

Contents

1 Topological Spaces and Continuous Maps 2

1.1 Elementary Topology 2

1.2 Topological Bases 5

1.3 Subspaces 8

1.4 Continuous Maps 9

1.5 Examples of Homeomorphisms 11

1 Topological Spaces and Continuous Maps

1.1 Elementary Topology

Given an inner product on an \mathbb{R} -vector space $\langle \cdot, \cdot \rangle$, one can define a norm $\|x\| = \sqrt{\langle x, x \rangle}$. Given a norm, one can define a metric $d(x, y) = \|x - y\|$. Given a metric d on a set X , one can define open sets in X :

given $a \in X$ and $r > 0$, $B(a, r) := \{x \in X \mid d(x, a) < r\}$. Then for $A \subseteq X$, we say A is open in X when $\forall a \in A \exists r > 0$ such that $B(a, r) \subseteq A$. Equivalently, for all $a \in A$, there is $b \in X$, $r > 0$ such that $a \in B(b, r) \subseteq A$.

Remark

The set of open sets on a metric space is called the *metric topology* on X .

Open sets in a metric space satisfy the following:

1. \emptyset and X are open
2. arbitrary unions of open sets are open
3. finite intersections of open sets are open

Notation

For a set of sets S , the union of S is

$$\bigcup S := \{x \mid \exists A \in S, x \in A\} = \bigcup_{A \in S} A$$

In the case that $S \neq \emptyset$, the intersection of S is

$$\bigcap S := \{x \mid \forall A \in S, x \in A\} = \bigcap_{A \in S} A$$

Note

$\bigcap S$ would contain all elements as the condition $\forall A \in \emptyset$ would be vacuously satisfied. If we are given a universal set X , and S is known to be a set of subsets of X , then $\bigcap \emptyset = X$.

Definition 1.1.1

Let X be a set. $\mathcal{T} \subseteq \mathcal{P}(X)$ is called a *topology* on X if

1. $\emptyset, X \in \mathcal{T}$
2. If $S \subseteq \mathcal{T}$ is nonempty, then $\bigcup S \in \mathcal{T}$
3. If $S \subseteq \mathcal{T}$ is nonempty and finite, then $\bigcap S \in \mathcal{T}$

The elements of \mathcal{T} are called the open sets of X . The closed sets are the compliments of the open sets.

Remark

To show 3 holds, it suffices to show the intersection of 2 open sets is open (by induction)

Definition 1.1.2

If X is a set, and \mathcal{T} is a topology on X , then (X, \mathcal{T}) is called a *topological space*

Remark

When $f : X \rightarrow Y$ is a map between metric spaces, f is continuous iff $f^{-1}(V)$ is open in X for every open set $V \subseteq Y$.

Definition 1.1.3

For a map $f : X \rightarrow Y$ between topological spaces, we say that f is continuous when $f^{-1}(V)$ is open in X for every open set $V \subseteq Y$.

Example 1.1.1

if $f : A \subseteq \mathbb{R}^n \rightarrow B \subseteq \mathbb{R}^m$ is an elementary function, then f is continuous.

Definition 1.1.4

When S, T are topologies on X with $S \subseteq T$, we say that S is coarser than T and T is finer than S . When $S \subsetneq T$, we use strictly coarser/finer.

Example 1.1.2

$\{\emptyset, X\}$ is a topology on X called the *trivial topology*

Example 1.1.3

$\mathcal{P}(X)$ is a topology on X called the *discrete topology*

Example 1.1.4

When $X = \emptyset$, $\mathcal{T} \subseteq \mathcal{P}(X) \Rightarrow \mathcal{T} \subseteq \{\emptyset\} \Rightarrow \mathcal{T} = \emptyset \vee \mathcal{T} = \{\emptyset\}$. Thus the only topology on \emptyset is $\{\emptyset\}$.

Example 1.1.5

When $X = \{a\}$ the only topology is $\mathcal{T} = \{\emptyset, \{a\}\}$

Exercise 1.1.1

Find all topologies on the 2 and 3 element sets.

Definition 1.1.5

Let X be a topological space. Let $A \subseteq X$.

1. The *interior* of A (in X) denoted by A° is the union of all open sets in X which are contained in A .
2. The *closure* of A denoted \overline{A} is the intersection of all closed sets in X which contain A .
3. The *boundary* of A , denoted by ∂A , given by $\partial A = \overline{A} \setminus A^\circ$

Note

The set of closed sets in a topological space is closed under arbitrary intersections and under finite unions. In particular \emptyset, X are closed

Theorem 1.1.1

Let X be a topological space, $A \subseteq X$.

1. A° is open, and is the largest open set which is contained in A
2. \overline{A} is closed, and is the smallest closed set which contains A
3. A is open iff $A = A^\circ$
4. A is closed iff $A = \overline{A}$
5. $A^{\circ\circ} = A^\circ$
6. $\overline{\overline{A}} = \overline{A}$

Definition 1.1.6

Let X be a topological space, let $A \subseteq X$, let $a \in X$.

1. We say that a is an *interior point* of A when $a \in A$ and there is an open set U such that $a \in U \subseteq A$
2. We say that a is a *limit point* of A when for every open set $U \ni a$ we have $U \cap (A \setminus \{a\}) \neq \emptyset$. The set of limit points of A is denoted by A'
3. We say that a is a *boundary point* of A when every open set $U \ni a$, we have $U \cap A \neq \emptyset$ and $U \cap A^c \neq \emptyset$

Theorem 1.1.2

Let X be a topological space and let $A \subseteq X$.

1. A° is equal to the set of all interior points
2. For $a \in X$,

$$a \in A' \iff a \in \overline{A \setminus \{a\}}$$

3. A is closed iff $A' \subseteq A$
4. $\overline{A} = A \cup A'$
5. \overline{A} is the disjoint union

$$\overline{A} = A^\circ \sqcup \partial A$$

6. ∂A is equal to the set of boundary points of A

1.2 Topological Bases**Theorem 1.2.1**

Let X be a set. Then the intersection of any set of topologies on X is also a topology on X .

Proof: Let $\{\mathcal{T}_\alpha\}_{\alpha \in I}$ be a collection of topologies on X . Let $\mathcal{T} = \bigcap_\alpha \mathcal{T}_\alpha$

1. Since $X, \emptyset \in \mathcal{T}_\alpha$ for all $\alpha \in I$. We have $X, \emptyset \in \mathcal{T}$
2. Let $\{U_i\} \subseteq \mathcal{T}$. For all $\alpha \in I$, we have each $U_i \in \mathcal{T}_\alpha$. Thus $\bigcup_i U_i \in \mathcal{T}_\alpha \implies \bigcup_i U_i \in \mathcal{T}$ as desired.
3. Let $U_1, \dots, U_n \in \mathcal{T}$. Then again for all $\alpha \in I$, we have each $U_i \in \mathcal{T}_\alpha$. Thus $\bigcap_{i=1}^n U_i \in \mathcal{T}_\alpha \implies \bigcap_{i=1}^n U_i \in \mathcal{T}$

□

Corollary 1.2.2

When X is a set and \mathcal{S} is any set of subsets of X (that is $\mathcal{S} \subseteq \mathcal{P}(X)$), there is a unique smallest (coarsest) topology \mathcal{T} on X which contains \mathcal{S} . Indeed \mathcal{T} is the intersection of (the set of) all topologies on X containing \mathcal{S} .

This topology \mathcal{T} is called the topology on X *generated by* \mathcal{S}

Definition 1.2.1

Let X be a set. A *basis of sets* on X is a set \mathcal{B} of subsets of X (So $\mathcal{B} \subseteq \mathcal{P}(X)$) such that

1. \mathcal{B} covers X , that is $\bigcup \mathcal{B} = X$
2. For every $C, D \in \mathcal{B}$ and $a \in C \cap D$. There is $B \in \mathcal{B}$ such that $a \in B \subseteq C \cap D$.

When \mathcal{B} is a basis of sets in X and \mathcal{T} is the topology on X generated by \mathcal{B} , we say that \mathcal{B} is a *basis for* \mathcal{T} . The elements in \mathcal{B} are called *basic open sets* in X .

Theorem 1.2.3**Characterization of Open Sets in Terms of Basic Open Sets**

Let X be a topological space, Let \mathcal{B} be a basis for the topology on X .

1. For $A \subseteq X$, A is open iff for every $a \in A$, there is $B \in \mathcal{B}$ such that $a \in B \subseteq A$ *
2. The open sets in X are the unions of (sets of) elements in \mathcal{B}

Equivalently,

1. $\mathcal{T} = \{A \subseteq X \mid \forall a \in A, \exists B \in \mathcal{B} \ a \in B \subseteq A\}$
2. $\mathcal{T} = \{\bigcup C \mid C \subseteq \mathcal{B}\}$

Proof: Let \mathcal{T} be the topology on X (generated by \mathcal{B}). Let \mathcal{S} be the set of all sets $A \subseteq X$ with property * ($\forall a \in A \exists B \in \mathcal{B} : a \in B \subseteq A$). And let \mathcal{R} be the set of (arbitrary) unions of (sets of) elements in \mathcal{B} . Recall that \mathcal{T} is the intersection of the set of all topologies on X which contain \mathcal{B} . Note that \mathcal{S} contains \mathcal{B} (obviously). Let us show that \mathcal{S} is a topology on X . We have $\emptyset \in \mathcal{S}$ vacuously and $X \in \mathcal{S}$ because \mathcal{B} covers X (given $a \in X$, we can choose $B \in \mathcal{B}$ with $a \in B$). When $U_k \in \mathcal{S}$ for every $k \in K$ (where K is any index set). Let $a \in \bigcup_k U_k$. Choose $\ell \in K$ so that $a \in U_\ell$. Since $U_\ell \in \mathcal{S}$, we can choose $B \in \mathcal{B}$ so that $a \in B \subseteq U_\ell$. Since $U_\ell \subseteq \bigcup_k U_k$, we have $a \in B \subseteq \bigcup_k U_k$. Thus $\bigcup_k U_k$ satisfies *, hence $\bigcup_k U_k \in \mathcal{S}$ as required. Suppose $U, V \in \mathcal{S}$. Let $a \in U \cap V$. Since $U \in \mathcal{S}$ we can choose $C \in \mathcal{B}$ with $a \in C \subseteq U$. Since $V \in \mathcal{S}$, we can choose $D \in \mathcal{B}$ with $a \in D \subseteq V$. Since \mathcal{B} is a basis, $C, D \in \mathcal{B}$ and $a \in C \cap D$, we can choose $B \in \mathcal{B}$ with $a \in B \subseteq C \cap D$. Then we have

$$a \in B \subseteq C \cap D \subseteq U \cap V$$

Thus $U \cap V$ satisfies * so that $U \cap V \in \mathcal{S}$ as required. Thus \mathcal{S} is a topology on X containing \mathcal{B} , hence $\mathcal{T} \subseteq \mathcal{S}$. Let us show that $\mathcal{S} \subseteq \mathcal{R}$ let $U \in \mathcal{S}$. For each $a \in U$, choose $B_a \in \mathcal{B}$ with $a \in B_a \subseteq U$. Then we have

$$U = \bigcup_{a \in U} B_a \in \mathcal{R}$$

Thus $\mathcal{S} \subseteq \mathcal{R}$. Finally note that $\mathcal{R} \subseteq \mathcal{T}$ because if $U = \bigcup_k B_k$ with $B_k \in \mathcal{B}$, then each $B_k \in \mathcal{T}$, and \mathcal{T} is a topology, so

$$U = \bigcup_{k \in K} B_k \in \mathcal{T}$$

□

Theorem 1.2.4**Characterization of a Basis in terms of the Open Sets**

Let X be a topological space with topology \mathcal{T} . Let $\mathcal{B} \subseteq \mathcal{T}$. Then \mathcal{B} is a basis for \mathcal{T} iff $\forall U \in \mathcal{T} \forall a \in U \exists B \in \mathcal{B} \ a \in B \subseteq U$. *

Proof: If \mathcal{B} is a basis for \mathcal{T} , then * holds by part 1 of the previous theorem. Suppose * holds. Let us show that \mathcal{B} is a basis of sets in X . Note that \mathcal{B} covers X since, taking $U = X$ in * we have $\forall a \in X \exists B \in \mathcal{B} \ a \in B \subseteq X$. Also note that given $C, D \in \mathcal{B}$ and $a \in C \cap D$, then by taking $U = C \cap D$ in * (noting that $C, D \in \mathcal{B} \subseteq \mathcal{T}$ so that $U = C \cap D \in \mathcal{T}$) we can choose $B \in \mathcal{B}$ with $a \in B \subseteq C \cap D$. Thus \mathcal{B} is a basis of sets in X . It remains to show that \mathcal{T} is the topology generated by \mathcal{B} . Let \mathcal{S} be the topology generated by \mathcal{B} . By part 1 of the previous theorem, \mathcal{S} is the set of all unions of

elements in \mathcal{B} . Also \mathcal{S} is the smallest topology which contains \mathcal{B} . Since $\mathcal{B} \subseteq \mathcal{T}$ and \mathcal{T} is a topology, we have $\mathcal{S} \subseteq \mathcal{T}$. Also we have $\mathcal{T} \subseteq \mathcal{S}$ because given $U \in \mathcal{T}$, by property *, for each $a \in U$, we can choose $B_a \in \mathcal{B}$ with $a \in B_a \subseteq U$, and then we have $U = \bigcup_{a \in U} B_a \in \mathcal{S}$ since it is a union of elements in \mathcal{B} \square

Example 1.2.1

When X is a metric space, the set \mathcal{B} of all open balls in X is a basis for the metric topology on X .

Remark

We can use a basis for testing various topological properties:

When X is a topological space, and \mathcal{B} is a basis for the topology on X , and $A \subseteq X$ and $a \in X$. Then

$$a \in A^\circ \iff \exists B \in \mathcal{B} \text{ with } a \in B \subseteq A$$

$$a \in \overline{A} \iff \forall B \in \mathcal{B} \text{ with } a \in B \quad B \cap A \neq \emptyset$$

$$a \in A' \iff \forall B \in \mathcal{B} \text{ with } a \in B \quad (B \setminus \{a\}) \cap A \neq \emptyset$$

$$a \in \partial A \iff \forall B \in \mathcal{B} \text{ with } a \in B \quad B \cap A \neq \emptyset \text{ and } B \cap (X \setminus A) \neq \emptyset$$

Definition 1.2.2

A topological space X is called *Hausdorff* when for all $a, b \in X$ with $a \neq b$, there exist disjoint open sets U and V in X with $a \in U$ and $b \in V$.

Example 1.2.2

Metric spaces are Hausdorff

1.3 Subspaces

Definition 1.3.1

Subspace Topology

Let Y be a topological space with topology \mathcal{S} , and $X \subseteq Y$ be a subset. Let

$$\mathcal{T} := \{V \cap X \mid V \in \mathcal{S}\}$$

Then \mathcal{T} is a topology on X :

Indeed $\emptyset \in \mathcal{S}$ so $\emptyset \cap X = \emptyset \in \mathcal{T}$ and $Y \in \mathcal{S}$ so $Y \cap X = X \in \mathcal{T}$. If K is any index set and $U_k \in \mathcal{T}$ for each $k \in K$, then for each $k \in K$ we can choose $V_k \in \mathcal{S}$ such that $U_k = V_k \cap X$ and then we have

$$\begin{aligned} \bigcup_{k \in K} U_k &= \bigcup_{k \in K} (V_k \cap X) \\ &= \left(\bigcup_{k \in K} V_k \right) \cap X \in \mathcal{T} \end{aligned}$$

since $\bigcup_{k \in K} V_k \in \mathcal{S}$. Similarly, when K is finite and $U_k \in \mathcal{T}$ for each $k \in K$ we have $\bigcap_{k \in K} U_k \in \mathcal{T}$. The topology \mathcal{T} on X is called the *subspace topology* on X (inherited from the topology on Y).

Theorem 1.3.1

Let Y be a topological space, let \mathcal{C} be a basis for the topology on Y . Let $X \subseteq Y$ be a subset. Then the set

$$\mathcal{B} = \{C \cap X \mid C \in \mathcal{C}\}$$

is a basis for the subspace topology on X .

Proof: Exercise □

Theorem 1.3.2

Let Z be a topological space, let $Y \subseteq Z$ be a subspace and $X \subseteq Y$ be a subset. Then the subspace topology on X inherited from Y is equal to the subspace topology on X inherited from Z .

Proof: Exercise □

Theorem 1.3.3

Let Y be a metric space, (using the metric topology) and let $X \subseteq Y$. Then the subspace topology on X (inherited from the topology on Y) is equal to the metric topology on X using the metric on X obtained by restricting the metric on Y .

Proof: Exercise □

1.4 Continuous Maps

Definition 1.4.1

Let X, Y be topological spaces.

1. For $f : X \rightarrow Y$ and $a \in X$, we say that f is *continuous at a* when for every open set $V \subseteq Y$ with $f(a) \in V$, there exists an open set $U \subseteq X$ with $a \in U \subseteq f^{-1}(V)$.
2. We say that f is *continuous* (in or on X) when for every open set $V \subseteq Y$, $f^{-1}(V)$ is open in X .
3. A *homeomorphism* from X to Y is a bijective map $f : X \rightarrow Y$ such that both f and its inverse $f^{-1} : Y \rightarrow X$ are continuous. We say that X and Y are *homeomorphic*, and we write $X \cong Y$, when there exists a homeomorphism $f : X \rightarrow Y$. (and we remark that $f^{-1} : Y \rightarrow X$ is also a homeomorphism).

Theorem 1.4.1

Constant maps and inclusion maps are continuous.

Proof: For $f : X \rightarrow Y$ given by $f(x) = c \in Y$ for all $x \in X$. When V is open in Y ,

$$f^{-1}(V) = \begin{cases} X & \text{if } c \in V \\ \emptyset & \text{if } c \notin V \end{cases}$$

When $X \subseteq Y$ is a subspace and $f : X \rightarrow Y$ is given by $f(x) = x$ for all $x \in X$, when V is open in Y .

$$\begin{aligned} f^{-1}(V) &= \{x \in X \mid f(x) \in V\} \\ &= \{x \in X \mid x \in V\} \\ &= V \cap X \end{aligned}$$

which is open in X . (when X uses the subspace topology) □

Remark

When Y is a topological space and $X \subseteq Y$ we shall assume, unless otherwise noted, that X uses the subspace topology.

Theorem 1.4.2

Equivalent Definitions of Continuity

Let $f : X \rightarrow Y$ be a map between topological spaces

1. f is continuous iff f is continuous at every $a \in X$
2. f is continuous iff for every closed set $K \subseteq Y$, $f^{-1}(K)$ is closed in X .
3. If \mathcal{C} is a basis for the topology on Y then f is continuous iff for every $C \in \mathcal{C}$, $f^{-1}(C)$ is open in X .

Proof of 1: Suppose f is continuous on X . Let $a \in X$. Let V be an open set in Y with $f(a) \in V$. Let $U = f^{-1}(V)$, then $f^{-1}(V)$ is open, since f is continuous and $a \in U \subseteq f^{-1}(V)$. Suppose, conversely, that f is continuous at every $a \in X$. Let V be an open set in Y . For each $a \in f^{-1}(V)$ since f is continuous at a with $f(a) \in V$, we can choose an open set U_a in X with $a \in U_a \subseteq f^{-1}(V)$. Then

$$f^{-1}(V) = \bigcup_{a \in f^{-1}(V)} U_a$$

which is open in X , since it is a union in open sets in X . □

Theorem 1.4.3

Let $f : X \rightarrow Y$, $g : Y \rightarrow Z$ be continuous maps between topological spaces, then the composite map $h = g \circ f : X \rightarrow Z$ is continuous.

Proof: Show that $h^{-1}(W) = f^{-1}(g^{-1}(W))$ □

Remark

Homeomorphism of topological spaces behaves like an equivalence relation on the class of all topological spaces. For topological spaces X, Y, Z

1. $X \cong X$ (since id_X is a homeomorphism – a special case of the inclusion map)
2. If $X \cong Y$ then $Y \cong X$ (when $f : X \rightarrow Y$ is a homeomorphism, so is $f^{-1} : Y \rightarrow X$)
3. If $X \cong Y \cong Z$ then $X \cong Z$ (if $f : X \rightarrow Y, g : Y \rightarrow Z$ are homeomorphisms then so is $g \circ f$)

Theorem 1.4.4

Restriction of Domain and Restriction or Expansion of Codomain

Let X, Y, Z be topological spaces. Suppose $f : X \rightarrow Y$ is continuous.

1. For any subspace $A \subseteq X$, the restriction $f|_A : A \rightarrow Y$ is continuous.
2. If $Y \subseteq Z$ is a subspace then $f : Y \rightarrow Z$ is continuous and if $B \subseteq Y$ with $f(X) \subseteq B$, then $f : X \rightarrow B$ is continuous.

Proof: Exercise □

Lemma 1.4.5

Glueing/Pasting Lemma

Let $f : X \rightarrow Y$ be a map between topological spaces

1. If $X = \bigcup_{k \in K} U_k$ where each U_k is open in X and if each restriction map $f|_{U_k} : U_k \rightarrow Y$ is continuous (where U_k is using the subspace topology), then f is continuous.
2. If $X = C_1 \cup \dots \cup C_n$ where each C_k is closed in X , and if each restriction $f|_{C_k} : C_k \rightarrow Y$ is continuous, then f is continuous.

Proof of 1: Suppose $X = \bigcup_{k \in K} U_k$ where each U_k is open in X and suppose each restriction $f|_{U_k}$ is continuous. Let $V \subseteq Y$ be open. Note that

$$\begin{aligned}
f^{-1}(V) &= \{x \in X \mid f(x) \in V\} \\
&= \bigcup_{k \in K} \{x \in U_k \mid f(x) \in V\} \\
&= \bigcup_{k \in K} \{x \in U_k \mid f|_{U_k}(x) \in V\} \\
&= \bigcup_{k \in K} f|_{U_k}^{-1}(V)
\end{aligned}$$

For each $k \in K$, since $f|_{U_k}$ is continuous, we know that $f|_{U_k}^{-1}(V)$ is open in U_k . Since U_k is using the subspace topology, we can choose an open W_k in X such that $f|_{U_k}^{-1}(V) = W_k \cap U_k$. This is open in X since W_k and U_k are both open in X . Since $f^{-1}(V) = \bigcup_{k \in K} f|_{U_k}^{-1}(V)$ it is a union of open sets in X , so it is open in X . Thus f is continuous. \square

Proof of 2: Exercise. First show that for $f : X \rightarrow Y$, f is continuous iff $f^{-1}(C)$ is closed in X for every closed set C in Y . And, show that when $A \subseteq X \subseteq Y$, A is closed in X (using the subspace topology from Y) iff $A = B \cap X$ for some closed set B in Y . \square

Example 1.4.1

The map $f : \mathbb{R} \rightarrow \mathbb{R}$ given by $f(x) = \begin{cases} 2x & x \leq 0 \\ x^2 & x > 0 \end{cases}$ is continuous.

1.5 Examples of Homeomorphisms

Example 1.5.1

The circle

$$\{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1\}$$

in \mathbb{R}^2 is homeomorphic to the ellipse

$$\left\{ (x, y) \in \mathbb{R}^2 \mid \frac{(x-a)^2}{A^2} + \frac{(y-b)^2}{B^2} = 1 \right\}$$

in \mathbb{R}^2

Example 1.5.2

$\mathbb{R} \cong (-1, 1) \subseteq \mathbb{R}$

Example 1.5.3

The standard unit n -sphere in \mathbb{R}^{n+1} is the set

$$\mathbb{S}^n = \{x \in \mathbb{R}^{n+1} \mid \|x\| = 1\}$$

Where p is the north pole

$$p = e_{n+1} = (0, \dots, 0, 1) \in \mathbb{S}^n$$

We have $\mathbb{S}^n \setminus \{p\} \cong \mathbb{R}^n$