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1. Topological Spaces and Continuous Maps

1.1. Elementary Topology

Given an inner product on an \mathbb{R} -vector space $\langle \cdot, \cdot \rangle$, one can define a norm $\|x\| = \sqrt{\langle x, x \rangle}$. Given a norm, one can define a metric $d(x, y) = \|x - y\|$. Given a metric d on a set X , one can define open sets in X :

given $a \in X$ and $r > 0$, $B(a, r) := \{x \in X \mid d(x, a) < r\}$. Then for $A \subseteq X$, we say A is open in X when $\forall a \in A \exists r > 0$ such that $B(a, r) \subseteq A$. Equivalently, for all $a \in A$, there is $b \in X$, $r > 0$ such that $a \in B(b, r) \subseteq A$.

Remark

The set of open sets on a metric space is called the *metric topology* on X .

Open sets in a metric space satisfy the following:

1. \emptyset and X are open
2. arbitrary unions of open sets are open
3. finite intersections of open sets are open

Notation

For a set of sets S , the union of S is

$$\bigcup S := \{x \mid \exists A \in S, x \in A\} = \bigcup_{A \in S} A$$

In the case that $S \neq \emptyset$, the intersection of S is

$$\bigcap S := \{x \mid \forall A \in S, x \in A\} = \bigcap_{A \in S} A$$

Note

$\bigcap S$ would contain all elements as the condition $\forall A \in \emptyset$ would be vacuously satisfied. If we are given a universal set X , and S is known to be a set of subsets of X , then $\bigcap \emptyset = X$.

Definition 1.1.1

Let X be a set. $\mathcal{T} \subseteq \mathcal{P}(X)$ is called a *topology* on X if

1. $\emptyset, X \in \mathcal{T}$
2. If $S \subseteq \mathcal{T}$ is nonempty, then $\bigcup S \in \mathcal{T}$
3. If $S \subseteq \mathcal{T}$ is nonempty and finite, then $\bigcap S \in \mathcal{T}$

The elements of \mathcal{T} are called the open sets of X . The closed sets are the compliments of the open sets.

Remark

To show 3 holds, it suffices to show the intersection of 2 open sets is open (by induction)

Definition 1.1.2

If X is a set, and \mathcal{T} is a topology on X , then (X, \mathcal{T}) is called a *topological space*

Remark

When $f : X \rightarrow Y$ is a map between metric spaces, f is continuous iff $f^{-1}(V)$ is open in X for every open set $V \subseteq Y$.

Definition 1.1.3

For a map $f : X \rightarrow Y$ between topological spaces, we say that f is continuous when $f^{-1}(V)$ is open in X for every open set $V \subseteq Y$.

Example 1.1.1

if $f : A \subseteq \mathbb{R}^n \rightarrow B \subseteq \mathbb{R}^m$ is an elementary function, then f is continuous.

Definition 1.1.4

When S, T are topologies on X with $S \subseteq T$, we say that S is coarser than T and T is finer than S . When $S \subsetneq T$, we use strictly coarser/finer.

Example 1.1.2

$\{\emptyset, X\}$ is a topology on X called the *trivial topology*

Example 1.1.3

$\mathcal{P}(X)$ is a topology on X called the *discrete topology*

Example 1.1.4

When $X = \emptyset$, $\mathcal{T} \subseteq \mathcal{P}(X) \Rightarrow \mathcal{T} \subseteq \{\emptyset\} \Rightarrow \mathcal{T} = \emptyset \vee \mathcal{T} = \{\emptyset\}$. Thus the only topology on \emptyset is $\{\emptyset\}$.

Example 1.1.5

When $X = \{a\}$ the only topology is $\mathcal{T} = \{\emptyset, \{a\}\}$

Exercise 1.1.1

Find all topologies on the 2 and 3 element sets.

Definition 1.1.5

Let X be a topological space. Let $A \subseteq X$.

1. The *interior* of A (in X) denoted by A° is the union of all open sets in X which are contained in A .
2. The *closure* of A denoted \overline{A} is the intersection of all closed sets in X which contain A .
3. The *boundary* of A , denoted by ∂A , given by $\partial A = \overline{A} \setminus A^\circ$

Note

The set of closed sets in a topological space is closed under arbitrary intersections and under finite unions. In particular \emptyset, X are closed

Theorem 1.1.1

Let X be a topological space, $A \subseteq X$.

1. A° is open, and is the largest open set which is contained in A
2. \overline{A} is closed, and is the smallest closed set which contains A
3. A is open iff $A = A^\circ$
4. A is closed iff $A = \overline{A}$
5. $A^{\circ\circ} = A^\circ$
6. $\overline{\overline{A}} = \overline{A}$

Definition 1.1.6

Let X be a topological space, let $A \subseteq X$, let $a \in X$.

1. We say that a is an *interior point* of A when $a \in A$ and there is an open set U such that $a \in U \subseteq A$
2. We say that a is a *limit point* of A when for every open set $U \ni a$ we have $U \cap (A \setminus \{a\}) \neq \emptyset$. The set of limit points of A is denoted by A'
3. We say that a is a *boundary point* of A when every open set $U \ni a$, we have $U \cap A \neq \emptyset$ and $U \cap A^c \neq \emptyset$

Theorem 1.1.2

Let X be a topological space and let $A \subseteq X$.

1. A° is equal to the set of all interior points
2. For $a \in X$,

$$a \in A' \iff a \in \overline{A \setminus \{a\}}$$

3. A is closed iff $A' \subseteq A$
4. $\overline{A} = A \cup A'$
5. \overline{A} is the disjoint union

$$\overline{A} = A^\circ \sqcup \partial A$$

6. ∂A is equal to the set of boundary points of A

1.2 Topological Bases**Theorem 1.2.1**

Let X be a set. Then the intersection of any set of topologies on X is also a topology on X .

Proof: Let $\{\mathcal{T}_\alpha\}_{\alpha \in I}$ be a collection of topologies on X . Let $\mathcal{T} = \cap_{\alpha \in I} \mathcal{T}_\alpha$

1. Since $X, \emptyset \in \mathcal{T}_\alpha$ for all $\alpha \in I$. We have $X, \emptyset \in \mathcal{T}$
2. Let $\{U_i\} \subseteq \mathcal{T}$. For all $\alpha \in I$, we have each $U_i \in \mathcal{T}_\alpha$. Thus $\cup_i U_i \in \mathcal{T}_\alpha \implies \cup_i U_i \in \mathcal{T}$ as desired.
3. Let $U_1, \dots, U_n \in \mathcal{T}$. Then again for all $\alpha \in I$, we have each $U_i \in \mathcal{T}_\alpha$. Thus $\cap_{i=1}^n U_i \in \mathcal{T}_\alpha \implies \cap_{i=1}^n U_i \in \mathcal{T}$

□

Corollary 1.2.2

When X is a set and \mathcal{S} is any set of subsets of X (that is $S \subseteq \mathcal{P}(X)$), there is a unique smallest (coarsest) topology \mathcal{T} on X which contains \mathcal{S} . Indeed \mathcal{T} is the intersection of (the set of) all topologies on X containing \mathcal{S} .

This topology \mathcal{T} is called the topology on X generated by \mathcal{S}

Definition 1.2.1

Let X be a set. A *basis of sets* on X is a set \mathcal{B} of subsets of X (So $\mathcal{B} \subseteq \mathcal{P}(X)$) such that

1. \mathcal{B} covers X , that is $\bigcup \mathcal{B} = X$
2. For every $C, D \in \mathcal{B}$ and $a \in C \cap D$. There is $B \in \mathcal{B}$ such that $a \in B \subseteq C \cap D$.

When \mathcal{B} is a basis of sets in X and \mathcal{T} is the topology on X generated by \mathcal{B} , we say that \mathcal{B} is a *basis for \mathcal{T}* . The elements in \mathcal{B} are called *basic open sets* in X .

Theorem 1.2.3**Characterization of Open Sets in Terms of Basic Open Sets**

Let X be a topological space, Let \mathcal{B} be a basis for the topology on X .

1. For $A \subseteq X$, A is open iff for every $a \in A$, there is $B \in \mathcal{B}$ such that $a \in B \subseteq A$ *
2. The open sets in X are the unions of (sets of) elements in \mathcal{B}

Equivalently,

1. $\mathcal{T} = \{A \subseteq X \mid \forall a \in A, \exists B \in \mathcal{B} \ a \in B \subseteq A\}$
2. $\mathcal{T} = \{\bigcup C \mid C \subseteq \mathcal{B}\}$

Proof: Let \mathcal{T} be the topology on X (generated by \mathcal{B}). Let \mathcal{S} be the set of all sets $A \subseteq X$ with property * ($\forall a \in A \exists B \in \mathcal{B} : a \in B \subseteq A$). And let \mathcal{R} be the set of (arbitrary) unions of (sets of) elements in \mathcal{B} . Recall that \mathcal{T} is the intersection of the set of all topologies on X which contain \mathcal{B} . Note that \mathcal{S} contains \mathcal{B} (obviously). Let us show that \mathcal{S} is a topology on X . We have $\emptyset \in \mathcal{S}$ vacuously and $X \in \mathcal{S}$ because \mathcal{B} covers X (given $a \in X$, we can choose $B \in \mathcal{B}$ with $a \in B$). When $U_k \in \mathcal{S}$ for every $k \in K$ (where K is any index set). Let $a \in \bigcup_k U_k$. Choose $\ell \in K$ so that $a \in U_\ell$. Since $U_\ell \in \mathcal{S}$, we can choose $B \in \mathcal{B}$ so that $a \in B \subseteq U_\ell$. Since $U_\ell \subseteq \bigcup_k U_k$, we have $a \in B \subseteq \bigcup_k U_k$. Thus $\bigcup_k U_k$ satisfies *, hence $\bigcup_k U_k \in \mathcal{S}$ as required. Suppose $U, V \in \mathcal{S}$. Let $a \in U \cap V$. Since $U \in \mathcal{S}$ we can choose $C \in \mathcal{B}$ with $a \in C \subseteq U$. Since $V \in \mathcal{S}$, we can choose $D \in \mathcal{B}$ with $a \in D \subseteq V$. Since \mathcal{B} is a basis, $C, D \in \mathcal{B}$ and $a \in C \cap D$, we can choose $B \in \mathcal{B}$ with $a \in B \subseteq C \cap D$. Then we have

$$a \in B \subseteq C \cap D \subseteq U \cap V$$

Thus $U \cap V$ satisfies * so that $U \cap V \in \mathcal{S}$ as required. Thus \mathcal{S} is a topology on X containing \mathcal{B} , hence $\mathcal{T} \subseteq \mathcal{S}$. Let us show that $\mathcal{S} \subseteq \mathcal{R}$ let $U \in \mathcal{S}$. For each $a \in U$, choose $B_a \in \mathcal{B}$ with $a \in B_a \subseteq U$. Then we have

$$U = \bigcup_{a \in U} B_a \in \mathcal{R}$$

Thus $\mathcal{S} \subseteq \mathcal{R}$. Finally note that $\mathcal{R} \subseteq \mathcal{T}$ because if $U = \bigcup_k B_k$ with $B_k \in \mathcal{B}$, then each $B_k \in \mathcal{T}$, and \mathcal{T} is a topology, so

$$U = \bigcup_{k \in K} B_k \in \mathcal{T}$$

□

Theorem 1.2.4**Characterization of a Basis in terms of the Open Sets**

Let X be a topological space with topology \mathcal{T} . Let $\mathcal{B} \subseteq \mathcal{T}$. Then \mathcal{B} is a basis for \mathcal{T} iff
 $\forall U \in \mathcal{T} \forall a \in U \exists B \in \mathcal{B} \quad a \in B \subseteq U$. *

Proof: If \mathcal{B} is a basis for \mathcal{T} , then * holds by part 1 of the previous theorem. Suppose * holds. Let us show that \mathcal{B} is a basis of sets in X . Note that \mathcal{B} covers X since, taking $U = X$ in * we have $\forall a \in X \exists B \in \mathcal{B} \quad a \in B \subseteq X$. Also note that given $C, D \in \mathcal{B}$ and $a \in C \cap D$, then by taking $U = C \cap D$ in * (noting that $C, D \in \mathcal{B} \subseteq \mathcal{T}$ so that $U = C \cap D \in \mathcal{T}$) we can choose $B \in \mathcal{B}$ with $a \in B \subseteq C \cap D$. Thus \mathcal{B} is a basis of sets in X . It remains to show that \mathcal{T} is the topology generated by \mathcal{B} . Let \mathcal{S} be the topology generated by \mathcal{B} . By part 1 of the previous theorem, \mathcal{S} is the set of all unions of elements in \mathcal{B} . Also \mathcal{S} is the smallest topology which contains \mathcal{B} . Since $\mathcal{B} \subseteq \mathcal{T}$ and \mathcal{T} is a topology, we have $\mathcal{S} \subseteq \mathcal{T}$. Also we have $\mathcal{T} \subseteq \mathcal{S}$ because given $U \in \mathcal{T}$, by property *, for each $a \in U$, we can choose $B_a \in \mathcal{B}$ with $a \in B_a \subseteq U$, and then we have $U = \bigcup_{a \in U} B_a \in \mathcal{S}$ since it is a union of elements in \mathcal{B} . □

Example 1.2.1

When X is a metric space, the set \mathcal{B} of all open balls in X is a basis for the metric topology on X .

Remark

We can use a basis for testing various topological properties:

When X is a topological space, and \mathcal{B} is a basis for the topology on X , and $A \subseteq X$ and $a \in X$. Then

$$a \in A^\circ \iff \exists B \in \mathcal{B} \text{ with } a \in B \subseteq A$$

$$a \in \overline{A} \iff \forall B \in \mathcal{B} \text{ with } a \in B \quad B \cap A \neq \emptyset$$

$$a \in A' \iff \forall B \in \mathcal{B} \text{ with } a \in B \quad (B \setminus \{a\}) \cap A \neq \emptyset$$

$$a \in \partial A \iff \forall B \in \mathcal{B} \text{ with } a \in B \quad B \cap A \neq \emptyset \text{ and } B \cap (X \setminus A) \neq \emptyset$$

Definition 1.2.2

A topological space X is called *Hausdorff* when for all $a, b \in X$ with $a \neq b$, there exist disjoint open sets U and V in X with $a \in U$ and $b \in V$.

Example 1.2.2

Metric spaces are Hausdorff

1.3 Subspaces

Definition 1.3.1

Subspace Topology

Let Y be a topological space with topology \mathcal{S} , and $X \subseteq Y$ be a subset. Let

$$\mathcal{T} := \{V \cap X \mid V \in \mathcal{S}\}$$

Then \mathcal{T} is a topology on X :

Indeed $\emptyset \in \mathcal{S}$ so $\emptyset \cap X = \emptyset \in \mathcal{T}$ and $Y \in \mathcal{S}$ so $Y \cap X = X \in \mathcal{T}$. If K is any index set and $U_k \in \mathcal{T}$ for each $k \in K$, then for each $k \in K$ we can choose $V_k \in \mathcal{S}$ such that $U_k = V_k \cap X$ and then we have

$$\begin{aligned} \bigcup_{k \in K} U_k &= \bigcup_{k \in K} (V_k \cap X) \\ &= \left(\bigcup_{k \in K} V_k \right) \cap X \in \mathcal{T} \end{aligned}$$

since $\bigcup_{k \in K} V_k \in \mathcal{S}$. Similarly, when K is finite and $U_k \in \mathcal{T}$ for each $k \in K$ we have $\bigcap_{k \in K} U_k \in \mathcal{T}$. The topology \mathcal{T} on X is called the *subspace topology* on X (inherited from the topology on Y).

Theorem 1.3.1

Let Y be a topological space, let \mathcal{C} be a basis for the topology on Y . Let $X \subseteq Y$ be a subset. Then the set

$$\mathcal{B} = \{C \cap X \mid C \in \mathcal{C}\}$$

is a basis for the subspace topology on X .

Proof: Exercise □

Theorem 1.3.2

Let Z be a topological space, let $Y \subseteq Z$ be a subspace and $X \subseteq Y$ be a subset. Then the subspace topology on X inherited from Y is equal to the subspace topology on X inherited from Z .

Proof: Exercise □

Theorem 1.3.3

Let Y be a metric space, (using the metric topology) and let $X \subseteq Y$. Then the subspace topology on X (inherited from the topology on Y) is equal to the metric topology on X using the metric on X obtained by restricting the metric on Y .

Proof: Exercise □