-1} \in \mathrm{SL}_n(\mathbb{R})$. By the subgroup test (Proposition 2.1.1), $\mathrm{SL}_n(\mathbb{R})$ is a subgroup of $\mathrm{GL}_n(\mathbb{R})$. We call $\mathrm{SL}_n(\mathbb{R})$ the *special linear group of order n over $\mathbb{R}$*

Definition 2.1.2

Given a group G, we define the *center of G* to be

$$Z(G) := \{z \in G \mid zg = gz \ \forall g \in G\}$$

Remark

$Z(G) = G$ iff G is abelian.

Proposition 2.1.2

$Z(G)$ is an abelian subgroup of G.

Proof: Note that $1 \in Z(G)$. Let $y, z \in Z(G)$ Then for all $g \in G$, we have

$$(yz)g = y(zg) = y(gz) = (yg)z = (gy)z = g(yz)$$

Thus $yz \in Z(G)$. Also, for $z \in Z(G)$, $g \in G$ we have

$$\begin{aligned} zg = gz &\iff z^{-1}(zg)z^{-1} = z^{-1}(gz)z^{-1} \\ &\iff gz^{-1} = z^{-1}g \end{aligned}$$

Thus $z^{-1} \in Z(G)$. By the subgroup test (Proposition 2.1.1), $Z(G)$ is a subgroup of G. Also, by the definition of $Z(G)$, we see that it is abelian. □

Proposition 2.1.3

Let H, K be subgroups of a group G. Then $H \cap G$ is also a subgroup.

Proof: Exercise □

Proposition 2.1.4 **Finite Subgroup Test**

If $H \neq \emptyset$ is a finite subset of a group G, then H is a subgroup of G iff H is closed under its operation.

Proof:
($\Longrightarrow$) obvious
($\Longleftarrow$) For $H \neq \emptyset$, let $h \in H$. Since H is closed under its operation, we have $h, h^2, h^3, \ldots \in H$. Since H is finite, these elements are not all distinct. Thus $h^n = h^{n+m}$ for some $n, m \in \mathbb{N}$. By cancellation, $h^m = 1$ and thus $1 \in H$. Also, $1 = h^{m-1}h$ implies that $h^{-1} = h^{m-1}$ and thus $h^{-1} \in H$. By the subgroup test, H is a subgroup of G. □

2.2 Alternating Groups

Definition 2.2.1

A *transposition* $\sigma \in S_n$ is a cycle of length 2. i.e. $\sigma = (ab)$ with $a, b \in \{1, 2, \ldots, n\}$ and $a \neq b$.

Example 2.2.1

Consider $(1245) \in S_5$. Also the composition $(12)(24)(45)$ can be computed as

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 2 & 3 & 5 & 4 \\ 1 & 4 & 3 & 5 & 2 \\ 2 & 4 & 3 & 5 & 1 \end{pmatrix}$$

Thus we have $(1245) = (12)(24)(45)$ Also we can show that

$$(1245) = (23)(12)(25)(13)(24)$$

We see from this example that the factorization into transpositions are NOT unique. However, one can prove (see Bonus 2)

Theorem 2.2.1 **Parity Theorem**

If a permutation σ has two factorizations

$$\sigma = \gamma_1 \gamma_2 \cdots \gamma_r = \mu_1 \mu_2 \cdots \mu_s$$

Where each γ_i and μ_j is a transposition, then $r \equiv s \pmod{2}$

Definition 2.2.2

A permutation σ is *even* (or *odd*) if it can be written as a product of an even (or odd) number of transpositions. By the previous theorem, a permutation is either even or odd, but not both.

Theorem 2.2.2

For $n \geq 2$, let A_n denote the set of all even permutations in S_n

1. $\varepsilon \in A_n$
2. If $\sigma, \tau \in A_n$, then $\sigma\tau \in A_n$ and $\sigma^{-1} \in A_n$
3. $|A_n| = \frac{1}{2}n!$

From (1) and (2), we see (A_n) is a subgroup of S_n called the *alternating group of degree* n.

Proof of 1: We can write $\varepsilon = (12)(12)$. Thus ε is even. □

Proof of 2: if $\sigma, \tau \in A_n$ we can write $\sigma = \sigma_1 \cdots \sigma_r$ and $\tau = \tau_1 \cdots \tau_s$ where σ_i, τ_j are transpositions and r, s are even integers. Then

$$\sigma\tau = \sigma_1 \cdots \sigma_r \tau_1 \cdots \tau_s$$

is a product of $(r + s)$ transpositions and thus $\sigma\tau \in A_n$. Also, we note that σ_i is a transposition, we have $\sigma_i^2 = \varepsilon$ and thus $\sigma_i^{-1} = \sigma_i$. It follows that

$$\sigma^{-1} = (\sigma_1 \cdots \sigma_r)^{-1} = \sigma_r^{-1} \cdots \sigma_1^{-1} = \sigma_r \cdots \sigma_1$$

which is an even permutation. □

Proof of 3: Let O_n denote the set of odd permutations in S_n. Thus $S_n = A_n \cup O_n$ and the parity theorem implies that $A_n \cap O_n = \emptyset$. Since $|S_n| = n!$, to prove $|A_n| = \frac{1}{2}n!$, it suffices to show that $|A_n| = |O_n|$. Let $\gamma = (12)$ and let $f : A_n \to O_n$ be defined by $f(\sigma) = \gamma\sigma$. Since σ is even, we have $\gamma\sigma$ is odd. Thus the map is well-defined. Also, if we have $\gamma\sigma_1 = \gamma\sigma_2$, then by cancellation, we get $\sigma_1 = \sigma_2$, thus f is injective. Finally, if $\tau \in O_n$, then $\sigma = \gamma\tau \in A_n$ and $f(\sigma) = \gamma\sigma = \gamma(\gamma\tau) = \gamma^2\tau = \tau$. Thus f is surjective. It follows that f is a bijection, thus $|A_n| = |O_n|$. It follows that $|A_n| = \frac{1}{2}n! = |O_n|$ □

2.3 Orders of Elements

Notation

If G is a group and $g \in G$, we denote

$$\langle g \rangle = \{g^k \mid k \in \mathbb{Z}\} = \{\ldots, g^{-1}, g^0 = 1, g, g^2, \ldots\}$$

Note that $1 = g^0 \in \langle g \rangle$. Also, if $x = g^m, y = g^n \in \langle g \rangle$ With $m, n \in \mathbb{Z}$, then $xy = g^n g^m = g^{n+m} \in \langle g \rangle$ and $x^{-1} = g^{-m} \in \langle g \rangle$. By the subgroup test, we have

Proposition 2.3.1

If G is a group and $g \in G$, then $\langle g \rangle$ is a subgroup of G.

Definition 2.3.1

Let G be a group with $g \in G$. We call $\langle g \rangle$ the *cyclic subgroup of* G *generated by* g. If $G = \langle g \rangle$ for some $g \in G$, then we say G is *cyclic* and g a *generator* of G.

Example 2.3.1

Consider $(\mathbb{Z},+)$ Note that for all $k \in \mathbb{Z}$, we can write $k = k \cdot 1$. Thus we can see $(\mathbb{Z},+) = \langle 1 \rangle$. Similarly, $(\mathbb{Z},+) = \langle -1 \rangle$. We observe, for any integer $n \in \mathbb{Z}$ with $n \neq \pm 1$ there exist no $k \in \mathbb{Z}$ such that $k \cdot n = 1$. Thus ± 1 are the only generators of $(\mathbb{Z},+)$.

Remark

Let G be a group and $g \in G$. Suppose there is $k \in \mathbb{Z}$ $k \neq 0$ such that $g^k = 1$ then $g^{-k} = \left(g^k\right)^{-1} = 1$. Thus we can assume $k \geq 1$. Then by the well-ordering principle, there exists the smallest positive integer n such that $g^n = 1$

Definition 2.3.2

Let G be a group and $g \in G$. If n is the smallest positive integer such that $g^n = 1$, then we say the *order* of g is n, denoted $o(g) = n$. If no such n exists, we say g has *infinite order* and write $o(g) = \infty$

Proposition 2.3.2

Let G be a group and $g \in G$ with $o(g) = n \in \mathbb{N}$. For $k \in \mathbb{Z}$ we have

1. $g^k = 1$ iff $n \mid k$
2. $g^k = g^m$ iff $k \equiv m \pmod{n}$
3. $\langle g \rangle = \{1, g, g^2, ..., g^{n-1}\}$ where $1, g, ..., g^{n-1}$ are all distinct. In particular, we have $|\langle g \rangle| = o(g)$

Proof of 1:
$(\Longleftarrow)$ if $n \mid k$, then $k = nq$ for some $q \in \mathbb{Z}$. Thus

$$g^k = g^{nq} = (g^n)^q = 1^q = 1$$

$(\Longrightarrow)$ By the division algorithm, we can write $k = nq + r$ with $q, r \in \mathbb{Z}$ and $0 \leq r < n$. Since $g^k = 1$ and $g^n = 1$, we have

$$g^r = g^{k-nq} = g^k (g^n)^{-q} = 1 \cdot 1^{-q} = 1$$

Since $0 \leq r < n$ and $o(g) = n$, we have $r = 0$ and hence $n \mid k$. □
Proof of 2: Note that $g^k = g^m$ iff $g^{km} = 1$. By (1), we have $n \mid (km)$ i.e. $k \equiv m \pmod{n}$ □
Proof of 3: It follows from (2) that $1, g, ..., g^{n-1}$ are all distinct. Clearly, we have $\{1, g, ..., g^{n-1}\} \subseteq \langle g \rangle$. To prove the other inclusion, let $g^k \in \langle g \rangle$ for some $k \in \mathbb{Z}$. Write $k = nq + r$ with $n, r \in \mathbb{Z}$ and $0 \leq r < n$. Then

$$g^k = g^{nq+r} = g^{nq} g^r = (g^n)^q g^r = 1^q g^r = g^r \in \{1, g, ..., g^{n-1}\}$$

Thus $\langle g \rangle = \{1, g, ..., g^{n-1}\}$ □

Proposition 2.3.3

Let G be a group and $g \in G$ with $o(g) = \infty$. For $k \in \mathbb{Z}$ we have

1. $g^k = 1$ iff $k = 0$
2. $g^k = g^m$ iff $k = m$
3. $\langle g \rangle = \{..., g^{-1}, g^0 = 1, g, ...\}$ where g^i are all distinct

Proposition 2.3.4

Let G be a group and $g \in G$ with $o(g) = n \in \mathbb{N}$. If $d \in \mathbb{N}$, then $o\left(g^d\right) = \frac{n}{\gcd(n,d)}$. In particular, if $d \mid n$, then $\gcd(n,d) = d$ and $o\left(g^d\right) = \frac{n}{d}$

Proof: Let $n_1 = \frac{n}{\gcd(n,d)}$ and $d_1 = \frac{d}{\gcd(n,d)}$. By a result from Math 135, we have $\gcd(n_1, d_1) = 1$. Note that

$$\left(g^d\right)^n_1 = \left(g^d\right)^{\frac{n}{\gcd(n,d)}} = (g^n)^{\frac{d}{\gcd(n,d)}} = 1$$

Thus it remains to show that n_1 is the smallest such positive integer. Suppose $\left(g^d\right)^r = 1$ with $r \in \mathbb{N}$. Since $o(g) = n$, by prop, we have $n \mid dr$. Thus there is $q \in \mathbb{Z}$ such that $dr = nq$. Dividing both sides by $\gcd(n,d)$ we get

$$d_1 r = \frac{d}{\gcd(n,d)} r = \frac{n}{\gcd(n,d)} q = n_1 q$$

Since $n_1 \mid d_1 r$ and $\gcd(n_1, d_1) = 1$, by a result from Math 135, we get $n_1 \mid r$ i.e. $r = n_1 \ell$ for some $\ell \in \mathbb{Z}$. Since $r_1, n_1 \in \mathbb{N}$, it follows that $\ell \in \mathbb{N}$. Since $\ell \geq 1$, we get $r \geq n_1$ □

2.4 Cyclic Groups

Remark

For a group G, if $G = \langle g \rangle$ for some $g \in G$, then G is a cyclic group. For $a, b \in G$, we have $a = g^n, b = g^m$ for some $m, n \in \mathbb{Z}$. We have

$$ab = g^n g^m = g^{n+m} = g^{m+n} = g^m g^n = ba$$

Proposition 2.4.1

Every cyclic group is abelian

Warning

The converse of the above prop is not true. For example the Klein 4 group is abelian, but not cyclic.

Proposition 2.4.2

Every subgroup of a cyclic group is cyclic.

Proof: Let $G = \langle g \rangle$ be cyclic and $H \subseteq G$ a subgroup. If $H = \{1\}$, then H is cyclic. Otherwise, there is $g^k \in H$ with $k \in \mathbb{Z} \setminus \{0\}$. Since H is a group, we have $g^{-k} \in H$. Thus we can assume that $k \in \mathbb{N}$. Let m be the smallest positive integer such that $g^m \in H$.
<u>*Claim:*</u> $H = \langle g^m \rangle$
Proof is exercise, by division algorithm. □

Proposition 2.4.3

Let $G = \langle g \rangle$ be a cyclic group with $o(g) = n$. Then $G = \langle g^k \rangle$ iff $\gcd(k, n) = 1$.

Proof: By prop,

$$o(g^k) = \frac{n}{\gcd(n, k)} = n$$

□

Theorem 2.4.4 **Fundamental Theorem of Finite Cyclic Groups**

Let $G = \langle g \rangle$ be a cyclic group with $o(g) = n \in \mathbb{N}$.

1. If H is a subgroup of G, then $G = \langle g^d \rangle$ for some $d \mid n$. It follows that $|H| \mid |G|$.
2. Conversely, if $k \mid n$, then $g^{\frac{n}{k}}$ is the unique subgroup of G with order k.

Proof of 1: By prop, H is cyclic. Write $H = \langle g^n \rangle$ for some $m \in \mathbb{N} \cup \{0\}$. Let $d = \gcd(m, n)$.
<u>*Claim:*</u> $H = \langle g^d \rangle$
Since $d \mid m$ we have $m = dk$ for some $k \in \mathbb{Z}$. Then

$$g^m = g^{dk} = \left(g^d\right)^k \in \langle g^d \rangle$$

Thus $H = \langle g^m \rangle \subseteq \langle g^d \rangle$. To prove the other incursion, since $d = \gcd(m, n)$, there is $x, y \in \mathbb{Z}$ such that $d = mx + ny$. Then

$$g^d = g^{mx+ny} = (g^m)^x (g^n)^y = (g^m)^x 1^y = (g^m)^x \in \langle g^m \rangle$$

Thus $\langle g^d \rangle \subseteq \langle g^m \rangle = H$. It follows that $H = \langle g^d \rangle$. Note that since $d = \gcd(m, n)$, we have $d \mid n$. By prop, we have

$$|H| = o(g^d) = \frac{n}{\gcd(n, d)} = \frac{n}{d}$$

Thus $|H| \mid |G|$ □

Proof of 2: By prop, the cyclic subgroup $\langle g^{\frac{n}{k}} \rangle$ is of order

$$\frac{n}{\gcd(n, \frac{n}{k})} = \frac{n}{n/k} = k$$

To show uniqueness, let K be a subgroup of G with order $k \mid n$. By 1, let $K = \langle g^d \rangle$ where $d \mid n$. Then by props, we have,

$$k = |K| = o(g^d) = \frac{n}{\gcd(n, d)} = \frac{n}{d}$$

It follows that $d = \frac{n}{k}$ and thus $K = \left\langle g^{\frac{n}{k}} \right\rangle$ □

2.5 Non-cyclic Groups

Definition 2.5.1

Let X be a non-empty subset of a group G, and let

$$\langle X \rangle := \left\{ x_1^{k_1} \cdots x_m^{k_m} \,\middle|\, x_i \in X, k_i \in \mathbb{Z}, m \geq 1 \right\}$$

denote the set of all products of powers of (not necessarily distinct) elements of X. Note that this is clearly a group. $\langle X \rangle$ is called the *subgroup of G generated by X*.

Example 2.5.1

The Klein-4 group $K_4 = \{1, a, b, c\}$ with $a^2 = b^2 = c^2 = 1$ and $ab = c$. Thus

$$K_4 = \langle a, b \mid a^2 = 1 = b^2 \text{ and } ab = ba \rangle$$

Example 2.5.2

The symmetric group of order 3 $S_3 = \left\{ \varepsilon, \sigma, \sigma^2, \tau, \tau\sigma, \tau\sigma^2 \right\}$ where $\sigma^3 = \varepsilon = \tau^2$ and $\sigma\tau = \tau\sigma^2$ (one can take $\tau = (12)$ and $\sigma = (123)$) Thus

$$\langle \sigma, \tau \mid \sigma^3 = \varepsilon = \tau^2 \text{ and } \sigma\tau = \tau\sigma^2 \rangle$$

We can also replace σ, τ be $\sigma, \tau\sigma$ or $\sigma, \tau\sigma^2, \ldots$, etc