

## Contents

1	Groups .....	2
1.1	Notation .....	2
1.2	Groups .....	2
1.3	Symmetric Groups .....	7
1.4	Cayley Tables .....	10
2	Subgroups .....	12
2.1	Subgroups .....	12
2.2	Alternating Groups .....	14
2.3	Orders of Elements .....	15
2.4	Cyclic Groups .....	17
2.5	Non-cyclic Groups .....	19
3	Normal Subgroups .....	20
3.1	Homomorphisms and Isomorphisms .....	20
3.2	Cosets and Lagrange's Theorem .....	21
3.3	Normal Subgroups .....	24
4	Isomorphism Theorems .....	28
4.1	Quotient Groups .....	28
4.2	Isomorphism Theorems .....	29
5	Group Actions .....	32
5.1	Cayley's Theorem .....	32
5.2	Group Actions .....	33
6	Sylow Theorems .....	37
6.1	$p$ -groups .....	37
6.2	Three Sylow Theorems .....	38
7	Finite Abelian Groups .....	41
7.1	Primary Decomposition .....	41
7.2	Structure Theorem of Finite Abelian Groups .....	42
8	Rings .....	44
8.1	Rings .....	44
8.2	Subrings .....	47
8.3	Ideals .....	48
8.4	Isomorphism Theorems .....	50
9	Commutative Rings .....	54
9.1	Integral Domains and Fields .....	54
9.2	Prime Ideals and Maximal Ideals .....	58
9.3	Fields of Fractions .....	60
10	Polynomial Rings .....	60
10.1	Polynomials .....	60
10.2	Polynomials Over a Field .....	62
10.3	Fermat's Last Theorem in $F[x]$ .....	69
10.4	Prime Number Theorem and Riemann Hypothesis in $\mathbb{Z}_p[t]$ .....	70

# 1 Groups

## 1.1 Notation

1.  $\mathbb{N} = \{1, 2, \dots\}$
2.  $\mathbb{Z} = \{\dots, -1, 0, 1, \dots\}$
3.  $\mathbb{Q} = \left\{ \frac{a}{b} : a \in \mathbb{Z}, b \in \mathbb{N} \right\}$
4.  $\mathbb{R}$  = real numbers
5.  $\mathbb{C} = \{a + bi : a, b \in \mathbb{R}, i^2 = -1\}$

For  $n \in \mathbb{N}$ ,  $\mathbb{Z}_n$  = integers modulo  $n = \{[0], \dots, [n-1]\}$  where  $[r] = \{z \in \mathbb{Z} : Z \equiv r \pmod{n}\}$

We note that the set  $S = \mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{R}, \mathbb{C}, \mathbb{Z}_n$  has 2 operations  $+, \cdot$ .

For  $n \in \mathbb{N}$ , an  $n \times n$  matrix over  $\mathbb{R}$  (or  $\mathbb{Q}$  or  $\mathbb{C}$ ) is an  $n \times n$  array

$$A = [a_{ij}] = \begin{bmatrix} a_{11} & \dots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \dots & a_{nn} \end{bmatrix}$$

with  $a_{ij} \in \mathbb{R}$ .

Note we can also do  $+, \cdot$ . For  $A, B \in M_n(\mathbb{R})$

$$A + B := [a_{ij} + b_{ij}] \quad A \cdot B := \left[ \sum_{k=1}^n a_{ik} b_{kj} \right]$$

## 1.2 Groups

### Definition 1.2.1

Let  $G$  be a set and  $* : G \times G \rightarrow G$ . We say  $G$  is a *group* if the following are satisfied:

1. Associativity: if  $a, b, c \in G$ , then  $a * (b * c) = (a * b) * c$
2. Identity: there is  $e \in G$  such that  $a * e = e * a = a$  for all  $a \in G$
3. Inverses: for all  $a \in G$ , there is  $a^{-1} \in G$  such that  $a * a^{-1} = a^{-1} * a = e$

### Definition 1.2.2

A group is called *abelian* if  $a * b = b * a$  for all  $a, b \in G$

### Exercise 1.2.1

Prove in the definition of a group, 1-sided identity and inverses are enough to have 2-sided identity and inverses

**Proposition 1.1**[previous exercise](#)

Suppose  $G$  is a set,  $* : G \times G \rightarrow G$  is associative. Suppose there is  $e \in G$  such that  $e * a = a$  for all  $a \in G$ . Further suppose that for every  $a \in G$ , there is  $a^{-1} \in G$  such that  $a^{-1} * a = e$ . Then for all  $a \in G$ ,

1.  $a * e = a$
2.  $a * a^{-1} = e$

**Proof of 1:** Let  $a \in G$ , then

$$a^{-1} * a * e = e * e = e = a^{-1} * a$$

Multiplying on the left by  $a^{-1}$  gives

$$\begin{aligned} a^{-1} * a^{-1} * a * e &= a^{-1} * a^{-1} * a \\ \implies e * a * e &= e * a \\ \implies a * e &= a \end{aligned}$$

□

**Proof of 2:** Let  $a \in G$ , then

$$a^{-1} * a * a^{-1} = e * a^{-1} = a^{-1}$$

Again multiplying on the left by  $a^{-1}$  gives

$$a * a^{-1} = e$$

□

**Proposition 1.2**

Let  $G$  be a group, let  $a \in G$ . Then

1. The group identity is unique
2. The inverse of  $a$  is unique

**Proof of 1:** Suppose  $e_1, e_2$  are both identities. Then

$$e_1 = e_1 * e_2 = e_2$$

□

**Proof of 2:** Suppose  $b_1, b_2$  are inverses of  $a$ . Then

$$b_1 = b_1 * e = b_1 * (a * b_2) = (b_1 * a) * b_2 = e * b_2 = b_2$$

□

**Example 1.2.1**

$(\mathbb{Z}, +), (\mathbb{Q}, +), (\mathbb{R}, +), (\mathbb{C}, +)$  are all abelian groups

**Example 1.2.2**

$(\mathbb{Z}, \cdot), (\mathbb{Q}, \cdot), (\mathbb{R}, \cdot), (\mathbb{C}, \cdot)$  are not groups as 0 has no inverse

**Example 1.2.3**

but  $(\mathbb{Q} \setminus \{0\}, \cdot), (\mathbb{R} \setminus \{0\}, \cdot), (\mathbb{C} \setminus \{0\}, \cdot)$  are abelian groups

**Definition 1.2.3**

For a set  $(S, \cdot)$  let  $S^* \subseteq S$  denote the set of all elements with inverses.

**Exercise 1.2.2**

what is  $\mathbb{Z}_n^*$ ?

**Example 1.2.4**

$(M_n(\mathbb{R}), +)$  is an abelian group.

**Example 1.2.5**

Consider  $(M_{n(\mathbb{R})}, \cdot)$ . The identity matrix is  $\begin{bmatrix} 1 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & 1 \end{bmatrix} \in M_n(\mathbb{R})$ . However, since not all  $M \in M_n(\mathbb{R})$  have multiplicative inverses,  $(M_n(\mathbb{R}), \cdot)$  is not a group.

**Notation**

$$\mathrm{GL}_n(\mathbb{R}) = \{M \in M_n(\mathbb{R}) : \det(M) \neq 0\}$$

**Note**

If  $A, B \in \mathrm{GL}_n(\mathbb{R})$ , then  $\det(AB) = \det(A)\det(B) \neq 0$ . Thus  $AB \in \mathrm{GL}_n(\mathbb{R})$ . The associativity of  $\mathrm{GL}_n(\mathbb{R})$  inherits from  $M_n(\mathbb{R})$ . Also the identity matrix satisfies  $\det(I) = 1 \neq 0$  and thus  $I \in \mathrm{GL}_n(\mathbb{R})$ . Finally, for  $M \in \mathrm{GL}_n(\mathbb{R})$ , there exists  $M^{-1} \in M_n(\mathbb{R})$  such that  $MM^{-1} = I = M^{-1}M$  since  $\det(M^{-1}) = \frac{1}{\det(M)} \neq 0$ , we have  $M^{-1} \in \mathrm{GL}_n(\mathbb{R})$ . Thus  $(\mathrm{GL}_n(\mathbb{R}), \cdot)$  is a group, called the *general linear group of degree n over  $\mathbb{R}$* .

**Note**

if  $n \geq 2$ , then  $\mathrm{GL}_n(\mathbb{R})$  is not abelian.

**Exercise 1.2.3**

What is  $(\mathrm{GL}_1(\mathbb{R}), \cdot)$ ?

**Example 1.2.6**

Let  $G, H$  be groups. The *direct product* is the set  $G \times H$  with the component wise operation defined by

$$(g_1, h_1) * (g_2, h_2) = (g_1 *_G g_2, h_1 *_H h_2)$$

One can check that  $G \times H$  is a group with identity  $(e_G, e_H)$  and the inverse of  $(g, h)$  is  $(g^{-1}, h^{-1})$

**Note**

One can show by induction that if  $G_1, \dots, G_n$  are groups, then  $G_1 \times \dots \times G_n$  is also a group.

**Notation**

Given a group  $G$  and  $g_1, g_2 \in G$ , we often denote  $g_1 * g_2$  by  $g_1 g_2$  and its identity by 1. Also the unique inverse of an element  $g \in G$  is denoted by  $g^{-1}$ . Also for  $n \in \mathbb{N}$ , we define  $g^n = g * g * \dots * g$  ( $n$ -times) and  $g^{-n} = (g^{-1})^n$ . Finally, we denote  $g^0 = 1$ .

**Proposition 1.3**

Let  $G$  be a group and  $g, h \in G$  we have

1.  $g^{-1-1} = g$
2.  $(gh)^{-1} = h^{-1}g^{-1}$
3.  $g^n g^m = g^{n+m}$  for all  $n, m \in \mathbb{Z}$
4.  $(g^n)^m = g^{nm}$  for all  $n, m \in \mathbb{Z}$

**Proof of 1:** Since

$$g^{-1}g = 1 = gg^{-1}$$

so  $g^{-1-1} = g$  □

**Proof of 2:**

$$(gh)(h^{-1}g^{-1}) = g(hh^{-1})g^{-1} = g1g^{-1} = 1$$

Similarly,

$$(h^{-1}g^{-1})(gh) = 1$$

Thus  $(gh)^{-1} = h^{-1}g^{-1}$  □

**Proof of 3:** We proceed by considering cases:

1. if  $n = 0$  then

$$g^n g^m = g^0 g^m = 1g^m = g^m = g^{0+m} = g^{n+m}$$

2. if  $n > 0$ , we will proceed by induction on  $n$ . Case 1 establishes the base case. Let  $m \in \mathbb{Z}, n \in \mathbb{Z}_{\geq 0}$ . Suppose that  $g^n g^m = g^{n+m}$  Then

$$g^{n+1}g^m = gg^n g^m = gg^{n+m} = g^{n+m+1}$$

3. if  $n < 0$ , then  $n = -k$  for some  $k \in \mathbb{N}$ . We have

$$g^k g^n g^m = g^{k+n} g^m = g^0 g^m = g^m$$

also

$$g^k g^{n+m} = g^{k+m+n} = g^m$$

Thus

$$g^k g^n g^m = g^k g^{n+m}$$

So

$$g^n g^m = g^{n+m}$$

as desired. □

**Proof of 4:** We proceed by considering cases:

1. if  $m = 0$ , then  $(g^n)^m = (g^n)^0 = 1 = g^0 = g^{n0} = g^{nm}$
2. if  $m > 0$ , then

$$(g^n)^m = \underbrace{g^n g^n \cdots g^n}_{m \text{ times}} = g^{nm}$$

3. if  $m < 0$ , then  $m = -k$  for some  $k \in \mathbb{N}$ . We will induct on  $k$ . For  $k = 1$  we see that  $(g^n)^{-1} = g^{-n}$  since

$$g^n g^{-n} = g^{n-n} = g^0 = 1$$

Suppose  $(g^n)^{-\ell} = g^{-n\ell}$  for all  $1 \leq \ell \leq k$ . Then

$$(g^n)^{-k-1} = (g^n)^{-k} (g^n)^{-1} = g^{-nk} g^{-n} = g^{-nk-n} = g^{-n(k+1)}$$

□

### Exercise 1.2.4

prove 3,4

#### Warning

In general, it is not the case that if  $g, h \in G$  then  $(gh)^n = g^n h^n$ , this is not true unless  $G$  is abelian

### Proposition 1.4

Let  $G$  be a group and  $g, h, f \in G$ . Then

1. They satisfy the left and right cancellation. More precisely,
  - a. if  $gh = gf$  then  $h = f$
  - b. if  $hg = fg$  then  $h = f$
2. Given  $a, b \in G$  the equations  $ax = b$  and  $ya = b$  have unique solutions for  $x, y \in G$

**Proof of 1-a:** By left-multiplying by  $g^{-1}$ , we have

$$gh = gf \iff g^{-1}gh = g^{-1}gf \iff h = f$$

□  
□

**Proof of 1-b:** similar to 1-a

**Proof of 2:** Let  $x = a^{-1}b$  then

$$ax = aa^{-1}b = b$$

If  $u$  is another solution, then  $au = b = ax$ . By 1-a,  $u = x$ . Similarly,  $y = ba^{-1}$  is the unique solution of  $ya = b$

□

## 1.3 Symmetric Groups

### Definition 1.3.1

Given a non-empty set  $L$ , a *permutation* of  $L$  is a bijection from  $L$  to  $L$ . The set of all permutations of  $L$  is denoted by  $S_L$

### Example 1.3.1

Consider the set  $L = \{1, 2, 3\}$  which has the following different permutations

$$\begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix}$$

Where  $\begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix}$  denotes the bijection

$$\sigma : \{1, 2, 3\} \longrightarrow \{1, 2, 3\}$$

$$\sigma(1) = 1, \sigma(2) = 2, \sigma(3) = 3$$

### Notation

For  $n \in \mathbb{N}$  we denote by  $S_n = S_{\{1, 2, \dots, n\}}$  the set of all permutations of  $\{1, 2, \dots, n\}$ . We have seen that the order of  $S_3 = 3! = 6$ . To consider the general  $S_n$ , we note that for a permutation  $\sigma \in S_n$ , there are  $n$  choices for  $\sigma(1)$ ,  $n - 1$  choices for  $\sigma(2), \dots$ , 1 choice for  $\sigma(n)$ . Thus

### Proposition 1.5

$$|S_n| = n!$$

**Note**

For Möbius quizzes, use “9 dots” for permutations.

**Remark**

Given  $\sigma, \tau \in S_n$  we can compose them to get a new element  $\sigma\tau$ , where

$\sigma\tau = \{1, 2, \dots, n\} \rightarrow \{1, 2, \dots, n\}$  given by  $x \mapsto \sigma(\tau(x))$  Since both  $\sigma, \tau$  are bijections,  $\sigma\tau \in S_n$

**Example 1.3.2**

Compute  $\sigma\tau$  and  $\tau\sigma$  if

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 1 & 2 \end{pmatrix}, \quad \tau = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 3 & 1 \end{pmatrix}$$

Then  $\sigma\tau(1) = \sigma(2) = 4, \dots$  Then  $\sigma\tau = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 2 & 1 & 3 \end{pmatrix}$ , and  $\tau\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 1 & 2 & 4 \end{pmatrix}$

We note that  $\sigma\tau \neq \tau\sigma$

**Note**

For any  $\sigma, \tau \in S_n$  we have that  $\tau\sigma, \sigma\tau \in S_n$  but  $\sigma\tau \neq \tau\sigma$  in general on the other hand, for any  $\sigma, \tau, \mu$  we have  $\sigma(\tau\mu) = (\sigma\tau)\mu$ . Also note the *identity permutation*  $\varepsilon \in S_n$  is defined as

$$\varepsilon = \begin{pmatrix} 1 & 2 & \cdots & n \\ 1 & 2 & \cdots & n \end{pmatrix}$$

Thus for any  $\sigma \in S_n$ , we have  $\sigma\varepsilon = \varepsilon\sigma = \sigma$

Finally, for  $\sigma \in S_n$ , since it is a bijection, there is a unique bijection  $\sigma^{-1} \in S_n$  called the *inverse permutation* of  $\sigma$  such that for all  $x, y \in \{1, 2, \dots, n\}$

$$\sigma^{-1}(x) = y \iff \sigma(y) = x$$

It follows that

$$\sigma(\sigma^{-1}(x)) = \sigma(y) = x$$

and

$$\sigma^{-1}(\sigma(y)) = y$$

i.e we have

$$\sigma\sigma^{-1} = \sigma^{-1}\sigma = \varepsilon$$

**Example 1.3.3**

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 4 & 5 & 1 & 2 & 3 \end{pmatrix}$$

Then

$$\sigma^{-1} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 4 & 5 & 1 & 2 \end{pmatrix}$$

From the above we have

**Proposition 1.6**

$(S_n, \circ)$  is a group, called the *symmetric group of degree n*

**Exercise 1.3.1**

Write down all rotations and reflections that fix an equilateral triangle. Then check why it is the “same” as  $S_3$

**Example 1.3.4**

Consider

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\ 3 & 1 & 7 & 6 & 9 & 4 & 2 & 5 & 8 & 10 \end{pmatrix} \in S_{10}$$

We note that  $1 \rightarrow 3 \rightarrow 7 \rightarrow 2 \rightarrow 1$  and  $4 \rightarrow 6 \rightarrow 4$  and  $5 \rightarrow 9 \rightarrow 8$  and  $10 \rightarrow 10$ . Thus  $\sigma$  can be *decomposed* into one 4-cycle  $(1372)$ , one 2-cycle  $(46)$ , and one 3-cycle  $(598)$  and one 1-cycle  $(10)$  (we usually do not write 1-cycles). Note that these cycles are *pairwise disjoint* and we have

$$\sigma = (1372)(46)(598)$$

We can also write  $\sigma = (46)(598)(1372)$ , or  $\sigma = (64)(985)(7213)$

**Theorem 1.7****Cycle Decomposition**

If Given  $\sigma \in S_n$  with  $\sigma \neq \varepsilon$ , then  $\sigma$  is a product of (one or more) disjoint cycles of length at least 2. This factorization is unique up to the order of the factors.

**Proof:** See bonus 1. □

**Convention**

Every permutation of  $S_n$  can be regarded as a permutation in  $S_{n+1}$  by fixing the number  $n + 1$ , thus

$$S_1 \subseteq S_2 \subseteq \cdots \subseteq S_n \subseteq S_{n+1}$$

## 1.4 Cayley Tables

### Definition 1.4.1

For a finite group  $G$ , defining its operation by means of a table is sometimes convenient. Given  $x, y \in G$ , the product  $xy$  is the entry of the table in the row corresponding to  $x$  and the column corresponding to  $y$ , such a table is a *Cayley table*.

### Remark

By cancellation, the entries in each row or column of a Cayley table are all distinct

### Example 1.4.1

Consider  $(\mathbb{Z}_2, +)$  its Cayley table is

$\mathbb{Z}_2$	[0]	[1]
[0]	[0]	[1]
[1]	[1]	[0]

### Example 1.4.2

Consider the group  $\mathbb{Z}^* = \{1, -1\}$ . Its Cayley table is

$\mathbb{Z}^*$	1	-1
1	1	-1
-1	-1	1

### Note

If we replace 1 by [0] and -1 by [1] the Cayley tables of  $\mathbb{Z}^*$  and  $\mathbb{Z}_2$  become the same. In this case, we say  $\mathbb{Z}^*$  and  $\mathbb{Z}_2$  are *isomorphic* denoted by

$$\mathbb{Z}^* \cong \mathbb{Z}_2$$

**Example 1.4.3**

For  $n \in \mathbb{N}$ , the *cyclic group of order n* is defined by

$$C_n = \{1, a, a^2, \dots, a^{n-1}\} \text{ with } a^n = 1 \text{ and } 1, a, \dots, a^{n-1} \text{ are distinct}$$

The Cayley table of  $C_n$  is as follows

$C_n$	1	$a$	$a^2$	$\dots$	$a^{n-2}$	$a^{n-1}$
1	1	$a$	$a^2$	$\dots$	$a^{n-2}$	$a^{n-1}$
$a$	$a$	$a^2$	$a^3$	$\dots$	$a^{n-1}$	1
$a^2$	$a^2$	$a^3$	$a^4$	$\dots$	1	$a$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$
$a^{n-2}$	$a^{n-2}$	$a^{n-1}$	1	$\dots$	$a^{n-4}$	$a^{n-3}$
$a^{n-1}$	$a^{n-1}$	1	$a$	$\dots$	$a^{n-3}$	$a^{n-2}$

**Proposition 1.8**

Let  $G$  be a group. Up to isomorphism, we have

1. If  $|G| = 1$ , then  $G \cong \{1\}$
2. If  $|G| = 2$ , then  $G \cong C_2$
3. If  $|G| = 3$ , then  $G \cong C_3$
4. If  $|G| = 4$ , then  $G \cong C_4$  or  $G \cong K_4 \cong C_2 \times C_2$

**Proof of 1:** obviously □

**Proof of 2:** If  $|G| = 2$  then  $G = \{1, g\}$  with  $g \neq 1$ . Then  $g^2 = g$  or  $g^2 = 1$ . We note that if  $g^2 = g$ , then  $g = 1$  contradiction. Thus  $g^2 = 1$ . Thus the Cayley table is as follows

$G$	1	$g$
1	1	$g$
$g$	$g$	1

which is the same as  $C_2$  □

**Proof of 3:** If  $|G| = 3$ , then  $G = \{1, g, h\}$  with  $g \neq 1, h \neq 1, g \neq h$ . By cancellation, we have  $gh \neq g, gh \neq h$ , thus  $gh = 1$ . Similarly, we have  $hg = 1$ . Also, on the row for  $g$ , we have  $g1 = g, gh = 1$ . Since all entries in this row are distinct, we have  $g^2 = h$ . Similarly, we have  $h^2 = g$ . Thus we obtain the following Cayley table

$G$	1	$g$	$h$
1	1	$g$	$h$
$g$	$g$	$h$	1
$h$	$h$	1	$g$

Which is the same as  $C_3$ . □

**Proof of 4:** See assignment 1 □

**Exercise 1.4.1**

Consider the symmetry group of a non-square rectangle. How is it related to  $K_4$ ?

**2 Subgroups****2.1 Subgroups****Definition 2.1.1**

Let  $G$  be a group and  $H \subseteq G$ . If  $H$  itself is a group, then we say  $H$  is a *subgroup* of  $G$ .

**Note**

We note that since  $G$  is a group, for  $h_1, h_2, h_3 \in H \subseteq G$ , we have

$$h_1(h_2h_3) = (h_1h_2)h_3$$

Thus

**Proposition 2.1****Subgroup Test**

Let  $G$  be a group,  $H \subseteq G$ . Then  $H$  is a subgroup of  $G$  if

1. If  $h_1, h_2 \in H$ , then  $h_1h_2 \in H$
2.  $1_H \in H$
3. If  $h \in H$ , then  $h^{-1} \in H$

**Exercise 2.1.1**

Prove that  $1_H = 1_G$

**Example 2.1.1**

Given a group  $G$ , then  $\{1\}, G$  are subgroups of  $G$

**Example 2.1.2**

We have a chain of groups

$$(\mathbb{Z}, +) \subseteq (\mathbb{Q}, +) \subseteq (\mathbb{R}, +) \subseteq (\mathbb{C}, +)$$

**Example 2.1.3**

Define

$$\mathrm{SL}_n(\mathbb{R}) = (\mathrm{SL}_n(\mathbb{R}), \cdot) := \{M \in M_n(\mathbb{R}), \det(M) = 1\} \subseteq \mathrm{GL}_n(\mathbb{R})$$

Note that the identity matrix  $I \in \mathrm{SL}_n(\mathbb{R})$ . Let  $A, B \in \mathrm{SL}_n(\mathbb{R})$ , then

$$\det(AB) = \det(A)\det(B) = 1 \cdot 1 = 1$$

and

$$\det(A^{-1}) = \frac{1}{\det(A)} = \frac{1}{1} = 1$$

i.e.  $AB, A^{-1} \in \mathrm{SL}_n(\mathbb{R})$ . By the subgroup test (Proposition 2.1),  $\mathrm{SL}_n(\mathbb{R})$  is a subgroup of  $\mathrm{GL}_n(\mathbb{R})$ . We call  $\mathrm{SL}_n(\mathbb{R})$  the *special linear group of order n over  $\mathbb{R}$*

**Definition 2.1.2**

Given a group  $G$ , we define the *center of  $G$*  to be

$$Z(G) := \{z \in G \mid zg = gz \ \forall g \in G\}$$

**Remark**

$Z(G) = G$  iff  $G$  is abelian.

**Proposition 2.2**

$Z(G)$  is an abelian subgroup of  $G$ .

**Proof:** Note that  $1 \in Z(G)$ . Let  $y, z \in Z(G)$ . Then for all  $g \in G$ , we have

$$(yz)g = y(zg) = y(gz) = (yg)z = (gy)z = g(yz)$$

Thus  $yz \in Z(G)$ . Also, for  $z \in Z(G)$ ,  $g \in G$  we have

$$\begin{aligned} zg = gz &\iff z^{-1}(zg)z^{-1} = z^{-1}(gz)z^{-1} \\ &\iff gz^{-1} = z^{-1}g \end{aligned}$$

Thus  $z^{-1} \in Z(G)$ . By the subgroup test (Proposition 2.1),  $Z(G)$  is a subgroup of  $G$ . Also, by the definition of  $Z(G)$ , we see that it is abelian.  $\square$

**Proposition 2.3**

Let  $H, K$  be subgroups of a group  $G$ . Then  $H \cap G$  is also a subgroup.

**Proof:** Exercise  $\square$

**Proposition 2.4****Finite Subgroup Test**

If  $H \neq \emptyset$  is a finite subset of a group  $G$ , then  $H$  is a subgroup of  $G$  iff  $H$  is closed under its operation.

**Proof:**

( $\Rightarrow$ ) obvious

( $\Leftarrow$ ) For  $H \neq \emptyset$ , let  $h \in H$ . Since  $H$  is closed under its operation, we have  $h, h^2, h^3, \dots \in H$ . Since  $H$  is finite, these elements are not all distinct. Thus  $h^n = h^{n+m}$  for some  $n, m \in \mathbb{N}$ . By cancellation,  $h^m = 1$  and thus  $1 \in H$ . Also,  $1 = h^{m-1}h$  implies that  $h^{-1} = h^{m-1}$  and thus  $h^{-1} \in H$ . By the subgroup test,  $H$  is a subgroup of  $G$ .  $\square$

## 2.2 Alternating Groups

**Definition 2.2.1**

A *transposition*  $\sigma \in S_n$  is a cycle of length 2. i.e.  $\sigma = (ab)$  with  $a, b \in \{1, 2, \dots, n\}$  and  $a \neq b$ .

**Example 2.2.1**

Consider  $(1245) \in S_5$ . Also the composition  $(12)(24)(45)$  can be computed as

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 2 & 3 & 5 & 4 \\ 1 & 4 & 3 & 5 & 2 \\ 2 & 4 & 3 & 5 & 1 \end{pmatrix}$$

Thus we have  $(1245) = (12)(24)(45)$  Also we can show that

$$(1245) = (23)(12)(25)(13)(24)$$

We see from this example that the factorization into transpositions are NOT unique. However, one can prove (see Bonus 2)

**Theorem 2.5****Parity Theorem**

If a permutation  $\sigma$  has two factorizations

$$\sigma = \gamma_1 \gamma_2 \cdots \gamma_r = \mu_1 \mu_2 \cdots \mu_s$$

Where each  $\gamma_i$  and  $\mu_j$  is a transposition, then  $r \equiv s \pmod{2}$

**Definition 2.2.2**

A permutation  $\sigma$  is *even* (or *odd*) if it can be written as a product of an even (or odd) number of transpositions. By the previous theorem, a permutation is either even or odd, but not both.

**Theorem 2.6**

For  $n \geq 2$ , let  $A_n$  denote the set of all even permutations in  $S_n$

1.  $\varepsilon \in A_n$
2. If  $\sigma, \tau \in A_n$ , then  $\sigma\tau \in A_n$  and  $\sigma^{-1} \in A_n$
3.  $|A_n| = \frac{1}{2}n!$

From (1) and (2), we see  $(A_n)$  is a subgroup of  $S_n$  called the *alternating group of degree n*.

**Proof of 1:** We can write  $\varepsilon = (12)(12)$ . Thus  $\varepsilon$  is even. □

**Proof of 2:** if  $\sigma, \tau \in A_n$  we can write  $\sigma = \sigma_1 \cdots \sigma_r$  and  $\tau = \tau_1 \cdots \tau_s$  where  $\sigma_i, \tau_j$  are transpositions and  $r, s$  are even integers. Then

$$\sigma\tau = \sigma_1 \cdots \sigma_r \tau_1 \cdots \tau_s$$

is a product of  $(r + s)$  transpositions and thus  $\sigma\tau \in A_n$ . Also, we note that  $\sigma_i$  is a transposition, we have  $\sigma_i^2 = \varepsilon$  and thus  $\sigma_i^{-1} = \sigma_i$ . It follows that

$$\sigma^{-1} = (\sigma_1 \cdots \sigma_r)^{-1} = \sigma_r^{-1} \cdots \sigma_1^{-1} = \sigma_r \cdots \sigma_1$$

which is an even permutation. □

**Proof of 3:** Let  $O_n$  denote the set of odd permutations in  $S_n$ . Thus  $S_n = A_n \cup O_n$  and the parity theorem implies that  $A_n \cap O_n = \emptyset$ . Since  $|S_n| = n!$ , to prove  $|A_n| = \frac{1}{2}n!$ , it suffices to show that  $|A_n| = |O_n|$ . Let  $\gamma = (12)$  and let  $f : A_n \rightarrow O_n$  be defined by  $f(\sigma) = \gamma\sigma$ . Since  $\sigma$  is even, we have  $\gamma\sigma$  is odd. Thus the map is well-defined. Also, if we have  $\gamma\sigma_1 = \gamma\sigma_2$ , then by cancellation, we get  $\sigma_1 = \sigma_2$ , thus  $f$  is injective. Finally, if  $\tau \in O_n$ , then  $\sigma = \gamma\tau \in A_n$  and  $f(\sigma) = \gamma\sigma = \gamma(\gamma\tau) = \gamma^2\tau = \tau$ . Thus  $f$  is surjective. It follows that  $f$  is a bijection, thus  $|A_n| = |O_n|$ . It follows that  $|A_n| = \frac{1}{2}n! = |O_n|$  □

## 2.3 Orders of Elements

**Notation**

If  $G$  is a group and  $g \in G$ , we denote

$$\langle g \rangle = \{g^k \mid k \in \mathbb{Z}\} = \{\dots, g^{-1}, g^0 = 1, g, g^2, \dots\}$$

Note that  $1 = g^0 \in \langle g \rangle$ . Also, if  $x = g^m, y = g^n \in \langle g \rangle$  With  $m, n \in \mathbb{Z}$ , then  $xy = g^n g^m = g^{n+m} \in \langle g \rangle$  and  $x^{-1} = g^{-m} \in \langle g \rangle$ . By the subgroup test, we have

**Proposition 2.7**

If  $G$  is a group and  $g \in G$ , then  $\langle g \rangle$  is a subgroup of  $G$ .

**Definition 2.3.1**

Let  $G$  be a group with  $g \in G$ . We call  $\langle g \rangle$  the *cyclic subgroup of  $G$  generated by  $g$* . If  $G = \langle g \rangle$  for some  $g \in G$ , then we say  $G$  is *cyclic* and  $g$  a *generator* of  $G$ .

**Example 2.3.1**

Consider  $(\mathbb{Z}, +)$ . Note that for all  $k \in \mathbb{Z}$ , we can write  $k = k \cdot 1$ . Thus we can see  $(\mathbb{Z}, +) = \langle 1 \rangle$ . Similarly,  $(\mathbb{Z}, +) = \langle -1 \rangle$ . We observe, for any integer  $n \in \mathbb{Z}$  with  $n \neq \pm 1$  there exist no  $k \in \mathbb{Z}$  such that  $k \cdot n = 1$ . Thus  $\pm 1$  are the only generators of  $(\mathbb{Z}, +)$ .

**Remark**

Let  $G$  be a group and  $g \in G$ . Suppose there is  $k \in \mathbb{Z}$   $k \neq 0$  such that  $g^k = 1$  then  $g^{-k} = (g^k)^{-1} = 1$ . Thus we can assume  $k \geq 1$ . Then by the well-ordering principle, there exists the smallest positive integer  $n$  such that  $g^n = 1$

**Definition 2.3.2**

Let  $G$  be a group and  $g \in G$ . If  $n$  is the smallest positive integer such that  $g^n = 1$ , then we say the *order* of  $g$  is  $n$ , denoted  $o(g) = n$ . If no such  $n$  exists, we say  $g$  has *infinite order* and write  $o(g) = \infty$

**Proposition 2.8**

Let  $G$  be a group and  $g \in G$  with  $o(g) = n \in \mathbb{N}$ . For  $k \in \mathbb{Z}$  we have

1.  $g^k = 1$  iff  $n \mid k$
2.  $g^k = g^m$  iff  $k \equiv m \pmod{n}$
3.  $\langle g \rangle = \{1, g, g^2, \dots, g^{n-1}\}$  where  $1, g, \dots, g^{n-1}$  are all distinct. In particular, we have  $|\langle g \rangle| = o(g)$

**Proof of 1:**

$(\Leftarrow)$  if  $n \mid k$ , then  $k = nq$  for some  $q \in \mathbb{Z}$ . Thus

$$g^k = g^{nq} = (g^n)^q = 1^q = 1$$

$(\Rightarrow)$  By the division algorithm, we can write  $k = nq + r$  with  $q, r \in \mathbb{Z}$  and  $0 \leq r < n$ . Since  $g^k = 1$  and  $g^n = 1$ , we have

$$g^r = g^{k-nq} = g^k(g^n)^{-q} = 1 \cdot 1^{-q} = 1$$

Since  $0 \leq r < n$  and  $o(g) = n$ , we have  $r = 0$  and hence  $n \mid k$ . □

**Proof of 2:** Note that  $g^k = g^m$  iff  $g^{km} = 1$ . By (1), we have  $n \mid (km)$  i.e.  $k \equiv m \pmod{n}$ . □

**Proof of 3:** It follows from (2) that  $1, g, \dots, g^{n-1}$  are all distinct. Clearly, we have  $\{1, g, \dots, g^{n-1}\} \subseteq \langle g \rangle$ .

To prove the other inclusion, let  $g^k \in \langle g \rangle$  for some  $k \in \mathbb{Z}$ . Write  $k = nq + r$  with  $n, r \in \mathbb{Z}$  and  $0 \leq r < n$ . Then

$$g^k = g^{nq+r} = g^{nq}g^r = (g^n)^qg^r = 1^qg^r = g^r \in \{1, g, \dots, g^{n-1}\}$$

Thus  $\langle g \rangle = \{1, g, \dots, g^{n-1}\}$  □

**Proposition 2.9**

Let  $G$  be a group and  $g \in G$  with  $o(g) = \infty$ . For  $k \in \mathbb{Z}$  we have

1.  $g^k = 1$  iff  $k = 0$
2.  $g^k = g^m$  iff  $k = m$
3.  $\langle g \rangle = \{..., g^{-1}, g^0 = 1, g, ...\}$  where  $g^i$  are all distinct

**Proposition 2.10**

Let  $G$  be a group and  $g \in G$  with  $o(g) = n \in \mathbb{N}$ . If  $d \in \mathbb{N}$ , then  $o(g^d) = \frac{n}{\gcd(n, d)}$ . In particular, if  $d \mid n$ , then  $\gcd(n, d) = d$  and  $o(g^d) = \frac{n}{d}$

**Proof:** Let  $n_1 = \frac{n}{\gcd(n, d)}$  and  $d_1 = \frac{d}{\gcd(n, d)}$ . By a result from Math 135, we have  $\gcd(n_1, d_1) = 1$ . Note that

$$(g^d)^{n_1} = (g^d)^{\frac{n}{\gcd(n, d)}} = (g^n)^{\frac{d}{\gcd(n, d)}} = 1$$

Thus it remains to show that  $n_1$  is the smallest such positive integer. Suppose  $(g^d)^r = 1$  with  $r \in \mathbb{N}$ . Since  $o(g) = n$ , by proposition, we have  $n \mid dr$ . Thus there is  $q \in \mathbb{Z}$  such that  $dr = nq$ . Dividing both sides by  $\gcd(n, d)$  we get

$$d_1 r = \frac{d}{\gcd(n, d)} r = \frac{n}{\gcd(n, d)} q = n_1 q$$

Since  $n_1 \mid d_1 r$  and  $\gcd(n_1, d_1) = 1$ , by a result from Math 135, we get  $n_1 \mid r$  i.e.  $r = n_1 \ell$  for some  $\ell \in \mathbb{Z}$ . Since  $r_1, n_1 \in \mathbb{N}$ , it follows that  $\ell \in \mathbb{N}$ . Since  $\ell \geq 1$ , we get  $r \geq n_1$  □

## 2.4 Cyclic Groups

**Remark**

For a group  $G$ , if  $G = \langle g \rangle$  for some  $g \in G$ , then  $G$  is a cyclic group. For  $a, b \in G$ , we have  $a = g^n, b = g^m$  for some  $m, n \in \mathbb{Z}$ . We have

$$ab = g^n g^m = g^{n+m} = g^{m+n} = g^m g^n = ba$$

**Proposition 2.11**

Every cyclic group is abelian

**Warning**

The converse of the above proposition is not true. For example the Klein 4 group is abelian, but not cyclic.

**Proposition 2.12**

Every subgroup of a cyclic group is cyclic.

**Proof:** Let  $G = \langle g \rangle$  be cyclic and  $H \subseteq G$  a subgroup. If  $H = \{1\}$ , then  $H$  is cyclic. Otherwise, there is  $g^k \in H$  with  $k \in \mathbb{Z} \setminus \{0\}$ . Since  $H$  is a group, we have  $g^{-k} \in H$ . Thus we can assume that  $k \in \mathbb{N}$ . Let  $m$  be the smallest positive integer such that  $g^m \in H$ .

Claim:  $H = \langle g^m \rangle$

Proof is exercise, by division algorithm.  $\square$

### Proposition 2.13

Let  $G = \langle g \rangle$  be a cyclic group with  $o(g) = n$ . Then  $G = \langle g^k \rangle$  iff  $\gcd(k, n) = 1$ .

**Proof:** By proposition,

$$o(g^k) = \frac{n}{\gcd(n, k)} = n$$

$\square$

### Theorem 2.14

### Fundamental Theorem of Finite Cyclic Groups

Let  $G = \langle g \rangle$  be a cyclic group with  $o(g) = n \in \mathbb{N}$ .

1. If  $H$  is a subgroup of  $G$ , then  $H = \langle g^d \rangle$  for some  $d \mid n$ . It follows that  $|H| \mid |G|$ .
2. Conversely, if  $k \mid n$ , then  $\langle g^{\frac{n}{k}} \rangle$  is the unique subgroup of  $G$  with order  $k$ .

**Proof of 1:** By proposition,  $H$  is cyclic. Write  $H = \langle g^m \rangle$  for some  $m \in \mathbb{N} \cup \{0\}$ . Let  $d = \gcd(m, n)$ .

Claim:  $H = \langle g^d \rangle$

Since  $d \mid m$  we have  $m = dk$  for some  $k \in \mathbb{Z}$ . Then

$$g^m = g^{dk} = (g^d)^k \in \langle g^d \rangle$$

Thus  $H = \langle g^m \rangle \subseteq \langle g^d \rangle$ . To prove the other inclusion, since  $d = \gcd(m, n)$ , there is  $x, y \in \mathbb{Z}$  such that  $d = mx + ny$ . Then

$$g^d = g^{mx+ny} = (g^m)^x (g^n)^y = (g^m)^x 1^y = (g^m)^x \in \langle g^m \rangle$$

Thus  $\langle g^d \rangle \subseteq \langle g^m \rangle = H$ . It follows that  $H = \langle g^d \rangle$ . Note that since  $d = \gcd(m, n)$ , we have  $d \mid n$ . By proposition, we have

$$|H| = o(g^d) = \frac{n}{\gcd(n, d)} = \frac{n}{d}$$

Thus  $|H| \mid |G|$   $\square$

**Proof of 2:** By proposition, the cyclic subgroup  $\langle g^{\frac{n}{k}} \rangle$  is of order

$$\frac{n}{\gcd(n, \frac{n}{k})} = \frac{n}{n/k} = k$$

To show uniqueness, let  $K$  be a subgroup of  $G$  with order  $k \mid n$ . By 1, let  $K = \langle g^d \rangle$  where  $d \mid n$ . Then by props, we have,

$$k = |K| = o(g^d) = \frac{n}{\gcd(n, d)} = \frac{n}{d}$$

It follows that  $d = \frac{n}{k}$  and thus  $K = \langle g^{\frac{n}{k}} \rangle$

□

## 2.5 Non-cyclic Groups

### Definition 2.5.1

Let  $X$  be a non-empty subset of a group  $G$ , and let

$$\langle X \rangle := \{x_1^{k_1} \cdots x_m^{k_m} \mid x_i \in X, k_i \in \mathbb{Z}, m \geq 1\}$$

denote the set of all products of powers of (not necessarily distinct) elements of  $X$ . Note that this is clearly a group.  $\langle X \rangle$  is called the *subgroup of  $G$  generated by  $X$* .

### Example 2.5.1

The Klein-4 group  $K_4 = \{1, a, b, c\}$  with  $a^2 = b^2 = c^2 = 1$  and  $ab = c$ . Thus

$$K_4 = \langle a, b \mid a^2 = 1 = b^2 \text{ and } ab = ba \rangle$$

### Example 2.5.2

The symmetric group of order 3  $S_3 = \{\varepsilon, \sigma, \sigma^2, \tau, \tau\sigma, \tau\sigma^2\}$  where  $\sigma^3 = \varepsilon = \tau^2$  and  $\sigma\tau = \tau\sigma^2$  (one can take  $\tau = (12)$  and  $\sigma = (123)$ ) Thus

$$\langle \sigma, \tau \mid \sigma^3 = \varepsilon = \tau^2 \text{ and } \sigma\tau = \tau\sigma^2 \rangle$$

We can also replace  $\sigma, \tau$  with  $\sigma, \tau\sigma$  or  $\sigma, \tau\sigma^2, \dots$ , etc

### Definition 2.5.2

For  $n \geq 2$  the *dihedral group of order  $2n$*  is defined by

$$D_{2n} = \{1, a, \dots, a^{n-1}, b, ba, \dots, ba^{n-1}\}$$

Where  $a^n = 1 = b^2$  and  $aba = b$ . Thus

$$D_{2n} = \langle a, b \mid a^n = 1 = b^2 \text{ and } aba = b \rangle$$

### Note

For  $n = 2$  or  $3$  we have

$$D_4 \cong K_4 \quad \text{and} \quad D_6 \cong S_3$$

### Exercise 2.5.1

For  $n \geq 3$ , consider a regular  $n$ -gon and its group of symmetries. How does it relate to  $D_{2n}$ ?

## 3 Normal Subgroups

### 3.1 Homomorphisms and Isomorphisms

#### Definition 3.1.1

Let  $G, H$  be groups. A mapping  $\alpha : G \rightarrow H$  is a *homomorphism* if

$$\alpha(a *_G b) = \alpha(a) *_H \alpha(b) \quad \forall a, b \in G$$

To simplify notation, we often write

$$\alpha(ab) = \alpha(a)\alpha(b) \quad \forall a, b \in G$$

#### Example 3.1.1

Consider the determinant map

$$\begin{aligned} \det : \mathrm{GL}_n(\mathbb{R}) &\longrightarrow \mathbb{R}^* \\ A &\longmapsto \det A \end{aligned}$$

Since  $\det AB = \det A \det B$ , the mapping  $\det$  is a homomorphism.

#### Proposition 3.1

Let  $\alpha : g \rightarrow H$  be a group homomorphism. Then

1.  $\alpha(1_G) = 1_H$
2.  $\alpha(g^{-1}) = \alpha(g)^{-1} \quad \forall g \in G$
3.  $\alpha(g^k) = \alpha(g)^k \quad \forall k \in \mathbb{Z}$

#### Definition 3.1.2

Let  $\alpha : G \rightarrow H$  be a mapping between groups. If  $\alpha$  is a homomorphism and  $\alpha$  is bijective, we say  $\alpha$  is an *isomorphism*. In this case, we say  $G, H$  are *isomorphic* and write  $G \cong H$ .

#### Proposition 3.2

We have

1. The identity map  $\mathrm{id} : G \rightarrow G$  is an isomorphism.
2. If  $\sigma : G \rightarrow H$  is an isomorphism, then the inverse map  $\sigma^{-1} : h \rightarrow G$  is also an isomorphism.
3. If  $\sigma : G \rightarrow H$  and  $\tau : H \rightarrow K$  is an isomorphism, the composite map  $\tau\sigma : G \rightarrow K$  is also an isomorphism.

So  $\cong$  is (sort-of) an equivalence relation

**Proof:** Exercise. □

**Example 3.1.2**

Let  $\mathbb{R}^+ = \{r \in \mathbb{R} \mid r > 0\}$ . Then  $(\mathbb{R}, +) \cong (\mathbb{R}^+, \cdot)$  since we see that

$$\begin{aligned}\sigma : \mathbb{R} &\rightarrow \mathbb{R}^+ \\ x &\mapsto e^x\end{aligned}$$

is a bijection. Moreover,  $\sigma(x + y) = e^{x+y} = e^x \cdot e^y = \sigma(x)\sigma(y)$  thus  $\sigma$  is an isomorphism.

**Example 3.1.3**

Claim:  $(\mathbb{Q}, +) \not\cong (\mathbb{Q}^*, \cdot)$  Suppose  $\tau : (\mathbb{Q}, +) \rightarrow (\mathbb{Q}^*, \cdot)$  is an isomorphism. Thus  $\tau$  is surjective. So there is some  $q \in \mathbb{Q}$  such that  $\tau(q) = 2$ . Then

$$\tau\left(\frac{q}{2}\right)^2 = \tau\left(\frac{q}{2}\right)\tau\left(\frac{q}{2}\right) = \tau\left(\frac{q}{2} + \frac{q}{2}\right) = \tau(q) = 2$$

Thus  $\tau\left(\frac{q}{2}\right)$  is a rational number whose square is 2, a contradiction.

## 3.2 Cosets and Lagrange's Theorem

**Definition 3.2.1**

Let  $H$  be a subgroup of a group  $G$ . If  $a \in G$ , we define

$$Ha = \{ha \mid h \in H\}$$

to be the *right coset of  $H$  generated by  $a$* . We define the left coset similarly.

**Remark**

Since  $1 \in H$ , we have  $H1 = H = 1H$ . Also  $a \in Ha$  and  $a \in aH$ . Note that in general  $Ha$  and  $aH$  are not subgroups of  $G$ , and  $aH \neq Ha$ . However, if  $G$  is abelian, then  $Ha = aH$ .

**Example 3.2.1**

Let  $K_4 = \{1, a, b, ab\}$ . Let  $H = \{1, a\}$  which is a subgroup of  $K_4$ . Note that since  $K_4$  is abelian, we have  $gH = Hg$  for all  $g \in K_4$ . Then the (right or left) cosets of  $H$  are

$$H1 = \{1, a\} = 1H$$

and

$$Hb = \{b, ab\} = Hab$$

Thus there are exactly two cosets of  $H$  in  $K_4$

**Example 3.2.2**

Let  $S_3 = \{\varepsilon, \sigma, \sigma^2, \tau, \tau\sigma, \tau\sigma^2\}$  with  $\sigma^3 = \varepsilon = \tau^2$  and  $\sigma\tau\sigma = \tau$ . Let  $H = \{\varepsilon, \tau\}$  which is a subgroup of  $S_3$ . Since  $\sigma\tau = \tau\sigma^{-1} = \tau\sigma^2$ , the right cosets of  $H$  are

$$\begin{aligned} H\varepsilon &= \{\varepsilon, \tau\} &= H\tau \\ H\sigma &= \{\sigma, \tau\sigma\} &= H\tau\sigma \\ H\sigma^2 &= \{\sigma^2, \tau\sigma^2\} &= H\tau\sigma^2 \end{aligned}$$

And the left cosets of  $H$  are

$$\begin{aligned} \varepsilon H &= \{\varepsilon, \tau\} &= \tau H \\ \sigma H &= \{\sigma, \tau\sigma^2\} &= \tau\sigma^2 H \\ \sigma^2 H &= \{\sigma^2, \tau\sigma\} &= \tau\sigma H \end{aligned}$$

Notice that  $H\sigma \neq \sigma H$  and  $H\sigma^2 \neq \sigma^2 H$

**Proposition 3.3**

Let  $H$  be a subgroup of a group  $G$  and let  $a, b \in G$ .

1.  $Ha = Hb$  if and only if  $ab^{-1} \in H$ . In particular, we have  $Ha = H$  if and only if  $a \in H$ .
2. If  $a \in Hb$ , then  $Ha = Hb$
3. Either  $Ha = Hb$  or  $Ha \cap Hb = \emptyset$ . Thus, the distinct right cosets of  $H$  forms a partition of  $G$ .

**Proof of 1:**

( $\Rightarrow$ ) If  $Ha = Hb$ , then  $a = 1a \in Ha = Hb$ . Thus  $a = hb$  for some  $h \in H$  and we have  $ab^{-1} = h \in H$ .

( $\Leftarrow$ ) Suppose  $ab^{-1} \in H$  for all  $h \in H$ . Then for all  $h \in H$ ,

$$ha = hab^{-1}b = h(ab^{-1})b \in Hb$$

Thus  $Ha \subseteq Hb$ . Note that if  $ab^{-1} \in H$ , since  $H$  is a subgroup, then

$$(ab^{-1})^{-1} = ba^{-1} \in H$$

Thus for all  $h \in H$ ,

$$hb = h(ba^{-1})a \in Ha$$

Thus  $Hb \subseteq Ha$ . It follows that  $Ha = Hb$ . □

**Proof of 2:** If  $a \in Hb$ , then  $ab^{-1} \in H$ . Thus, by (1), we have  $Ha = Hb$ . □

**Proof of 3:** Two cases:

1. If  $Ha \cap Hb = \emptyset$ , then we are done.
2. If  $Ha \cap Hb \neq \emptyset$ , then there exists  $x \in Ha \cap Hb$ . Since  $x \in Hb$ , by (2), we have  $Hb = Hx$ . Thus

$$Ha = Hx = Hb$$

□

**Remark**

The analogues of the previous proposition also holds for left cosets

1.  $aH = bH$  if and only if  $b^{-1}a \in H$

**Exercise 3.2.1**

Let  $G$  be a group and  $H$  a subset of  $G$ . For  $a, b \in G$ , do we still have  $Ha = Hb$ , or  $Ha \cap Hb = \emptyset$  if  $H$  is not a subgroup of  $G$ .

**Definition 3.2.2**

By the previous proposition, we see that  $G$  can be written as a disjoint union of right cosets of  $H$ . We define the *index*  $[G : H]$  to be the number of disjoint right (or left) cosets of  $H$  in  $G$ . (Note that  $[G : H]$  could be infinite).

**Theorem 3.4****Lagrange's Theorem**

Let  $H$  be a subgroup of a finite group  $G$ . We have  $|H| \mid |G|$  and

$$[G : H] = \frac{|G|}{|H|}$$

**Proof:** Write  $k = [G : H]$  and let  $Ha_1, \dots, Ha_k$  be the distinct right cosets of  $H$  in  $G$ . By prop

$$G = Ha_1 \sqcup \cdots \sqcup Ha_k$$

is a disjoint union. Since  $|Ha_i| = |H|$  for each  $i$ , we have

$$|G| = |Ha_1| + \cdots + |Ha_k| = k|H|$$

It follows that  $|H| \mid |G|$  and  $[G : H] = k = \frac{|G|}{|H|}$ . □

**Corollary 3.5**

1. If  $G$  is a finite group and  $g \in G$  then  $o(g) \mid |G|$
2. If  $G$  is a finite group with  $|G| = n$ , then for all  $g \in G$ , we have  $g^n = 1$

**Proof of 1:** Take  $H = \langle g \rangle$  in the theorem. Note that  $|H| = o(g)$  □

**Proof of 2:** Let  $o(g) = m$  then by (1), we have  $m \mid n$ . Thus

$$g^n = (g^m)^{\frac{n}{m}} = 1^{\frac{n}{m}} = 1$$

**Example 3.2.3**

For  $n \in \mathbb{N}$  with  $n \geq 2$ , let  $\mathbb{Z}_n^*$  be the set of (multiplicative) invertible elements in  $\mathbb{Z}_n$ . Let the *Euler's  $\varphi$ -function*  $\varphi(n)$ , denote the order of  $\mathbb{Z}_n^*$ , i.e.

$$\varphi(n) = |\{[k] \in \mathbb{Z}_n \mid k \in \{0, 1, \dots, n-1\} \text{ and } \gcd(k, n) = 1\}|$$

As a direct consequence of the corollary, we see that if  $a \in \mathbb{Z}$  with  $\gcd(a, n) = 1$ , then  $a^{\varphi(n)} \equiv 1 \pmod{n}$ . This is Euler's Theorem. If  $n = p$ , a prime number, then Euler's Theorem implies that  $a^{p-1} \equiv 1 \pmod{p}$ , which is Fermat's little theorem.

**Recall**

If  $|G| = 2$  then  $G \cong C_2$ , and  $|G| = 3$  then  $G \cong C_3$ .

**Corollary 3.6**

If  $G$  is a group with  $|G| = p$  a prime, then  $G \cong C_p$ , the cyclic group of order  $p$ .

**Proof:** Let  $g \in G$  with  $g \neq 1$ . Then by corollary, we have  $o(g) \mid p$ . Since  $g \neq 1$  and  $p$  is a prime, we have  $o(g) = p$ . By proposition, we have

$$|\langle g \rangle| = o(g) = p$$

It follows that  $G \cong \langle g \rangle \cong C_p$  □

**Corollary 3.7**

Let  $H$  and  $K$  be finite subgroups of a group  $G$ . If  $\gcd(|H|, |K|) = 1$ , then  $H \cap K = \{1\}$ .

**Proof:** Note  $H \cap K$  is a subgroup of  $H$  and  $K$ . So by Lagrange's Theorem, we have  $|H \cap K| \mid |H|$  and  $|H \cap K| \mid |K|$ . It follows that  $|H \cap K| \mid \gcd(|H|, |K|)$ , i.e.  $|H \cap K| = 1$ . Thus  $H \cap K = \{1\}$ . □

### 3.3 Normal Subgroups

**Definition 3.3.1**

Let  $H$  be a subgroup of a group  $G$ . If  $gH = Hg$  for all  $g \in G$ , we say  $H$  is *normal*, denoted by  $H \triangleleft G$ .

**Example 3.3.1**

We have  $\{1\} \triangleleft G$  and  $G \triangleleft G$ .

**Example 3.3.2**

The center  $Z(G)$  of  $G$  is an abelian subgroup of  $G$ . By its definition,  $Z(G) \triangleleft G$ . Thus every subgroup of  $Z(G)$  is normal in  $G$ .

**Example 3.3.3**

If  $G$  is an abelian group, then every subgroup of  $G$  is normal in  $G$ . Note the converse is false (see assignment 3)

**Proposition 3.8****Normality Test**

Let  $H$  be a subgroup of a group  $G$ . The following are equivalent:

1.  $H \triangleleft G$
2.  $gHg^{-1} \subseteq H$  for all  $g \in G$ . We call  $gHg^{-1}$  a *conjugate* of  $H$
3.  $gHg^{-1} = H$  for all  $g \in G$ . (Thus  $H \triangleleft G$  if and only if  $H$  is the only conjugate of  $H$ )

**Proof of (1)  $\implies$  (2):** Let  $ghg^{-1} \in gHg^{-1}$  for some  $h \in H$ . Then by (1),  $gh \in gH = Hg$ , say  $gh = h_1g$  for some  $h_1 \in H$ . Then  $ghg^{-1} = h_1gg^{-1} = h_1 \in H$ .  $\square$

**Proof of (2)  $\implies$  (3):** If  $g \in G$ , then by (2),  $gHg^{-1} \subseteq H$ . Taking  $g^{-1}$  in place of  $g$  in (2), we get  $g^{-1}Hg \subseteq H$ . Thus implies that  $H \subseteq gHg^{-1}$ . Thus  $H = gHg^{-1}$ .  $\square$

**Proof of (3)  $\implies$  (1):** If  $gHg^{-1} = H$ , then  $gH = Hg$ .  $\square$

**Example 3.3.4**

Let  $G = \mathrm{GL}_n(\mathbb{R})$  and  $H = \mathrm{SL}_n(\mathbb{R})$ . For  $A \in G$  and  $B \in H$ , we have

$$\det(ABA^{-1}) = \det A \det B \det A^{-1} = \det B = 1$$

Thus  $ABA^{-1} \in H$  and it follows that  $AHA^{-1} \subseteq H$  for all  $A \in G$ , so by the normality test,  $\mathrm{SL}_n(\mathbb{R}) \triangleleft \mathrm{GL}_n(\mathbb{R})$ .

**Proposition 3.9**

If  $H$  is a subgroup of a group  $G$  with  $[G : H] = 2$ , then  $H \triangleleft G$ .

**Proof:** Let  $g \in G$ . If  $g \in H$ , then  $Hg = H = gH$ . If  $g \notin H$ , since  $[G : H] = 2$ , then  $G = H \sqcup Hg$ , a disjoint union. Then  $Hg = G \setminus H$ . Similarly,  $gH = G \setminus H$ . Thus  $gH = Hg$  for all  $g \in G$  i.e.  $H \triangleleft G$ .  $\square$

**Example 3.3.5**

Let  $A_n$  be the alternating group contained in  $S_n$ . Since  $[S_n : A_n] = 2$ . By proposition, we have  $A_n \triangleleft S_n$ .

**Example 3.3.6**

Let  $D_{2n} = \langle a, b \mid a^n = 1 = b^2 \text{ and } aba = b \rangle$  be the dihedral group of order  $2n$ . Since  $[D_{2n} : \langle a \rangle] = 2$ , by proposition,  $\langle a \rangle \triangleleft D_{2n}$

Let  $H$  and  $K$  be subgroups of a group  $G$ . Then the intersection  $H \cap K$  is the largest subgroup of  $G$  that contained in both  $H$  and  $K$ .

Question: What is the smallest subgroup containing  $H$  and  $K$ ? Note that  $H \cup K$  is the smallest subset

containing  $H$  and  $K$ , but  $H \cup K$  is a subgroup if and only if  $H \subseteq K$  or  $H \supseteq K$ . A more useful subset to consider is the *product*  $HK$  of  $H$  and  $K$  defined as follows

### Definition 3.3.2

$$HK = \{hk \mid h \in H, k \in K\}$$

### Remark

The product of 2 subgroups is not always a subgroup.

### Lemma 3.10

Let  $H$  and  $K$  be subgroups of a group  $G$ , then the following are equivalent:

1.  $HK$  is a subgroup of  $G$
2.  $HK = KH$
3.  $KH$  is a subgroup of  $G$ .

**Proof of (1  $\iff$  2):** Note that (2  $\iff$  3) will follow after exchanging  $H$  and  $K$ . Suppose (2) holds, we have  $1 = 1 \cdot 1 \in HK$ . Also if  $hk \in HK$ , then  $(hk)^{-1} = k^{-1}h^{-1} \in KH = HK$ . Also for  $hk, h_1, k_1 \in HK$ , we have  $kh_1 \in KH = HK$ , say  $kh_1 = h_2k_2$ , it follows that

$$(hk)(h_1k_1) = h(kh_1)k_1 = h(h_2k_2)k_1 = (hh_2)(k_2k_1) \in HK$$

By the subgroup test,  $HK$  is a subgroup of  $G$ . Suppose conversely that (1) holds. Let  $kh \in KH$  with  $k \in K, h \in H$ . Since  $H$  and  $K$  are subgroups of  $G$ , we have  $h^{-1} \in H$ , and  $k^{-1} \in K$ . Since  $HK$  is a subgroup of  $G$ , we have

$$kh = (h^{-1}k^{-1})^{-1} \in HK$$

Thus  $KH \subseteq HK$ , similarly, one can show  $HK \subseteq KH$ . Thus  $HK = KH$ . □

### Proposition 3.11

Let  $H$  and  $K$  be subgroups of a group  $G$ . Then

1. If  $H \triangleleft G$  or  $K \triangleleft G$ , then  $HK = KH$  is a subgroup of  $G$
2. If  $H \triangleleft G$  and  $K \triangleleft G$ , then  $KH \triangleleft G$

**Proof of 1:** Suppose  $H \triangleleft G$  then

$$HK = \bigcup_{k \in K} Hk = \bigcup_{k \in K} kH = KH$$

By lemma,  $HK = KH$  is a subgroup of  $G$ . □

**Proof of 2:** If  $g \in G$  and  $hk \in HK$ , since  $H \triangleleft G$  and  $K \triangleleft G$  we have

$$g^{-1}(hk)g = (g^{-1}hg)(g^{-1}kg) \in HK$$

Thus  $g^{-1}HKg \subseteq HK$  and  $HK \triangleleft G$ . □

**Definition 3.3.3**

Let  $H$  be a subgroup of a group  $G$ . The *normalizer* of  $H$ , denoted by  $N_G(H)$  is defined to be

$$N_G(H) = \{g \in G \mid gH = Hg\}$$

We see that  $H \triangleleft G$  if and only if  $N_G(H) = G$

**Note**

In the proof of the previous proposition, we do not need the full assumption that  $H \triangleleft G$ . We only need  $kH = Hk$  for all  $k \in K$ , i.e.  $k \in N_G(H)$ . Thus

**Corollary 3.12**

Let  $H$  and  $K$  be subgroups of a group  $G$ . If  $K \subseteq N_G(H)$  (or  $H \subseteq N_G(K)$ ) then  $HK = KH$  is a subgroup of  $G$ .

**Theorem 3.13**

If  $H \triangleleft G$  and  $K \triangleleft G$  satisfy  $H \cap K = \{1\}$ , then  $HK \cong H \times K$ .

**Proof:**

Claim: If  $H \triangleleft G$  and  $K \triangleleft G$  satisfy  $H \cap K = \{1\}$  then  $hk = kh$  for all  $h \in H$  and  $k \in K$ .

Consider  $x = hk(kh)^{-1} = hkh^{-1}k^{-1}$ . Note that  $kh^{-1}k^{-1} \in kKh^{-1} = H$  (since  $H \triangleleft G$ ). Thus  $x \in H$ .

Similarly, since  $hkh^{-1} \in hKh^{-1} = K$ , we have  $x \in K$ . Since  $x \in H \cap K = \{1\}$ , we have

$hkh^{-1}k^{-1} = 1$  i.e.  $hk = kh$ .

Since  $H \triangleleft G$ , by proposition,  $HK$  is a subgroup of  $G$ . Define  $\sigma : H \times K \rightarrow HK$  by  $\sigma(h, k) = hk$ .

Claim:  $\sigma$  is an isomorphism.

Let  $(h, k), (h_1, k_1) \in H \times K$ . By claim 1, we have  $h_1k = kh_1$ . Thus

$$\sigma((h, k) \cdot (h_1, k_1)) = \sigma(hh_1, kk_1) = hh_1kk_1 = hkh_1k_1 = \sigma(h, k) \cdot \sigma(h_1, k_1)$$

Thus  $\sigma$  is a homomorphism. Note that by the definition of  $HK$ ,  $\sigma$  is surjective. Also, if

$\sigma(h, k) = \sigma(h_1, k_1)$ , we have  $hk = h_1k_1$ . Thus  $h_1^{-1}h = k_1k^{-1} \in H \cap K = \{1\}$ . Thus

$h_1^{-1}h = 1 = k_1k^{-1}$  i.e.  $h_1 = h$  and  $k_1 = k$ . Thus  $\sigma$  is injective. So  $\sigma$  is an isomorphism and we have

$HK \cong H \times K$ . □

**Corollary 3.14**

Let  $G$  be a finite group, and let  $H$  and  $K$  be normal subgroups such that  $H \cap K = \{1\}$  and  $|H||K| = |G|$ . Then  $G \cong H \times K$ .

**Proof:**

$$|HK| = \frac{|H||K|}{|H \cap K|} = |H||K| = |G|$$

Thus  $HK = G$ , and so a direct application of the theorem gives  $G = HK \cong H \times K$ . □

**Example 3.3.7**

Let  $m, n \in \mathbb{N}$  with  $\gcd(m, n) = 1$ . Let  $G$  be a cyclic group of order  $mn$ . Write  $G = \langle a \rangle$  with  $o(a) = mn$ . Let  $H = \langle a^n \rangle$  and  $K = \langle a^m \rangle$ . Thus  $|H| = o(a^n) = m$  and  $|K| = o(a^m) = n$ . It follows that  $|H||K| = mn = |G|$ . Since  $\gcd(m, n) = 1$ , by corollary, we have  $H \cap K = \{1\}$ . Also, since  $G$  is cyclic and thus abelian, we have  $H \triangleleft G$  and  $K \triangleleft G$ . Then by corollary, we have  $G \cong H \times K$ , i.e.  $C_{mn} \cong C_m \times C_n$ . Hence, to consider finite cyclic groups, it suffices to consider cyclic groups of prime power order.

## 4 Isomorphism Theorems

### 4.1 Quotient Groups

**Remark**

Let  $K$  be a subgroup of  $G$ . Consider the set of right cosets of  $K$ , i.e.  $\{Ka \mid a \in G\}$ . To make it a group, a natural way is to define

$$Ka \cdot Kb = Kab \quad \forall a, b \in G \quad (*)$$

Note that we could have  $Ka = Ka_1$  and  $Kb = Kb_1$  with  $a \neq a_1$  and  $b \neq b_1$ . Thus in order for  $(*)$  to make sense, a necessary condition is

$$Ka = Ka_1 \text{ and } Kb = Kb_1 \implies Kab = Ka_1b_1$$

In this case, we say that the multiplication is *well-defined*.

**Lemma 4.1**

Let  $K$  be a subgroup of a group  $G$ , the following are equivalent:

1.  $K \triangleleft G$
2. For  $a, b \in G$ , the multiplication  $Ka \cdot Kb = Kab$  is well-defined.

**Proof of  $(1 \Rightarrow 2)$ :** Let  $Ka = Ka_1$  and  $Kb = Kb_1$ . Thus  $aa_1^{-1} \in K$  and  $bb_1^{-1} \in K$ . To get  $Kab = Ka_1b_1$ , we need  $ab(a_1b_1)^{-1} \in K$ . Note that since  $K \triangleleft G$ , we have  $aKa^{-1} = K$ . Thus

$$ab(a_1b_1)^{-1} = abb_1^{-1}a_1^{-1} = (abb_1^{-1}a^{-1})(aa_1^{-1}) \in K$$

Thus  $Kab = Ka_1b_1$ . □

**Proof of  $(2 \Rightarrow 1)$ :** If  $a \in G$ , to show  $K \triangleleft G$ , we need  $aka^{-1} \in K$  for all  $k \in K$ . Since  $Ka = Ka$  and  $Kk = K1$ , by (2), we have  $Kak = Ka1$  i.e.  $Kak = Ka$ . It follows that  $aka^{-1} \in K$ . Thus  $K \triangleleft G$ . □

**Proposition 4.2**

Let  $K \triangleleft G$  and write  $G/K = \{Ka \mid a \in G\}$  for the set of all cosets of  $K$ . Then

1.  $G/K$  is a group under the operation  $Ka * Kb = Kab$ .
2. The mapping  $\varphi : G \rightarrow G/K$  given by  $\varphi(a) = Ka$  is a surjective homomorphism.
3. If  $[G : K]$  is finite, then  $|G/K| = [G : K]$ . In particular, if  $|G|$  is finite, then  $|G/K| = \frac{|G|}{|K|}$

**Proof of 1:** By other proposition, the operation is well defined and  $G/K$  is closed under operation. The identity of  $G/K$  is  $K \cdot 1 = K$ . Also, the inverse of  $Ka$  is  $Ka^{-1}$ . Finally, by the associativity of  $G$ , we have

$$Ka(KbKc) = (KaKb)Kc.$$

It follows that  $G/K$  is a group. □

**Proof of 2:**  $\varphi$  is clearly surjective. Also, for  $a, b \in G$ , we have

$$\varphi(a)\varphi(b) = KaKb = Kab = \varphi(ab)$$

so  $\varphi$  is a homomorphism. □

**Proof of 3:** If  $[G : K]$  is finite, by the definition of index,  $|G/K| = [G : K]$ . Also, if  $|G|$  is finite, by Lagrange's Theorem,  $|G/K| = [G : K] = \frac{|G|}{|K|}$  □

### Definition 4.1.1

Let  $K \triangleleft G$ . The group  $G/K$  of all cosets of  $K$  in  $G$  is called the *quotient group of  $G$  by  $K$* . Also, the mapping  $\varphi : G \rightarrow G/K$  given by  $\varphi(a) = Ka$  is called the *coset map*.

### Exercise 4.1.1

List all normal subgroups of  $D_{10}$  and all quotient groups of  $D_{10}/K$ .

## 4.2 Isomorphism Theorems

### Definition 4.2.1

Let  $\alpha : G \rightarrow H$  be a group homomorphism. The *kernel of  $\alpha$*  is defined by

$$\ker \alpha = \{g \in G \mid \alpha(g) = 1_H\} \subseteq G$$

and the *image of  $\alpha$*  is defined by

$$\text{im } \alpha = \alpha(G) = \{\alpha(g) \mid g \in G\} \subseteq H$$

### Proposition 4.3

Let  $\alpha : G \rightarrow H$  be a group homomorphism

1.  $\text{im } \alpha$  is a subgroup of  $H$
2.  $\ker \alpha$  is a normal subgroup of  $G$

**Proof of 1:** Note that  $1_H = \alpha(1_G) \in \text{im } \alpha$ . Also, for  $h_1 = \alpha(g_1), h_2 = \alpha(g_2) \in \text{im } \alpha$ , we have

$$h_1 h_2 = \alpha(g_1) \alpha(g_2) = \alpha(g_1 g_2) \in \text{im } \alpha$$

Also, by proposition,  $\alpha(g)^{-1} = \alpha(g^{-1}) \in \text{im } \alpha$ . By the subgroup test,  $\text{im } \alpha$  is a subgroup of  $H$ . □

**Proof of 2:** For  $\ker \alpha$ , note that  $\alpha(1_G) = 1_H$ . Also, for  $k_1, k_2 \in \ker \alpha$ , then

$$\alpha(k_1 k_2) = \alpha(k_1) \alpha(k_2) = 1 \cdot 1 = 1$$

and

$$\alpha(k_1^{-1}) = \alpha(k_1)^{-1} = 1^{-1} = 1$$

By the subgroup test,  $\ker \alpha$  is a subgroup of  $G$ . Note that if  $g \in H$  and  $k \in \ker \alpha$ , then

$$\alpha(gkg^{-1}) = \alpha(g)\alpha(k)\alpha(g^{-1}) = \alpha(g)1\alpha(g)^{-1} = 1$$

Thus  $g(\ker \alpha)g^{-1} \subseteq \ker \alpha$ . By the normality test,  $\ker \alpha \triangleleft G$ .  $\square$

### Example 4.2.1

Consider the determinant map  $\det : \mathrm{GL}_n(\mathbb{R}) \rightarrow \mathbb{R}^*$  defined by  $A \mapsto \det A$ . Then  $\ker(\det) = \mathrm{SL}_n(\mathbb{R})$ . Thus, we get another proof that  $\mathrm{SL}_n(\mathbb{R}) \triangleleft \mathrm{GL}_n(\mathbb{R})$ .

### Example 4.2.2

Define the *sign* of a permutation  $\sigma \in S_n$  by

$$\mathrm{sgn}(\sigma) = \begin{cases} 1 & \text{if } \sigma \text{ is even} \\ -1 & \text{if } \sigma \text{ is odd} \end{cases}$$

Note that  $\mathrm{sgn} : S_n \rightarrow (\pm 1, \cdot)$  defined by  $\sigma \mapsto \mathrm{sgn}(\sigma)$  is a homomorphism. Also,  $\ker(\mathrm{sgn}) = A_n$ . Thus we have another proof that  $A_n \triangleleft S_n$ .

### Theorem 4.4

### First Isomorphism Theorem

Let  $\alpha : G \rightarrow H$  be a group homomorphism. Then

$$G / \ker \alpha \cong \mathrm{im} \alpha$$

**Proof:** Let  $K = \ker \alpha$ . Since  $K \triangleleft G$ ,  $G/K$  is a group. Define the map

$$\begin{aligned} \bar{\alpha} : G/K &\longrightarrow \mathrm{im} \alpha \\ Kg &\longmapsto \alpha(g) \end{aligned}$$

Note that

$$Kg = Kg_1 \iff gg_1^{-1} \in K \iff \alpha(gg_1^{-1}) = 1 \iff \alpha(g) = \alpha(g_1)$$

Thus,  $\bar{\alpha}$  is well-defined and injective. Also  $\bar{\alpha}$  is clearly surjective. For  $g, h \in G$ , we have

$$\bar{\alpha}(KgKh) = \bar{\alpha}(Kgh) = \alpha(gh) = \alpha(g)\alpha(h) = \bar{\alpha}(Kg)\bar{\alpha}(Kh)$$

Thus  $\bar{\alpha}$  is a group isomorphism and we have  $G / \ker \alpha \cong \mathrm{im} \alpha$ .  $\square$

**Remark**

Let  $\alpha : G \rightarrow H$  be a group homomorphism and  $K = \ker \alpha$ . Let  $\varphi : G \rightarrow G/K$  be the coset map and let  $\bar{\alpha}$  be defined as in the previous proof. We have the following diagram:

$$\begin{array}{ccc} G & \xrightarrow{\alpha} & \text{im } \alpha \\ \varphi \downarrow & \nearrow \bar{\alpha} & \\ G/K & & \end{array}$$

Note that for  $g \in G$ , we have

$$\bar{\alpha}\varphi(g) = \bar{\alpha}(\varphi(g)) = \bar{\alpha}(Kg) = \alpha(g)$$

Thus  $\alpha = \bar{\alpha}\varphi$  on the other hand, if we have  $\alpha = \bar{\alpha}\varphi$ , then the action of  $\bar{\alpha}$  is determined by  $\alpha$  and  $\varphi$  as

$$\bar{\alpha}(Kg) = \bar{\alpha}(\varphi(g)) = \bar{\alpha}\varphi(g) = \alpha(g)$$

Thus  $\bar{\alpha}$  is the only homomorphism  $G/K \rightarrow H$  satisfying  $\bar{\alpha}\varphi = \alpha$ .

**Proposition 4.5**

Let  $\alpha : G \rightarrow H$  be group homomorphism and  $K = \ker \alpha$ . Then  $\alpha$  factors uniquely as  $\alpha = \bar{\alpha}\varphi$  where  $\varphi : g \rightarrow G/K$  is the coset map and  $\bar{\alpha} : G/K \rightarrow H$  is defined by  $\bar{\alpha}(Kg) = \alpha(g)$ . Note that  $\varphi$  is surjective and  $\bar{\alpha}$  is injective.

**Example 4.2.3**

We have seen that  $(\mathbb{Z}, +) = \langle \pm 1 \rangle$  and for  $n \in \mathbb{N}$ ,  $(\mathbb{Z}_n, +) = \langle [1] \rangle$  are cyclic groups. In the following, we will show that these are the only cyclic groups.

Let  $G = \langle g \rangle$  be a cyclic group. Consider  $\alpha : (\mathbb{Z}, +) \rightarrow G$  defined by  $\alpha(k) = g^k$  for all  $k \in \mathbb{Z}$ , which is a group homomorphism. By the definition of  $\langle g \rangle$ ,  $\alpha$  is surjective. Note that  $\ker \alpha = \{k \in \mathbb{Z} \mid g^k = 1\}$ , we have two cases:

1. If  $o(g) = \infty$ , then  $\ker \alpha = \{0\}$ . By the first isomorphism theorem, we have

$$G \cong \mathbb{Z}/\{0\} \cong \mathbb{Z}$$

2. If  $o(g) = n$ , by proposition,  $\ker \alpha = n\mathbb{Z}$ . By the fist isomorphism theorem,

$$G \cong \mathbb{Z}/n\mathbb{Z} \cong \mathbb{Z}_n$$

By (1) and (2), we can conclude that if  $G$  is cyclic, then  $G \cong \mathbb{Z}$  or  $G \cong \mathbb{Z}_n$ .

**Theorem 4.6****Second Isomorphism Theorem**

Let  $H$  and  $K$  be subgroups of a group  $G$  with  $K \triangleleft G$ . Then  $HK$  is a subgroup of  $G$ ,  $K \triangleleft HK$ ,  $H \cap K \triangleleft H$  and  $HK/K \cong H/H \cap K$ .

**Proof:** Since  $K \triangleleft G$ , by proposition,  $HK$  is a subgroup,  $HK = KH$  and  $K \triangleleft HK$ . Consider  $\alpha : H \rightarrow HK/K$  defined by  $\alpha(h) = Kh$ . (note that  $h \in H \subseteq HK$ ). Then  $\alpha$  is a homomorphism (exercise). Also, if  $x \in HK = KH$ , say  $x = kh$ , then

$$Kx = K(kh) = Kh = \alpha(h)$$

Thus  $\alpha$  is surjective. Finally, by proposition,

$$\ker \alpha = \{h \in H \mid Kh = K\} = \{h \in H \mid h \in K\} = H \cap K$$

By the first isomorphism theorem,

$$H/H \cap K \cong HK/K$$

□

**Theorem 4.7****Third Isomorphism Theorem**

Let  $K \subseteq H \subseteq G$  be groups with  $K \triangleleft G$  and  $H \triangleleft G$ . Then  $H/K \triangleleft G/K$  and

$$(G/K)/(H/K) \cong G/H$$

**Proof:** Define  $\alpha : G/K \rightarrow G/H$  by  $\alpha(Kg) = Hg$  for all  $g \in G$ . Note that if  $Kg = Kg_1$ , then  $gg_1^{-1} \in K \subseteq H$ . Thus  $Hg = Hg_1$  and  $\alpha$  is well defined. Clearly,  $\alpha$  is surjective. Note that

$$\ker \alpha = \{Kg \mid Hg = H\} = \{Kg \mid g \in H\} = H/K$$

By the first isomorphism theorem,

$$(G/K)/(H/K) \cong G/H$$

□

## 5 Group Actions

### 5.1 Cayley's Theorem

**Theorem 5.1****Cayley's Theorem**

If  $G$  is a finite group of order  $n$ , then  $G$  is isomorphic to a subgroup of  $S_n$ .

**Proof:** Let  $G = \langle g_1, \dots, g_n \rangle$  and let  $S_G$  be the permutation group of  $G$ . By identifying  $g_i$  with  $i$ , we see that  $S_G \cong S_n$ . Thus it suffices to find a injective homomorphism  $\sigma : G \rightarrow S_G$ . For  $a \in G$ , define  $\mu_a : G \rightarrow G$  by  $\mu_a(g) = ag$  for all  $g \in G$ . Note that  $ag = ag_1$  implies  $g = g_1$  and  $a(a^{-1}g) = g$ . Hence  $\mu_a$  is a bijection and  $\mu_a \in S_G$ . Define  $\sigma : G \rightarrow S_G$  by  $\sigma(a) = \mu_a$ . For  $a, b \in G$ , we have  $\mu_a \mu_b = \mu_{ab}$  and  $\sigma$  is a homomorphism. Also, if  $\mu_a = \mu_b$ , then  $a = \mu_a(1) = \mu_b(1) = b$ . Thus, by the first isomorphism theorem, we have  $G \cong \text{im } \sigma$ , a subgroup of  $S_G \cong S_n$ . □

**Example 5.1.1**

Let  $H$  be a subgroup of a group  $G$  with  $[G : H] = m < \infty$ . Let  $X = \{g_1H, g_2H, \dots, g_mH\}$  be the set of all distinct left cosets of  $H$  in  $G$ . For  $a \in G$ , define  $\lambda_a : X \rightarrow X$  by  $\lambda_a(gH) = agH$  for all  $gH \in X$ . Note that  $agH = ag_1H$  implies that  $gH = g_1H$  and  $a(a^{-1}gH) = gH$ . Hence  $\lambda_a$  is a bijection and thus  $\lambda_a \in S_X$ . Consider  $\tau : G \rightarrow S_X$  defined by  $\tau(a) = \lambda_a$ . For  $a, b \in G$ , we have  $\lambda_{ab} = \lambda_a \lambda_b$  and thus  $\tau$  is a homomorphism. Note that if  $a \in \ker \tau$ , then  $\lambda_a$  is the identity permutation. In particular,  $aH = \lambda_a(H) = H$ . In particular,  $a \in H$ . Thus  $\ker \tau \subseteq H$ .

**Theorem 5.2****Extended Cayley's Theorem**

Let  $H$  be a subgroup of a group  $G$  with  $[G : H] = m < \infty$ . If  $G$  has no normal subgroup contained in  $H$  except for  $\{1\}$ , then  $G$  is isomorphic to a subgroup of  $S_m$ .

**Proof:** Let  $X$  be the set of all distinct left cosets of  $H$  in  $G$ . We have  $|X| = m$  and  $S_X \cong S_m$ . We have seen from the above example that there exist a group homomorphism  $\tau : G \rightarrow S_X$  with  $K = \ker \tau \subseteq H$ . By the first isomorphism theorem, we have  $G/K \cong \text{im } \tau$ . Since  $K \subseteq H$  and  $K \triangleleft G$ , by the assumption, we have  $K = \{1\}$ . It follows that  $G \cong \text{im } \tau$ , a subgroup of  $S_X \cong S_m$ .  $\square$

**Corollary 5.3**

Let  $G$  be a finite group and  $p$  the smallest prime dividing  $|G|$ . If  $H$  is a subgroup of  $G$  with  $[G : H] = p$  then  $H \triangleleft G$ .

**Proof:** Let  $X$  be the set of all distinct left cosets of  $H$  in  $G$ . We have  $|X| = p$  and  $S_X \cong S_p$ . Let  $\tau : G \rightarrow S_X \cong S_p$  be the group homomorphism defined in the above example with  $K := \ker \tau \subseteq H$ . By the first isomorphism theorem, we have  $G/K \cong \text{im } \tau \subseteq S_p$ . Thus  $G/K$  is isomorphic to a subgroup of  $S_p$ . By Lagrange's Theorem, we have  $|G/K| \mid p!$ . Also, since  $K \subseteq H$ , if  $[H : K] = k$ , then

$$|G/K| = \frac{|G|}{|K|} = \frac{|G|}{|H|} \frac{|H|}{|K|} = pk.$$

Thus  $pk \mid p!$  and hence  $k \mid (p-1)!$ . Since  $k \mid |H|$ , which divides  $|G|$  and  $p$  is the smallest prime dividing  $|G|$ , we see every prime divisor of  $k$  must be  $\geq p$  unless  $k = 1$ . Combining this with  $k \mid (p-1)!$ , this forces  $k = 1$ , which implies  $K = H$ , thus  $H \triangleleft G$ .  $\square$

## 5.2 Group Actions

**Definition 5.2.1**

Let  $G$  be a group and  $X$  a non-empty set. A (left) *group action of  $G$  on  $X$*  is a mapping  $G \times X \rightarrow X$  denoted  $(a, x) \mapsto a \cdot x$  such that

1.  $1 \cdot x = x$  for all  $x \in X$
2.  $a \cdot (b \cdot x) = (ab) \cdot x$  for all  $a, b \in G$  and  $x \in X$

In this case, we say  $G$  *acts on  $X$* .

**Remark**

Let  $G$  be a group acting on a set  $X \neq \emptyset$ . For  $a, b \in G$  and  $x, y \in X$ , by (1) and (2), we have

$$a \cdot x = b \cdot y \iff (b^{-1}a) \cdot x = y$$

In particular, we have  $a \cdot x = a \cdot y$  if and only if  $x = y$ .

**Example 5.2.1**

If  $G$  is group, let  $G$  act on itself by conjugation. i.e.  $X = G$ , by  $a \cdot x = axa^{-1}$  for all  $a, x \in G$ . Note that

$$1 \cdot x = 1x1^{-1} = x$$

and

$$a \cdot (b \cdot x) = a(bxb^{-1})a^{-1} = (ab)x(ab)^{-1} = (ab) \cdot x$$

So it is indeed a group action.

**Remark**

For  $a \in G$ , define  $\sigma_a : X \rightarrow X$  by  $\sigma_a(x) = a \cdot x$  for all  $x \in X$ . Then one can show

1.  $\sigma_a \in S_X$ , the permutation group of  $X$
2. The function  $\theta : G \rightarrow S_X$  give  $\theta(a) = \sigma_a$  is a group homomorphism with  
 $\ker \theta = \{a \in G \mid ax = x \ \forall x \in X\}$

Note that the group homomorphism  $\theta : G \rightarrow S_X$  gives an equivalent definition of group action of  $G$  on  $X$ . If  $X = G$  with  $|G| = n$  and  $\ker \theta = \{1\}$ , the map  $\theta : G \rightarrow S_n$  shows that  $G$  is isomorphic to a subgroup of  $S_n$ , which is Cayley's Theorem. Thus, the notion of group action can be viewed as a generalization of the proof of Cayley's Theorem.

**Definition 5.2.2**

Let  $G$  be a group acting on  $X \neq \emptyset$ . Let  $x \in X$ . We call

1.  $G \cdot x = \{g \cdot x \mid g \in G\} \subseteq X$  *The orbit of x*
2.  $S(x) = \{g \in G \mid g \cdot x = x\} \subseteq G$  *The stabilizer of x*

**Proposition 5.4**

Let  $G$  be a group acting on a set  $X \neq \emptyset$  and let  $x \in X$ . Then

1.  $S(x)$  is a subgroup of  $G$ .
2. There exists a bijection from  $G \cdot x$  to  $\{gS(x) \mid g \in G\}$  and thus  $|G \cdot x| = [G : S(x)]$

**Proof of 1:** Since  $1 \cdot x = x$ , we have  $1 \in S(x)$ . Also, if  $g, h \in S(x)$ , then

$$gh \cdot (x) = g \cdot (h \cdot x) = g \cdot x = x$$

and

$$g^{-1} \cdot x = g^{-1} \cdot (g \cdot x) = (g^{-1}g) \cdot x = 1 \cdot x = x$$

Thus  $gh, g^{-1} \in S(x)$ . By the subgroup test,  $S(x)$  is a subgroup of  $G$ . □

**Proof of 2:** Consider the map  $\varphi : G \rightarrow \{gS(x) \mid g \in G\}$  defined by  $\varphi(g \cdot x) = gS(x)$ . Note that

$$g \cdot x = h \cdot x \iff (h^{-1}g) \cdot x = x \iff h^{-1}g \in S(x) \iff hS(x) = gS(x)$$

Thus  $\varphi$  is well-defined and injective. Since  $\varphi$  is clearly surjective,  $\varphi$  is a bijection. It follows that

$$|G \cdot x| = |\{gS(x) \mid g \in G\}| = [G : S(x)]$$

□

### Theorem 5.5

### Orbit Decomposition Theorem

Let  $G$  be a group acting on a finite set  $X \neq \emptyset$ . Let

$$X_f = \{x \in X \mid a \cdot x = x \ \forall a \in G\}$$

(Note that  $x \in X_f$  iff  $|G \cdot x| = 1$ ) Let  $G \cdot x_1, G \cdot x_2, \dots, G \cdot x_n$  denote the distinct non-singleton orbits (i.e.  $|G \cdot x_i| > 1$ ) Then

$$|X| = |X_f| + \sum_{i=1}^n [G : S(x_i)]$$

**Proof:** Note that for  $a, b \in G$  and  $x, y \in X$ ,

$$a \cdot x = b \cdot y \iff (b^{-1}a) \cdot x = y \iff y \in G \cdot x \iff G \cdot y = G \cdot x$$

Thus two orbits are either disjoint, or the same. It follows that the orbits form a disjoint union of  $X$ . Since  $x \in X_f$  iff  $|G \cdot x| = 1$ , the set  $X \setminus X_f$  contains all non-singleton orbits, which are disjoint. Thus by proposition 5.4, we have

$$\begin{aligned} |X| &= |X_f| + \sum_{i=1}^n |G \cdot x_i| \\ &= |X_f| + \sum_{i=1}^n [G : S(x_i)] \end{aligned}$$

□

**Example 5.2.2**

Let  $G$  be a group acting on itself by conjugation i.e.  $g \cdot x = gxg^{-1}$ . Then

$$\begin{aligned} G_f &= \{x \in G \mid gxg^{-1} = x \ \forall g \in G\} \\ &= \{x \in G \mid gx = xg \ \forall g \in G\} \\ &= Z(G) \end{aligned}$$

Also, for  $x \in G$ ,

$$S(x) = \{g \in G \mid gxg^{-1} = x\} = \{g \in G \mid gx = xg\}$$

This set is called the *centralizer* of  $x$  and is denoted by  $S(x) = C_G(x)$ . Finally in this case, the orbit

$$G \cdot x = \{gxg^{-1} \mid g \in G\}$$

is called the *conjugacy class of  $x$* .

By Theorem 5.5,

**Corollary 5.6****Class Equation**

Let  $G$  be a finite group and let  $\{gx_1g^{-1} \mid g \in G\}, \dots, \{gx_ng^{-1} \mid g \in G\}$  denote the distinct non-singleton conjugacy classes, then

$$|G| = |Z(G)| + \sum_{i=1}^n [G : C_G(x_i)]$$

**Lemma 5.7**

Let  $p$  be a prime and  $m \in \mathbb{N}$ . Let  $G$  be a group of order  $p^m$  acting on a finite set  $X \neq \emptyset$ . Let  $X_f$  be defined as in Theorem 5.5. Then we have

$$|X| \equiv |X_f| \pmod{p}$$

**Proof:** By Theorem 5.5, we have

$$|X| = |X_f| + \sum_{i=1}^n [G : S(x_i)] \text{ with } [g : S(x_i)] > 1$$

Since  $[G : S(x_i)]$  divides  $|G| = p^m$  and  $[G : S(x_i)] > 1$ . We have  $p \mid [G : S(x_i)]$  for all  $i$ . It follows that

$$|X| \equiv |X_f| \pmod{p}$$

□

**Theorem 5.8****Cauchy's Theorem**

Let  $p$  be a prime and  $G$  a finite group. If  $p \mid |G|$ , then  $G$  contains an element of order  $p$ .

**Proof:** Define  $X = \{(a_1, \dots, a_p) \mid a_i \in G \text{ and } a_1 \cdots a_p = 1\}$ . Since  $a_p$  is uniquely determined by  $a_1, \dots, a_{p-1}$ , if  $|G| = n$ , we have  $|X| = n^{p-1}$ . Since  $p \mid n$ , we have  $|X| \equiv 0 \pmod{p}$ . Let the group  $\mathbb{Z}_p = (\mathbb{Z}_p, +)$  acts on  $X$  by “cycling”, i.e. for  $k \in \mathbb{Z}_p$ ,

$$k \cdot (a_1, \dots, a_p) = (a_{k+1}, \dots, a_p, a_1, \dots, a_k)$$

One can verify that this action is well defined. Let  $X_f$  be defined as in theorem 5.5. Then  $(a_1, \dots, a_p) \in X_f$  iff  $a_1 = a_2 = \dots = a_p$ . Clearly  $(1, 1, \dots, 1) \in X_f$  and hence  $|X_f| \geq 1$ . Since  $|\mathbb{Z}_p| = p$ , by lemma 5.7, we have

$$|X_f| \equiv |X| \equiv 0 \pmod{p}$$

Since  $|X_f| \equiv 0 \pmod{p}$  and  $|X_f| \geq 1$ . It follows that  $|X_f| \geq p$ . Therefore, there exists  $a \neq 1$  st  $(a, \dots, a) \in X_f$  which implies that  $a^p = 1$ . Since  $p$  is prime and  $a \neq 1$ , the order of  $a$  is  $p$ .  $\square$

## 6 Sylow Theorems

### 6.1 $p$ -groups

#### Definition 6.1.1

Let  $p$  be a prime. A group in which every element has order of a non-negative power of  $p$  is called a  $p$ -group

#### Remark

As a direct consequence of Cauchy's Theorem we have

#### Corollary 6.1

A finite group  $G$  is a  $p$ -group if and only if  $|G|$  is a power of  $p$

#### Lemma 6.2

The center  $Z(G)$  of a non-trivial finite  $p$ -group  $G$  contains more than one element.

**Proof:** The class equation of  $G$  (Cor 5.6) states that

$$|G| = |Z(G)| + \sum_{i=1}^m [G : C_G(x_i)]$$

where  $[G : C_G(x_i)] > 1$ . Since  $G$  is a  $p$ -group, by Cor 6.1,  $p \mid |G|$ . By lemma 5.7,  $|Z(G)| \equiv |G| \equiv 0 \pmod{p}$ . It follows that  $p \mid |Z(G)|$ . Since  $1 \in Z(G)$  and  $|Z(G)| \geq 1$ ,  $Z(G)$  has at least  $p$  elements.  $\square$

**Recall**

If  $H$  is a subgroup of a group  $G$ , then  $N_G(H) = \{g \in G \mid gHg^{-1} = H\}$  is the *normalizer* of  $H$  in  $G$ . In particular,  $H \triangleleft N_G(H)$ .

**Lemma 6.3**

If  $H$  is a  $p$ -subgroup of a finite group  $G$ , then

$$[N_G(H) : H] \equiv [G : H] \pmod{p}$$

**Proof:** Let  $X$  be the set of all left cosets of  $H$  in  $G$ . Hence  $|X| = [G : H]$ . Let  $H$  act on  $X$  by left multiplication. Then for  $x \in G$ , we have

$$\begin{aligned} xH \in X_f &\iff hxH = xH \quad \forall h \in H \\ &\iff x^{-1}hxH = H \quad \forall h \in H \\ &\iff x^{-1}Hx = H \\ &\iff x \in N_G(H) \end{aligned}$$

Thus  $|X_f|$  is the number of cosets  $xH$  with  $x \in N_G(H)$  and hence  $|X_f| = [N_G(H) : H]$ . By lemma 5.7,

$$[N_G(H) : H] = |X_f| \equiv |X| = [G : H] \pmod{p}$$

□

**Corollary 6.4**

Let  $H$  be a  $p$ -subgroup of a finite group  $G$ . If  $p \mid [G : H]$  then  $p \mid [N_G(H) : H]$  and  $N_G(H) \neq H$ .

**Proof:** Since  $p \mid [G : H]$ , by lemma 6.3, we have

$$[N_G(H) : H] \equiv [G : H] \equiv 0 \pmod{p}$$

Since  $p \mid [N_G(H) : H]$  and  $[N_G(H) : H] \geq 1$ , we have  $[N_G(H) : H] \geq p$ . Thus  $N_G(H) \neq H$ .

□

**6.2 Three Sylow Theorems****Recall**

Cauchy's theorem states that if  $p \mid |G|$ , then  $G$  contains an element of order  $p$ . Thus  $|\langle a \rangle| = p$ . The following first Sylow Theorem can be viewed as a generalization of Cauchy's Theorem.

**Theorem 6.5****First Sylow Theorem**

Let  $G$  be a group of order  $p^n m$  where  $p$  is a prime,  $n \geq 1$  and  $\gcd(p, m) = 1$ . Then  $G$  contains a subgroup of order  $p^i$  for all  $1 \leq i \leq n$ . Moreover, every subgroup of  $G$  of order  $p^i$  ( $i < n$ ) is normal in some subgroup of order  $p^{i+1}$ .

**Proof:** We prove this theorem by induction on  $i$ . For  $i = 1$ , since  $p \mid |G|$ , by Cauchy's theorem,  $G$  contains an element  $a$  of order  $p$ , i.e.  $|\langle a \rangle| = p$ . Suppose that the statement holds for some  $1 \leq i < n$ .

Say  $H$  is a subgroup of  $G$  of order  $p^i$ . Then  $p \mid [G : H]$ , by Cor 6.4,  $p \mid [N_G(H) : H]$  and  $[N_G(H) : H] \geq p$ ,  $p \mid [G : H]$ . Then by Cauchy's theorem,  $N_G(H)/H$  contains a subgroup of order  $p$ . Such a group is of the form  $H_1/H$ , where  $H_1$  is a subgroup of  $N_G(H)$  containing  $H$ . Since  $H \triangleleft N_G(H)$ , we have  $H \triangleleft H_1$ . Finally,  $|H_1| = |H||H_1/H| = p^i \cdot p = p^{i+1}$ .  $\square$

### Definition 6.2.1

A subgroup  $P$  of a group  $G$  is said to be a *Sylow p-subgroup* of  $G$  if  $P$  is a maximal  $p$ -group of  $G$  i.e. if  $P \subseteq H \subseteq G$  with  $H$  a  $p$ -group, then  $P = H$ .

As a direct consequence of theorem 6.5,

### Corollary 6.6

Let  $G$  be a group of order  $p^n m$  where  $p$  is a prime,  $n \geq 1$  and  $\gcd(p, m) = 1$ . Let  $H$  be a  $p$ -subgroup of  $G$ .

1.  $H$  is a Sylow  $p$ -subgroup iff  $|H| = p^n$
2. Every conjugate of a Sylow  $p$ -subgroup is a Sylow  $p$ -subgroup.
3. If there is only one Sylow  $p$ -subgroup  $P$ , then  $P \triangleleft G$ .

### Theorem 6.7

### Second Sylow Theorem

If  $H$  is a  $p$ -subgroup of a finite group  $G$ , and  $P$  is any Sylow  $p$ -subgroup of  $G$ , then there exists  $g \in G$  such that  $H \subseteq gPg^{-1}$ . In particular, any two Sylow  $p$ -subgroups are conjugate.

**Proof:** Let  $X$  be the set of all left cosets of  $P$  in  $G$ , and let  $H$  act on  $X$  by left multiplication. By lemma 5.7, we have  $|X_f| \equiv |X| = [G : P] \pmod{p}$ . Since  $p \nmid [G : P]$ , we have  $|X_f| \neq 0$ . Thus there exists  $gP \in X_f$  for some  $g \in G$ . Note that

$$\begin{aligned} gP \in X_f &\iff hgP = gP \quad \forall h \in H \\ &\iff g^{-1}hgP = P \quad \forall h \in H \\ &\iff g^{-1}Hg \subseteq P \\ &\iff H \subseteq gPg^{-1} \end{aligned}$$

If  $H$  is Sylow  $p$ -subgroup, then  $|H| = |P| = |gHg^{-1}|$ , thus  $H = gPg^{-1}$ .  $\square$

### Theorem 6.8

### Third Sylow Theorem

If  $G$  is a finite group and  $p$  a prime with  $p \mid |G|$ , then the number of Sylow  $p$ -subgroups of  $G$  divides  $|G|$  and is of the form  $kp + 1$  for some  $k \in \mathbb{N} \cup \{0\}$ .

**Proof:** By theorem 6.7, the number of Sylow  $p$ -subgroups of  $G$  is the number of conjugates of any of them, say  $P$ . This number is  $[G : N_G(P)]$ . Which is a divisor of  $|G|$ . Let  $X$  be the set of all Sylow  $p$ -subgroups of  $G$  and let  $P$  act on  $X$  by conjugation. Then  $Q \in X_f$  iff  $gQg^{-1} = Q$  for all  $g \in P$ . The latter condition holds iff  $P \subseteq N_G(Q)$ . Both  $P$  and  $Q$  are Sylow  $p$ -subgroups of  $G$  and hence  $N_G(Q)$ . Thus by Cor 6.6, they are conjugate in  $N_G(Q)$ . Since  $Q \triangleleft N_G(Q)$ , this can only occur if  $Q = P$  and  $X_f = \{P\}$ . By lemma 5.7,  $|X| \equiv |X_f| \equiv 1 \pmod{p}$ . Thus  $|X| = kp + 1$  for some  $k \in \mathbb{N} \cup \{0\}$ .  $\square$

**Remark**

Suppose that  $G$  is a group with  $|G| = p^n m$  and  $\gcd(p, m) = 1$ . Let  $n_p$  be the number of  $p$ -subgroups of  $G$ . By the third Sylow theorem, we have  $n_p \mid p^n m$  and  $n_p \equiv 1 \pmod{p}$ . Since  $p \nmid n_p$ , we have  $n_p \mid m$ .

**Example 6.2.1**

Claim: every group of order 15 is cyclic.

Let  $n_p$  be the number of Sylow  $p$ -subgroups of  $G$ . By the third Sylow theorem, we have  $n_3 \mid 5$  and  $n_3 \equiv 1 \pmod{3}$ . Thus  $n_3 = 1$ . Similarly, we have  $n_5 \mid 3$  and  $n_5 \equiv 1 \pmod{5}$ , Thus  $n_5 = 1$ . It follows that there is only one Sylow 3-subgroup and Sylow 5-subgroup, say  $P_3$  and  $P_5$  respectively. Thus  $P_3, P_5 \triangleleft G$ . Consider  $|P_3 \cap P_5|$ , which divides 3 and 5. Thus  $|P_3 \cap P_5| = 1$  and  $P_3 \cap P_5 = \{1\}$ . Also  $|P_3 P_5| = 15 = |G|$ . Thus

$$G \cong P_3 \times P_5 \cong \mathbb{Z}_3 \times \mathbb{Z}_5 \cong \mathbb{Z}_{15}$$

**Example 6.2.2**

Claim: there are two isomorphism classes of groups of order 21.

Let  $G$  be a group of order  $21 = 3 \cdot 7$ . Let  $n_p$  be the number of Sylow  $p$ -subgroups of  $G$ . By the third Sylow theorem, we have  $n_3 \mid 7$  and  $n_3 \equiv 1 \pmod{3}$ . Thus  $n_3 = 1$  or 7. Also we have  $n_7 \mid 3$  and  $n_7 \equiv 1 \pmod{7}$ . Thus  $n_7 = 1$ . It follows that  $G$  has a unique Sylow 7-subgroup, say  $P_7$ . Note that  $P_7 \triangleleft G$  and  $P_7$  is cyclic, say  $P_7 = \langle x : x^7 = 1 \rangle$ . Let  $H$  be a Sylow 3-subgroup. Since  $|H| = 3$ ,  $H$  is cyclic and  $H = \langle y : y^3 = 1 \rangle$ . Since  $P_7 \triangleleft G$ , we have  $yxy^{-1} = x^i$  for some  $0 \leq i \leq 6$ . It follows that

$$x = y^3 xy^{-3} = y^2(yxy^{-1})y^{-2} = y^2 x^i y^{-2} = y(yx^i y^{-1})y^{-1} = yx^{i^2} y^{-1} = x^{i^3}$$

Since  $x^{i^3} = x$  and  $x^7 = 1$ , we have  $i^3 - 1 \equiv 0 \pmod{7}$ . Since  $0 \leq i \leq 6$ , we have  $i = 1, 2, 4$ .

1. If  $i = 1$ , then  $yxy^{-1} = x$ , i.e.  $yx = xy$ . Thus  $G$  is an abelian group. Since  $P_3 \triangleleft G$ ,  $P_7 \triangleleft G$ ,  $P_3 \cap P_7 = \{1\}$  and  $|G| = |P_3 P_7|$ , we have

$$G \cong P_3 \times P_7 \cong \mathbb{Z}_3 \times \mathbb{Z}_7 \cong \mathbb{Z}_{21}$$

2. If  $i = 2$ , then  $yxy^{-1} = x^2$ . Thus

$$G = \{x^i y^j : 0 \leq i \leq 6, 0 \leq j \leq 2, yxy^{-1} = x^2\}$$

3. If  $i = 4$ , then  $yxy^{-1} = x^4$ . Note that

$$\begin{aligned} y^2 xy^{-2} &= y(yxy^{-1})y^{-1} \\ &= yx^4 y^{-1} \\ &= x^{16} = x^2 \end{aligned}$$

Note that  $y^2$  is also a generator of  $H$ . Thus by replacing  $y$  by  $y^2$ , we get back to case 2. It follows that there are two isomorphism classes of groups of order 21.

## 7 Finite Abelian Groups

### 7.1 Primary Decomposition

#### Notation

Let  $G$  be a group and  $m \in \mathbb{Z}$  we define

$$G^{(m)} = \{g \in G \mid g^m = 1\}$$

#### Proposition 7.1

Let  $G$  be an abelian group. Then  $G^{(m)}$  is a subgroup of  $G$ .

**Proof:** We have  $1 = 1^m \in G^{(m)}$ . Also if  $g, h \in G^{(m)}$ , since  $G$  is abelian, we have  $(gh)^m = g^m h^m = 1$  and thus  $gh \in G^{(m)}$ . Finally, if  $g \in G^{(m)}$ , we have

$$(g^{-1})^m = g^{-m} = (g^m)^{-1} = 1$$

and thus  $g^{-1} \in G^{(m)}$ . By the subgroup test,  $G^{(m)}$  is a subgroup of  $G$ .  $\square$

#### Proposition 7.2

Let  $G$  be a finite abelian group with  $|G| = mk$  with  $\gcd(m, k) = 1$ . Then

1.  $G \cong G^{(m)} \times G^{(k)}$
2.  $|G^{(m)}| = m$  and  $|G^{(k)}| = k$

**Proof of 1:** Since  $G$  is abelian, we have  $G^{(m)} \triangleleft (G)$  and  $G^{(k)} \triangleleft G$ . Also, since  $\gcd(m, k) = 1$ , there exist  $x, y \in \mathbb{Z}$  such that  $1 = mx + ky$

Claim:  $G^{(m)} \cap G^{(k)} = \{1\}$

If  $g \in G^{(m)} \cap G^{(k)}$ , then  $g^m = 1 = g^k$ . We have

$$g = g^{mx+ky} = (g^m)^x (g^k)^y = 1$$

Claim:  $G = G^{(m)}G^{(k)}$

If  $g \in G$ , then

$$1 = g^{mk} = (g^m)^k = (g^k)^m$$

It follows that  $g^k \in G^{(m)}$  and  $g^m \in G^{(k)}$ . Thus

$$g = g^{mx+ky} = (g^k)^y (g^m)^x \in G^{(m)}G^{(k)}$$

Combining both claims, by Theorem 3.13, we have

$$G \cong G^{(m)}G^{(k)}$$

$\square$

**Proof of 2:** Write  $|G^{(m)}| = m'$  and  $|G^{(k)}| = k'$ . By (1), we have  $mk = |G| = m'k'$

Claim:  $\gcd(m, k') = 1$

Suppose that  $\gcd(m, k') \neq 1$ . Then there exists a prime  $p$  such that  $p \mid m$  and  $p \mid k'$ . By Cauchy's

theorem, there exists  $g \in G^{(k)}$  with  $o(g) = p$ . Since  $p \mid m$ , we have  $g^m = (g^p)^{\frac{m}{p}} = 1$ , i.e.  $g \in G^{(m)}$ . By (1), we have  $g \in G^{(m)} \cap G^{(k)} = \{1\}$ , which gives a contradiction since  $o(g) = p$ . Thus we have  $\gcd(m, k') = 1$ . Note that since  $m \mid m'k'$  and  $\gcd(m, k') = 1$ , we have  $m \mid m'$ . Similarly, we have  $k \mid k'$ . Since  $mk = m'k'$ , it follows that  $m = m'$  and  $k = k'$ .  $\square$

As a direct consequence of proposition 7.2, we have

### Theorem 7.3

### Primary Decomposition Theorem

Let  $G$  be a finite abelian group with  $|G| = p_1^{n_1} \cdots p_k^{n_k}$  where  $p_1, \dots, p_k$  are distinct primes and  $n_1, \dots, n_k \in \mathbb{N}$ . Then we have

1.  $G \cong G^{(p_1^{n_1})} \times \cdots \times G^{(p_k^{n_k})}$
2.  $|G^{(p_i^{n_i})}| = p_i^{n_i} \quad (1 \leq i \leq k)$ .

### Example 7.1.1

Let  $G = \mathbb{Z}_{13}^*$ . Then  $|G| = 12 = 2^2 3$ . Note that

$$\begin{aligned} G^{(3)} &= \{a \in \mathbb{Z}_{13}^* \mid a^3 = 1\} = \{1, 3, 9\} \\ G^{(4)} &= \{a \in \mathbb{Z}_{13}^* \mid a^4 = 1\} = \{1, 5, 8, 12\} \end{aligned}$$

By theorem 7.3, we have

$$\mathbb{Z}_{13}^* \cong \{1, 5, 8, 12\} \times \{1, 3, 9\}$$

## 7.2 Structure Theorem of Finite Abelian Groups

We have seen that if  $|G| = p$  (a prime), then  $G \cong C_p$ . Also, if  $|G| = p^2$ , then  $G \cong C_{p^2}$  or  $G \cong C_p \times C_p$ . Question How about abelian groups of order  $p^3, p^4$  and  $p^n$  for general  $n \in \mathbb{N}$ .

### Proposition 7.4

Let  $G$  be a finite abelian  $p$ -group that contains only one subgroup of order  $p$ , then  $G$  is cyclic. In other words, if a finite abelian  $p$ -group  $G$  is not cyclic, then  $G$  has at least two subgroups of order  $p$ .

**Proof:** Let  $y \in G$  be of maximum order, i.e.  $o(y) \geq o(x) \forall x \in G$ .

Claim:  $G = \langle y \rangle$ .

Suppose that  $G \neq \langle y \rangle$ . Then the quotient group  $G/\langle y \rangle$  is a nontrivial  $p$ -group, which contains an element  $z$  of order  $p$  by Cauchy's theorem. In particular  $z \neq 1$ . Consider the coset map  $\pi : G \rightarrow G/\langle y \rangle$ . Let  $x \in G$  such that  $\pi(x) = z$ . Since  $\pi(x^p) = \pi(x)^p = z^p = 1$ , we see that  $x^p \in \langle y \rangle$ . Thus  $x^p = y^m$  for some  $m \in \mathbb{Z}$ . Two cases:

1. If  $p \nmid m$  since  $o(y) = p^r$  for some  $r \in \mathbb{N}$ , by prop 2.11,  $o(y^m) = o(y)$ . Since  $y$  is of maximum order, we have  $o(x^p) < o(x) \leq o(y) = o(y^m) = o(x^p)$  which is a contradiction.
2. If  $p \mid m$ , then  $m = pk$  for some  $k \in \mathbb{Z}$ . Thus we have  $x^p = y^m = y^{pk}$ . Since  $G$  is abelian, we have  $(xy^{-k})^p = 1$ . Thus  $xy^{-k}$  belongs to the one and only subgroup of order  $p$ , say  $H$ . On the other hand, the cyclic group  $\langle y \rangle$  contains a subgroup of order  $p$ , which must be the one and only  $H$ . Thus  $xy^{-k} \in \langle y \rangle$ , which implies that  $x \in \langle y \rangle$ . It follows that  $z = \pi(x) = 1$ , a contradiction.

By combining the above two cases, we see that  $G = \langle y \rangle$ . □

### Proposition 7.5

Let  $G \neq \{1\}$  be a finite abelian  $p$ -group. Let  $C$  be a cyclic subgroup of maximum order. Then  $G$  contains a subgroup  $B$  such that

$$G = CB \text{ and } C \cap B = \{1\}$$

### Theorem 7.6

Let  $G \neq 1$  be a finite abelian  $p$ -group. Then  $G$  is isomorphic to a direct product of cyclic groups.

**Proof:** By prop 7.5, there exists a cyclic group  $C_1$  and a subgroup  $B_1$  of  $G$  such that  $G \cong C_1 \times B_1$ . Since  $|B_1| \mid |G|$  by Lagrange's theorem, the group  $B_1$  is also a  $p$ -group. Thus if  $B_1 \neq \{1\}$ , by prop 7.5, there exists a cyclic group  $C_2$  and a subgroup  $B_2$  such that  $B_1 \cong C_2 \times B_2$ . Continue in this way to get cyclic groups  $C_1, \dots, C_k$  until we get  $B_k = \{1\}$  for some  $k \in \mathbb{N}$ . Then  $G \cong C_1 \times \dots \times C_k$ . □

### Remark

One can show that the decomposition of a finite abelian  $p$ -group into a direct product of cyclic groups is unique up to its order.

Combining the remark, theorem 7.6 and theorem 7.3, we have

### Theorem 7.7

### Structure Theorem of Finite Abelian Groups

If  $G$  is a finite abelian group, then

$$G \cong \mathbb{Z}_{p_1^{n_1}} \times \dots \times \mathbb{Z}_{p_k^{n_k}}$$

Where  $\mathbb{Z}_{p_i^{n_i}} = (\mathbb{Z}_{p_i^{n_i}}, +) \cong C_{p_i^{n_i}}$  are cyclic groups of order  $p_i^{n_i}$  ( $1 \leq i \leq k$ ). Note that  $p_i$  are not necessarily distinct. The numbers  $p_i^{n_i}$  are uniquely determined up to their order.

Note that if  $p_1$  and  $p_2$  are distinct primes, then  $C_{p_1^{n_1}} \times C_{p_2^{n_2}} \cong C_{p_1^{n_1} p_2^{n_2}}$ . Thus by combining suitable coprime factors together,

### Theorem 7.8

### Invariant Factor Decomposition of Finite Abelian Groups

Let  $G$  be a finite abelian group. Then

$$G \cong \mathbb{Z}_{n_1} \times \dots \times \mathbb{Z}_{n_r}$$

where  $n_i \in \mathbb{N}, n_1 > 1$  and  $n_1 \mid n_2 \mid \dots \mid n_r$ .

**Example 7.2.1**

Let  $G$  be an abelian group of order 48. Since  $48 = 2^4 \cdot 3$ , by theorem 7.3,  $G \cong H \times \mathbb{Z}_3$ , where  $H$  is an abelian group of order  $2^4$ . The options for  $H$  are  $\mathbb{Z}_{2^4}, \mathbb{Z}_{2^3} \times \mathbb{Z}_2, \mathbb{Z}_{2^2} \times \mathbb{Z}_{2^2}, \mathbb{Z}_{2^2} \times \mathbb{Z}_2 \times \mathbb{Z}_2$  and  $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ . Thus we have

$$\begin{aligned} G &\cong \mathbb{Z}_{2^4} \times \mathbb{Z}_3 \cong \mathbb{Z}_{48} \\ G &\cong \mathbb{Z}_{2^3} \times \mathbb{Z}_2 \times \mathbb{Z}_3 \cong \mathbb{Z}_2 \times \mathbb{Z}_{24} \\ G &\cong \mathbb{Z}_{2^2} \times \mathbb{Z}_{2^2} \times \mathbb{Z}_3 \cong \mathbb{Z}_{2^2} \times \mathbb{Z}_{12} \\ G &\cong \mathbb{Z}_{2^2} \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_3 \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_{12} \\ G &\cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_3 \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_6 \end{aligned}$$

There are 5 non-isomorphic groups in total.

## 8 Rings

### 8.1 Rings

**Definition 8.1.1**

A set  $R$  is a (unitary) *ring* if it has two operations, addition  $+$  and multiplication  $\cdot$  such that  $(R, +)$  is an abelian group and  $(R, \cdot)$  satisfies the closure, associativity and identity properties of a group, in addition to a distributive law. More precisely, if  $R$  is a ring, then for all  $a, b, c \in R$

1.  $a + b \in R$
2.  $a + (b + c) = (a + b) + c$
3. There exists  $0 \in R$  such that  $a + 0 = a = 0 + a$  ( $0$  is called the *zero* of  $R$ )
4. There exists  $-a \in R$  such that  $a + (-a) = 0 = (-a) + a$  ( $-a$  is called the *negative* of  $a$ )
5.  $a + b = b + a$
6.  $ab = a \cdot b \in R$
7.  $a(bc) = (ab)c$
8. There exists  $1 \in R$  such that  $a \cdot 1 = a = 1 \cdot a$  ( $1$  is called the *unity* of  $R$ )
9.  $a(b + c) = ab + ac$  and  $(b + c)a = ba + ca$  (distributive law)

The ring  $R$  is called a *commutative ring* if it also satisfies  $ab = ba$ .

**Example 8.1.1**

$\mathbb{Z}, \mathbb{Q}, \mathbb{R}, \mathbb{C}$  are commutative rings.

**Example 8.1.2**

For  $n \in \mathbb{N}, n \geq 2, \mathbb{Z}_n$  is a commutative ring.

**Example 8.1.3**

For  $n \in \mathbb{N}, n \geq 2, M_n(\mathbb{R})$  is a (non commutative) ring

**Warning**

Note that since  $(R, \cdot)$  is not a group, there is no left or right cancellation. For example, in  $\mathbb{Z}$ ,  $0 \cdot x = 0 \cdot y$  does not imply  $x = y$ .

**Notation**

Given a ring  $R$ , to distinguish the difference between multiples in addition and in multiplication, for  $n \in \mathbb{N}$  and  $a \in R$ , we write

$$\begin{aligned} na &:= \underbrace{a + a + \cdots + a}_{n \text{ times}} \\ a^n &:= \underbrace{a \cdot a \cdot \cdots \cdot a}_{n \text{ times}} \end{aligned}$$

**Recall**

For a group  $G$  and  $g \in G$ , we have  $g^0 = 1$ ,  $g^1 = g$  and  $(g^{-1})^{-1} = g$ . Thus for addition, we have, for a ring  $R$  and  $a \in R$

1.  $\underbrace{0}_{\text{integer}} \cdot a = \underbrace{0}_{\text{zero of } R}$
2.  $\underbrace{1}_{\text{integer}} a = a$
3.  $-(-a) = a$

**Notation**

For  $n \in \mathbb{N}$ , we define

$$(-n)a := \underbrace{(-a) + \cdots + (-a)}_{n \text{ times}}$$

Also, we define  $a^0 = 1$ . If the multiplicative inverse of  $a$  exists,

$$a^{-n} = (a^{-1})^n$$

**Remark**

By Prop 1.2 for  $n, m \in \mathbb{Z}$ , we have

1.  $(na) + (ma) = (n + m)a$
2.  $n(ma) = (nm)a$
3.  $n(a + b) = na + nb$

**Proposition 8.1**

Let  $R$  be a ring and  $r, s \in R$ .

1. If  $0$  is the zero of  $R$ , then

$$0r = 0 = r0$$

2.  $(-r)s = r(-s) = -(rs)$
3.  $(-r)(-s) = rs$
4. For any  $m, n \in \mathbb{Z}$ ,

$$(mr)(ns) = (mn)(rs)$$

**Definition 8.1.2**

A *trivial ring* is a ring of only one element. In this case, we have  $1 = 0$ .

**Remark**

If  $R$  is a ring with  $R \neq \{0\}$ , since  $r = r1$  for all  $r \in R$ , we have  $1 \neq 0$ .

**Example 8.1.4**

Let  $R_1, \dots, R_n$  be rings. We define component-wise operations on the product  $R_1 \times \dots \times R_n$  as follows:

$$\begin{aligned} (r_1, \dots, r_n) + (s_1, \dots, s_n) &= (r_1 + s_1, \dots, r_n + s_n) \\ (r_1, \dots, r_n) \cdot (s_1, \dots, s_n) &= (r_1 s_1, \dots, r_n s_n) \end{aligned}$$

One can check that  $R_1 \times \dots \times R_n$  is a ring. This set is called the *direct product* of  $R_1, \dots, R_n$ .

**Definition 8.1.3**

If  $R$  is a ring, we define the *characteristic* of  $R$  denoted by  $\text{ch}(R)$ , in terms of the order of  $1_R$  in the additive group  $(R, +)$ :

$$\text{ch}(R) = \begin{cases} n & \text{if } o(1_R) = n \in \mathbb{N} \text{ in } (R, +) \\ 0 & \text{if } o(1_R) = \infty \text{ in } (R, +) \end{cases}$$

**Remark**

For  $k \in \mathbb{Z}$ , we write  $kR = 0$  to mean that  $kr = 0$  for all  $r \in R$ .

By Prop 8.1, we have

$$kr = k(1_R r) = (k1_R)r$$

Thus  $kR = 0$  if and only if  $k1_R = 0$ . By Prop 2.6 and 2.7,

**Proposition 8.2**

Let  $R$  be a ring and  $k \in \mathbb{Z}$ .

1. If  $\text{ch}(R) = n \in \mathbb{N}$ , then  $kR = 0$  iff  $n \mid k$
2. If  $\text{ch}(R) = 0$ , then  $kR = 0$  iff  $k = 0$

**Example 8.1.5**

Each of  $\mathbb{Z}, \mathbb{Q}, \mathbb{R}, \mathbb{C}$  has characteristic 0. For  $n \in \mathbb{N}$  with  $n \geq 2$ , the ring  $\mathbb{Z}_n$  has characteristic  $n$ .

**8.2 Subrings****Definition 8.2.1**

A subset  $S$  of a ring  $R$  is a *subring* if  $S$  is a ring itself with  $1_S = 1_R$  (with the same addition and multiplication). Note that properties (2),(3),(7), and (9) of a ring are automatically satisfied. Thus to show that  $S$  is a subring, it suffices to show

Subring Test:

$S \subseteq R$  is a subring if

1.  $1_R \in S$
2. If  $s, t \in S$ , then  $s - t, st \in S$ .

Note that if (2) holds, then  $0 = s - s \in S$  and  $-t = 0 - t \in S$

**Example 8.2.1**

We have a chain of commutative rings

$$\mathbb{Z} \subseteq \mathbb{Q} \subseteq \mathbb{R} \subseteq \mathbb{C}$$

**Example 8.2.2**

If  $R$  is a ring, the *center*  $Z(R)$  of  $R$  is defined to be

$$Z(R) = \{z \in R \mid zr = rz \ \forall r \in R\}$$

Note that  $1_R \in Z(R)$ . Also, if  $s, t \in Z(R)$ , then for  $r \in R$ ,

$$\begin{aligned} (s - t)r &= sr - tr = rs - rt = r(s - t) \\ (st)r &= s(tr) = s(rt) = (sr)t = (rs)t = r(st) \end{aligned}$$

By the subring test,  $Z(R)$  is a subring of  $R$ .

**Example 8.2.3**

Let

$$\mathbb{Z}[i] = \{a + bi \mid a, b \in \mathbb{Z} \text{ and } i^2 = -1\} \subseteq \mathbb{C}.$$

Then one can show that  $\mathbb{Z}[i]$  is a subring of  $\mathbb{C}$ , called the *ring of Gaussian integers*.

## 8.3 Ideals

**Note**

Let  $R$  be a ring and  $A$  an additive subgroup of  $R$ . Since  $(R, +)$  is abelian, we have  $A \triangleleft R$ . Thus we have the additive quotient group

$$R/A = \{r + A \mid r \in R\} \text{ with } r + A = \{r + a \mid a \in A\}$$

Using the known properties about cosets and quotient groups, we have

**Proposition 8.3**

Let  $R$  be a ring and  $A$  an additive subgroup of  $R$ . For  $r, s \in R$ , we have

1.  $r + A = s + A$  iff  $(r - s) \in A$
2.  $(r + A) + (s + A) = (r + s) + A$
3.  $0 + A = A$  is the (additive) identity of  $R/A$
4.  $-(r + A) = (-r) + A$  is the (additive) inverse of  $r + A$
5.  $k(r + A) = kr + A$  for all  $k \in \mathbb{Z}$

**Remark**

Since  $R$  is a ring, it is natural to ask if we could make  $R/A$  a ring. A natural way to define multiplication in  $R/A$  is that

$$(r + A)(s + A) = (rs + A) \quad \forall r, s \in R \quad (*)$$

Note that we could have  $r + A = r_1 + A$  and  $s + A = s_1 + A$  with  $r \neq r_1$  and  $s \neq s_1$ . Thus in order for  $(*)$  to make sense, a necessary condition is

$$r + A = r_1 + A \text{ and } s + A = s_1 + A \implies rs + A = r_1s_1 + A$$

In this case, we say that multiplication  $(r + A)(s + A)$  is *well-defined*.

### Proposition 8.4

Let  $A$  be an additive subgroup of a ring  $R$ . For  $a \in A$  define

$$Ra = \{ra \mid r \in R\} \text{ and } aR = \{ar \mid r \in R\}$$

Then the following are equivalent:

1.  $Ra \subseteq A$  and  $aR \subseteq A \quad \forall a \in A$
2. For  $r, s \in R$ , the multiplication  $(r + A)(s + A)$  is well-defined in  $R/A$ .

**Proof of (1)  $\Rightarrow$  (2):** If  $r + A = r_1 + A$  and  $s + A = s_1 + A$ , we need to show that  $rs + A = r_1s_1 + A$ . Since  $(r - r_1) \in A$  and  $(s - s_1) \in A$ , by (1), we have

$$rs - r_1s_1 = rs - r_1s + r_1s - r_1s_1 = (r - r_1)s + r_1(s - s_1) \in A$$

By proposition 8.3,  $rs + A = r_1s_1 + A$ . □

**Proof of (2)  $\Rightarrow$  (1):** Let  $r \in R$  and  $a \in A$ . By prop 8.1, we have

$$ra + A = (r + A)(a + A) = (r + A)(0 + A) = r0 + A = 0 + A = A$$

Thus  $ra \in A$  and we have  $Ra \subseteq A$ . Similarly, we can show  $aR \subseteq A$ . □

### Definition 8.3.1

An additive subgroup  $A$  of a ring  $R$  is an *ideal* of  $R$  if  $Ra \subseteq R$  and  $aR \subseteq R$ .

#### Ideal Test:

1.  $0 \in A$
2. For  $a, b \in A$  and  $r \in R$ , we have  $a - b \in A$  and  $ra, ar \in A$

### Example 8.3.1

If  $R$  is a ring, then  $\{0\}$  and  $R$  are ideals of  $R$ .

### Example 8.3.2

Let  $R$  be a commutative ring and  $a_1, \dots, a_n \in R$ . Consider the set  $I$  generated by  $a_1, \dots, a_n$  i.e.

$$I = \langle a_1, \dots, a_n \rangle = \{r_1a_1 + \dots + r_na_n \mid r_i \in R\}$$

Then one can show that  $I$  is an ideal.

### Proposition 8.5

Let  $A$  be an ideal of a ring  $R$ . If  $1_R \in A$ , then  $A = R$ .

**Proof:** For every  $r \in R$ , since  $A$  is an ideal and  $1_R \in A$ , we have  $r = r1_R \in A$ . It follows that  $R \subseteq A \subseteq R$  and hence  $R = A$ . □

From the above discussion, we have

**Proposition 8.6**

Let  $A$  be an ideal of a ring  $R$ . Then the additive quotient group  $R/A$  is a ring with multiplication  $(r+A)(s+A) = rs+A$ . The unity of  $R/A$  is  $1+A$ .

**Definition 8.3.2**

Let  $A$  be an ideal of a ring  $R$ . The ring  $R/A$  is called a *quotient ring of  $R$  by  $A$* .

**Definition 8.3.3**

Let  $R$  be a commutative ring and  $A$  an ideal of  $R$ . If  $A = aR = Ra$  for some  $a \in R$ , we say  $A$  is a *principal ideal generated by  $a$*  and is denoted by  $A = \langle a \rangle$ .

**Example 8.3.3**

If  $n \in \mathbb{Z}$ , then  $\langle n \rangle = n\mathbb{Z}$  is an ideal of  $\mathbb{Z}$ .

**Proposition 8.7**

All ideals of  $\mathbb{Z}$  are of the form  $\langle z \rangle$  for some  $z \in \mathbb{Z}$ . If  $\langle n \rangle \neq \{0\}$  and  $n \in \mathbb{N}$ , then the generator is uniquely determined.

**Proof:** Let  $A$  be an ideal of  $\mathbb{Z}$ . If  $A = \{0\}$ , then  $A = \langle 0 \rangle$ . Otherwise, choose  $a \in A$  with  $a \neq 0$  and  $|a|$  minimum. Clearly,  $\langle a \rangle \subseteq A$ . To prove the other inclusion, let  $b \in A$ . By the division algorithm, we have  $b = qa + r$  with  $q, r \in \mathbb{Z}$  and  $0 \leq r < |a|$ . If  $r \neq 0$ , since  $A$  is an ideal, and  $a, b \in A$ , we have  $r = b - qa \in A$  with  $|r| < |a|$ , a contradiction. Thus  $r = 0$  and  $b = qa$ , i.e.  $b \in \langle a \rangle$ . It follows that  $A = \langle a \rangle$ .  $\square$

## 8.4 Isomorphism Theorems

**Definition 8.4.1**

Let  $R, S$  be rings. A mapping  $\theta : R \rightarrow S$  is a *ring homomorphism* if for all  $a, b \in R$

1.  $\theta(a+b) = \theta(a) + \theta(b)$
2.  $\theta(ab) = \theta(a)\theta(b)$
3.  $\theta(1_R) = 1_S$

**Example 8.4.1**

The mapping  $k \mapsto [k]$  from  $\mathbb{Z}$  to  $\mathbb{Z}_n$  is a surjective ring homomorphism.

**Example 8.4.2**

If  $R_1, R_2$  are rings, the projection  $\pi_1 : R_1 \times R_2 \rightarrow R_1$  defined by  $\pi_1(r_1, r_2) = r_1$  is a surjective ring homomorphism. Similarly for  $\pi_2$ .

### Proposition 8.8

Let  $\theta : R \rightarrow S$  be a ring homomorphism.

1.  $\theta(0_R) = 0_S$
2.  $\theta(-r) = -\theta(r)$
3.  $\theta(kr) = k\theta(r)$  for all  $k \in \mathbb{Z}$
4.  $\theta(r^n) = \theta(r)^n$  for all  $n \in \mathbb{N} \cup \{0\}$
5. If  $a \in R^*$  (the set of elements in  $R$  which have multiplicative inverses, such  $a$  is called a *unit* of  $R$ ) then  $\theta(a^k) = \theta(a)^k$  for all  $k \in \mathbb{Z}$ .

### Definition 8.4.2

A *ring isomorphism* is a bijective homomorphism. If there exists an isomorphism between rings  $R$  and  $S$ , we say  $R$  and  $S$  are isomorphic, denoted  $R \cong S$ .

### Exercise 8.4.1

Let  $\theta : R \rightarrow S$  be a bijection of rings with  $\theta(rr') = \theta(r)\theta(r')$  for all  $r, r' \in R$ . Write  $\theta(1_R) = e$ . Prove that  $se = es = s$  for all  $s \in S$  (hence condition 3 for a ring homomorphism can be omitted in this case).

### Definition 8.4.3

Let  $\theta : R \rightarrow S$  be a ring homomorphism. The *kernel* of  $\theta$  is defined by

$$\ker \theta = \{r \in R \mid \theta(r) = 0\} \subseteq R$$

and the *image* of  $\theta$  is defined by

$$\text{im } \theta = \theta(R) = \{\theta(r) \mid r \in R\} \subseteq S$$

We have seen earlier that  $\ker \theta$  and  $\text{im } \theta$  are additive subgroups of  $R$  and  $S$  respectively.

### Proposition 8.9

Let  $\theta : R \rightarrow S$  be a ring homomorphism. Then

1.  $\text{im } \theta$  is a subring of  $S$
2.  $\ker \theta$  is an ideal of  $R$

**Proof of 1:** Since  $\text{im } \theta$  is an additive subgroup of  $S$ , it suffices to show that  $\theta(R)$  is closed under multiplication, and  $1_S \in \theta(R)$ . Note that  $1_S = \theta(1_R) \in \theta(R)$ . Also if  $s_1 = \theta(r_1)$  and  $s_2 = \theta(r_2)$ , then

$$s_1 s_2 = \theta(r_1)\theta(r_2) = \theta(r_1 r_2) \in \theta(R)$$

By the subring test,  $\text{im } \theta$  is a subring of  $S$ . □

**Proof of 2:** Since  $\ker \theta$  is an additive subgroup of  $R$ , it suffices to show that  $ra, ar \in \ker \theta$  for all  $r \in R$ ,  $a \in \ker \theta$ . If  $r \in R$  and  $a \in \ker \theta$ , then

$$\theta(ra) = \theta(r)\theta(a) = \theta(r) \cdot 0 = 0$$

Thus  $ra \in \ker \theta$ . Similarly, one can show  $ar \in \ker \theta$ . Thus  $\ker \theta$  is an ideal of  $R$ .  $\square$

### Theorem 8.10

### First Isomorphism Theorem

Let  $\theta : R \rightarrow S$  be a ring homomorphism. We have  $R/\ker \theta \cong \text{im } \theta$ .

**Proof:** Let  $A = \ker \theta$ . Since  $A$  is an ideal of  $R$ ,  $R/A$  is a ring. Define the map

$$\begin{aligned}\bar{\theta} : R/A &\longrightarrow \text{im } \theta \\ r + A &\longmapsto \theta(r)\end{aligned}$$

Note that  $r + A = s + A \iff r - s \in A \iff \theta(r - s) = 0 \iff \theta(r) = \theta(s)$ . Thus  $\bar{\theta}$  is well defined and injective. Also,  $\bar{\theta}$  is clearly surjective. One can show that  $\bar{\theta}$  is a homomorphism. It follows that  $\bar{\theta}$  is a ring isomorphism and  $\text{im } \theta \cong R/\ker \theta$   $\square$

### Remark

Let  $A, B$  be subsets of a ring  $R$ . If  $A$  and  $B$  are both subrings, then  $A \cap B$  is the largest subring of  $R$  contained in both  $A$  and  $B$ .

### Notation

To consider the smallest subring of  $R$  containing both  $A$  and  $B$  ( $A, B$  not necessarily subrings), we define the *sum*  $A + B$  to be

$$A + B = \{a + b \mid a \in A \text{ and } b \in B\}$$

One can show

### Proposition 8.11

If  $R$  is a ring, then we have

1. If  $A, B$  are subrings of  $R$  (with  $1_A = 1_B = 1_R$ ) then  $A \cap B$  is a subring of  $R$ .
2. If  $A$  is a subring and  $B$  is an ideal of  $R$ , then  $A + B$  is a subring of  $R$
3. If  $A$  and  $B$  are ideals of  $R$ , then  $A + B$  is an ideal of  $R$ .

Using the first isomorphism theorem, one can show (see A8)

### Theorem 8.12

### Second Isomorphism Theorem

Let  $A$  be a subring and  $B$  an ideal of a ring  $R$ . Then  $A + B$  is a subring of  $R$ ,  $B$  is an ideal of  $A + B$ ,  $A \cap B$  is an ideal of  $A$  and

$$(A + B)/B \cong A/(A \cap B)$$

**Theorem 8.13****Third Isomorphism Theorem**

Let  $A$  and  $B$  be ideals of a ring  $R$  with  $A \subseteq B$ . Then  $B/A$  is an ideal in  $R/A$  and

$$(R/A)/(B/A) \cong R/B$$

**Corollary 8.14****Correspondence Theorem / Fourth Isomorphism Theorem**

Let  $R$  be a ring and  $A$  an ideal. There exists a bijection between the set of ideals  $B$  of  $R$  that contains  $A$  and the set of ideals of  $R/A$ .

**Example 8.4.3**

Combining the third isomorphism theorem and the fact that all ideals of  $\mathbb{Z}$  are principal, all ideals of  $\mathbb{Z}_n$  are principal.

**Theorem 8.15****Chinese Remainder Theorem**

Let  $A, B$  be ideals of  $R$

1. If  $A + B = R$  then  $R/(A \cap B) \cong R/A \times R/B$
2. If  $A + B = R$  and  $A \cap B = \{0\}$ , then  $R \cong R/A \times R/B$

**Proof:** (2) obviously follows from (1), so we prove (1). Define  $\theta : R \rightarrow R/A \times R/B$  by  $\theta(r) = (r + A, r + B)$ . Then  $\theta$  is a ring homomorphism with  $\ker \theta = A \cap B$ . To show  $\theta$  is surjective, let  $(s + A, t + B) \in R/A \times R/B$  with  $s, t \in R$ . Since  $A + B = R$ , there exists  $a \in A$  and  $b \in B$  such that  $1 = a + b$ . Let  $r = sb + ta$ . Then

$$s - r = s - sb - ta = s(1 - b) - ta = sa - ta = (s - t)a \in A$$

Thus  $s + A = r + A$ . Similarly, we have  $t + B = r + B$ . Thus  $\theta(r) = (r + A, r + B) = (s + A, t + B)$ . Thus  $\text{im } \theta = R/A \times R/B$ . By the first isomorphism theorem, we have

$$R/(A \cap B) \cong R/A \times R/B$$

□

**Example 8.4.4**

Let  $m, n \in \mathbb{N}$  with  $\gcd(m, n) = 1$ . By Bézout's Lemma, we have  $1 = mr + ns$  for some  $r, s \in \mathbb{Z}$ . Thus  $1 \in m\mathbb{Z} + n\mathbb{Z}$  and hence  $m\mathbb{Z} + n\mathbb{Z} = \mathbb{Z}$ . Also, since  $\gcd(m, n) = 1$ , we have  $m\mathbb{Z} \cap n\mathbb{Z} = mn\mathbb{Z}$ . By CRT,

**Corollary 8.16**

1. If  $m, n \in \mathbb{N}$  with  $\gcd(m, n) = 1$ , then

$$\mathbb{Z}_{mn} \cong \mathbb{Z}_m \times \mathbb{Z}_n$$

2. If  $m, n \in \mathbb{N}$  with  $m, n \geq 2$  and  $\gcd(m, n) \neq 1$ ,  $\varphi(mn) = \varphi(m)\varphi(n)$ , where  $\varphi(m) = |\mathbb{Z}_m^*|$  is the Euler  $\varphi$ -function

**Proof of 2:** From (1), we have

$$(\mathbb{Z}_{mn})^* \cong (\mathbb{Z}_m \times \mathbb{Z}_n)^* \cong \mathbb{Z}_m^* \times \mathbb{Z}_n^*$$

Since  $|\mathbb{Z}_m^*| = \varphi(m)$ , we have  $\varphi(mn) = \varphi(m)\varphi(n)$  □

**Remark**

Let  $m, n \in \mathbb{Z}$  with  $\gcd(m, n) = 1$ . For  $a, b \in \mathbb{Z}$ , by Cor 8.16 and the proof of Thm 8.15, for  $[a] \in \mathbb{Z}_m$  and  $[b] \in \mathbb{Z}_n$ , there exists a unique  $[c] \in \mathbb{Z}_{mn}$  such that  $[c] = [a]$  in  $\mathbb{Z}_m$  and  $[c] = [b]$  in  $\mathbb{Z}_n$ . In other words, the simultaneous congruences  $x \equiv a \pmod{m}$  and  $x \equiv b \pmod{n}$  has a unique solution  $x \equiv c \pmod{mn}$ , which is CRT in Math 135.

**Proposition 8.17**

If  $R$  is a ring with  $|R| = p$  a prime, then  $R \cong \mathbb{Z}_p$ .

**Proof:** Define  $\theta : \mathbb{Z}_p \rightarrow R$  by  $\theta[k] = k1_R$ . Note that since  $R$  is an additive group and  $|R| = p$ , by Lagrange,  $o(1_R) = 1$  or  $p$ . Since  $1_R \neq 0$ , we have  $o(1_R) = p$ . Thus

$$[k] = [m] \iff p \mid (k - m) \iff (k - m)1_R = 0 \iff k1_R = m1_R \text{ in } R$$

Thus  $\theta$  is well-defined and injective. Since  $|\mathbb{Z}_p| = p = |R|$  and  $\theta$  is injective,  $\theta$  is also surjective. Finally, one can prove that  $\theta$  is a ring homomorphism. It follows that  $\theta$  is a ring isomorphism and  $R \cong \mathbb{Z}_p$ . □

**Exercise 8.4.2**

What are the possible rings  $R$  with  $|R| = p^2$  where  $p$  is a prime.

## 9 Commutative Rings

### 9.1 Integral Domains and Fields

**Definition 9.1.1**

Let  $R$  be a ring. We say  $u \in R$  is a *unit* if  $u$  has a multiplicative inverse in  $R$ . Denoted by  $u^{-1}$ . We have  $uu^{-1} = 1 = u^{-1}u$ . Note that if  $u$  is a unit in  $R$ , and  $r, s \in R$  we have

$$ur = us \implies s = s \quad \text{and} \quad ru = su \implies r = s$$

Let  $R^*$  denote the set of all units in  $R$ . One can show that  $(R, \cdot)$  is group called the *group of units* of  $R$ .

**Example 9.1.1**

Note that 2 is a unit in  $\mathbb{Q}$ , but not a unit in  $\mathbb{Z}$ . We have  $\mathbb{Q}^* = \mathbb{Q} \setminus \{0\}$  and  $\mathbb{Z}^* = \{\pm 1\}$ .

**Exercise 9.1.1**

Consider the ring of Gaussian integers  $\mathbb{Z}[i]$ . One can show  $\mathbb{Z}[i]^* = \{\pm 1, \pm i\}$

Hint: Prove that  $|xy| = |x||y|$  and  $|x| = 1 \iff x$  is a unit.

**Definition 9.1.2**

A ring  $R \neq \{0\}$  is a *division ring* if  $R^* = R \setminus \{0\}$  i.e. every nonzero element of  $R$  is a unit of  $R$ . A commutative division ring is called a *field*.

**Example 9.1.2**

$\mathbb{Q}, \mathbb{R}, \mathbb{C}$  are fields, but  $\mathbb{Z}$  is not a field.

**Example 9.1.3**

We recall that the equation  $[a][x] = [1]$  in  $\mathbb{Z}_n$  has a solution iff  $\gcd(a, n) = 1$  for all  $a \in \{1, 2, \dots, p-1\}$ . Thus  $\mathbb{Z}_p^* = \mathbb{Z}_p \setminus \{0\}$  and  $\mathbb{Z}_p$  is a field. However, if  $n$  is not prime, say  $n = ab$  with  $1 < a, b < n$ . Then the nonzero congruence classes  $[a], [b]$  are not units in  $\mathbb{Z}_n$  as there is no solution for  $[a][x] = [1]$  and hence  $\mathbb{Z}_n^* \neq \mathbb{Z}_n \setminus \{0\}$ . Thus  $\mathbb{Z}_n$  is a field iff  $n$  is a prime.

**Remark**

If  $R$  is a division ring or a field, then its only ideals are  $\{0\}$  or  $R$  since if  $A \neq \{0\}$  is an ideal of  $R$ , then  $0 \neq a \in A$  implies that  $1 = aa^{-1} \in A$ . By prop 8.5,  $A = R$ . As a consequence, if we have a ring homomorphism a field  $F$  to a ring  $S$ , since  $\ker \theta$  is an ideal,  $\ker \theta = \{0\}$  or  $F$ . Hence  $\theta$  is either injective or the zero map.

**Exercise 9.1.2**

(This is quite hard) Prove that every finite division ring is a field.

**Note**

For  $r, s \in \mathbb{R}$ , we have  $rs = 0$  implies that  $r = 0$  or  $s = 0$ . This property is useful in solving equations, say if  $x^2 - x - 6 = 0$  i.e.  $(x-3)(x-2) = 0$ , then  $x = 3$  or  $x = 2$ . However, such property is not always true. For example,  $[2][3] = [6] = [0]$  in  $\mathbb{Z}_6$ , but  $[2] \neq [0]$  and  $[3] \neq [0]$ .

**Exercise 9.1.3**

Solve  $[(x-2)(x-3)] = [0]$  in  $\mathbb{Z}_6$ .

**Definition 9.1.3**

Let  $R \neq \{0\}$  be a ring. For  $0 \neq a \in R$ , we say  $a$  is a *zero divisor* if there exists  $0 \neq b \in R$  such that  $ab = 0$ .

**Example 9.1.4**

In  $\mathbb{Z}_6$ ,  $[2], [3], [4]$  are zero divisors since  $[2][3] = [0] = [4][3]$ .

**Example 9.1.5**

Note that in  $M_2(\mathbb{R})$ , we have

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Thus  $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$  is a zero divisor.

**Proposition 9.1**

Given a ring  $R$ , for all  $a, b, c \in R$ , the following are equivalent:

1. If  $ab = 0$ , then  $a = 0$  or  $b = 0$
2. If  $ab = ac$  and  $a \neq 0$ , then  $b = c$
3. If  $ba = ca$  and  $a \neq 0$ , then  $b = c$

**Proof:** We prove  $(1) \iff (2)$  and the proof of  $(1) \iff (3)$  is similar.

$(1) \implies (2)$ : Let  $ab = ac$  with  $a \neq 0$ . Then  $a(b - c) = 0$ . By (1), since  $a \neq 0$ , we have  $b - c = 0$  i.e.  $b = c$ .

$(2) \implies (1)$ : Let  $ab = 0$  in  $R$ . Two cases:

1. If  $a = 0$ , then we are done
2. If  $a \neq 0$ , then  $ab = 0 = a0$ . By (2), since  $a \neq 0$ , we have  $b = 0$ .

□

**Definition 9.1.4**

A commutative ring  $R \neq \{0\}$  is an *integral domain* if it has no zero divisors i.e. if  $ab = 0$  in  $R$ , then  $a = 0$  or  $b = 0$ .

**Example 9.1.6**

$\mathbb{Z}$  is an integral domain since for  $a, b \in \mathbb{Z}$ ,  $ab = 0$  implies  $a = 0$  or  $b = 0$ .

**Example 9.1.7**

Note that if  $p$  is a prime, if  $p \mid ab$  then  $p \mid a$  or  $p \mid b$  i.e.  $[a][b] = [0]$  in  $\mathbb{Z}_p$  implies that  $[a] = [0]$  or  $[b] = [0]$ . Thus  $\mathbb{Z}_p$  is an integral domain. However, if  $n = ab$  with  $1 < a, b < n$ , then  $[a][b] = [0]$  with  $[a] \neq [0]$  and  $[b] \neq [0]$ . Thus  $\mathbb{Z}_n$  is an integral domain iff  $n$  is a prime.

**Proposition 9.2**

Every field is an integral domain.

**Proof:** Let  $ab = 0$  in a field  $R$ . We need to show that  $a = 0$  or  $b = 0$ . Two cases:

1. If  $a = 0$ , then we are done
2. If  $a \neq 0$ , since  $R$  is a field,  $a \in R^*$  and  $a^{-1} \in R$  exists. Then

$$b = 1 \cdot b = (a^{-1}ab) = a^{-1}(ab) = a^{-1}0 = 0$$

Thus  $R$  is an integral domain. □

**Remark**

Using the above proof, one can show that every subring of a field is an integral domain.

**Remark**

The converse of Prop 9.2 is not true, for example,  $\mathbb{Z}$  is an integral domain but not a field.

**Example 9.1.8**

The Gaussian integers  $\mathbb{Z}[i]$  is an integral domain, but not a field.

**Proposition 9.3**

Every finite integral domain is a field.

**Proof:** Let  $R$  be a finite integral domain and  $0 \neq a \in R$ . Consider the map

$$\begin{aligned}\theta : R &\longrightarrow R \\ r &\longmapsto ar\end{aligned}$$

Since  $R$  is an integral domain,  $ar = as$  and  $a \neq 0$  implies that  $r = s$ . Hence  $\theta$  is injective. Since  $R$  is finite,  $\theta$  is surjective. In particular, there is  $b \in R$  such that  $ab = 1$ . Since  $R$  is commutative, we have  $ab = ba$ , i.e.  $a$  is a unit. Hence  $R^* = R \setminus \{0\}$  and  $R$  is a field. □

**Recall**

The characteristic of a ring  $R$ , denoted by  $\text{ch}(R)$  is the order of  $1_R$  in  $(R, +)$ . We write  $\text{ch}(R) = 0$  if  $o(1_R) = \infty$  and  $\text{ch}(R) = n$  if  $o(1_R) = n \in \mathbb{N}$ .

### Proposition 9.4

The characteristic of any integral domain is either 0 or a prime  $p$ .

**Proof:** Let  $R$  be an integral domain. Two cases:

1. If  $\text{ch}(R) = \infty$ , then we are done.
2. Note that since  $R \neq \{0\}$ , we have  $n \neq 1$ . If  $\text{ch}(R) = n \in \mathbb{N} \setminus \{1\}$ , suppose that  $n$  is not prime, say  $n = ab$  with  $1 < a, b < n$ . If 1 is the unity of  $R$ , then by Prop 8.1, we have

$$(a \cdot 1)(b \cdot 1) = (ab)(1 \cdot 1) = n \cdot 1 = 0$$

Since  $R$  is an integral domain, we have  $a \cdot 1 = 0$  or  $b \cdot 1 = 0$ , which leads to a contradiction since  $o(1) = n$ . Thus  $n$  is prime.

□

### Remark

Let  $R$  be an integral domain with  $\text{ch}(R) = p$ , a prime. For  $a, b \in R$ , we have

$$(a + b)^p = a^p + \binom{p}{1}a^{p-1}b + \binom{p}{2}a^{p-2}b^2 + \dots + \binom{p}{p-1}ab^{p-1} + b^p$$

Since  $p$  is a prime,  $p \mid \binom{p}{i}$  for all  $1 \leq i \leq (p-1)$ . Since  $\text{ch}(R) = p$ , we have

$$(a + b)^p = a^p + b^p$$

## 9.2 Prime Ideals and Maximal Ideals

Let  $p$  be a prime and  $a, b \in \mathbb{Z}$ . We recall from Math 135 that  $p \mid ab$  implies  $p \mid a$  or  $p \mid b$ . In other words, if  $ab \in p\mathbb{Z}$ , then  $a \in p\mathbb{Z}$  or  $b \in p\mathbb{Z}$ .

### Definition 9.2.1

Let  $R$  be a commutative ring. An ideal  $P \neq R$  of  $R$  is a *prime ideal* if whenever  $r, s \in R$  satisfy  $rs \in P$ , then  $r \in P$  or  $s \in P$ .

### Example 9.2.1

$\{0\}$  is prime ideal of  $\mathbb{Z}$

### Example 9.2.2

For  $n \in \mathbb{N}$  with  $n \geq 2$ ,  $n\mathbb{Z}$  is a prime ideal of  $\mathbb{Z}$  if and only if  $n$  is prime.

### Proposition 9.5

If  $R$  is a commutative ring, then an ideal  $P$  of  $R$  is a prime ideal if and only if  $R/P$  is an integral domain.

**Proof:** Since  $R$  is a commutative ring, so is  $R/P$ . Note that

$$R/P \neq \{0\} \iff 0 + P \neq 1 + P \iff 1 \notin P \iff P \neq R.$$

Also, for  $r, s \in R$ , we have

$$\begin{aligned} P \text{ is a prime ideal} &\iff rs \in P \text{ implies that } r \in P \text{ or } s \in P \\ &\iff (r + P)(s + P) = 0 + P \text{ implies that} \\ &\quad r + P = 0 + P \text{ or } s + P = 0 + P \\ &\iff R/P \text{ is an integral domain} \end{aligned}$$

□

### Definition 9.2.2

Let  $R$  be a commutative ring. An ideal  $M \neq R$  of  $R$  is a *maximal ideal* if whenever  $A$  is an ideal such that  $M \subseteq A \subseteq R$ , then  $A = M$  or  $A = R$ .

### Remark

Let  $M$  be a maximal ideal of  $R$  and  $r \notin M$ . Then  $M \subseteq \langle r \rangle + M \subseteq R$ . Since  $M \neq \langle r \rangle + M$ , we have  $\langle r \rangle + M = R$ .

### Proposition 9.6

If  $R$  is a commutative ring, then an ideal  $M$  of  $R$  is a maximal ideal if and only if  $R/M$  is a field.

**Proof:** Since  $R$  is a commutative ring, so is  $R/M$ . Note that

$$R/M \neq \{0\} \iff 0 + M \neq 1 + M \iff 1 \notin M \iff M \neq R$$

Also, for  $r \in R$ , note that  $r \notin M$  iff  $r + M \neq 0 + M$ . Thus we have

$$\begin{aligned} M \text{ is a maximal ideal} &\iff \langle r \rangle + M = R \text{ for any } r \notin M \\ &\iff 1 \in \langle r \rangle + M \text{ for all } r \notin M \\ &\iff \forall r \notin M, \text{ there exists } s \in R \text{ s.t. } 1 + M = rs + M \\ &\iff \forall r + M \neq 0 + M, \text{ there exists } s + M \in R/M \text{ s.t. } (r + M)(s + M) = 1 + M \\ &\iff R/M \text{ is a field} \end{aligned}$$

□

Combining Prop 9.2, 9.5 and 9.6, we have

### Corollary 9.7

Every maximal ideal of a commutative ring is a prime ideal.

**Remark**

The converse of Cor 9.7 is not true. For example, in  $\mathbb{Z}$ ,  $\{0\}$  is a prime ideal, but not a max ideal.

**Example 9.2.3**

Consider the ideal  $\langle x^2 + 1 \rangle$  in the ring  $\mathbb{Z}[x]$ . The map  $\theta : \mathbb{Z}[x] \rightarrow \mathbb{Z}[i]$  defined by  $\theta(f(x)) = f(i)$  is surjective since  $\theta(a + bx) = a + bi$ . Also, one can check that the kernel of the map is  $\langle x^2 + 1 \rangle$  (see Piazza). By the first isomorphism theorem, we have  $\mathbb{Z}[x]/\langle x^2 + 1 \rangle \cong \mathbb{Z}[i]$ . Since  $\mathbb{Z}[i]$  is an integral domain, but not a field, we conclude that the ideal  $\langle x^2 + 1 \rangle$  is prime, but not maximal. Note that  $\langle x^2 + 1 \rangle \subsetneq \langle x^2 + 1, 3 \rangle \subsetneq \mathbb{Z}[x]$ . We have  $\mathbb{Z}[x]/\langle x^2 + 1, 3 \rangle \cong \mathbb{Z}_3[x]/\langle x^2 + 1 \rangle$  and  $x^2 + 1$  is irreducible in  $\mathbb{Z}_3[x]$ .

### 9.3 Fields of Fractions

Let  $R$  be an integral domain. We now construct a field  $F$  of all fractions  $\frac{r}{s}$  from  $R$ .

Let  $R$  be an integral domain and let  $D = R \setminus \{0\}$ . Consider the set  $X = R \times D$ . We say  $(r, s) \equiv (r_1, s_1)$  on  $X$  iff  $rs_1 = r_1s$ . One can show that  $\equiv$  is an equivalence relation on  $X$ . Motivated by the case  $R = \mathbb{Z}$ , we now define the fraction  $\frac{r}{s}$  to be the equivalence class  $[(r, s)]$  of the pair  $(r, s)$  on  $X$ . Let  $F$  denote the set of all these fractions, i.e.

$$F = \left\{ \frac{r}{s} \mid r \in R, s \in D \right\} = \left\{ \frac{r}{s} \mid r, s \in R, s \neq 0 \right\}$$

The addition and multiplication of  $F$  are defined by

$$\frac{r}{s} + \frac{r_1}{s_1} = \frac{rs_1 + r_1s}{ss_1} \quad \text{and} \quad \frac{r}{s} \cdot \frac{r_1}{s_1} = \frac{rr_1}{ss_1}$$

Note that  $ss_1 \neq 0$  since  $R$  is an integral domain and thus these operations are well-defined. Then one can show that with the above defined addition and multiplication,  $F$  becomes a field with the zero being  $\frac{0}{1}$ , the unity  $\frac{1}{1}$ , the negative of  $\frac{r}{s}$  is  $\frac{-r}{s}$ . Moreover, if  $\frac{r}{s} \neq 0$  in  $F$ , then  $r \neq 0$  and thus  $\frac{s}{r} \in F$  and we have

$$\frac{r}{s} \cdot \frac{s}{r} = \frac{rs}{sr} = \frac{rs}{rs} = \frac{1}{1} \in F$$

In addition, we have  $R \cong R'$  where  $R' = \left\{ \frac{r}{1} \mid r \in R \right\} \subseteq F$ . Thus we have

**Theorem 9.8**

Let  $R$  be an integral domain. Then there exists a field  $F$  consisting of fractions  $\frac{r}{s}$  with  $r, s \in R$  and  $s \neq 0$ . By identifying  $r = \frac{r}{1}$  for all  $r \in R$ , we can view  $R$  as a subring of  $F$ . The field  $F$  is called the *field of fractions of  $R$* .

## 10 Polynomial Rings

### 10.1 Polynomials

Let  $R$  be a ring and  $x$  a variable. Let

$$R[x] = \{f(x) = a_0 + a_1x + \cdots + a_mx^m \mid m \in \mathbb{N} \cup \{0\} \text{ and } a_i \in R \ (0 \leq i \leq m)\}$$

Such  $f(x)$  is called a *polynomial in  $x$  over  $R$* . If  $a_m \neq 0$ , we say  $f(x)$  has degree  $m$ , denoted by  $\deg f = m$ , and we say the  $a_m$  is *leading coefficient* of  $f(x)$ . If the leading coefficient  $a_m = 1$ , we say  $f(x)$  is *monic*. If  $\deg f = 0$ , then  $f(x) = a_0 \in R \setminus \{0\}$ . In this case, we say  $f(x)$  is a *constant polynomial*. Note that

$$f(x) = 0 \iff a_0 = 0, a_1 = 0, \dots$$

0 is also a constant polynomial and we define  $\deg 0 = -\infty$ . Let  $f(x) = a_0 + a_1x + \cdots + a_mx^m \in R[x]$  and  $g(x) = b_0 + b_1x + \cdots + b_nx^n \in R[x]$  with  $m \leq n$ . Then we write  $a_i = 0$  for all  $m+1 \leq i \leq n$ . We can define addition and multiplication on  $R[x]$  as follows:

$$f(x) + g(x) = (a_0 + b_0) + (a_1 + b_1)x + \cdots + (a_n + b_n)x^n$$

and

$$\begin{aligned} f(x)g(x) &= (a_0 + a_1x + \cdots + a_mx^m)(b_0 + b_1x + \cdots + b_nx^n) \\ &= a_0b_0 + (a_0b_1 + a_1b_0)x + (a_2b_0 + a_1b_1 + a_0b_2)x^2 + \cdots + (a_mb_n)x^{m+n} \\ &= c_0 + c_1x + c_2x^2 + \cdots + c_{m+n}x^{m+n} \end{aligned}$$

where

$$c_i = a_0b_i + a_1b_{i-1} + \cdots + a_{i-1}b_1 + a_ib_0$$

### Proposition 10.1

Let  $R$  be a ring and  $x$  a variable

1.  $R[x]$  is a ring
2.  $R$  is a subring of  $R[x]$
3. If  $Z = Z(R)$  is the center of  $R$ , then  $Z(R[x]) = Z[x]$

**Proof:** (1) and (2) are left as exercises. □

**Proof of 3:** Let  $f(x) = a_0 + a_1x + \cdots + a_mx^m \in Z[x]$  and  $g(x) = b_0 + b_1x + \cdots + b_nx^n \in R[x]$ . We have

$$f(x)g(x) = c_0 + c_1x + c_2x^2 + \cdots + c_{m+n}x^{m+n}$$

with

$$c_i = a_0b_i + a_1b_{i-1} + \cdots + a_{i-1}b_1 + a_ib_0$$

Since  $a_i \in Z$ , we have  $a_i b_j = a_j b_i$  for all  $i, j$ . Thus we get  $f(x)g(x) = g(x)f(x)$  for all  $g(x) \in R[x]$  and hence  $Z[x] \subseteq Z(R[x])$ . To show the other inclusion, if  $h(x) = c_0 + c_1x + \cdots + c_sx^s \in Z(R[x])$ , then for all  $r \in R$ , we have  $h(x)r = rh(x)$ . Thus,  $c_i r = r c_i$  for all  $r \in R$  and  $0 \leq i \leq s$ . Thus,  $c_i \in Z$  and  $Z(R[x]) \subseteq Z[x]$ . It follows that  $Z(R[x]) = Z[x]$ . □

**Warning**

Although  $f(x) \in R[x]$  can be used to define a function from  $R$  to  $R$ , the polynomial is not the same as the function it defines. For example, if  $R = \mathbb{Z}_2$ , there are only 4 different functions from  $\mathbb{Z}_2$  to  $\mathbb{Z}_2$ . However, the polynomial ring  $\mathbb{Z}_2[x]$  is an infinite set.

**Proposition 10.2**

Let  $R$  be an integral domain. Then

1.  $R[x]$  is an integral domain.
2. If  $f \neq 0$  and  $g \neq 0$  in  $R[x]$ , then

$$\deg(fg) = \deg(f) + \deg(g) \quad (\text{product formula})$$

3. The units in  $R[x]$  are  $R^*$ , the units in  $R$ .

**Proof of 1,2:** Suppose that  $f(x) \neq 0$  and  $g(x) \neq 0$  are polynomials in  $R[x]$ , say

$f(x) = a_0 + a_1x + \dots + a_mx^m$  and  $g(x) = b_0 + b_1x + \dots + b_nx^n$  with  $a_m \neq 0$  and  $a_n \neq 0$ . Then

$$f(x)g(x) = (a_m b_n)x^{m+n} + \dots + a_0 b_0$$

Since  $R$  is an integral domain,  $a_m b_n \neq 0$  and thus  $f(x)g(x) \neq 0$ . It follows that  $R[x]$  is an integral domain. Moreover,

$$\deg(fg) = \deg(f) + \deg(g)$$

Thus (1) and (2) follow. □

**Proof of 3:** Let  $u(x) \in R[x]$  be a unit with inverse  $v(x)$ . Since  $u(x)v(x) = 1$ , by (2) we have

$$\deg(u) + \deg(v) = \deg 1 = 0$$

Since  $u(x)v(x) = 1$ , we have  $u(x) \neq 0$  and  $v(x) \neq 0$ . Since  $\deg u \geq 0$  and  $\deg v \geq 0$ , the above equation implies that  $\deg u = 0 = \deg v$ . Thus  $u(x), v(x)$  are units in  $R$  and hence  $R[x]^* \subseteq R^*$ . Since  $R^* \subseteq R[x]^*$ , we have  $R[x]^* = R^*$ . □

**Remark**

Note that in  $\mathbb{Z}_4[x]$ , we have  $2x \cdot 2x = 4x^2 = 0$  thus the product formula doesn't hold here since  $\mathbb{Z}_4$  is not an integral domain.

**Remark**

To extend the product formula in Prop 10.2 to 0, we define  $\deg 0 = \pm\infty$ .

## 10.2 Polynomials Over a Field

**Definition 10.2.1**

Let  $F$  be a field and  $f(x), g(x) \in F[x]$ . We say  $f(x)$  divides  $g(x)$ , denoted by  $f(x) | g(x)$ , if there exists  $q(x) \in F[x]$  such that  $g(x) = q(x)f(x)$ .

### Proposition 10.3

Let  $F$  be a field.  $f(x), g(x), h(x) \in F[x]$ .

1. If  $f(x) | g(x)$  and  $g(x) | h(x)$ , then  $f(x) | h(x)$ . (transitivity of divisibility)
2. If  $f(x) | g(x)$  and  $f(x) | g(x)$ , then  $f(x) | (g(x)u(x) + h(x)v(x))$  for any  $u(x), v(x) \in F[x]$  (divisibility of integer combinations)

### Recall

For  $a, b \in \mathbb{Z}$  if  $a | b$  and  $b | a$  and  $a, b > 0$ , then  $a = b$ . The following is its analogue in  $F[x]$

### Proposition 10.4

Let  $F$  be a field and  $f(x), g(x) \in F[x]$  be monic polynomials. If  $f(x) | g(x)$  and  $g(x) | f(x)$ , then  $f(x) = g(x)$ .

**Proof:** Since  $f(x) | g(x)$  and  $g(x) | f(x)$ , we have  $g(x) = r(x)f(x)$  and  $f(x) = s(x)g(x)$  for some  $r(x), s(x) \in F[x]$ . Then  $f(x) = s(x)r(x)f(x)$ . By Prop 10.2, we have  $\deg f = \deg s + \deg r + \deg f$ , which implies that  $\deg s = \deg r = 0$ . Thus  $f(x) = sg(x)$  for some  $s \in F$ . Since both  $f(x)$  and  $g(x)$  are monic, we have  $s = 1$  and hence  $f(x) = g(x)$ .  $\square$

### Proposition 10.5

### Division Algorithm

Let  $F$  be a field and  $f(x), g(x) \in F[x]$  with  $f(x) \neq 0$ . Then there exist unique  $q(x), r(x) \in F[x]$  such that

$$g(x) = q(x)f(x) + r(x) \quad \text{with } \deg r < \deg f$$

Note that this includes the case for  $r = 0$  (this explains why we define  $\deg 0 = -\infty$ ).

**Proof:** We first prove by induction that such  $q(x)$  and  $r(x)$  exist. Write  $m = \deg f$  and  $n = \deg g$ . If  $n < m$ , then  $g(x) = 0 \cdot f(x) + g(x)$ . Suppose  $n \geq m$  and the result holds for all  $g(x) \in F[x]$  with  $\deg g < n$ . Write  $f(x) = a_0 + a_1x + \dots + a_mx^m$  with  $a_m \neq 0$  and  $g(x) = b_0 + b_1x + \dots + b_nx^n$ . Since  $F$  is a field,  $a_m^{-1}$  exists. Consider

$$\begin{aligned} g_1(x) &= g(x) - b_n a_m^{-1} x^{n-m} f(x) \\ &= (b_n x^n + b_{n-1} x^{n-1} + \dots + b_0) - b_n a_m^{-1} x^{n-m} (a_m x^m + \dots + a_1 x + a_0) \\ &= 0 \cdot x^n + (b_{n-1} - b_n a_m^{-1} a_{m-1}) x^{n-1} + \dots \end{aligned}$$

Since  $\deg g_1 < n$ , by induction, there exist  $q_1(x), r_1(x) \in F[x]$  such that  $g_1(x) = q_1(x)f(x) + r_1(x)$  with  $\deg r_1 < \deg f$ . It follows that

$$\begin{aligned} g(x) &= q_1(x) + b_n a_m^{-1} x^{n-m} f(x) \\ &= (q_1(x)f(x) + r_1(x)) + b_n a_m^{-1} x^{n-m} f(x) \\ &= (q_1(x) + b_n a_m^{-1} x^{n-m})f(x) + r_1(x) \end{aligned}$$

By taking  $q(x) = q_1(x) + b_n a_m^{-1} x^{n-m}$  and  $r(x) = r_1(x)$ , we have

$$g(x) = q(x)f(x) + r(x) \quad \text{with } \deg r < \deg f$$

To prove uniqueness, suppose that we have  $g(x) = q_1(x)f(x) + r_1(x)$  with  $\deg r_1 < \deg f$ . Then

$$r(x) - r_1(x) = (q_1(x) - q(x))f(x).$$

If  $q_1(x) - q(x) \neq 0$ , we get

$$\deg(r - r_1) = \deg((q_1 - q)f) = \deg(q_1 - q) + \deg f \geq \deg f$$

which leads to a contradiction since  $\deg(r - r_1) < \deg f$ . Thus  $q_1(x) - q(x) = 0$  and hence  $r(x) - r_1(x) = 0$ . It follows that  $q_1(x) = q(x)$  and  $r_1(x) = r(x)$ .  $\square$

### Note

For  $a, b \in \mathbb{Z} \setminus \{0\}$ , the Bézout Lemma states that  $\gcd(a, b) = ax + by$  for some  $x, y \in \mathbb{Z}$ .

### Proposition 10.6

Let  $F$  be a field and  $f(x), g(x) \in F[x] \setminus \{0\}$ . Then there exists  $d(x) \in F[x]$  which satisfies the following conditions:

1.  $d(x)$  is monic
2.  $d(x) \mid f(x)$  and  $d(x) \mid g(x)$
3. If  $e(x) \mid f(x)$  and  $e(x) \mid g(x)$ , then  $e(x) \mid d(x)$
4.  $d(x) = u(x)f(x) + v(x)g(x)$  for some  $u(x), v(x) \in F[x]$

Note that if both  $d(x)$  and  $d_1(x)$  satisfy the above conditions, since  $d(x) \mid d_1(x)$  and  $d_1(x) \mid d(x)$  and both of them are monic, By prop 10.4, we have  $d(x) = d_1(x)$ . We call such  $d(x)$  the greatest common divisor of  $f(x)$  and  $g(x)$  denote it by  $d(x) = \gcd(f(x), g(x))$

**Proof:** Consider the set  $X = \{u(x)f(x) + v(x)g(x), u(x), v(x) \in F[x]\}$ . Since  $f(x) \in X$ , the set  $X$  contains nonzero polynomials and thus monic polynomials (since  $F$  is a field, if  $h(x) \in X$  with leading coefficient  $a$ , then  $a^{-1}h(x) \in X$  is monic). Among all monic polynomials in  $X$ , choose  $d(x) = u(x)f(x) + v(x)g(x)$  of minimal degree. Then (1) and (4) are satisfied. For (3), if  $e(x) \mid f(x)$  and  $e(x) \mid g(x)$ , since  $d(x) = u(x)f(x) + v(x)g(x)$  by prop 10.3,  $e(x) \mid d(x)$ . It remains to prove (2). By the division algorithm, write  $f(x) = q(x)d(x) + r(x)$  with  $\deg r < \deg d$ . Then

$$\begin{aligned} r(x) &= f(x) - q(x)d(x) \\ &= f(x) - q(x)(u(x)f(x) + v(x)g(x)) \\ &= (1 - q(x)u(x))f(x) - (q(x)v(x))g(x) \end{aligned}$$

Note that if  $r(x) \neq 0$ , write  $c \neq 0$  be the leading coefficient of  $r(x)$ . Since  $F$  is a field,  $c^{-1}$  exists. The above expression shows that  $c^{-1}r(x)$  is a monic polynomial of  $X$  with  $\deg(c^{-1}r) = \deg r < \deg d$ , which contradicts the choice of  $d(x)$ . Thus  $r(x) = 0$  and we have  $d(x) \mid f(x)$ . Similarly, we can show  $d(x) \mid g(x)$ . Thus (2) follows.  $\square$

**Recall**

$p \in \mathbb{Z}$  is a prime if  $p \geq 2$  and whenever  $p = ab$  with  $a, b \in \mathbb{Z}$ , then  $a = \pm 1$  or  $b = \pm 1$  (note that  $\pm 1$  are the units of  $\mathbb{Z}$ ).

**Definition 10.2.2**

If  $F$  is a field, a polynomial  $\ell(x) \neq 0$  in  $F[x]$  is *irreducible* if  $\deg \ell \geq 1$  and whenever  $\ell(x) = \ell_1(x)\ell_2(x)$  in  $F[x]$ , we have  $\deg \ell_1 = 0$  or  $\deg \ell_2 = 0$  ( $\deg 0$  polynomials are the units in  $F[x]$ ). Polynomials that are not irreducible are *reducible*.

**Example 10.2.1**

If  $\ell(x) \in F[x]$  satisfies  $\deg \ell = 1$ , then  $\ell(x)$  is irreducible.

**Exercise 10.2.1**

One can show that if  $\deg f \in \{2, 3\}$ , then  $f$  is irreducible iff  $f(d) \neq 0$  for all  $d \in F$ .

**Example 10.2.2**

Let  $\ell(x), f(x) \in F[x]$ . If  $\ell(x)$  is irreducible and  $\ell(x) \nmid f(x)$  then  $\gcd(\ell(x), f(x)) = 1$

**Recall**

Given a prime  $p \in \mathbb{Z}$  and  $a, b \in \mathbb{Z}$ , Euclid's Lemma states that if  $p \mid ab$  then  $p \mid a$  or  $p \mid b$ .

**Proposition 10.7**

Let  $F$  be a field and  $f(x), g(x) \in F[x]$ . If  $\ell(x) \in F[x]$  is irreducible and  $\ell \mid f(x)g(x)$ , then  $\ell \mid f(x)$  or  $\ell \mid g(x)$ .

**Proof:** Suppose  $\ell(x) \mid f(x)g(x)$ . Two cases:

1. If  $\ell(x) \mid f(x)$  then we are done.
2. If  $\ell(x) \nmid f(x)$ , then  $d(x) = \gcd(\ell(x), f(x)) = 1$ . By Prop 10.6, we have

$$1 = \ell(x)u(x) + f(x)v(x) \quad \text{for some } u(x), v(x) \in F[x]$$

Then

$$g(x) = g(x)\ell(x)u(x) + g(x)f(x)v(x)$$

Since  $\ell(x) \mid \ell(x)$  and  $\ell(x) \mid f(x)g(x)$ , By prop 10.3, we have  $\ell(x) \mid g(x)$ .

□

**Remark**

Let  $f_1(x), \dots, f_n(x) \in F[x]$  and let  $\ell(x) \in F[x]$  be irreducible. If  $\ell(x) \mid f_1(x) \cdots f_n(x)$ , by applying Prop 10.7 repeatedly, we get  $\ell(x) \mid f_i(x)$  for some  $i$ .

**Recall**

For an integer  $n \in \mathbb{Z}$  with  $|n| \geq 2$ , up to  $\pm 1$  sign,  $n$  can be written uniquely as a product of primes. By induction and Prop 10.7, we have the following analogous result in  $F[x]$ .

**Theorem 10.8****Unique Factorization Theorem**

Let  $F$  be a field and  $f(x) \in F[x]$  with  $\deg f \geq 1$ . Then we can write  $f(x) = c\ell_1(x) \cdots \ell_m(x)$  where  $c \in F^*$  and  $\ell_i(x)$  are monic, irreducible polynomials (not necessarily distinct.) The formulation is unique up to the order of  $\ell_i$ .

**Exercise 10.2.2**

Use Theorem 10.8 to prove there are infinitely many irreducible polynomials in  $F[x]$ .

**Recall**

In  $\mathbb{Z}$ , all ideals are of the form  $\langle n \rangle = n\mathbb{Z}$  and if  $n \in \mathbb{N}$ , then  $n$  is uniquely determined.

**Proposition 10.9**

Let  $F$  be a field. Then all ideals of  $F[x]$  are of the form  $\langle h(x) \rangle = h(x)F[x]$  for some  $h(x) \in F[x]$ . If  $\langle h(x) \rangle \neq 0$  and  $h(x)$  is monic, then the generator is uniquely determined.

**Proof:** Let  $A$  be an ideal of  $F[x]$ . If  $A = \{0\}$ , then  $A = \langle 0 \rangle$ . If  $A \neq \{0\}$ , then it contains a monic polynomial (since  $F$  is a field, if  $f \in A$  with leading coefficient  $a$ , then  $a^{-1}f \in A$ ). Among all monic polynomials in  $A$ , choose  $h(x) \in A$  of minimum degree. Then  $\langle h(x) \rangle \subseteq A$ . To prove the other inclusion, let  $f(x) \in A$ . By the division algorithm, we have  $f(x) = q(x)h(x) + r(x)$  with  $q(x), r(x) \in F[x]$  and  $\deg r < \deg h$ . If  $r(x) \neq 0$ , let  $u \neq 0$  be its leading coefficient. Since  $A$  is an ideal and  $f(x), h(x) \in A$  we have  $u^{-1}r(x) = u^{-1}(f(x) - q(x)h(x)) = u^{-1}f(x) - u^{-1}q(x)h(x) \in A$  which is a monic polynomial in  $A$  with  $\deg(u^{-1}r) < \deg h$ . This contradicts the minimum property of  $\deg h$ . Thus  $r(x) = 0$  and  $f(x) = q(x)h(x)$ . It follows that  $f(x) \in \langle h(x) \rangle$  and hence  $A = \langle h(x) \rangle$ . To prove uniqueness, suppose  $A = \langle h(x) \rangle = \langle h_1(x) \rangle$ . Since  $h(x) \mid h_1(x)$  and  $h_1(x) \mid h(x)$  and both of them are monic, by Prop 10.4, we have  $h(x) = h_1(x)$ . □

**Recall**

We have seen in  $\mathbb{Z}$  that all ideals are of the form  $\langle n \rangle$  for some  $n \in \mathbb{Z}$ . For  $n \geq 2$ , if we divide an integer by  $n$ , the remainder  $r \in \{0, 1, \dots, n-1\}$ . Write  $\langle n \rangle = n\mathbb{Z}$ . Then we have

$$\mathbb{Z}_n = \mathbb{Z}/\langle n \rangle = \{0 + \langle n \rangle, \dots, (n-1) + \langle n \rangle\} = \{[0], \dots, [n-1]\}$$

We now consider its analogue in  $F[x]$ . Let  $F$  be a field. By Prop 10.9, all ideals of  $F[x]$  are of the form  $\langle h(x) \rangle$ . Suppose that  $h(x)$  is monic and  $\deg h = m \geq 1$ . Consider the quotient ring  $R = F[x]/\langle h(x) \rangle$  and thus

$$R = \left\{ \overline{f(x)} := f(x) + \langle h(x) \rangle \mid f(x) \in F[x] \right\}$$

Write  $t = \bar{x} = x + \langle h(x) \rangle$ . We have  $h(t) = 0$  in  $R$  (exercise). By the division algorithm, we can write  $f(x) = q(x)h(x) + r(x)$  with  $\deg r < \deg h = m$ . Thus one can show that

$$R = \{ \overline{a_0} + \overline{a_1}t + \cdots + \overline{a_{m-1}}t^{m-1} \mid a_i \in F \text{ and } h(t) = 0 \}$$

Consider the map  $\theta : F \rightarrow R$  given by  $\theta(a) = \bar{a}$ . Since  $\theta$  is not the zero map and  $\ker \theta$  is an ideal of  $F$ , we have  $\ker \theta = \{0\}$ . Thus  $\theta$  is an injective ring homomorphism. Since  $F \cong \theta(F)$ , by identifying  $F$  with  $\theta(F)$ , we have

$$R = \{ a_0 + a_1t + \cdots + a_{m-1}t^{m-1} \mid a_i \in F \text{ and } h(t) = 0 \}$$

Note that in  $R$  we have

$$a_0 + a_1t + \cdots + a_{m-1}t^{m-1} = b_0 + b_1t + \cdots + b_{m-1}t^{m-1} \iff a_i = b_i \ (\forall 0 \leq i \leq m-1)$$

Hence this representation of the elements in  $R$  is unique.

### Proposition 10.10

Let  $F$  be a field and  $h(x) \in F[x]$  be monic with  $\deg h = m \geq 1$ . Then the quotient ring  $R = F[x]/\langle h(x) \rangle$  is given by

$$R = \{ a_0 + a_1t + \cdots + a_{m-1}t^{m-1} \mid a_i \in F \text{ and } h(t) = 0 \}$$

in which an element of  $R$  can be uniquely represented by the above form.

### Example 10.2.3

Consider the ring  $\mathbb{R}[x]$ . Let  $h(x) = x^2 + 1 \in \mathbb{R}[x]$ . By Prop 10.10, we have

$$\mathbb{R}[x]/\langle x^2 + 1 \rangle = \{ a + bt \mid a, b \in \mathbb{R} \text{ and } t^2 + 1 = 0 \} \cong \mathbb{C}$$

### Proposition 10.11

Let  $F$  be a field and  $h(x) \in F[x]$  with  $\deg h \geq 1$ . The following are equivalent:

1.  $F[x]/\langle h(x) \rangle$  is a field.
2.  $F[x]/\langle h(x) \rangle$  is an integral domain.
3.  $h(x)$  is irreducible in  $F[x]$

**Proof:** Write  $A = \langle h(x) \rangle$

(1  $\implies$  2) Every field is an integral domain.

(2  $\implies$  3) If  $h(x) = f(x)g(x)$  with  $f(x), g(x) \in F[x]$ , then

$$(f(x) + A)(g(x) + A) = f(x)g(x) + A = h(x) + A = 0 + A \text{ in } F[x]/A$$

By (2), either  $f(x) + A = 0 + A$  or  $g(x) + A = 0 + A$ , i.e. either  $f(x) \in A$  or  $g(x) \in A$ . If  $f(x) \in A = \langle h(x) \rangle$ , then  $f(x) = q(x)h(x)$  for some  $q(x) \in F[x]$ . Thus  $h(x) = f(x)g(x) = q(x)h(x)g(x)$ . Since  $F[x]$  is an integral domain, this implies that  $q(x)h(x) = 1$ , which implies that  $\deg g = 0$ . Similarly, if  $g(x) \in A$ , then  $\deg f = 0$ . Thus  $h(x)$  is irreducible in  $F[x]$ . (3  $\Rightarrow$  1) Note that  $F[x]/A$  is a commutative ring. Thus to show that it is a field, it suffices to show that every nonzero element of  $F[x]/A$  has an inverse. Let  $f(x) + A \neq 0 + A$  with  $f(x) \in F[x]$ . Then  $f(x) \notin A$  and hence  $h(x) \nmid f(x)$ . Since  $h(x)$  is irreducible and  $h(x) \nmid f(x)$ , we have  $\gcd(f(x), h(x)) = 1$ . By Prop 10.6, there exist  $u(x), v(x) \in F[x]$  such that  $1 = u(x)h(x) + v(x)f(x)$ . Thus  $(v(x) + A)(f(x) + A) = 1 + A$  (since  $h(x) \in A$ ). It follows that  $f(x) + A$  has an inverse in  $F[x]/A$  and hence  $F[x]/A$  is a field.  $\square$

### Example 10.2.4

Since  $\mathbb{R}[x]/\langle x^2 + 1 \rangle \cong \mathbb{C}$ , which is a field, the polynomial  $x^2 + 1$  is irreducible in  $\mathbb{R}[x]$ .

### Example 10.2.5

Since  $x^2 + x + 1$  has no root in  $\mathbb{Z}_2$  (since 0 and 1 are not roots), it is irreducible in  $\mathbb{Z}_2[x]$ . Thus  $\mathbb{Z}_2[x]/\langle x^2 + x + 1 \rangle = \{a + bt \mid a, b \in \mathbb{Z}_2 \text{ and } t^2 + t + 1 = 0\}$  is a field of 4 elements.

### Remark

Before the previous example, the only finite fields we know are of the form  $\mathbb{Z}_p$  where  $p$  is a prime. We have seen before that there are infinitely many irreducible polynomials in  $\mathbb{Z}_p[x]$ . One can show that for any  $n \in \mathbb{N}$ , there exists at least one irreducible polynomial of degree  $n$  in  $\mathbb{Z}_p[x]$ , say  $f_n(x)$ . Since  $f_n(x)$  is irreducible,  $\mathbb{Z}_p[x]/\langle f_n(x) \rangle$  is a field of order  $p^n$ . Note that  $\mathbb{Z}_{p^n}$  is NOT a field if  $n \geq 2$ .

### Analogies Between $\mathbb{Z}$ and $F[x]$

	$\mathbb{Z}$	$F[x]$
elements	$m$	$f(x)$
size	$ m  = \text{absolute value}$	$\deg f$
units	$\pm 1; \mathbb{Z} \setminus \{0\}/\{\pm 1\} = \mathbb{N}$	$F^*; F[x] \setminus \{0\}/F^* = \{\text{monic polynomials}\}$
unique factorization	$m = \pm 1 p_1^{\alpha_1} \cdots p_n^{\alpha_n}, p_i \text{ prime}$	$f = c \ell_1^{\alpha_1} \cdots \ell_r^{\alpha_r}, c \in F^*, \ell_i = \ell_i(t) = \text{monic, irreducible}$
ideals	$\langle n \rangle$ (unique if $n \in \mathbb{N}$ )	$\langle h(x) \rangle$ (unique if $h(x)$ monic)
ideals	$\mathbb{Z}/\langle n \rangle$ is a field iff $n$ is a prime	$F[x]/\langle h(x) \rangle$ is a field iff $h(x)$ is irreducible

### 10.3 Fermat's Last Theorem in $F[x]$

#### The Pythagorean Theorem

We have  $x^2 + y^2 = z^2$ . The triples  $(x, y, z) = (3, 4, 5)$  and  $(5, 12, 13)$  are positive integer solutions.

#### Proposition 10.12

#### Euclid's Formula

For  $a, b \in \mathbb{N}$  with  $a > b$ , we can take

$$x = a^2 - b^2, \quad y = 2ab, \quad z = a^2 + b^2$$

Then the triple  $(x, y, z)$  satisfies  $x^2 + y^2 + z^2$  and we have infinitely many positive integer solutions.

Question: How about positive integer solutions for

$$x^n + y^n + z^n \quad \text{with } n \in \mathbb{N} \text{ and } n \geq 3$$

#### Theorem 10.13

#### Fermat's Last Theorem

For  $n \in \mathbb{N}$  with  $n \geq 3$ , the equation  $x^n + y^n = z^n$  has no solution with  $x, y, z \in \mathbb{N}$ .

- The result was first claimed by Fermat in the mid 17<sup>th</sup> century without proof
- The theorem was proved by Wiles and others in 1994
- The proof involves the use of elliptic curves, modular forms and Galois representations

Let  $F$  be a field and  $n \in \mathbb{N}$ . Consider

$$f(x)^n + g(x)^n = h(x)^n$$

with  $f(x), g(x), h(x) \in F[x]$ . We say a solution  $(f, g, h)$  is *non-trivial* if  $\deg(f), \deg(g), \deg(h) \geq 1$ .

#### Theorem 10.14

Let  $F$  be a field with  $\text{ch}(F) = 0$ . For  $n \in \mathbb{N}$  with  $n \geq 3$ , there is no non-trivial solutions to the equation

$$f(x)^n + g(x)^n = h(x)^n$$

with  $f(x), g(x), h(x) \in F[x]$ .

**Proof:** Suppose we have a non-trivial solution  $f^n + g^n = h^n$ . By dividing all common divisors of  $f, g, h$ , we can assume  $(f, g, h)$  is “coprime”, i.e.  $\gcd(f, g) = \gcd(f, h) = \gcd(g, h) = 1$ . Without loss of generality, suppose

$$\deg f = \deg h \geq \deg g$$

Write  $f' = df/dx$ . Since  $f^n + g^n = h^n$ , by the chain rule, we have

$$nf^{n-1}f' + ng^{n-1}g' = nh^{n-1}h'.$$

Since  $\text{ch}(F) = 0$ , we have  $n \cdot 1 \neq 0$ . By multiplying both sides by  $h$  we get

$$\begin{aligned} f^{n-1}f'h + g^{n-1}g'h &= f^n h' + g^n h' \\ \implies f^{n-1}(f'h - fh') &= g^{n-1}(gh' - g'h) \end{aligned}$$

Since  $f$  and  $g$  are coprime, the above equation implies that  $f^{n-1} \mid (gh' - g'h)$ . Thus

$$(n-1) \deg f \leq \deg g + \deg h - 1$$

Since  $\deg f = \deg h \geq \deg g$ , we have

$$(n-2) \deg g \leq (n-2) \deg f \leq \deg g - 1$$

which is a contradiction if  $n \geq 3$ .  $\square$

### Note

Let  $p$  be a prime and consider  $\mathbb{Z}_p[x]$ . We have

$$(f+g)^p = f^p + g^p$$

Let  $h = f + g$ . Then the equation  $f^p + g^p = h^p$  has infinitely many non-trivial solutions.

### Theorem 10.15

Let  $F$  be a field with  $\text{ch}(F) = p$ . For  $n \in \mathbb{N}$  with  $n \geq 3$  and  $\gcd(n, p) = 1$ , there is no non-trivial solutions for

$$f^n + g^n = h^n$$

with  $f, g, h \in F[x]$ .

## 10.4 Prime Number Theorem and Riemann Hypothesis in $\mathbb{Z}_p[t]$

Define  $\pi(x) = \#\{p \leq x \mid p \text{ prime}\}$

Conjecture:

$$\pi(x) \sim \text{li } x = \int_2^x \frac{dt}{\log t} \sim \frac{x}{\log x}$$

Hence the probability of primes  $\leq x$  is  $\frac{1}{\log x}$ . For example, about 1% of  $n \in \mathbb{N}$  with  $n < e^{100}$  are primes.

### Definition 10.4.1

For  $s \in \mathbb{C}$ , the *Riemann zeta function* is defined to be

$$\zeta(s) = \sum_{n \in \mathbb{N}} \frac{1}{n^s} = \prod_{p \text{ prime}} \left(1 + \frac{1}{p^s} + \frac{1}{p^{2s}} + \dots\right) = \prod_{p \text{ prime}} \left(1 - \frac{1}{p^s}\right)^{-1}$$

- Absolutely convergent for  $\text{Re}(s) > 1$
- Functional equation: relate  $\zeta(s)$  with  $\zeta(1-s)$
- Analytic continuation: extend  $\zeta(s)$  to  $s \in \mathbb{C}$

Riemann Hypothesis:

All “non-trivial zeros” of  $\zeta(s)$  lie on  $\operatorname{Re}(s) = \frac{1}{2}$

**Theorem 10.16**

For any  $n \in \mathbb{N}$ , we have

$$\pi(x) = \operatorname{li}(x) + O\left(\frac{x}{(\log x)^n}\right)$$

**Theorem 10.17**

For any  $\varepsilon > 0$ , assuming RH,

$$\pi(x) = \operatorname{li}(x) + O\left(x^{\frac{1}{2}+\varepsilon}\right)$$

Let's consider the Prime Number Theorem in  $\mathbb{Z}_p[t]$ . For  $f \in \mathbb{Z}_p[t]$ , define  $|f| = p^{\deg f}$ . For  $s \in \mathbb{C}$ , the *zeta function* in  $\mathbb{Z}_p[t]$  is defined to be

$$\zeta_p(s) = \sum_{f \text{ monic}} \frac{1}{|f|^s} = \prod_{\substack{\ell \text{ monic} \\ \text{irreducible}}} \left(1 - \frac{1}{|\ell|^s}\right)^{-1}$$

Note that

$$\#\{f \mid f \text{ monic, } \deg f = d\} = p^d$$

So

$$\zeta_p(s) = \sum_{d=0}^{\infty} \frac{p^d}{(p^d)^s} = \frac{1}{1 - p^{1-s}}$$

- Functional equation: relate  $\zeta_p(s)$  with  $\zeta_p(1-s)$
- Analytic continuation: extend  $\zeta_p(s)$  to  $s \in \mathbb{C}$

Define  $a_d = \#\{\ell \mid \ell \text{ monic, irreducible, } \deg \ell = d\}$ . Then we have

$$\zeta_p(s) = \prod_{d=1}^{\infty} \left(1 - \frac{1}{(p^d)^s}\right)^{-a_d}$$

Write  $T = p^{-s}$ . Then

$$\zeta_p(s) = \frac{1}{1 - pT} = \prod_{d=1}^{\infty} (1 - T^d)^{-a_d}$$

Note that by taking logarithmic derivative,

$$\left(\log\left(\frac{1}{1 - pT}\right)\right)' = \frac{p}{1 - pT}$$

By taking logarithmic derivatives and multiplying by  $T$ , we have

$$\frac{pT}{1-pT} = \sum_{d=1}^{\infty} \frac{da_d T^d}{1-T^d}$$

By expanding both sides into power series and comparing the coefficients of  $T^n$ , we have,

$$p^n = \sum_{d|n} da_d \quad (*)$$

### Definition 10.4.2

Now define the *Möbius function* on  $d \in \mathbb{N}$

$$\mu(d) = \begin{cases} 1 & \text{if } d = 1 \\ (-1)^r & \text{if } d \text{ is a product of distinct } r \text{ primes} \\ 0 & \text{otherwise} \end{cases}$$

### Proposition 10.18

### Möbius inversion formula

$$f(n) = \sum_{d|n} g(d) \iff g(n) = \sum_{d|n} \mu(d) f\left(\frac{n}{d}\right)$$

By  $(*)$  and the Möbius inversion formula, we have

$$da_d = p^d + O\left(p^{\frac{d}{2}}\right)$$

i.e.

$$a_d = \frac{p^d}{d} + O\left(\frac{p^{\frac{d}{2}}}{d}\right)$$

Define  $\pi_p(x) = \#\{\ell \mid \ell \text{ monic, irreducible, } |\ell| \leq x\}$  By the estimates of  $a_d$ ,

$$\pi_p(x) = \frac{p}{p-1} \cdot \frac{x}{\log x} + O\left(x^{\frac{1}{2}+\varepsilon}\right)$$

Which is RH in  $\mathbb{Z}_p[t]$