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# 1 Groups

## 1.1 Notation

1.  $\mathbb{N} = \{1, 2, \dots\}$
2.  $\mathbb{Z} = \{\dots, -1, 0, 1, \dots\}$
3.  $\mathbb{Q} = \left\{ \frac{a}{b} : a \in \mathbb{Z}, b \in \mathbb{N} \right\}$
4.  $\mathbb{R}$  = real numbers
5.  $\mathbb{C} = \{a + bi : a, b \in \mathbb{R}, i^2 = -1\}$

For  $n \in \mathbb{N}$ ,  $\mathbb{Z}_n$  = integers modulo  $n = \{[0], \dots, [n-1]\}$  where  $[r] = \{z \in \mathbb{Z} : Z \equiv r \pmod{n}\}$

We note that the set  $S = \mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{R}, \mathbb{C}, \mathbb{Z}_n$  has 2 operations  $+, \cdot$ .

For  $n \in \mathbb{N}$ , an  $n \times n$  matrix over  $\mathbb{R}$  (or  $\mathbb{Q}$  or  $\mathbb{C}$ ) is an  $n \times n$  array

$$A = [a_{ij}] = \begin{bmatrix} a_{11} & \dots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \dots & a_{nn} \end{bmatrix}$$

with  $a_{ij} \in \mathbb{R}$ .

Note we can also do  $+, \cdot$ . For  $A, B \in M_n(\mathbb{R})$

$$A + B := [a_{ij} + b_{ij}] \quad A \cdot B := \left[ \sum_{k=1}^n a_{ik} b_{kj} \right]$$

## 1.2 Groups

### Definition 1.2.1

Let  $G$  be a set and  $* : G \times G \rightarrow G$ . We say  $G$  is a *group* if the following are satisfied:

1. Associativity: if  $a, b, c \in G$ , then  $a * (b * c) = (a * b) * c$
2. Identity: there is  $e \in G$  such that  $a * e = e * a = a$  for all  $a \in G$
3. Inverses: for all  $a \in G$ , there is  $a^{-1} \in G$  such that  $a * a^{-1} = a^{-1} * a = e$

### Definition 1.2.2

A group is called *abelian* if  $a * b = b * a$  for all  $a, b \in G$

### Exercise 1.2.1

Prove in the definition of a group, 1-sided identity and inverses are enough to have 2-sided identity and inverses

**Proposition 1.1**[previous exercise](#)

Suppose  $G$  is a set,  $* : G \times G \rightarrow G$  is associative. Suppose there is  $e \in G$  such that  $e * a = a$  for all  $a \in G$ . Further suppose that for every  $a \in G$ , there is  $a^{-1} \in G$  such that  $a^{-1} * a = e$ . Then for all  $a \in G$ ,

1.  $a * e = a$
2.  $a * a^{-1} = e$

**Proof of 1:** Let  $a \in G$ , then

$$a^{-1} * a * e = e * e = e = a^{-1} * a$$

Multiplying on the left by  $a^{-1}$  gives

$$\begin{aligned} a^{-1} * a^{-1} * a * e &= a^{-1} * a^{-1} * a \\ \implies e * a * e &= e * a \\ \implies a * e &= a \end{aligned}$$

□

**Proof of 2:** Let  $a \in G$ , then

$$a^{-1} * a * a^{-1} = e * a^{-1} = a^{-1}$$

Again multiplying on the left by  $a^{-1}$  gives

$$a * a^{-1} = e$$

□

**Proposition 1.2**

Let  $G$  be a group, let  $a \in G$ . Then

1. The group identity is unique
2. The inverse of  $a$  is unique

**Proof of 1:** Suppose  $e_1, e_2$  are both identities. Then

$$e_1 = e_1 * e_2 = e_2$$

□

**Proof of 2:** Suppose  $b_1, b_2$  are inverses of  $a$ . Then

$$b_1 = b_1 * e = b_1 * (a * b_2) = (b_1 * a) * b_2 = e * b_2 = b_2$$

□

**Example 1.2.1**

$(\mathbb{Z}, +), (\mathbb{Q}, +), (\mathbb{R}, +), (\mathbb{C}, +)$  are all abelian groups

**Example 1.2.2**

$(\mathbb{Z}, \cdot), (\mathbb{Q}, \cdot), (\mathbb{R}, \cdot), (\mathbb{C}, \cdot)$  are not groups as 0 has no inverse

**Example 1.2.3**

but  $(\mathbb{Q} \setminus \{0\}, \cdot), (\mathbb{R} \setminus \{0\}, \cdot), (\mathbb{C} \setminus \{0\}, \cdot)$  are abelian groups

**Definition 1.2.3**

For a set  $(S, \cdot)$  let  $S^* \subseteq S$  denote the set of all elements with inverses.

**Exercise 1.2.2**

what is  $\mathbb{Z}_n^*$ ?

**Example 1.2.4**

$(M_n(\mathbb{R}), +)$  is an abelian group.

**Example 1.2.5**

Consider  $(M_{n(\mathbb{R})}, \cdot)$ . The identity matrix is  $\begin{bmatrix} 1 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & 1 \end{bmatrix} \in M_n(\mathbb{R})$ . However, since not all  $M \in M_n(\mathbb{R})$  have multiplicative inverses,  $(M_n(\mathbb{R}), \cdot)$  is not a group.

**Notation**

$$\mathrm{GL}_n(\mathbb{R}) = \{M \in M_n(\mathbb{R}) : \det(M) \neq 0\}$$

**Note**

If  $A, B \in \mathrm{GL}_n(\mathbb{R})$ , then  $\det(AB) = \det(A)\det(B) \neq 0$ . Thus  $AB \in \mathrm{GL}_n(\mathbb{R})$ . The associativity of  $\mathrm{GL}_n(\mathbb{R})$  inherits from  $M_n(\mathbb{R})$ . Also the identity matrix satisfies  $\det(I) = 1 \neq 0$  and thus  $I \in \mathrm{GL}_n(\mathbb{R})$ . Finally, for  $M \in \mathrm{GL}_n(\mathbb{R})$ , there exists  $M^{-1} \in M_n(\mathbb{R})$  such that  $MM^{-1} = I = M^{-1}M$  since  $\det(M^{-1}) = \frac{1}{\det(M)} \neq 0$ , we have  $M^{-1} \in \mathrm{GL}_n(\mathbb{R})$ . Thus  $(\mathrm{GL}_n(\mathbb{R}), \cdot)$  is a group, called the *general linear group of degree n over  $\mathbb{R}$* .

**Note**

if  $n \geq 2$ , then  $\mathrm{GL}_n(\mathbb{R})$  is not abelian.

**Exercise 1.2.3**

What is  $(\mathrm{GL}_1(\mathbb{R}), \cdot)$ ?

**Example 1.2.6**

Let  $G, H$  be groups. The *direct product* is the set  $G \times H$  with the component wise operation defined by

$$(g_1, h_1) * (g_2, h_2) = (g_1 *_G g_2, h_1 *_H h_2)$$

One can check that  $G \times H$  is a group with identity  $(e_G, e_H)$  and the inverse of  $(g, h)$  is  $(g^{-1}, h^{-1})$

**Note**

One can show by induction that if  $G_1, \dots, G_n$  are groups, then  $G_1 \times \dots \times G_n$  is also a group.

**Notation**

Given a group  $G$  and  $g_1, g_2 \in G$ , we often denote  $g_1 * g_2$  by  $g_1 g_2$  and its identity by 1. Also the unique inverse of an element  $g \in G$  is denoted by  $g^{-1}$ . Also for  $n \in \mathbb{N}$ , we define

$g^n = g * g * \dots * g$  ( $n$ -times) and  $g^{-n} = (g^{-1})^n$ . Finally, we denote  $g^0 = 1$ .

**Proposition 1.3**

Let  $G$  be a group and  $g, h \in G$  we have

1.  $g^{-1-1} = g$
2.  $(gh)^{-1} = h^{-1}g^{-1}$
3.  $g^n g^m = g^{n+m}$  for all  $n, m \in \mathbb{Z}$
4.  $(g^n)^m = g^{nm}$  for all  $n, m \in \mathbb{Z}$

**Proof of 1:** Since

$$g^{-1}g = 1 = gg^{-1}$$

so  $g^{-1-1} = g$  □

**Proof of 2:**

$$(gh)(h^{-1}g^{-1}) = g(hh^{-1})g^{-1} = g1g^{-1} = 1$$

Similarly,

$$(h^{-1}g^{-1})(gh) = 1$$

Thus  $(gh)^{-1} = h^{-1}g^{-1}$  □

**Proof of 3:** We proceed by considering cases:

1. if  $n = 0$  then

$$g^n g^m = g^0 g^m = 1g^m = g^m = g^{0+m} = g^{n+m}$$

2. if  $n > 0$ , we will proceed by induction on  $n$ . Case 1 establishes the base case. Let  $m \in \mathbb{Z}, n \in \mathbb{Z}_{\geq 0}$ . Suppose that  $g^n g^m = g^{n+m}$  Then

$$g^{n+1}g^m = gg^n g^m = gg^{n+m} = g^{n+m+1}$$

3. if  $n < 0$ , then  $n = -k$  for some  $k \in \mathbb{N}$ . We have

$$g^k g^n g^m = g^{k+n} g^m = g^0 g^m = g^m$$

also

$$g^k g^{n+m} = g^{k+m+n} = g^m$$

Thus

$$g^k g^n g^m = g^k g^{n+m}$$

So

$$g^n g^m = g^{n+m}$$

as desired. □

**Proof of 4:** We proceed by considering cases:

1. if  $m = 0$ , then  $(g^n)^m = (g^n)^0 = 1 = g^0 = g^{n0} = g^{nm}$
2. if  $m > 0$ , then

$$(g^n)^m = \underbrace{g^n g^n \cdots g^n}_{m \text{ times}} = g^{nm}$$

3. if  $m < 0$ , then  $m = -k$  for some  $k \in \mathbb{N}$ . We will induct on  $k$ . For  $k = 1$  we see that  $(g^n)^{-1} = g^{-n}$  since

$$g^n g^{-n} = g^{n-n} = g^0 = 1$$

Suppose  $(g^n)^{-\ell} = g^{-n\ell}$  for all  $1 \leq \ell \leq k$ . Then

$$(g^n)^{-k-1} = (g^n)^{-k} (g^n)^{-1} = g^{-nk} g^{-n} = g^{-nk-n} = g^{-n(k+1)}$$

□

### Exercise 1.2.4

prove 3,4

#### Warning

In general, it is not the case that if  $g, h \in G$  then  $(gh)^n = g^n h^n$ , this is not true unless  $G$  is abelian

### Proposition 1.4

Let  $G$  be a group and  $g, h, f \in G$ . Then

1. They satisfy the left and right cancellation. More precisely,
  - a. if  $gh = gf$  then  $h = f$
  - b. if  $hg = fg$  then  $h = f$
2. Given  $a, b \in G$  the equations  $ax = b$  and  $ya = b$  have unique solutions for  $x, y \in G$

**Proof of 1-a:** By left-multiplying by  $g^{-1}$ , we have

$$gh = gf \iff g^{-1}gh = g^{-1}gf \iff h = f$$

□  
□

**Proof of 1-b:** similar to 1-a

**Proof of 2:** Let  $x = a^{-1}b$  then

$$ax = aa^{-1}b = b$$

If  $u$  is another solution, then  $au = b = ax$ . By 1-a,  $u = x$ . Similarly,  $y = ba^{-1}$  is the unique solution of  $ya = b$

□

## 1.3 Symmetric Groups

### Definition 1.3.1

Given a non-empty set  $L$ , a *permutation* of  $L$  is a bijection from  $L$  to  $L$ . The set of all permutations of  $L$  is denoted by  $S_L$

### Example 1.3.1

Consider the set  $L = \{1, 2, 3\}$  which has the following different permutations

$$\begin{pmatrix} 123 \\ 123 \end{pmatrix}, \begin{pmatrix} 123 \\ 132 \end{pmatrix}, \begin{pmatrix} 123 \\ 213 \end{pmatrix}, \begin{pmatrix} 123 \\ 231 \end{pmatrix}, \begin{pmatrix} 123 \\ 312 \end{pmatrix}, \begin{pmatrix} 123 \\ 321 \end{pmatrix}$$

Where  $\begin{pmatrix} 123 \\ 123 \end{pmatrix}$  denotes the bijection

$$\sigma : \{1, 2, 3\} \longrightarrow \{1, 2, 3\}$$

$$\sigma(1) = 1, \sigma(2) = 2, \sigma(3) = 3$$

### Notation

For  $n \in \mathbb{N}$  we denote by  $S_n = S_{\{1, 2, \dots, n\}}$  the set of all permutations of  $\{1, 2, \dots, n\}$ . We have seen that the order of  $S_3 = 3! = 6$ . To consider the general  $S_n$ , we note that for a permutation  $\sigma \in S_n$ , there are  $n$  choices for  $\sigma(1)$ ,  $n - 1$  choices for  $\sigma(2), \dots$ , 1 choice for  $\sigma(n)$ . Thus

### Proposition 1.5

$$|S_n| = n!$$

**Note**

For Möbius quizzes, use “9 dots” for permutations.

**Remark**

Given  $\sigma, \tau \in S_n$  we can compose them to get a new element  $\sigma\tau$ , where

$\sigma\tau = \{1, 2, \dots, n\} \rightarrow \{1, 2, \dots, n\}$  given by  $x \mapsto \sigma(\tau(x))$  Since both  $\sigma, \tau$  are bijections,  $\sigma\tau \in S_n$

**Example 1.3.2**

Compute  $\sigma\tau$  and  $\tau\sigma$  if

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 1 & 2 \end{pmatrix}, \quad \tau = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 3 & 1 \end{pmatrix}$$

Then  $\sigma\tau(1) = \sigma(2) = 4, \dots$  Then  $\sigma\tau = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 2 & 1 & 3 \end{pmatrix}$ , and  $\tau\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 1 & 2 & 4 \end{pmatrix}$

We note that  $\sigma\tau \neq \tau\sigma$

**Note**

For any  $\sigma, \tau \in S_n$  we have that  $\tau\sigma, \sigma\tau \in S_n$  but  $\sigma\tau \neq \tau\sigma$  in general on the other hand, for any  $\sigma, \tau, \mu$  we have  $\sigma(\tau\mu) = (\sigma\tau)\mu$ . Also note the *identity permutation*  $\varepsilon \in S_n$  is defined as

$$\varepsilon = \begin{pmatrix} 1 & 2 & \dots & n \\ 1 & 2 & \dots & n \end{pmatrix}$$

Thus for any  $\sigma \in S_n$ , we have  $\sigma\varepsilon = \varepsilon\sigma = \sigma$

Finally, for  $\sigma \in S_n$ , since it is a bijection, there is a unique bijection  $\sigma^{-1} \in S_n$  called the *inverse permutation* of  $\sigma$  such that for all  $x, y \in \{1, 2, \dots, n\}$

$$\sigma^{-1}(x) = y \iff \sigma(y) = x$$

It follows that

$$\sigma(\sigma^{-1}(x)) = \sigma(y) = x$$

and

$$\sigma^{-1}(\sigma(y)) = y$$

i.e we have

$$\sigma\sigma^{-1} = \sigma^{-1}\sigma = \varepsilon$$

**Example 1.3.3**

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 4 & 5 & 1 & 2 & 3 \end{pmatrix}$$

Then

$$\sigma^{-1} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 4 & 5 & 1 & 2 \end{pmatrix}$$

From the above we have

**Proposition 1.6**

$(S_n, \circ)$  is a group, called the *symmetric group of degree n*

**Exercise 1.3.1**

Write down all rotations and reflections that fix an equilateral triangle. Then check why it is the “same” as  $S_3$

**Example 1.3.4**

Consider

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\ 3 & 1 & 7 & 6 & 9 & 4 & 2 & 5 & 8 & 10 \end{pmatrix} \in S_{10}$$

We note that  $1 \rightarrow 3 \rightarrow 7 \rightarrow 2 \rightarrow 1$  and  $4 \rightarrow 6 \rightarrow 4$  and  $5 \rightarrow 9 \rightarrow 8$  and  $10 \rightarrow 10$ . Thus  $\sigma$  can be *decomposed* into one 4-cycle  $(1372)$ , one 2-cycle  $(46)$ , and one 3-cycle  $(598)$  and one 1-cycle  $(10)$  (we usually do not write 1-cycles). Note that these cycles are *pairwise disjoint* and we have

$$\sigma = (1372)(46)(598)$$

We can also write  $\sigma = (46)(598)(1372)$ , or  $\sigma = (64)(985)(7213)$

**Theorem 1.7****Cycle Decomposition**

If Given  $\sigma \in S_n$  with  $\sigma \neq \varepsilon$ , then  $\sigma$  is a product of (one or more) disjoint cycles of length at least 2. This factorization is unique up to the order of the factors.

**Proof:** See bonus 1. □

**Convention**

Every permutation of  $S_n$  can be regarded as a permutation in  $S_{n+1}$  by fixing the number  $n + 1$ , thus

$$S_1 \subseteq S_2 \subseteq \cdots \subseteq S_n \subseteq S_{n+1}$$

## 1.4 Cayley Tables

### Definition 1.4.1

For a finite group  $G$ , defining its operation by means of a table is sometimes convenient. Given  $x, y \in G$ , the product  $xy$  is the entry of the table in the row corresponding to  $x$  and the column corresponding to  $y$ , such a table is a *Cayley table*.

### Remark

By cancellation, the entries in each row or column of a Cayley table are all distinct

### Example 1.4.1

Consider  $(\mathbb{Z}_2, +)$  its Cayley table is

$\mathbb{Z}_2$	[0]	[1]
[0]	[0]	[1]
[1]	[1]	[0]

### Example 1.4.2

Consider the group  $\mathbb{Z}^* = \{1, -1\}$ . Its Cayley table is

$\mathbb{Z}^*$	1	-1
1	1	-1
-1	-1	1

### Note

If we replace 1 by [0] and -1 by [1] the Cayley tables of  $\mathbb{Z}^*$  and  $\mathbb{Z}_2$  become the same. In this case, we say  $\mathbb{Z}^*$  and  $\mathbb{Z}_2$  are *isomorphic* denoted by

$$\mathbb{Z}^* \cong \mathbb{Z}_2$$

**Example 1.4.3**

For  $n \in \mathbb{N}$ , the *cyclic group of order n* is defined by

$$C_n = \{1, a, a^2, \dots, a^{n-1}\} \text{ with } a^n = 1 \text{ and } 1, a, \dots, a^{n-1} \text{ are distinct}$$

The Cayley table of  $C_n$  is as follows

$C_n$	1	$a$	$a^2$	$\dots$	$a^{n-2}$	$a^{n-1}$
1	1	$a$	$a^2$	$\dots$	$a^{n-2}$	$a^{n-1}$
$a$	$a$	$a^2$	$a^3$	$\dots$	$a^{n-1}$	1
$a^2$	$a^2$	$a^3$	$a^4$	$\dots$	1	$a$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$
$a^{n-2}$	$a^{n-2}$	$a^{n-1}$	1	$\dots$	$a^{n-4}$	$a^{n-3}$
$a^{n-1}$	$a^{n-1}$	1	$a$	$\dots$	$a^{n-3}$	$a^{n-2}$

**Proposition 1.8**

Let  $G$  be a group. Up to isomorphism, we have

1. If  $|G| = 1$ , then  $G \cong \{1\}$
2. If  $|G| = 2$ , then  $G \cong C_2$
3. If  $|G| = 3$ , then  $G \cong C_3$
4. If  $|G| = 4$ , then  $G \cong C_4$  or  $G \cong K_4 \cong C_2 \times C_2$

**Proof of 1:** obviously □

**Proof of 2:** If  $|G| = 2$  then  $G = \{1, g\}$  with  $g \neq 1$ . Then  $g^2 = g$  or  $g^2 = 1$ . We note that if  $g^2 = g$ , then  $g = 1$  contradiction. Thus  $g^2 = 1$ . Thus the Cayley table is as follows

$G$	1	$g$
1	1	$g$
$g$	$g$	1

which is the same as  $C_2$  □

**Proof of 3:** If  $|G| = 3$ , then  $G = \{1, g, h\}$  with  $g \neq 1, h \neq 1, g \neq h$ . By cancellation, we have  $gh \neq g, gh \neq h$ , thus  $gh = 1$ . Similarly, we have  $hg = 1$ . Also, on the row for  $g$ , we have  $g1 = g, gh = 1$ . Since all entries in this row are distinct, we have  $g^2 = h$ . Similarly, we have  $h^2 = g$ . Thus we obtain the following Cayley table

$G$	1	$g$	$h$
1	1	$g$	$h$
$g$	$g$	$h$	1
$h$	$h$	1	$g$

Which is the same as  $C_3$ . □

**Proof of 4:** See assignment 1 □

**Exercise 1.4.1**

Consider the symmetry group of a non-square rectangle. How is it related to  $K_4$ ?

**2 Subgroups****2.1 Subgroups****Definition 2.1.1**

Let  $G$  be a group and  $H \subseteq G$ . If  $H$  itself is a group, then we say  $H$  is a *subgroup* of  $G$ .

**Note**

We note that since  $G$  is a group, for  $h_1, h_2, h_3 \in H \subseteq G$ , we have

$$h_1(h_2h_3) = (h_1h_2)h_3$$

Thus

**Proposition 2.1****Subgroup Test**

Let  $G$  be a group,  $H \subseteq G$ . Then  $H$  is a subgroup of  $G$  if

1. If  $h_1, h_2 \in H$ , then  $h_1h_2 \in H$
2.  $1_H \in H$
3. If  $h \in H$ , then  $h^{-1} \in H$

**Exercise 2.1.1**

Prove that  $1_H = 1_G$

**Example 2.1.1**

Given a group  $G$ , then  $\{1\}, G$  are subgroups of  $G$

**Example 2.1.2**

We have a chain of groups

$$(\mathbb{Z}, +) \subseteq (\mathbb{Q}, +) \subseteq (\mathbb{R}, +) \subseteq (\mathbb{C}, +)$$

**Example 2.1.3**

Define

$$\mathrm{SL}_n(\mathbb{R}) = (\mathrm{SL}_n(\mathbb{R}), \cdot) := \{M \in M_n(\mathbb{R}), \det(M) = 1\} \subseteq \mathrm{GL}_n(\mathbb{R})$$

Note that the identity matrix  $I \in \mathrm{SL}_n(\mathbb{R})$ . Let  $A, B \in \mathrm{SL}_n(\mathbb{R})$ , then

$$\det(AB) = \det(A)\det(B) = 1 \cdot 1 = 1$$

and

$$\det(A^{-1}) = \frac{1}{\det(A)} = \frac{1}{1} = 1$$

i.e.  $AB, A^{-1} \in \mathrm{SL}_n(\mathbb{R})$ . By the subgroup test (Proposition 2.1),  $\mathrm{SL}_n(\mathbb{R})$  is a subgroup of  $\mathrm{GL}_n(\mathbb{R})$ . We call  $\mathrm{SL}_n(\mathbb{R})$  the *special linear group of order n over  $\mathbb{R}$*

**Definition 2.1.2**

Given a group  $G$ , we define the *center of  $G$*  to be

$$Z(G) := \{z \in G \mid zg = gz \ \forall g \in G\}$$

**Remark**

$Z(G) = G$  iff  $G$  is abelian.

**Proposition 2.2**

$Z(G)$  is an abelian subgroup of  $G$ .

**Proof:** Note that  $1 \in Z(G)$ . Let  $y, z \in Z(G)$ . Then for all  $g \in G$ , we have

$$(yz)g = y(zg) = y(gz) = (yg)z = (gy)z = g(yz)$$

Thus  $yz \in Z(G)$ . Also, for  $z \in Z(G)$ ,  $g \in G$  we have

$$\begin{aligned} zg = gz &\iff z^{-1}(zg)z^{-1} = z^{-1}(gz)z^{-1} \\ &\iff gz^{-1} = z^{-1}g \end{aligned}$$

Thus  $z^{-1} \in Z(G)$ . By the subgroup test (Proposition 2.1),  $Z(G)$  is a subgroup of  $G$ . Also, by the definition of  $Z(G)$ , we see that it is abelian.  $\square$

**Proposition 2.3**

Let  $H, K$  be subgroups of a group  $G$ . Then  $H \cap G$  is also a subgroup.

**Proof:** Exercise  $\square$

**Proposition 2.4****Finite Subgroup Test**

If  $H \neq \emptyset$  is a finite subset of a group  $G$ , then  $H$  is a subgroup of  $G$  iff  $H$  is closed under its operation.

**Proof:**

( $\Rightarrow$ ) obvious

( $\Leftarrow$ ) For  $H \neq \emptyset$ , let  $h \in H$ . Since  $H$  is closed under its operation, we have  $h, h^2, h^3, \dots \in H$ . Since  $H$  is finite, these elements are not all distinct. Thus  $h^n = h^{n+m}$  for some  $n, m \in \mathbb{N}$ . By cancellation,  $h^m = 1$  and thus  $1 \in H$ . Also,  $1 = h^{m-1}h$  implies that  $h^{-1} = h^{m-1}$  and thus  $h^{-1} \in H$ . By the subgroup test,  $H$  is a subgroup of  $G$ .  $\square$

## 2.2 Alternating Groups

**Definition 2.2.1**

A *transposition*  $\sigma \in S_n$  is a cycle of length 2. i.e.  $\sigma = (ab)$  with  $a, b \in \{1, 2, \dots, n\}$  and  $a \neq b$ .

**Example 2.2.1**

Consider  $(1245) \in S_5$ . Also the composition  $(12)(24)(45)$  can be computed as

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 2 & 3 & 5 & 4 \\ 1 & 4 & 3 & 5 & 2 \\ 2 & 4 & 3 & 5 & 1 \end{pmatrix}$$

Thus we have  $(1245) = (12)(24)(45)$  Also we can show that

$$(1245) = (23)(12)(25)(13)(24)$$

We see from this example that the factorization into transpositions are NOT unique. However, one can prove (see Bonus 2)

**Theorem 2.5****Parity Theorem**

If a permutation  $\sigma$  has two factorizations

$$\sigma = \gamma_1 \gamma_2 \cdots \gamma_r = \mu_1 \mu_2 \cdots \mu_s$$

Where each  $\gamma_i$  and  $\mu_j$  is a transposition, then  $r \equiv s \pmod{2}$

**Definition 2.2.2**

A permutation  $\sigma$  is *even* (or *odd*) if it can be written as a product of an even (or odd) number of transpositions. By the previous theorem, a permutation is either even or odd, but not both.

**Theorem 2.6**

For  $n \geq 2$ , let  $A_n$  denote the set of all even permutations in  $S_n$

1.  $\varepsilon \in A_n$
2. If  $\sigma, \tau \in A_n$ , then  $\sigma\tau \in A_n$  and  $\sigma^{-1} \in A_n$
3.  $|A_n| = \frac{1}{2}n!$

From (1) and (2), we see  $(A_n)$  is a subgroup of  $S_n$  called the *alternating group of degree n*.

**Proof of 1:** We can write  $\varepsilon = (12)(12)$ . Thus  $\varepsilon$  is even. □

**Proof of 2:** if  $\sigma, \tau \in A_n$  we can write  $\sigma = \sigma_1 \cdots \sigma_r$  and  $\tau = \tau_1 \cdots \tau_s$  where  $\sigma_i, \tau_j$  are transpositions and  $r, s$  are even integers. Then

$$\sigma\tau = \sigma_1 \cdots \sigma_r \tau_1 \cdots \tau_s$$

is a product of  $(r + s)$  transpositions and thus  $\sigma\tau \in A_n$ . Also, we note that  $\sigma_i$  is a transposition, we have  $\sigma_i^2 = \varepsilon$  and thus  $\sigma_i^{-1} = \sigma_i$ . It follows that

$$\sigma^{-1} = (\sigma_1 \cdots \sigma_r)^{-1} = \sigma_r^{-1} \cdots \sigma_1^{-1} = \sigma_r \cdots \sigma_1$$

which is an even permutation. □

**Proof of 3:** Let  $O_n$  denote the set of odd permutations in  $S_n$ . Thus  $S_n = A_n \cup O_n$  and the parity theorem implies that  $A_n \cap O_n = \emptyset$ . Since  $|S_n| = n!$ , to prove  $|A_n| = \frac{1}{2}n!$ , it suffices to show that  $|A_n| = |O_n|$ . Let  $\gamma = (12)$  and let  $f : A_n \rightarrow O_n$  be defined by  $f(\sigma) = \gamma\sigma$ . Since  $\sigma$  is even, we have  $\gamma\sigma$  is odd. Thus the map is well-defined. Also, if we have  $\gamma\sigma_1 = \gamma\sigma_2$ , then by cancellation, we get  $\sigma_1 = \sigma_2$ , thus  $f$  is injective. Finally, if  $\tau \in O_n$ , then  $\sigma = \gamma\tau \in A_n$  and  $f(\sigma) = \gamma\sigma = \gamma(\gamma\tau) = \gamma^2\tau = \tau$ . Thus  $f$  is surjective. It follows that  $f$  is a bijection, thus  $|A_n| = |O_n|$ . It follows that  $|A_n| = \frac{1}{2}n! = |O_n|$  □

## 2.3 Orders of Elements

**Notation**

If  $G$  is a group and  $g \in G$ , we denote

$$\langle g \rangle = \{g^k \mid k \in \mathbb{Z}\} = \{\dots, g^{-1}, g^0 = 1, g, g^2, \dots\}$$

Note that  $1 = g^0 \in \langle g \rangle$ . Also, if  $x = g^m, y = g^n \in \langle g \rangle$  With  $m, n \in \mathbb{Z}$ , then  $xy = g^n g^m = g^{n+m} \in \langle g \rangle$  and  $x^{-1} = g^{-m} \in \langle g \rangle$ . By the subgroup test, we have

**Proposition 2.7**

If  $G$  is a group and  $g \in G$ , then  $\langle g \rangle$  is a subgroup of  $G$ .

**Definition 2.3.1**

Let  $G$  be a group with  $g \in G$ . We call  $\langle g \rangle$  the *cyclic subgroup of  $G$  generated by  $g$* . If  $G = \langle g \rangle$  for some  $g \in G$ , then we say  $G$  is *cyclic* and  $g$  a *generator* of  $G$ .

**Example 2.3.1**

Consider  $(\mathbb{Z}, +)$ . Note that for all  $k \in \mathbb{Z}$ , we can write  $k = k \cdot 1$ . Thus we can see  $(\mathbb{Z}, +) = \langle 1 \rangle$ . Similarly,  $(\mathbb{Z}, +) = \langle -1 \rangle$ . We observe, for any integer  $n \in \mathbb{Z}$  with  $n \neq \pm 1$  there exist no  $k \in \mathbb{Z}$  such that  $k \cdot n = 1$ . Thus  $\pm 1$  are the only generators of  $(\mathbb{Z}, +)$ .

**Remark**

Let  $G$  be a group and  $g \in G$ . Suppose there is  $k \in \mathbb{Z}$   $k \neq 0$  such that  $g^k = 1$  then  $g^{-k} = (g^k)^{-1} = 1$ . Thus we can assume  $k \geq 1$ . Then by the well-ordering principle, there exists the smallest positive integer  $n$  such that  $g^n = 1$

**Definition 2.3.2**

Let  $G$  be a group and  $g \in G$ . If  $n$  is the smallest positive integer such that  $g^n = 1$ , then we say the *order* of  $g$  is  $n$ , denoted  $o(g) = n$ . If no such  $n$  exists, we say  $g$  has *infinite order* and write  $o(g) = \infty$

**Proposition 2.8**

Let  $G$  be a group and  $g \in G$  with  $o(g) = n \in \mathbb{N}$ . For  $k \in \mathbb{Z}$  we have

1.  $g^k = 1$  iff  $n \mid k$
2.  $g^k = g^m$  iff  $k \equiv m \pmod{n}$
3.  $\langle g \rangle = \{1, g, g^2, \dots, g^{n-1}\}$  where  $1, g, \dots, g^{n-1}$  are all distinct. In particular, we have  $|\langle g \rangle| = o(g)$

**Proof of 1:**

$(\Leftarrow)$  if  $n \mid k$ , then  $k = nq$  for some  $q \in \mathbb{Z}$ . Thus

$$g^k = g^{nq} = (g^n)^q = 1^q = 1$$

$(\Rightarrow)$  By the division algorithm, we can write  $k = nq + r$  with  $q, r \in \mathbb{Z}$  and  $0 \leq r < n$ . Since  $g^k = 1$  and  $g^n = 1$ , we have

$$g^r = g^{k-nq} = g^k(g^n)^{-q} = 1 \cdot 1^{-q} = 1$$

Since  $0 \leq r < n$  and  $o(g) = n$ , we have  $r = 0$  and hence  $n \mid k$ . □

**Proof of 2:** Note that  $g^k = g^m$  iff  $g^{km} = 1$ . By (1), we have  $n \mid (km)$  i.e.  $k \equiv m \pmod{n}$  □

**Proof of 3:** It follows from (2) that  $1, g, \dots, g^{n-1}$  are all distinct. Clearly, we have  $\{1, g, \dots, g^{n-1}\} \subseteq \langle g \rangle$ .

To prove the other inclusion, let  $g^k \in \langle g \rangle$  for some  $k \in \mathbb{Z}$ . Write  $k = nq + r$  with  $n, r \in \mathbb{Z}$  and  $0 \leq r < n$ . Then

$$g^k = g^{nq+r} = g^{nq}g^r = (g^n)^qg^r = 1^qg^r = g^r \in \{1, g, \dots, g^{n-1}\}$$

Thus  $\langle g \rangle = \{1, g, \dots, g^{n-1}\}$  □

**Proposition 2.9**

Let  $G$  be a group and  $g \in G$  with  $o(g) = \infty$ . For  $k \in \mathbb{Z}$  we have

1.  $g^k = 1$  iff  $k = 0$
2.  $g^k = g^m$  iff  $k = m$
3.  $\langle g \rangle = \{..., g^{-1}, g^0 = 1, g, ...\}$  where  $g^i$  are all distinct

**Proposition 2.10**

Let  $G$  be a group and  $g \in G$  with  $o(g) = n \in \mathbb{N}$ . If  $d \in \mathbb{N}$ , then  $o(g^d) = \frac{n}{\gcd(n, d)}$ . In particular, if  $d \mid n$ , then  $\gcd(n, d) = d$  and  $o(g^d) = \frac{n}{d}$

**Proof:** Let  $n_1 = \frac{n}{\gcd(n, d)}$  and  $d_1 = \frac{d}{\gcd(n, d)}$ . By a result from Math 135, we have  $\gcd(n_1, d_1) = 1$ . Note that

$$(g^d)^{n_1} = (g^d)^{\frac{n}{\gcd(n, d)}} = (g^n)^{\frac{d}{\gcd(n, d)}} = 1$$

Thus it remains to show that  $n_1$  is the smallest such positive integer. Suppose  $(g^d)^r = 1$  with  $r \in \mathbb{N}$ . Since  $o(g) = n$ , by proposition, we have  $n \mid dr$ . Thus there is  $q \in \mathbb{Z}$  such that  $dr = nq$ . Dividing both sides by  $\gcd(n, d)$  we get

$$d_1 r = \frac{d}{\gcd(n, d)} r = \frac{n}{\gcd(n, d)} q = n_1 q$$

Since  $n_1 \mid d_1 r$  and  $\gcd(n_1, d_1) = 1$ , by a result from Math 135, we get  $n_1 \mid r$  i.e.  $r = n_1 \ell$  for some  $\ell \in \mathbb{Z}$ . Since  $r_1, n_1 \in \mathbb{N}$ , it follows that  $\ell \in \mathbb{N}$ . Since  $\ell \geq 1$ , we get  $r \geq n_1$  □

## 2.4 Cyclic Groups

**Remark**

For a group  $G$ , if  $G = \langle g \rangle$  for some  $g \in G$ , then  $G$  is a cyclic group. For  $a, b \in G$ , we have  $a = g^n, b = g^m$  for some  $m, n \in \mathbb{Z}$ . We have

$$ab = g^n g^m = g^{n+m} = g^{m+n} = g^m g^n = ba$$

**Proposition 2.11**

Every cyclic group is abelian

**Warning**

The converse of the above proposition is not true. For example the Klein 4 group is abelian, but not cyclic.

**Proposition 2.12**

Every subgroup of a cyclic group is cyclic.

**Proof:** Let  $G = \langle g \rangle$  be cyclic and  $H \subseteq G$  a subgroup. If  $H = \{1\}$ , then  $H$  is cyclic. Otherwise, there is  $g^k \in H$  with  $k \in \mathbb{Z} \setminus \{0\}$ . Since  $H$  is a group, we have  $g^{-k} \in H$ . Thus we can assume that  $k \in \mathbb{N}$ . Let  $m$  be the smallest positive integer such that  $g^m \in H$ .

Claim:  $H = \langle g^m \rangle$

Proof is exercise, by division algorithm.  $\square$

### Proposition 2.13

Let  $G = \langle g \rangle$  be a cyclic group with  $o(g) = n$ . Then  $G = \langle g^k \rangle$  iff  $\gcd(k, n) = 1$ .

**Proof:** By proposition,

$$o(g^k) = \frac{n}{\gcd(n, k)} = n$$

$\square$

### Theorem 2.14

### Fundamental Theorem of Finite Cyclic Groups

Let  $G = \langle g \rangle$  be a cyclic group with  $o(g) = n \in \mathbb{N}$ .

1. If  $H$  is a subgroup of  $G$ , then  $H = \langle g^d \rangle$  for some  $d \mid n$ . It follows that  $|H| \mid |G|$ .
2. Conversely, if  $k \mid n$ , then  $\langle g^{\frac{n}{k}} \rangle$  is the unique subgroup of  $G$  with order  $k$ .

**Proof of 1:** By proposition,  $H$  is cyclic. Write  $H = \langle g^m \rangle$  for some  $m \in \mathbb{N} \cup \{0\}$ . Let  $d = \gcd(m, n)$ .

Claim:  $H = \langle g^d \rangle$

Since  $d \mid m$  we have  $m = dk$  for some  $k \in \mathbb{Z}$ . Then

$$g^m = g^{dk} = (g^d)^k \in \langle g^d \rangle$$

Thus  $H = \langle g^m \rangle \subseteq \langle g^d \rangle$ . To prove the other inclusion, since  $d = \gcd(m, n)$ , there is  $x, y \in \mathbb{Z}$  such that  $d = mx + ny$ . Then

$$g^d = g^{mx+ny} = (g^m)^x (g^n)^y = (g^m)^x 1^y = (g^m)^x \in \langle g^m \rangle$$

Thus  $\langle g^d \rangle \subseteq \langle g^m \rangle = H$ . It follows that  $H = \langle g^d \rangle$ . Note that since  $d = \gcd(m, n)$ , we have  $d \mid n$ . By proposition, we have

$$|H| = o(g^d) = \frac{n}{\gcd(n, d)} = \frac{n}{d}$$

Thus  $|H| \mid |G|$   $\square$

**Proof of 2:** By proposition, the cyclic subgroup  $\langle g^{\frac{n}{k}} \rangle$  is of order

$$\frac{n}{\gcd(n, \frac{n}{k})} = \frac{n}{n/k} = k$$

To show uniqueness, let  $K$  be a subgroup of  $G$  with order  $k \mid n$ . By 1, let  $K = \langle g^d \rangle$  where  $d \mid n$ . Then by props, we have,

$$k = |K| = o(g^d) = \frac{n}{\gcd(n, d)} = \frac{n}{d}$$

It follows that  $d = \frac{n}{k}$  and thus  $K = \langle g^{\frac{n}{k}} \rangle$

□

## 2.5 Non-cyclic Groups

### Definition 2.5.1

Let  $X$  be a non-empty subset of a group  $G$ , and let

$$\langle X \rangle := \{x_1^{k_1} \cdots x_m^{k_m} \mid x_i \in X, k_i \in \mathbb{Z}, m \geq 1\}$$

denote the set of all products of powers of (not necessarily distinct) elements of  $X$ . Note that this is clearly a group.  $\langle X \rangle$  is called the *subgroup of  $G$  generated by  $X$* .

### Example 2.5.1

The Klein-4 group  $K_4 = \{1, a, b, c\}$  with  $a^2 = b^2 = c^2 = 1$  and  $ab = c$ . Thus

$$K_4 = \langle a, b \mid a^2 = 1 = b^2 \text{ and } ab = ba \rangle$$

### Example 2.5.2

The symmetric group of order 3  $S_3 = \{\varepsilon, \sigma, \sigma^2, \tau, \tau\sigma, \tau\sigma^2\}$  where  $\sigma^3 = \varepsilon = \tau^2$  and  $\sigma\tau = \tau\sigma^2$  (one can take  $\tau = (12)$  and  $\sigma = (123)$ ) Thus

$$\langle \sigma, \tau \mid \sigma^3 = \varepsilon = \tau^2 \text{ and } \sigma\tau = \tau\sigma^2 \rangle$$

We can also replace  $\sigma, \tau$  with  $\sigma, \tau\sigma$  or  $\sigma, \tau\sigma^2, \dots$ , etc

### Definition 2.5.2

For  $n \geq 2$  the *dihedral group of order  $2n$*  is defined by

$$D_{2n} = \{1, a, \dots, a^{n-1}, b, ba, \dots, ba^{n-1}\}$$

Where  $a^n = 1 = b^2$  and  $aba = b$ . Thus

$$D_{2n} = \langle a, b \mid a^n = 1 = b^2 \text{ and } aba = b \rangle$$

### Note

For  $n = 2$  or  $3$  we have

$$D_4 \cong K_4 \quad \text{and} \quad D_6 \cong S_3$$

### Exercise 2.5.1

For  $n \geq 3$ , consider a regular  $n$ -gon and its group of symmetries. How does it relate to  $D_{2n}$ ?

## 3 Normal Subgroups

### 3.1 Homomorphisms and Isomorphisms

#### Definition 3.1.1

Let  $G, H$  be groups. A mapping  $\alpha : G \rightarrow H$  is a *homomorphism* if

$$\alpha(a *_G b) = \alpha(a) *_H \alpha(b) \quad \forall a, b \in G$$

To simplify notation, we often write

$$\alpha(ab) = \alpha(a)\alpha(b) \quad \forall a, b \in G$$

#### Example 3.1.1

Consider the determinant map

$$\begin{aligned} \det : \mathrm{GL}_n(\mathbb{R}) &\longrightarrow \mathbb{R}^* \\ A &\longmapsto \det A \end{aligned}$$

Since  $\det AB = \det A \det B$ , the mapping  $\det$  is a homomorphism.

#### Proposition 3.1

Let  $\alpha : g \rightarrow H$  be a group homomorphism. Then

1.  $\alpha(1_G) = 1_H$
2.  $\alpha(g^{-1}) = \alpha(g)^{-1} \quad \forall g \in G$
3.  $\alpha(g^k) = \alpha(g)^k \quad \forall k \in \mathbb{Z}$

#### Definition 3.1.2

Let  $\alpha : G \rightarrow H$  be a mapping between groups. If  $\alpha$  is a homomorphism and  $\alpha$  is bijective, we say  $\alpha$  is an *isomorphism*. In this case, we say  $G, H$  are *isomorphic* and write  $G \cong H$ .

#### Proposition 3.2

We have

1. The identity map  $\mathrm{id} : G \rightarrow G$  is an isomorphism.
2. If  $\sigma : G \rightarrow H$  is an isomorphism, then the inverse map  $\sigma^{-1} : h \rightarrow G$  is also an isomorphism.
3. If  $\sigma : G \rightarrow H$  and  $\tau : H \rightarrow K$  is an isomorphism, the composite map  $\tau\sigma : G \rightarrow K$  is also an isomorphism.

So  $\cong$  is (sort-of) an equivalence relation

**Proof:** Exercise. □

**Example 3.1.2**

Let  $\mathbb{R}^+ = \{r \in \mathbb{R} \mid r > 0\}$ . Then  $(\mathbb{R}, +) \cong (\mathbb{R}^+, \cdot)$  since we see that

$$\begin{aligned}\sigma : \mathbb{R} &\rightarrow \mathbb{R}^+ \\ x &\mapsto e^x\end{aligned}$$

is a bijection. Moreover,  $\sigma(x + y) = e^{x+y} = e^x \cdot e^y = \sigma(x)\sigma(y)$  thus  $\sigma$  is an isomorphism.

**Example 3.1.3**

Claim:  $(\mathbb{Q}, +) \not\cong (\mathbb{Q}^*, \cdot)$  Suppose  $\tau : (\mathbb{Q}, +) \rightarrow (\mathbb{Q}^*, \cdot)$  is an isomorphism. Thus  $\tau$  is surjective. So there is some  $q \in \mathbb{Q}$  such that  $\tau(q) = 2$ . Then

$$\tau\left(\frac{q}{2}\right)^2 = \tau\left(\frac{q}{2}\right)\tau\left(\frac{q}{2}\right) = \tau\left(\frac{q}{2} + \frac{q}{2}\right) = \tau(q) = 2$$

Thus  $\tau\left(\frac{q}{2}\right)$  is a rational number whose square is 2, a contradiction.

## 3.2 Cosets and Lagrange's Theorem

**Definition 3.2.1**

Let  $H$  be a subgroup of a group  $G$ . If  $a \in G$ , we define

$$Ha = \{ha \mid h \in H\}$$

to be the *right coset of  $H$  generated by  $a$* . We define the left coset similarly.

**Remark**

Since  $1 \in H$ , we have  $H1 = H = 1H$ . Also  $a \in Ha$  and  $a \in aH$ . Note that in general  $Ha$  and  $aH$  are not subgroups of  $G$ , and  $aH \neq Ha$ . However, if  $G$  is abelian, then  $Ha = aH$ .

**Example 3.2.1**

Let  $K_4 = \{1, a, b, ab\}$ . Let  $H = \{1, a\}$  which is a subgroup of  $K_4$ . Note that since  $K_4$  is abelian, we have  $gH = Hg$  for all  $g \in K_4$ . Then the (right or left) cosets of  $H$  are

$$H1 = \{1, a\} = 1H$$

and

$$Hb = \{b, ab\} = Hab$$

Thus there are exactly two cosets of  $H$  in  $K_4$

**Example 3.2.2**

Let  $S_3 = \{\varepsilon, \sigma, \sigma^2, \tau, \tau\sigma, \tau\sigma^2\}$  with  $\sigma^3 = \varepsilon = \tau^2$  and  $\sigma\tau\sigma = \tau$ . Let  $H = \{\varepsilon, \tau\}$  which is a subgroup of  $S_3$ . Since  $\sigma\tau = \tau\sigma^{-1} = \tau\sigma^2$ , the right cosets of  $H$  are

$$\begin{aligned} H\varepsilon &= \{\varepsilon, \tau\} &= H\tau \\ H\sigma &= \{\sigma, \tau\sigma\} &= H\tau\sigma \\ H\sigma^2 &= \{\sigma^2, \tau\sigma^2\} &= H\tau\sigma^2 \end{aligned}$$

And the left cosets of  $H$  are

$$\begin{aligned} \varepsilon H &= \{\varepsilon, \tau\} &= \tau H \\ \sigma H &= \{\sigma, \tau\sigma^2\} &= \tau\sigma^2 H \\ \sigma^2 H &= \{\sigma^2, \tau\sigma\} &= \tau\sigma H \end{aligned}$$

Notice that  $H\sigma \neq \sigma H$  and  $H\sigma^2 \neq \sigma^2 H$

**Proposition 3.3**

Let  $H$  be a subgroup of a group  $G$  and let  $a, b \in G$ .

1.  $Ha = Hb$  if and only if  $ab^{-1} \in H$ . In particular, we have  $Ha = H$  if and only if  $a \in H$ .
2. If  $a \in Hb$ , then  $Ha = Hb$
3. Either  $Ha = Hb$  or  $Ha \cap Hb = \emptyset$ . Thus, the distinct right cosets of  $H$  forms a partition of  $G$ .

**Proof of 1:**

( $\Rightarrow$ ) If  $Ha = Hb$ , then  $a = 1a \in Ha = Hb$ . Thus  $a = hb$  for some  $h \in H$  and we have  $ab^{-1} = h \in H$ .

( $\Leftarrow$ ) Suppose  $ab^{-1} \in H$  for all  $h \in H$ . Then for all  $h \in H$ ,

$$ha = hab^{-1}b = h(ab^{-1})b \in Hb$$

Thus  $Ha \subseteq Hb$ . Note that if  $ab^{-1} \in H$ , since  $H$  is a subgroup, then

$$(ab^{-1})^{-1} = ba^{-1} \in H$$

Thus for all  $h \in H$ ,

$$hb = h(ba^{-1})a \in Ha$$

Thus  $Hb \subseteq Ha$ . It follows that  $Ha = Hb$ . □

**Proof of 2:** If  $a \in Hb$ , then  $ab^{-1} \in H$ . Thus, by (1), we have  $Ha = Hb$ . □

**Proof of 3:** Two cases:

1. If  $Ha \cap Hb = \emptyset$ , then we are done.
2. If  $Ha \cap Hb \neq \emptyset$ , then there exists  $x \in Ha \cap Hb$ . Since  $x \in Hb$ , by (2), we have  $Hb = Hx$ . Thus

$$Ha = Hx = Hb$$

□

**Remark**

The analogues of the previous proposition also holds for left cosets

1.  $aH = bH$  if and only if  $b^{-1}a \in H$

**Exercise 3.2.1**

Let  $G$  be a group and  $H$  a subset of  $G$ . For  $a, b \in G$ , do we still have  $Ha = Hb$ , or  $Ha \cap Hb = \emptyset$  if  $H$  is not a subgroup of  $G$ .

**Definition 3.2.2**

By the previous proposition, we see that  $G$  can be written as a disjoint union of right cosets of  $H$ . We define the *index*  $[G : H]$  to be the number of disjoint right (or left) cosets of  $H$  in  $G$ . (Note that  $[G : H]$  could be infinite).

**Theorem 3.4****Lagrange's Theorem**

Let  $H$  be a subgroup of a finite group  $G$ . We have  $|H| \mid |G|$  and

$$[G : H] = \frac{|G|}{|H|}$$

**Proof:** Write  $k = [G : H]$  and let  $Ha_1, \dots, Ha_k$  be the distinct right cosets of  $H$  in  $G$ . By prop

$$G = Ha_1 \sqcup \dots \sqcup Ha_k$$

is a disjoint union. Since  $|Ha_i| = |H|$  for each  $i$ , we have

$$|G| = |Ha_1| + \dots + |Ha_k| = k|H|$$

It follows that  $|H| \mid |G|$  and  $[G : H] = k = \frac{|G|}{|H|}$ . □

**Corollary 3.5**

1. If  $G$  is a finite group and  $g \in G$  then  $o(g) \mid |G|$
2. If  $G$  is a finite group with  $|G| = n$ , then for all  $g \in G$ , we have  $g^n = 1$

**Proof of 1:** Take  $H = \langle g \rangle$  in the theorem. Note that  $|H| = o(g)$  □

**Proof of 2:** Let  $o(g) = m$  then by (1), we have  $m \mid n$ . Thus

$$g^n = (g^m)^{\frac{n}{m}} = 1^{\frac{n}{m}} = 1$$

**Example 3.2.3**

For  $n \in \mathbb{N}$  with  $n \geq 2$ , let  $\mathbb{Z}_n^*$  be the set of (multiplicative) invertible elements in  $\mathbb{Z}_n$ . Let the *Euler's  $\varphi$ -function*  $\varphi(n)$ , denote the order of  $\mathbb{Z}_n^*$ , i.e.

$$\varphi(n) = |\{[k] \in \mathbb{Z}_n \mid k \in \{0, 1, \dots, n-1\} \text{ and } \gcd(k, n) = 1\}|$$

As a direct consequence of the corollary, we see that if  $a \in \mathbb{Z}$  with  $\gcd(a, n) = 1$ , then  $a^{\varphi(n)} \equiv 1 \pmod{n}$ . This is Euler's Theorem. If  $n = p$ , a prime number, then Euler's Theorem implies that  $a^{p-1} \equiv 1 \pmod{p}$ , which is Fermat's little theorem.

**Recall**

If  $|G| = 2$  then  $G \cong C_2$ , and  $|G| = 3$  then  $G \cong C_3$ .

**Corollary 3.6**

If  $G$  is a group with  $|G| = p$  a prime, then  $G \cong C_p$ , the cyclic group of order  $p$ .

**Proof:** Let  $g \in G$  with  $g \neq 1$ . Then by corollary, we have  $o(g) \mid p$ . Since  $g \neq 1$  and  $p$  is a prime, we have  $o(g) = p$ . By proposition, we have

$$|\langle g \rangle| = o(g) = p$$

It follows that  $G \cong \langle g \rangle \cong C_p$  □

**Corollary 3.7**

Let  $H$  and  $K$  be finite subgroups of a group  $G$ . If  $\gcd(|H|, |K|) = 1$ , then  $H \cap K = \{1\}$ .

**Proof:** Note  $H \cap K$  is a subgroup of  $H$  and  $K$ . So by Lagrange's Theorem, we have  $|H \cap K| \mid |H|$  and  $|H \cap K| \mid |K|$ . It follows that  $|H \cap K| \mid \gcd(|H|, |K|)$ , i.e.  $|H \cap K| = 1$ . Thus  $H \cap K = \{1\}$ . □

### 3.3 Normal Subgroups

**Definition 3.3.1**

Let  $H$  be a subgroup of a group  $G$ . If  $gH = Hg$  for all  $g \in G$ , we say  $H$  is *normal*, denoted by  $H \triangleleft G$ .

**Example 3.3.1**

We have  $\{1\} \triangleleft G$  and  $G \triangleleft G$ .

**Example 3.3.2**

The center  $Z(G)$  of  $G$  is an abelian subgroup of  $G$ . By its definition,  $Z(G) \triangleleft G$ . Thus every subgroup of  $Z(G)$  is normal in  $G$ .

**Example 3.3.3**

If  $G$  is an abelian group, then every subgroup of  $G$  is normal in  $G$ . Note the converse is false (see assignment 3)

**Proposition 3.8****Normality Test**

Let  $H$  be a subgroup of a group  $G$ . The following are equivalent:

1.  $H \triangleleft G$
2.  $gHg^{-1} \subseteq H$  for all  $g \in G$ . We call  $gHg^{-1}$  a *conjugate* of  $H$
3.  $gHg^{-1} = H$  for all  $g \in G$ . (Thus  $H \triangleleft G$  if and only if  $H$  is the only conjugate of  $H$ )

**Proof of (1)  $\implies$  (2):** Let  $ghg^{-1} \in gHg^{-1}$  for some  $h \in H$ . Then by (1),  $gh \in gH = Hg$ , say  $gh = h_1g$  for some  $h_1 \in H$ . Then  $ghg^{-1} = h_1gg^{-1} = h_1 \in H$ .  $\square$

**Proof of (2)  $\implies$  (3):** If  $g \in G$ , then by (2),  $gHg^{-1} \subseteq H$ . Taking  $g^{-1}$  in place of  $g$  in (2), we get  $g^{-1}Hg \subseteq H$ . Thus implies that  $H \subseteq gHg^{-1}$ . Thus  $H = gHg^{-1}$ .  $\square$

**Proof of (3)  $\implies$  (1):** If  $gHg^{-1} = H$ , then  $gH = Hg$ .  $\square$

**Example 3.3.4**

Let  $G = \mathrm{GL}_n(\mathbb{R})$  and  $H = \mathrm{SL}_n(\mathbb{R})$ . For  $A \in G$  and  $B \in H$ , we have

$$\det(ABA^{-1}) = \det A \det B \det A^{-1} = \det B = 1$$

Thus  $ABA^{-1} \in H$  and it follows that  $AHA^{-1} \subseteq H$  for all  $A \in G$ , so by the normality test,  $\mathrm{SL}_n(\mathbb{R}) \triangleleft \mathrm{GL}_n(\mathbb{R})$ .

**Proposition 3.9**

If  $H$  is a subgroup of a group  $G$  with  $[G : H] = 2$ , then  $H \triangleleft G$ .

**Proof:** Let  $g \in G$ . If  $g \in H$ , then  $Hg = H = gH$ . If  $g \notin H$ , since  $[G : H] = 2$ , then  $G = H \sqcup Hg$ , a disjoint union. Then  $Hg = G \setminus H$ . Similarly,  $gH = G \setminus H$ . Thus  $gH = Hg$  for all  $g \in G$  i.e.  $H \triangleleft G$ .  $\square$

**Example 3.3.5**

Let  $A_n$  be the alternating group contained in  $S_n$ . Since  $[S_n : A_n] = 2$ . By proposition, we have  $A_n \triangleleft S_n$ .

**Example 3.3.6**

Let  $D_{2n} = \langle a, b \mid a^n = 1 = b^2 \text{ and } aba = b \rangle$  be the dihedral group of order  $2n$ . Since  $[D_{2n} : \langle a \rangle] = 2$ , by proposition,  $\langle a \rangle \triangleleft D_{2n}$

Let  $H$  and  $K$  be subgroups of a group  $G$ . Then the intersection  $H \cap K$  is the largest subgroup of  $G$  that contained in both  $H$  and  $K$ .

Question: What is the smallest subgroup containing  $H$  and  $K$ ? Note that  $H \cup K$  is the smallest subset

containing  $H$  and  $K$ , but  $H \cup K$  is a subgroup if and only if  $H \subseteq K$  or  $H \supseteq K$ . A more useful subset to consider is the *product*  $HK$  of  $H$  and  $K$  defined as follows

### Definition 3.3.2

$$HK = \{hk \mid h \in H, k \in K\}$$

#### Remark

The product of 2 subgroups is not always a subgroup.

### Lemma 3.10

Let  $H$  and  $K$  be subgroups of a group  $G$ , then the following are equivalent:

1.  $HK$  is a subgroup of  $G$
2.  $HK = KH$
3.  $KH$  is a subgroup of  $G$ .

**Proof of (1  $\iff$  2):** Note that (2  $\iff$  3) will follow after exchanging  $H$  and  $K$ . Suppose (2) holds, we have  $1 = 1 \cdot 1 \in HK$ . Also if  $hk \in HK$ , then  $(hk)^{-1} = k^{-1}h^{-1} \in KH = HK$ . Also for  $hk, h_1, k_1 \in HK$ , we have  $kh_1 \in KH = HK$ , say  $kh_1 = h_2k_2$ , it follows that

$$(hk)(h_1k_1) = h(kh_1)k_1 = h(h_2k_2)k_1 = (hh_2)(k_2k_1) \in HK$$

By the subgroup test,  $HK$  is a subgroup of  $G$ . Suppose conversely that (1) holds. Let  $kh \in KH$  with  $k \in K, h \in H$ . Since  $H$  and  $K$  are subgroups of  $G$ , we have  $h^{-1} \in H$ , and  $k^{-1} \in K$ . Since  $HK$  is a subgroup of  $G$ , we have

$$kh = (h^{-1}k^{-1})^{-1} \in HK$$

Thus  $KH \subseteq HK$ , similarly, one can show  $HK \subseteq KH$ . Thus  $HK = KH$ . □

### Proposition 3.11

Let  $H$  and  $K$  be subgroups of a group  $G$ . Then

1. If  $H \triangleleft G$  or  $K \triangleleft G$ , then  $HK = KH$  is a subgroup of  $G$
2. If  $H \triangleleft G$  and  $K \triangleleft G$ , then  $KH \triangleleft G$

**Proof of 1:** Suppose  $H \triangleleft G$  then

$$HK = \bigcup_{k \in K} Hk = \bigcup_{k \in K} kH = KH$$

By lemma,  $HK = KH$  is a subgroup of  $G$ . □

**Proof of 2:** If  $g \in G$  and  $hk \in HK$ , since  $H \triangleleft G$  and  $K \triangleleft G$  we have

$$g^{-1}(hk)g = (g^{-1}hg)(g^{-1}kg) \in HK$$

Thus  $g^{-1}HKg \subseteq HK$  and  $HK \triangleleft G$ . □

**Definition 3.3.3**

Let  $H$  be a subgroup of a group  $G$ . The *normalizer* of  $H$ , denoted by  $N_G(H)$  is defined to be

$$N_G(H) = \{g \in G \mid gH = Hg\}$$

We see that  $H \triangleleft G$  if and only if  $N_G(H) = G$

**Note**

In the proof of the previous proposition, we do not need the full assumption that  $H \triangleleft G$ . We only need  $kH = Hk$  for all  $k \in K$ , i.e.  $k \in N_G(H)$ . Thus

**Corollary 3.12**

Let  $H$  and  $K$  be subgroups of a group  $G$ . If  $K \subseteq N_G(H)$  (or  $H \subseteq N_G(K)$ ) then  $HK = KH$  is a subgroup of  $G$ .

**Theorem 3.13**

If  $H \triangleleft G$  and  $K \triangleleft G$  satisfy  $H \cap K = \{1\}$ , then  $HK \cong H \times K$ .

**Proof:**

Claim: If  $H \triangleleft G$  and  $K \triangleleft G$  satisfy  $H \cap K = \{1\}$  then  $hk = kh$  for all  $h \in H$  and  $k \in K$ .

Consider  $x = hk(kh)^{-1} = hkh^{-1}k^{-1}$ . Note that  $kh^{-1}k^{-1} \in kKh^{-1} = H$  (since  $H \triangleleft G$ ). Thus  $x \in H$ .

Similarly, since  $hkh^{-1} \in hKh^{-1} = K$ , we have  $x \in K$ . Since  $x \in H \cap K = \{1\}$ , we have

$hkh^{-1}k^{-1} = 1$  i.e.  $hk = kh$ .

Since  $H \triangleleft G$ , by proposition,  $HK$  is a subgroup of  $G$ . Define  $\sigma : H \times K \rightarrow HK$  by  $\sigma(h, k) = hk$ .

Claim:  $\sigma$  is an isomorphism.

Let  $(h, k), (h_1, k_1) \in H \times K$ . By claim 1, we have  $h_1k = kh_1$ . Thus

$$\sigma((h, k) \cdot (h_1, k_1)) = \sigma(hh_1, kk_1) = hh_1kk_1 = hkh_1k_1 = \sigma(h, k) \cdot \sigma(h_1, k_1)$$

Thus  $\sigma$  is a homomorphism. Note that by the definition of  $HK$ ,  $\sigma$  is surjective. Also, if

$\sigma(h, k) = \sigma(h_1, k_1)$ , we have  $hk = h_1k_1$ . Thus  $h_1^{-1}h = k_1k^{-1} \in H \cap K = \{1\}$ . Thus

$h_1^{-1}h = 1 = k_1k^{-1}$  i.e.  $h_1 = h$  and  $k_1 = k$ . Thus  $\sigma$  is injective. So  $\sigma$  is an isomorphism and we have

$HK \cong H \times K$ . □

**Corollary 3.14**

Let  $G$  be a finite group, and let  $H$  and  $K$  be normal subgroups such that  $H \cap K = \{1\}$  and  $|H||K| = |G|$ . Then  $G \cong H \times K$ .

**Proof:**

$$|HK| = \frac{|H||K|}{|H \cap K|} = |H||K| = |G|$$

Thus  $HK = G$ , and so a direct application of the theorem gives  $G = HK \cong H \times K$ . □

**Example 3.3.7**

Let  $m, n \in \mathbb{N}$  with  $\gcd(m, n) = 1$ . Let  $G$  be a cyclic group of order  $mn$ . Write  $G = \langle a \rangle$  with  $o(a) = mn$ . Let  $H = \langle a^n \rangle$  and  $K = \langle a^m \rangle$ . Thus  $|H| = o(a^n) = m$  and  $|K| = o(a^m) = n$ . It follows that  $|H||K| = mn = |G|$ . Since  $\gcd(m, n) = 1$ , by corollary, we have  $H \cap K = \{1\}$ . Also, since  $G$  is cyclic and thus abelian, we have  $H \triangleleft G$  and  $K \triangleleft G$ . Then by corollary, we have  $G \cong H \times K$ , i.e.  $C_{mn} \cong C_m \times C_n$ . Hence, to consider finite cyclic groups, it suffices to consider cyclic groups of prime power order.

## 4 Isomorphism Theorems

### 4.1 Quotient Groups

**Remark**

Let  $K$  be a subgroup of  $G$ . Consider the set of right cosets of  $K$ , i.e.  $\{Ka \mid a \in G\}$ . To make it a group, a natural way is to define

$$Ka \cdot Kb = Kab \quad \forall a, b \in G \quad (*)$$

Note that we could have  $Ka = Ka_1$  and  $Kb = Kb_1$  with  $a \neq a_1$  and  $b \neq b_1$ . Thus in order for  $(*)$  to make sense, a necessary condition is

$$Ka = Ka_1 \text{ and } Kb = Kb_1 \implies Kab = Ka_1b_1$$

In this case, we say that the multiplication is *well-defined*.

**Lemma 4.1**

Let  $K$  be a subgroup of a group  $G$ , the following are equivalent:

1.  $K \triangleleft G$
2. For  $a, b \in G$ , the multiplication  $Ka \cdot Kb = Kab$  is well-defined.

**Proof of  $(1 \Rightarrow 2)$ :** Let  $Ka = Ka_1$  and  $Kb = Kb_1$ . Thus  $aa_1^{-1} \in K$  and  $bb_1^{-1} \in K$ . To get  $Kab = Ka_1b_1$ , we need  $ab(a_1b_1)^{-1} \in K$ . Note that since  $K \triangleleft G$ , we have  $aKa^{-1} = K$ . Thus

$$ab(a_1b_1)^{-1} = abb_1^{-1}a_1^{-1} = (abb_1^{-1}a^{-1})(aa_1^{-1}) \in K$$

Thus  $Kab = Ka_1b_1$ . □

**Proof of  $(2 \Rightarrow 1)$ :** If  $a \in G$ , to show  $K \triangleleft G$ , we need  $aka^{-1} \in K$  for all  $k \in K$ . Since  $Ka = Ka$  and  $Kk = K1$ , by (2), we have  $Kak = Ka1$  i.e.  $Kak = Ka$ . It follows that  $aka^{-1} \in K$ . Thus  $K \triangleleft G$ . □

**Proposition 4.2**

Let  $K \triangleleft G$  and write  $G/K = \{Ka \mid a \in G\}$  for the set of all cosets of  $K$ . Then

1.  $G/K$  is a group under the operation  $Ka * Kb = Kab$ .
2. The mapping  $\varphi : G \rightarrow G/K$  given by  $\varphi(a) = Ka$  is a surjective homomorphism.
3. If  $[G : K]$  is finite, then  $|G/K| = [G : K]$ . In particular, if  $|G|$  is finite, then  $|G/K| = \frac{|G|}{|K|}$

**Proof of 1:** By other proposition, the operation is well defined and  $G/K$  is closed under operation. The identity of  $G/K$  is  $K \cdot 1 = K$ . Also, the inverse of  $Ka$  is  $Ka^{-1}$ . Finally, by the associativity of  $G$ , we have

$$Ka(KbKc) = (KaKb)Kc.$$

It follows that  $G/K$  is a group. □

**Proof of 2:**  $\varphi$  is clearly surjective. Also, for  $a, b \in G$ , we have

$$\varphi(a)\varphi(b) = KaKb = Kab = \varphi(ab)$$

so  $\varphi$  is a homomorphism. □

**Proof of 3:** If  $[G : K]$  is finite, by the definition of index,  $|G/K| = [G : K]$ . Also, if  $|G|$  is finite, by Lagrange's Theorem,  $|G/K| = [G : K] = \frac{|G|}{|K|}$  □

### Definition 4.1.1

Let  $K \triangleleft G$ . The group  $G/K$  of all cosets of  $K$  in  $G$  is called the *quotient group of  $G$  by  $K$* . Also, the mapping  $\varphi : G \rightarrow G/K$  given by  $\varphi(a) = Ka$  is called the *coset map*.

### Exercise 4.1.1

List all normal subgroups of  $D_{10}$  and all quotient groups of  $D_{10}/K$ .

## 4.2 Isomorphism Theorems

### Definition 4.2.1

Let  $\alpha : G \rightarrow H$  be a group homomorphism. The *kernel of  $\alpha$*  is defined by

$$\ker \alpha = \{g \in G \mid \alpha(g) = 1_H\} \subseteq G$$

and the *image of  $\alpha$*  is defined by

$$\text{im } \alpha = \alpha(G) = \{\alpha(g) \mid g \in G\} \subseteq H$$

### Proposition 4.3

Let  $\alpha : G \rightarrow H$  be a group homomorphism

1.  $\text{im } \alpha$  is a subgroup of  $H$
2.  $\ker \alpha$  is a normal subgroup of  $G$

**Proof of 1:** Note that  $1_H = \alpha(1_G) \in \text{im } \alpha$ . Also, for  $h_1 = \alpha(g_1), h_2 = \alpha(g_2) \in \text{im } \alpha$ , we have

$$h_1 h_2 = \alpha(g_1) \alpha(g_2) = \alpha(g_1 g_2) \in \text{im } \alpha$$

Also, by proposition,  $\alpha(g)^{-1} = \alpha(g^{-1}) \in \text{im } \alpha$ . By the subgroup test,  $\text{im } \alpha$  is a subgroup of  $H$ . □

**Proof of 2:** For  $\ker \alpha$ , note that  $\alpha(1_G) = 1_H$ . Also, for  $k_1, k_2 \in \ker \alpha$ , then

$$\alpha(k_1 k_2) = \alpha(k_1) \alpha(k_2) = 1 \cdot 1 = 1$$

and

$$\alpha(k_1^{-1}) = \alpha(k_1)^{-1} = 1^{-1} = 1$$

By the subgroup test,  $\ker \alpha$  is a subgroup of  $G$ . Note that if  $g \in H$  and  $k \in \ker \alpha$ , then

$$\alpha(gkg^{-1}) = \alpha(g)\alpha(k)\alpha(g^{-1}) = \alpha(g)1\alpha(g)^{-1} = 1$$

Thus  $g(\ker \alpha)g^{-1} \subseteq \ker \alpha$ . By the normality test,  $\ker \alpha \triangleleft G$ .  $\square$

### Example 4.2.1

Consider the determinant map  $\det : \mathrm{GL}_n(\mathbb{R}) \rightarrow \mathbb{R}^*$  defined by  $A \mapsto \det A$ . Then  $\ker(\det) = \mathrm{SL}_n(\mathbb{R})$ . Thus, we get another proof that  $\mathrm{SL}_n(\mathbb{R}) \triangleleft \mathrm{GL}_n(\mathbb{R})$ .

### Example 4.2.2

Define the *sign* of a permutation  $\sigma \in S_n$  by

$$\mathrm{sgn}(\sigma) = \begin{cases} 1 & \text{if } \sigma \text{ is even} \\ -1 & \text{if } \sigma \text{ is odd} \end{cases}$$

Note that  $\mathrm{sgn} : S_n \rightarrow (\pm 1, \cdot)$  defined by  $\sigma \mapsto \mathrm{sgn}(\sigma)$  is a homomorphism. Also,  $\ker(\mathrm{sgn}) = A_n$ . Thus we have another proof that  $A_n \triangleleft S_n$ .

### Theorem 4.4

### First Isomorphism Theorem

Let  $\alpha : G \rightarrow H$  be a group homomorphism. Then

$$G / \ker \alpha \cong \mathrm{im} \alpha$$

**Proof:** Let  $K = \ker \alpha$ . Since  $K \triangleleft G$ ,  $G/K$  is a group. Define the map

$$\begin{aligned} \bar{\alpha} : G/K &\longrightarrow \mathrm{im} \alpha \\ Kg &\longmapsto \alpha(g) \end{aligned}$$

Note that

$$Kg = Kg_1 \iff gg_1^{-1} \in K \iff \alpha(gg_1^{-1}) = 1 \iff \alpha(g) = \alpha(g_1)$$

Thus,  $\bar{\alpha}$  is well-defined and injective. Also  $\bar{\alpha}$  is clearly surjective. For  $g, h \in G$ , we have

$$\bar{\alpha}(KgKh) = \bar{\alpha}(Kgh) = \alpha(gh) = \alpha(g)\alpha(h) = \bar{\alpha}(Kg)\bar{\alpha}(Kh)$$

Thus  $\bar{\alpha}$  is a group isomorphism and we have  $G / \ker \alpha \cong \mathrm{im} \alpha$ .  $\square$

**Remark**

Let  $\alpha : G \rightarrow H$  be a group homomorphism and  $K = \ker \alpha$ . Let  $\varphi : G \rightarrow G/K$  be the coset map and let  $\bar{\alpha}$  be defined as in the previous proof. We have the following diagram:

$$\begin{array}{ccc} G & \xrightarrow{\alpha} & \text{im } \alpha \\ \varphi \downarrow & \nearrow \bar{\alpha} & \\ G/K & & \end{array}$$

Note that for  $g \in G$ , we have

$$\bar{\alpha}\varphi(g) = \bar{\alpha}(\varphi(g)) = \bar{\alpha}(Kg) = \alpha(g)$$

Thus  $\alpha = \bar{\alpha}\varphi$  on the other hand, if we have  $\alpha = \bar{\alpha}\varphi$ , then the action of  $\bar{\alpha}$  is determined by  $\alpha$  and  $\varphi$  as

$$\bar{\alpha}(Kg) = \bar{\alpha}(\varphi(g)) = \bar{\alpha}\varphi(g) = \alpha(g)$$

Thus  $\bar{\alpha}$  is the only homomorphism  $G/K \rightarrow H$  satisfying  $\bar{\alpha}\varphi = \alpha$ .

**Proposition 4.5**

Let  $\alpha : G \rightarrow H$  be group homomorphism and  $K = \ker \alpha$ . Then  $\alpha$  factors uniquely as  $\alpha = \bar{\alpha}\varphi$  where  $\varphi : g \rightarrow G/K$  is the coset map and  $\bar{\alpha} : G/K \rightarrow H$  is defined by  $\bar{\alpha}(Kg) = \alpha(g)$ . Note that  $\varphi$  is surjective and  $\bar{\alpha}$  is injective.

**Example 4.2.3**

We have seen that  $(\mathbb{Z}, +) = \langle \pm 1 \rangle$  and for  $n \in \mathbb{N}$ ,  $(\mathbb{Z}_n, +) = \langle [1] \rangle$  are cyclic groups. In the following, we will show that these are the only cyclic groups.

Let  $G = \langle g \rangle$  be a cyclic group. Consider  $\alpha : (\mathbb{Z}, +) \rightarrow G$  defined by  $\alpha(k) = g^k$  for all  $k \in \mathbb{Z}$ , which is a group homomorphism. By the definition of  $\langle g \rangle$ ,  $\alpha$  is surjective. Note that  $\ker \alpha = \{k \in \mathbb{Z} \mid g^k = 1\}$ , we have two cases:

1. If  $o(g) = \infty$ , then  $\ker \alpha = \{0\}$ . By the first isomorphism theorem, we have

$$G \cong \mathbb{Z}/\{0\} \cong \mathbb{Z}$$

2. If  $o(g) = n$ , by proposition,  $\ker \alpha = n\mathbb{Z}$ . By the fist isomorphism theorem,

$$G \cong \mathbb{Z}/n\mathbb{Z} \cong \mathbb{Z}_n$$

By (1) and (2), we can conclude that if  $G$  is cyclic, then  $G \cong \mathbb{Z}$  or  $G \cong \mathbb{Z}_n$ .

**Theorem 4.6****Second Isomorphism Theorem**

Let  $H$  and  $K$  be subgroups of a group  $G$  with  $K \triangleleft G$ . Then  $HK$  is a subgroup of  $G$ ,  $K \triangleleft HK$ ,  $H \cap K \triangleleft H$  and  $HK/K \cong H/H \cap K$ .

**Proof:** Since  $K \triangleleft G$ , by proposition,  $HK$  is a subgroup,  $HK = KH$  and  $K \triangleleft HK$ . Consider  $\alpha : H \rightarrow HK/K$  defined by  $\alpha(h) = Kh$ . (note that  $h \in H \subseteq HK$ ). Then  $\alpha$  is a homomorphism (exercise). Also, if  $x \in HK = KH$ , say  $x = kh$ , then

$$Kx = K(kh) = Kh = \alpha(h)$$

Thus  $\alpha$  is surjective. Finally, by proposition,

$$\ker \alpha = \{h \in H \mid Kh = K\} = \{h \in H \mid h \in K\} = H \cap K$$

By the first isomorphism theorem,

$$H/H \cap K \cong HK/K$$

□

**Theorem 4.7****Third Isomorphism Theorem**

Let  $K \subseteq H \subseteq G$  be groups with  $K \triangleleft G$  and  $H \triangleleft G$ . Then  $H/K \triangleleft G/K$  and

$$(G/K)/(H/K) \cong G/H$$

**Proof:** Define  $\alpha : G/K \rightarrow G/H$  by  $\alpha(Kg) = Hg$  for all  $g \in G$ . Note that if  $Kg = Kg_1$ , then  $gg_1^{-1} \in K \subseteq H$ . Thus  $Hg = Hg_1$  and  $\alpha$  is well defined. Clearly,  $\alpha$  is surjective. Note that

$$\ker \alpha = \{Kg \mid Hg = H\} = \{Kg \mid g \in H\} = H/K$$

By the first isomorphism theorem,

$$(G/K)/(H/K) \cong G/H$$

□

## 5 Group Actions

### 5.1 Cayley's Theorem

**Theorem 5.1****Cayley's Theorem**

If  $G$  is a finite group of order  $n$ , then  $G$  is isomorphic to a subgroup of  $S_n$ .

**Proof:** Let  $G = \langle g_1, \dots, g_n \rangle$  and let  $S_G$  be the permutation group of  $G$ . By identifying  $g_i$  with  $i$ , we see that  $S_G \cong S_n$ . Thus it suffices to find a injective homomorphism  $\sigma : G \rightarrow S_G$ . For  $a \in G$ , define  $\mu_a : G \rightarrow G$  by  $\mu_a(g) = ag$  for all  $g \in G$ . Note that  $ag = ag_1$  implies  $g = g_1$  and  $a(a^{-1}g) = g$ . Hence  $\mu_a$  is a bijection and  $\mu_a \in S_G$ . Define  $\sigma : G \rightarrow S_G$  by  $\sigma(a) = \mu_a$ . For  $a, b \in G$ , we have  $\mu_a \mu_b = \mu_{ab}$  and  $\sigma$  is a homomorphism. Also, if  $\mu_a = \mu_b$ , then  $a = \mu_a(1) = \mu_b(1) = b$ . Thus, by the first isomorphism theorem, we have  $G \cong \text{im } \sigma$ , a subgroup of  $S_G \cong S_n$ . □

**Example 5.1.1**

Let  $H$  be a subgroup of a group  $G$  with  $[G : H] = m < \infty$ . Let  $X = \{g_1H, g_2H, \dots, g_mH\}$  be the set of all distinct left cosets of  $H$  in  $G$ . For  $a \in G$ , define  $\lambda_a : X \rightarrow X$  by  $\lambda_a(gH) = agH$  for all  $gH \in X$ . Note that  $agH = ag_1H$  implies that  $gH = g_1H$  and  $a(a^{-1}gH) = gH$ . Hence  $\lambda_a$  is a bijection and thus  $\lambda_a \in S_X$ . Consider  $\tau : G \rightarrow S_X$  defined by  $\tau(a) = \lambda_a$ . For  $a, b \in G$ , we have  $\lambda_{ab} = \lambda_a \lambda_b$  and thus  $\tau$  is a homomorphism. Note that if  $a \in \ker \tau$ , then  $\lambda_a$  is the identity permutation. In particular,  $aH = \lambda_a(H) = H$ . In particular,  $a \in H$ . Thus  $\ker \tau \subseteq H$ .

**Theorem 5.2****Extended Cayley's Theorem**

Let  $H$  be a subgroup of a group  $G$  with  $[G : H] = m < \infty$ . If  $G$  has no normal subgroup contained in  $H$  except for  $\{1\}$ , then  $G$  is isomorphic to a subgroup of  $S_m$ .

**Proof:** Let  $X$  be the set of all distinct left cosets of  $H$  in  $G$ . We have  $|X| = m$  and  $S_X \cong S_m$ . We have seen from the above example that there exist a group homomorphism  $\tau : G \rightarrow S_X$  with  $K = \ker \tau \subseteq H$ . By the first isomorphism theorem, we have  $G/K \cong \text{im } \tau$ . Since  $K \subseteq H$  and  $K \triangleleft G$ , by the assumption, we have  $K = \{1\}$ . It follows that  $G \cong \text{im } \tau$ , a subgroup of  $S_X \cong S_m$ .  $\square$

**Corollary 5.3**

Let  $G$  be a finite group and  $p$  the smallest prime dividing  $|G|$ . If  $H$  is a subgroup of  $G$  with  $[G : H] = p$  then  $H \triangleleft G$ .

**Proof:** Let  $X$  be the set of all distinct left cosets of  $H$  in  $G$ . We have  $|X| = p$  and  $S_X \cong S_p$ . Let  $\tau : G \rightarrow S_X \cong S_p$  be the group homomorphism defined in the above example with  $K := \ker \tau \subseteq H$ . By the first isomorphism theorem, we have  $G/K \cong \text{im } \tau \subseteq S_p$ . Thus  $G/K$  is isomorphic to a subgroup of  $S_p$ . By Lagrange's Theorem, we have  $|G/K| \mid p!$ . Also, since  $K \subseteq H$ , if  $[H : K] = k$ , then

$$|G/K| = \frac{|G|}{|K|} = \frac{|G|}{|H|} \frac{|H|}{|K|} = pk.$$

Thus  $pk \mid p!$  and hence  $k \mid (p-1)!$ . Since  $k \mid |H|$ , which divides  $|G|$  and  $p$  is the smallest prime dividing  $|G|$ , we see every prime divisor of  $k$  must be  $\geq p$  unless  $k = 1$ . Combining this with  $k \mid (p-1)!$ , this forces  $k = 1$ , which implies  $K = H$ , thus  $H \triangleleft G$ .  $\square$

## 5.2 Group Actions

**Definition 5.2.1**

Let  $G$  be a group and  $X$  a non-empty set. A (left) *group action of  $G$  on  $X$*  is a mapping  $G \times X \rightarrow X$  denoted  $(a, x) \mapsto a \cdot x$  such that

1.  $1 \cdot x = x$  for all  $x \in X$
2.  $a \cdot (b \cdot x) = (ab) \cdot x$  for all  $a, b \in G$  and  $x \in X$

In this case, we say  $G$  *acts on  $X$* .

**Remark**

Let  $G$  be a group acting on a set  $X \neq \emptyset$ . For  $a, b \in G$  and  $x, y \in X$ , by (1) and (2), we have

$$a \cdot x = b \cdot y \iff (b^{-1}a) \cdot x = y$$

In particular, we have  $a \cdot x = a \cdot y$  if and only if  $x = y$ .

**Example 5.2.1**

If  $G$  is group, let  $G$  act on itself by conjugation. i.e.  $X = G$ , by  $a \cdot x = axa^{-1}$  for all  $a, x \in G$ . Note that

$$1 \cdot x = 1x1^{-1} = x$$

and

$$a \cdot (b \cdot x) = a(bxb^{-1})a^{-1} = (ab)x(ab)^{-1} = (ab) \cdot x$$

So it is indeed a group action.

**Remark**

For  $a \in G$ , define  $\sigma_a : X \rightarrow X$  by  $\sigma_a(x) = a \cdot x$  for all  $x \in X$ . Then one can show

1.  $\sigma_a \in S_X$ , the permutation group of  $X$
2. The function  $\theta : G \rightarrow S_X$  give  $\theta(a) = \sigma_a$  is a group homomorphism with  
 $\ker \theta = \{a \in G \mid ax = x \ \forall x \in X\}$

Note that the group homomorphism  $\theta : G \rightarrow S_X$  gives an equivalent definition of group action of  $G$  on  $X$ . If  $X = G$  with  $|G| = n$  and  $\ker \theta = \{1\}$ , the map  $\theta : G \rightarrow S_n$  shows that  $G$  is isomorphic to a subgroup of  $S_n$ , which is Cayley's Theorem. Thus, the notion of group action can be viewed as a generalization of the proof of Cayley's Theorem.

**Definition 5.2.2**

Let  $G$  be a group acting on  $X \neq \emptyset$ . Let  $x \in X$ . We call

1.  $G \cdot x = \{g \cdot x \mid g \in G\} \subseteq X$  *The orbit of x*
2.  $S(x) = \{g \in G \mid g \cdot x = x\} \subseteq G$  *The stabilizer of x*

**Proposition 5.4**

Let  $G$  be a group acting on a set  $X \neq \emptyset$  and let  $x \in X$ . Then

1.  $S(x)$  is a subgroup of  $G$ .
2. There exists a bijection from  $G \cdot x$  to  $\{gS(x) \mid g \in G\}$  and thus  $|G \cdot x| = [G : S(x)]$

**Proof of 1:** Since  $1 \cdot x = x$ , we have  $1 \in S(x)$ . Also, if  $g, h \in S(x)$ , then

$$gh \cdot (x) = g \cdot (h \cdot x) = g \cdot x = x$$

and

$$g^{-1} \cdot x = g^{-1} \cdot (g \cdot x) = (g^{-1}g) \cdot x = 1 \cdot x = x$$

Thus  $gh, g^{-1} \in S(x)$ . By the subgroup test,  $S(x)$  is a subgroup of  $G$ . □

**Proof of 2:** Consider the map  $\varphi : G \rightarrow \{gS(x) \mid g \in G\}$  defined by  $\varphi(g \cdot x) = gS(x)$ . Note that

$$g \cdot x = h \cdot x \iff (h^{-1}g) \cdot x = x \iff h^{-1}g \in S(x) \iff hS(x) = gS(x)$$

Thus  $\varphi$  is well-defined and injective. Since  $\varphi$  is clearly surjective,  $\varphi$  is a bijection. It follows that

$$|G \cdot x| = |\{gS(x) \mid g \in G\}| = [G : S(x)]$$

□

### Theorem 5.5

### Orbit Decomposition Theorem

Let  $G$  be a group acting on a finite set  $X \neq \emptyset$ . Let

$$X_f = \{x \in X \mid a \cdot x = x \ \forall a \in G\}$$

(Note that  $x \in X_f$  iff  $|G \cdot x| = 1$ ) Let  $G \cdot x_1, G \cdot x_2, \dots, G \cdot x_n$  denote the distinct non-singleton orbits (i.e.  $|G \cdot x_i| > 1$ ) Then

$$|X| = |X_f| + \sum_{i=1}^n [G : S(x_i)]$$

**Proof:** Note that for  $a, b \in G$  and  $x, y \in X$ ,

$$a \cdot x = b \cdot y \iff (b^{-1}a) \cdot x = y \iff y \in G \cdot x \iff G \cdot y = G \cdot x$$

Thus two orbits are either disjoint, or the same. It follows that the orbits form a disjoint union of  $X$ . Since  $x \in X_f$  iff  $|G \cdot x| = 1$ , the set  $X \setminus X_f$  contains all non-singleton orbits, which are disjoint. Thus by proposition 5.4, we have

$$\begin{aligned} |X| &= |X_f| + \sum_{i=1}^n |G \cdot x_i| \\ &= |X_f| + \sum_{i=1}^n [G : S(x_i)] \end{aligned}$$

□

**Example 5.2.2**

Let  $G$  be a group acting on itself by conjugation i.e.  $g \cdot x = gxg^{-1}$ . Then

$$\begin{aligned} G_f &= \{x \in G \mid gxg^{-1} = x \ \forall g \in G\} \\ &= \{x \in G \mid gx = xg \ \forall g \in G\} \\ &= Z(G) \end{aligned}$$

Also, for  $x \in G$ ,

$$S(x) = \{g \in G \mid gxg^{-1} = x\} = \{g \in G \mid gx = xg\}$$

This set is called the *centralizer* of  $x$  and is denoted by  $S(x) = C_G(x)$ . Finally in this case, the orbit

$$G \cdot x = \{gxg^{-1} \mid g \in G\}$$

is called the *conjugacy class of  $x$* .

By Theorem 5.5,

**Corollary 5.6****Class Equation**

Let  $G$  be a finite group and let  $\{gx_1g^{-1} \mid g \in G\}, \dots, \{gx_ng^{-1} \mid g \in G\}$  denote the distinct non-singleton conjugacy classes, then

$$|G| = |Z(G)| + \sum_{i=1}^n [G : C_G(x_i)]$$

**Lemma 5.7**

Let  $p$  be a prime and  $m \in \mathbb{N}$ . Let  $G$  be a group of order  $p^m$  acting on a finite set  $X \neq \emptyset$ . Let  $X_f$  be defined as in Theorem 5.5. Then we have

$$|X| \equiv |X_f| \pmod{p}$$

**Proof:** By Theorem 5.5, we have

$$|X| = |X_f| + \sum_{i=1}^n [G : S(x_i)] \text{ with } [g : S(x_i)] > 1$$

Since  $[G : S(x_i)]$  divides  $|G| = p^m$  and  $[G : S(x_i)] > 1$ . We have  $p \mid [G : S(x_i)]$  for all  $i$ . It follows that

$$|X| \equiv |X_f| \pmod{p}$$

□

**Theorem 5.8****Cauchy's Theorem**

Let  $p$  be a prime and  $G$  a finite group. If  $p \mid |G|$ , then  $G$  contains an element of order  $p$ .

**Proof:** Define  $X = \{(a_1, \dots, a_p) \mid a_i \in G \text{ and } a_1 \cdots a_p = 1\}$ . Since  $a_p$  is uniquely determined by  $a_1, \dots, a_{p-1}$ , if  $|G| = n$ , we have  $|X| = n^{p-1}$ . Since  $p \mid n$ , we have  $|X| \equiv 0 \pmod{p}$ . Let the group  $\mathbb{Z}_p = (\mathbb{Z}_p, +)$  acts on  $X$  by “cycling”, i.e. for  $k \in \mathbb{Z}_p$ ,

$$k \cdot (a_1, \dots, a_p) = (a_{k+1}, \dots, a_p, a_1, \dots, a_k)$$

One can verify that this action is well defined. Let  $X_f$  be defined as in theorem 5.5. Then  $(a_1, \dots, a_p) \in X_f$  iff  $a_1 = a_2 = \dots = a_p$ . Clearly  $(1, 1, \dots, 1) \in X_f$  and hence  $|X_f| \geq 1$ . Since  $|\mathbb{Z}_p| = p$ , by lemma 5.7, we have

$$|X_f| \equiv |X| \equiv 0 \pmod{p}$$

Since  $|X_f| \equiv 0 \pmod{p}$  and  $|X_f| \geq 1$ . It follows that  $|X_f| \geq p$ . Therefore, there exists  $a \neq 1$  st  $(a, \dots, a) \in X_f$  which implies that  $a^p = 1$ . Since  $p$  is prime and  $a \neq 1$ , the order of  $a$  is  $p$ .  $\square$

## 6 Sylow Theorems

### 6.1 $p$ -groups

#### Definition 6.1.1

Let  $p$  be a prime. A group in which every element has order of a non-negative power of  $p$  is called a  $p$ -group

#### Remark

As a direct consequence of Cauchy's Theorem we have

#### Corollary 6.1

A finite group  $G$  is a  $p$ -group if and only if  $|G|$  is a power of  $p$

#### Lemma 6.2

The center  $Z(G)$  of a non-trivial finite  $p$ -group  $G$  contains more than one element.

**Proof:** The class equation of  $G$  (Cor 5.6) states that

$$|G| = |Z(G)| + \sum_{i=1}^m [G : C_G(x_i)]$$

where  $[G : C_G(x_i)] > 1$ . Since  $G$  is a  $p$ -group, by Cor 6.1,  $p \mid |G|$ . By lemma 5.7,  $|Z(G)| \equiv |G| \equiv 0 \pmod{p}$ . It follows that  $p \mid |Z(G)|$ . Since  $1 \in Z(G)$  and  $|Z(G)| \geq 1$ ,  $Z(G)$  has at least  $p$  elements.  $\square$

**Recall**

If  $H$  is a subgroup of a group  $G$ , then  $N_G(H) = \{g \in G \mid gHg^{-1} = H\}$  is the *normalizer* of  $H$  in  $G$ . In particular,  $H \triangleleft N_G(H)$ .

**Lemma 6.3**

If  $H$  is a  $p$ -subgroup of a finite group  $G$ , then

$$[N_G(H) : H] \equiv [G : H] \pmod{p}$$

**Proof:** Let  $X$  be the set of all left cosets of  $H$  in  $G$ . Hence  $|X| = [G : H]$ . Let  $H$  act on  $X$  by left multiplication. Then for  $x \in G$ , we have

$$\begin{aligned} xH \in X_f &\iff hxH = xH \quad \forall h \in H \\ &\iff x^{-1}hxH = H \quad \forall h \in H \\ &\iff x^{-1}Hx = H \\ &\iff x \in N_G(H) \end{aligned}$$

Thus  $|X_f|$  is the number of cosets  $xH$  with  $x \in N_G(H)$  and hence  $|X_f| = [N_G(H) : H]$ . By lemma 5.7,

$$[N_G(H) : H] = |X_f| \equiv |X| = [G : H] \pmod{p}$$

□

**Corollary 6.4**

Let  $H$  be a  $p$ -subgroup of a finite group  $G$ . If  $p \mid [G : H]$  then  $p \mid [N_G(H) : H]$  and  $N_G(H) \neq H$ .

**Proof:** Since  $p \mid [G : H]$ , by lemma 6.3, we have

$$[N_G(H) : H] \equiv [G : H] \equiv 0 \pmod{p}$$

Since  $p \mid [N_G(H) : H]$  and  $[N_G(H) : H] \geq 1$ , we have  $[N_G(H) : H] \geq p$ . Thus  $N_G(H) \neq H$ .

□

**6.2 Three Sylow Theorems****Recall**

Cauchy's theorem states that if  $p \mid |G|$ , then  $G$  contains an element of order  $p$ . Thus  $|\langle a \rangle| = p$ . The following first Sylow Theorem can be viewed as a generalization of Cauchy's Theorem.

**Theorem 6.5****First Sylow Theorem**

Let  $G$  be a group of order  $p^n m$  where  $p$  is a prime,  $n \geq 1$  and  $\gcd(p, m) = 1$ . Then  $G$  contains a subgroup of order  $p^i$  for all  $1 \leq i \leq n$ . Moreover, every subgroup of  $G$  of order  $p^i$  ( $i < n$ ) is normal in some subgroup of order  $p^{i+1}$ .

**Proof:** We prove this theorem by induction on  $i$ . For  $i = 1$ , since  $p \mid |G|$ , by Cauchy's theorem,  $G$  contains an element  $a$  of order  $p$ , i.e.  $|\langle a \rangle| = p$ . Suppose that the statement holds for some  $1 \leq i < n$ .

Say  $H$  is a subgroup of  $G$  of order  $p^i$ . Then  $p \mid [G : H]$ , by Cor 6.4,  $p \mid [N_G(H) : H]$  and  $[N_G(H) : H] \geq p$ ,  $p \mid [G : H]$ . Then by Cauchy's theorem,  $N_G(H)/H$  contains a subgroup of order  $p$ . Such a group is of the form  $H_1/H$ , where  $H_1$  is a subgroup of  $N_G(H)$  containing  $H$ . Since  $H \triangleleft N_G(H)$ , we have  $H \triangleleft H_1$ . Finally,  $|H_1| = |H||H_1/H| = p^i \cdot p = p^{i+1}$ .  $\square$

### Definition 6.2.1

A subgroup  $P$  of a group  $G$  is said to be a *Sylow p-subgroup* of  $G$  if  $P$  is a maximal  $p$ -group of  $G$  i.e. if  $P \subseteq H \subseteq G$  with  $H$  a  $p$ -group, then  $P = H$ .

As a direct consequence of theorem 6.5,

### Corollary 6.6

Let  $G$  be a group of order  $p^n m$  where  $p$  is a prime,  $n \geq 1$  and  $\gcd(p, m) = 1$ . Let  $H$  be a  $p$ -subgroup of  $G$ .

1.  $H$  is a Sylow  $p$ -subgroup iff  $|H| = p^n$
2. Every conjugate of a Sylow  $p$ -subgroup is a Sylow  $p$ -subgroup.
3. If there is only one Sylow  $p$ -subgroup  $P$ , then  $P \triangleleft G$ .

### Theorem 6.7

### Second Sylow Theorem

If  $H$  is a  $p$ -subgroup of a finite group  $G$ , and  $P$  is any Sylow  $p$ -subgroup of  $G$ , then there exists  $g \in G$  such that  $H \subseteq gPg^{-1}$ . In particular, any two Sylow  $p$ -subgroups are conjugate.

**Proof:** Let  $X$  be the set of all left cosets of  $P$  in  $G$ , and let  $H$  act on  $X$  by left multiplication. By lemma 5.7, we have  $|X_f| \equiv |X| = [G : P] \pmod{p}$ . Since  $p \nmid [G : P]$ , we have  $|X_f| \neq 0$ . Thus there exists  $gP \in X_f$  for some  $g \in G$ . Note that

$$\begin{aligned} gP \in X_f &\iff hgP = gP \quad \forall h \in H \\ &\iff g^{-1}hgP = P \quad \forall h \in H \\ &\iff g^{-1}Hg \subseteq P \\ &\iff H \subseteq gPg^{-1} \end{aligned}$$

If  $H$  is Sylow  $p$ -subgroup, then  $|H| = |P| = |gHg^{-1}|$ , thus  $H = gPg^{-1}$ .  $\square$

### Theorem 6.8

### Third Sylow Theorem

If  $G$  is a finite group and  $p$  a prime with  $p \mid |G|$ , then the number of Sylow  $p$ -subgroups of  $G$  divides  $|G|$  and is of the form  $kp + 1$  for some  $k \in \mathbb{N} \cup \{0\}$ .

**Proof:** By theorem 6.7, the number of Sylow  $p$ -subgroups of  $G$  is the number of conjugates of any of them, say  $P$ . This number is  $[G : N_G(P)]$ . Which is a divisor of  $|G|$ . Let  $X$  be the set of all Sylow  $p$ -subgroups of  $G$  and let  $P$  act on  $X$  by conjugation. Then  $Q \in X_f$  iff  $gQg^{-1} = Q$  for all  $g \in P$ . The latter condition holds iff  $P \subseteq N_G(Q)$ . Both  $P$  and  $Q$  are Sylow  $p$ -subgroups of  $G$  and hence  $N_G(Q)$ . Thus by Cor 6.6, they are conjugate in  $N_G(Q)$ . Since  $Q \triangleleft N_G(Q)$ , this can only occur if  $Q = P$  and  $X_f = \{P\}$ . By lemma 5.7,  $|X| \equiv |X_f| \equiv 1 \pmod{p}$ . Thus  $|X| = kp + 1$  for some  $k \in \mathbb{N} \cup \{0\}$ .  $\square$

**Remark**

Suppose that  $G$  is a group with  $|G| = p^n m$  and  $\gcd(p, m) = 1$ . Let  $n_p$  be the number of  $p$ -subgroups of  $G$ . By the third Sylow theorem, we have  $n_p \mid p^n m$  and  $n_p \equiv 1 \pmod{p}$ . Since  $p \nmid n_p$ , we have  $n_p \mid m$ .

**Example 6.2.1**

Claim: every group of order 15 is cyclic.

Let  $n_p$  be the number of Sylow  $p$ -subgroups of  $G$ . By the third Sylow theorem, we have  $n_3 \mid 5$  and  $n_3 \equiv 1 \pmod{3}$ . Thus  $n_3 = 1$ . Similarly, we have  $n_5 \mid 3$  and  $n_5 \equiv 1 \pmod{5}$ , Thus  $n_5 = 1$ . It follows that there is only one Sylow 3-subgroup and Sylow 5-subgroup, say  $P_3$  and  $P_5$  respectively. Thus  $P_3, P_5 \triangleleft G$ . Consider  $|P_3 \cap P_5|$ , which divides 3 and 5. Thus  $|P_3 \cap P_5| = 1$  and  $P_3 \cap P_5 = \{1\}$ . Also  $|P_3 P_5| = 15 = |G|$ . Thus

$$G \cong P_3 \times P_5 \cong \mathbb{Z}_3 \times \mathbb{Z}_5 \cong \mathbb{Z}_{15}$$

**Example 6.2.2**

Claim: there are two isomorphism classes of groups of order 21.

Let  $G$  be a group of order  $21 = 3 \cdot 7$ . Let  $n_p$  be the number of Sylow  $p$ -subgroups of  $G$ . By the third Sylow theorem, we have  $n_3 \mid 7$  and  $n_3 \equiv 1 \pmod{3}$ . Thus  $n_3 = 1$  or 7. Also we have  $n_7 \mid 3$  and  $n_7 \equiv 1 \pmod{7}$ . Thus  $n_7 = 1$ . It follows that  $G$  has a unique Sylow 7-subgroup, say  $P_7$ . Note that  $P_7 \triangleleft G$  and  $P_7$  is cyclic, say  $P_7 = \langle x : x^7 = 1 \rangle$ . Let  $H$  be a Sylow 3-subgroup. Since  $|H| = 3$ ,  $H$  is cyclic and  $H = \langle y : y^3 = 1 \rangle$ . Since  $P_7 \triangleleft G$ , we have  $yxy^{-1} = x^i$  for some  $0 \leq i \leq 6$ . It follows that

$$x = y^3 xy^{-3} = y^2(yxy^{-1})y^{-2} = y^2 x^i y^{-2} = y(yx^i y^{-1})y^{-1} = yx^{i^2} y^{-1} = x^{i^3}$$

Since  $x^{i^3} = x$  and  $x^7 = 1$ , we have  $i^3 - 1 \equiv 0 \pmod{7}$ . Since  $0 \leq i \leq 6$ , we have  $i = 1, 2, 4$ .

1. If  $i = 1$ , then  $yxy^{-1} = x$ , i.e.  $yx = xy$ . Thus  $G$  is an abelian group. Since  $P_3 \triangleleft G$ ,  $P_7 \triangleleft G$ ,  $P_3 \cap P_7 = \{1\}$  and  $|G| = |P_3 P_7|$ , we have

$$G \cong P_3 \times P_7 \cong \mathbb{Z}_3 \times \mathbb{Z}_7 \cong \mathbb{Z}_{21}$$

2. If  $i = 2$ , then  $yxy^{-1} = x^2$ . Thus

$$G = \{x^i y^j : 0 \leq i \leq 6, 0 \leq j \leq 2, yxy^{-1} = x^2\}$$

3. If  $i = 4$ , then  $yxy^{-1} = x^4$ . Note that

$$\begin{aligned} y^2 xy^{-2} &= y(yxy^{-1})y^{-1} \\ &= yx^4 y^{-1} \\ &= x^{16} = x^2 \end{aligned}$$

Note that  $y^2$  is also a generator of  $H$ . Thus by replacing  $y$  by  $y^2$ , we get back to case 2. It follows that there are two isomorphism classes of groups of order 21.

## 7 Finite Abelian Groups

### 7.1 Primary Decomposition

#### Notation

Let  $G$  be a group and  $m \in \mathbb{Z}$  we define

$$G^{(m)} = \{g \in G \mid g^m = 1\}$$

#### Proposition 7.1

Let  $G$  be an abelian group. Then  $G^{(m)}$  is a subgroup of  $G$ .

**Proof:** We have  $1 = 1^m \in G^{(m)}$ . Also if  $g, h \in G^{(m)}$ , since  $G$  is abelian, we have  $(gh)^m = g^m h^m = 1$  and thus  $gh \in G^{(m)}$ . Finally, if  $g \in G^{(m)}$ , we have

$$(g^{-1})^m = g^{-m} = (g^m)^{-1} = 1$$

and thus  $g^{-1} \in G^{(m)}$ . By the subgroup test,  $G^{(m)}$  is a subgroup of  $G$ .  $\square$

#### Proposition 7.2

Let  $G$  be a finite abelian group with  $|G| = mk$  with  $\gcd(m, k) = 1$ . Then

1.  $G \cong G^{(m)} \times G^{(k)}$
2.  $|G^{(m)}| = m$  and  $|G^{(k)}| = k$

**Proof of 1:** Since  $G$  is abelian, we have  $G^{(m)} \triangleleft (G)$  and  $G^{(k)} \triangleleft G$ . Also, since  $\gcd(m, k) = 1$ , there exist  $x, y \in \mathbb{Z}$  such that  $1 = mx + ky$

Claim:  $G^{(m)} \cap G^{(k)} = \{1\}$

If  $g \in G^{(m)} \cap G^{(k)}$ , then  $g^m = 1 = g^k$ . We have

$$g = g^{mx+ky} = (g^m)^x (g^k)^y = 1$$

Claim:  $G = G^{(m)}G^{(k)}$

If  $g \in G$ , then

$$1 = g^{mk} = (g^m)^k = (g^k)^m$$

It follows that  $g^k \in G^{(m)}$  and  $g^m \in G^{(k)}$ . Thus

$$g = g^{mx+ky} = (g^k)^y (g^m)^x \in G^{(m)}G^{(k)}$$

Combining both claims, by Theorem 3.13, we have

$$G \cong G^{(m)}G^{(k)}$$

$\square$

**Proof of 2:** Write  $|G^{(m)}| = m'$  and  $|G^{(k)}| = k'$ . By (1), we have  $mk = |G| = m'k'$

Claim:  $\gcd(m, k') = 1$

Suppose that  $\gcd(m, k') \neq 1$ . Then there exists a prime  $p$  such that  $p \mid m$  and  $p \mid k'$ . By Cauchy's

theorem, there exists  $g \in G^{(k)}$  with  $o(g) = p$ . Since  $p \mid m$ , we have  $g^m = (g^p)^{\frac{m}{p}} = 1$ , i.e.  $g \in G^{(m)}$ . By (1), we have  $g \in G^{(m)} \cap G^{(k)} = \{1\}$ , which gives a contradiction since  $o(g) = p$ . Thus we have  $\gcd(m, k') = 1$ . Note that since  $m \mid m'k'$  and  $\gcd(m, k') = 1$ , we have  $m \mid m'$ . Similarly, we have  $k \mid k'$ . Since  $mk = m'k'$ , it follows that  $m = m'$  and  $k = k'$ .  $\square$

As a direct consequence of proposition 7.2, we have

### Theorem 7.3

### Primary Decomposition Theorem

Let  $G$  be a finite abelian group with  $|G| = p_1^{n_1} \cdots p_k^{n_k}$  where  $p_1, \dots, p_k$  are distinct primes and  $n_1, \dots, n_k \in \mathbb{N}$ . Then we have

1.  $G \cong G^{(p_1^{n_1})} \times \cdots \times G^{(p_k^{n_k})}$
2.  $|G^{(p_i^{n_i})}| = p_i^{n_i} \quad (1 \leq i \leq k)$ .

### Example 7.1.1

Let  $G = \mathbb{Z}_{13}^*$ . Then  $|G| = 12 = 2^2 3$ . Note that

$$\begin{aligned} G^{(3)} &= \{a \in \mathbb{Z}_{13}^* \mid a^3 = 1\} = \{1, 3, 9\} \\ G^{(4)} &= \{a \in \mathbb{Z}_{13}^* \mid a^4 = 1\} = \{1, 5, 8, 12\} \end{aligned}$$

By theorem 7.3, we have

$$\mathbb{Z}_{13}^* \cong \{1, 5, 8, 12\} \times \{1, 3, 9\}$$

## 7.2 Structure Theorem of Finite Abelian Groups

We have seen that if  $|G| = p$  (a prime), then  $G \cong C_p$ . Also, if  $|G| = p^2$ , then  $G \cong C_{p^2}$  or  $G \cong C_p \times C_p$ . Question How about abelian groups of order  $p^3, p^4$  and  $p^n$  for general  $n \in \mathbb{N}$ .

### Proposition 7.4

Let  $G$  be a finite abelian  $p$ -group that contains only one subgroup of order  $p$ , then  $G$  is cyclic. In other words, if a finite abelian  $p$ -group  $G$  is not cyclic, then  $G$  has at least two subgroups of order  $p$ .

**Proof:** Let  $y \in G$  be of maximum order, i.e.  $o(y) \geq o(x) \forall x \in G$ .

Claim:  $G = \langle y \rangle$ .

Suppose that  $G \neq \langle y \rangle$ . Then the quotient group  $G/\langle y \rangle$  is a nontrivial  $p$ -group, which contains an element  $z$  of order  $p$  by Cauchy's theorem. In particular  $z \neq 1$ . Consider the coset map  $\pi : G \rightarrow G/\langle y \rangle$ . Let  $x \in G$  such that  $\pi(x) = z$ . Since  $\pi(x^p) = \pi(x)^p = z^p = 1$ , we see that  $x^p \in \langle y \rangle$ . Thus  $x^p = y^m$  for some  $m \in \mathbb{Z}$ . Two cases:

1. If  $p \nmid m$  since  $o(y) = p^r$  for some  $r \in \mathbb{N}$ , by prop 2.11,  $o(y^m) = o(y)$ . Since  $y$  is of maximum order, we have  $o(x^p) < o(x) \leq o(y) = o(y^m) = o(x^p)$  which is a contradiction.
2. If  $p \mid m$ , then  $m = pk$  for some  $k \in \mathbb{Z}$ . Thus we have  $x^p = y^m = y^{pk}$ . Since  $G$  is abelian, we have  $(xy^{-k})^p = 1$ . Thus  $xy^{-k}$  belongs to the one and only subgroup of order  $p$ , say  $H$ . On the other hand, the cyclic group  $\langle y \rangle$  contains a subgroup of order  $p$ , which must be the one and only  $H$ . Thus  $xy^{-k} \in \langle y \rangle$ , which implies that  $x \in \langle y \rangle$ . It follows that  $z = \pi(x) = 1$ , a contradiction.

By combining the above two cases, we see that  $G = \langle y \rangle$ . □

### Proposition 7.5

Let  $G \neq \{1\}$  be a finite abelian  $p$ -group. Let  $C$  be a cyclic subgroup of maximum order. Then  $G$  contains a subgroup  $B$  such that

$$G = CB \text{ and } C \cap B = \{1\}$$

### Theorem 7.6

Let  $G \neq 1$  be a finite abelian  $p$ -group. Then  $G$  is isomorphic to a direct product of cyclic groups.

**Proof:** By prop 7.5, there exists a cyclic group  $C_1$  and a subgroup  $B_1$  of  $G$  such that  $G \cong C_1 \times B_1$ . Since  $|B_1| \mid |G|$  by Lagrange's theorem, the group  $B_1$  is also a  $p$ -group. Thus if  $B_1 \neq \{1\}$ , by prop 7.5, there exists a cyclic group  $C_2$  and a subgroup  $B_2$  such that  $B_1 \cong C_2 \times B_2$ . Continue in this way to get cyclic groups  $C_1, \dots, C_k$  until we get  $B_k = \{1\}$  for some  $k \in \mathbb{N}$ . Then  $G \cong C_1 \times \dots \times C_k$ . □

### Remark

One can show that the decomposition of a finite abelian  $p$ -group into a direct product of cyclic groups is unique up to its order.

Combining the remark, theorem 7.6 and theorem 7.3, we have

### Theorem 7.7

### Structure Theorem of Finite Abelian Groups

If  $G$  is a finite abelian group, then

$$G \cong \mathbb{Z}_{p_1^{n_1}} \times \dots \times \mathbb{Z}_{p_k^{n_k}}$$

Where  $\mathbb{Z}_{p_i^{n_i}} = (\mathbb{Z}_{p_i^{n_i}}, +) \cong C_{p_i^{n_i}}$  are cyclic groups of order  $p_i^{n_i}$  ( $1 \leq i \leq k$ ). Note that  $p_i$  are not necessarily distinct. The numbers  $p_i^{n_i}$  are uniquely determined up to their order.

Note that if  $p_1$  and  $p_2$  are distinct primes, then  $C_{p_1^{n_1}} \times C_{p_2^{n_2}} \cong C_{p_1^{n_1} p_2^{n_2}}$ . Thus by combining suitable coprime factors together,

### Theorem 7.8

### Invariant Factor Decomposition of Finite Abelian Groups

Let  $G$  be a finite abelian group. Then

$$G \cong \mathbb{Z}_{n_1} \times \dots \times \mathbb{Z}_{n_r}$$

where  $n_i \in \mathbb{N}, n_1 > 1$  and  $n_1 \mid n_2 \mid \dots \mid n_r$ .

**Example 7.2.1**

Let  $G$  be an abelian group of order 48. Since  $48 = 2^4 \cdot 3$ , by theorem 7.3,  $G \cong H \times \mathbb{Z}_3$ , where  $H$  is an abelian group of order  $2^4$ . The options for  $H$  are  $\mathbb{Z}_{2^4}, \mathbb{Z}_{2^3} \times \mathbb{Z}_2, \mathbb{Z}_{2^2} \times \mathbb{Z}_{2^2}, \mathbb{Z}_{2^2} \times \mathbb{Z}_2 \times \mathbb{Z}_2$  and  $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ . Thus we have

$$\begin{aligned} G &\cong \mathbb{Z}_{2^4} \times \mathbb{Z}_3 \cong \mathbb{Z}_{48} \\ G &\cong \mathbb{Z}_{2^3} \times \mathbb{Z}_2 \times \mathbb{Z}_3 \cong \mathbb{Z}_2 \times \mathbb{Z}_{24} \\ G &\cong \mathbb{Z}_{2^2} \times \mathbb{Z}_{2^2} \times \mathbb{Z}_3 \cong \mathbb{Z}_{2^2} \times \mathbb{Z}_{12} \\ G &\cong \mathbb{Z}_{2^2} \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_3 \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_{12} \\ G &\cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_3 \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_6 \end{aligned}$$

There are 5 non-isomorphic groups in total.

## 8 Rings

### 8.1 Rings

**Definition 8.1.1**

A set  $R$  is a (unitary) *ring* if it has two operations, addition  $+$  and multiplication  $\cdot$  such that  $(R, +)$  is an abelian group and  $(R, \cdot)$  satisfies the closure, associativity and identity properties of a group, in addition to a distributive law. More precisely, if  $R$  is a ring, then for all  $a, b, c \in R$

1.  $a + b \in R$
2.  $a + (b + c) = (a + b) + c$
3. There exists  $0 \in R$  such that  $a + 0 = a = 0 + a$  ( $0$  is called the *zero* of  $R$ )
4. There exists  $-a \in R$  such that  $a + (-a) = 0 = (-a) + a$  ( $-a$  is called the *negative* of  $a$ )
5.  $a + b = b + a$
6.  $ab = a \cdot b \in R$
7.  $a(bc) = (ab)c$
8. There exists  $1 \in R$  such that  $a \cdot 1 = a = 1 \cdot a$  ( $1$  is called the *unity* of  $R$ )
9.  $a(b + c) = ab + ac$  and  $(b + c)a = ba + ca$  (distributive law)

The ring  $R$  is called a *commutative ring* if it also satisfies  $ab = ba$ .

**Example 8.1.1**

$\mathbb{Z}, \mathbb{Q}, \mathbb{R}, \mathbb{C}$  are commutative rings.

**Example 8.1.2**

For  $n \in \mathbb{N}, n \geq 2, \mathbb{Z}_n$  is a commutative ring.

**Example 8.1.3**

For  $n \in \mathbb{N}, n \geq 2, M_n(\mathbb{R})$  is a (non commutative) ring

**Warning**

Note that since  $(R, \cdot)$  is not a group, there is no left or right cancellation. For example, in  $\mathbb{Z}$ ,  $0 \cdot x = 0 \cdot y$  does not imply  $x = y$ .

**Notation**

Given a ring  $R$ , to distinguish the difference between multiples in addition and in multiplication, for  $n \in \mathbb{N}$  and  $a \in R$ , we write

$$\begin{aligned} na &:= \underbrace{a + a + \cdots + a}_{n \text{ times}} \\ a^n &:= \underbrace{a \cdot a \cdot \cdots \cdot a}_{n \text{ times}} \end{aligned}$$

**Recall**

For a group  $G$  and  $g \in G$ , we have  $g^0 = 1$ ,  $g^1 = g$  and  $(g^{-1})^{-1} = g$ . Thus for addition, we have, for a ring  $R$  and  $a \in R$

1.  $\underbrace{0}_{\text{integer}} \cdot a = \underbrace{0}_{\text{zero of } R}$
2.  $\underbrace{1}_{\text{integer}} a = a$
3.  $-(-a) = a$

**Notation**

For  $n \in \mathbb{N}$ , we define

$$(-n)a := \underbrace{(-a) + \cdots + (-a)}_{n \text{ times}}$$

Also, we define  $a^0 = 1$ . If the multiplicative inverse of  $a$  exists,

$$a^{-n} = (a^{-1})^n$$

**Remark**

By Prop 1.2 for  $n, m \in \mathbb{Z}$ , we have

1.  $(na) + (ma) = (n + m)a$
2.  $n(ma) = (nm)a$
3.  $n(a + b) = na + nb$

**Proposition 8.1**

Let  $R$  be a ring and  $r, s \in R$ .

1. If  $0$  is the zero of  $R$ , then

$$0r = 0 = r0$$

2.  $(-r)s = r(-s) = -(rs)$
3.  $(-r)(-s) = rs$
4. For any  $m, n \in \mathbb{Z}$ ,

$$(mr)(ns) = (mn)(rs)$$

**Definition 8.1.2**

A *trivial ring* is a ring of only one element. In this case, we have  $1 = 0$ .

**Remark**

If  $R$  is a ring with  $R \neq \{0\}$ , since  $r = r1$  for all  $r \in R$ , we have  $1 \neq 0$ .

**Example 8.1.4**

Let  $R_1, \dots, R_n$  be rings. We define component-wise operations on the product  $R_1 \times \dots \times R_n$  as follows:

$$\begin{aligned} (r_1, \dots, r_n) + (s_1, \dots, s_n) &= (r_1 + s_1, \dots, r_n + s_n) \\ (r_1, \dots, r_n) \cdot (s_1, \dots, s_n) &= (r_1 s_1, \dots, r_n s_n) \end{aligned}$$

One can check that  $R_1 \times \dots \times R_n$  is a ring. This set is called the *direct product* of  $R_1, \dots, R_n$ .

**Definition 8.1.3**

If  $R$  is a ring, we define the *characteristic* of  $R$  denoted by  $\text{ch}(R)$ , in terms of the order of  $1_R$  in the additive group  $(R, +)$ :

$$\text{ch}(R) = \begin{cases} n & \text{if } o(1_R) = n \in \mathbb{N} \text{ in } (R, +) \\ 0 & \text{if } o(1_R) = \infty \text{ in } (R, +) \end{cases}$$

**Remark**

For  $k \in \mathbb{Z}$ , we write  $kR = 0$  to mean that  $kr = 0$  for all  $r \in R$ .

By Prop 8.1, we have

$$kr = k(1_R r) = (k1_R)r$$

Thus  $kR = 0$  if and only if  $k1_R = 0$ . By Prop 2.6 and 2.7,

**Proposition 8.2**

Let  $R$  be a ring and  $k \in \mathbb{Z}$ .

1. If  $\text{ch}(R) = n \in \mathbb{N}$ , then  $kR = 0$  iff  $n \mid k$
2. If  $\text{ch}(R) = 0$ , then  $kR = 0$  iff  $k = 0$

**Example 8.1.5**

Each of  $\mathbb{Z}, \mathbb{Q}, \mathbb{R}, \mathbb{C}$  has characteristic 0. For  $n \in \mathbb{N}$  with  $n \geq 2$ , the ring  $\mathbb{Z}_n$  has characteristic  $n$ .

**8.2 Subrings****Definition 8.2.1**

A subset  $S$  of a ring  $R$  is a *subring* if  $S$  is a ring itself with  $1_S = 1_R$  (with the same addition and multiplication). Note that properties (2),(3),(7), and (9) of a ring are automatically satisfied. Thus to show that  $S$  is a subring, it suffices to show

Subring Test:

$S \subseteq R$  is a subring if

1.  $1_R \in S$
2. If  $s, t \in S$ , then  $s - t, st \in S$ .

Note that if (2) holds, then  $0 = s - s \in S$  and  $-t = 0 - t \in S$

**Example 8.2.1**

We have a chain of commutative rings

$$\mathbb{Z} \subseteq \mathbb{Q} \subseteq \mathbb{R} \subseteq \mathbb{C}$$

**Example 8.2.2**

If  $R$  is a ring, the *center*  $Z(R)$  of  $R$  is defined to be

$$Z(R) = \{z \in R \mid zr = rz \ \forall r \in R\}$$

Note that  $1_R \in Z(R)$ . Also, if  $s, t \in Z(R)$ , then for  $r \in R$ ,

$$\begin{aligned} (s - t)r &= sr - tr = rs - rt = r(s - t) \\ (st)r &= s(tr) = s(rt) = (sr)t = (rs)t = r(st) \end{aligned}$$

By the subring test,  $Z(R)$  is a subring of  $R$ .

**Example 8.2.3**

Let

$$\mathbb{Z}[i] = \{a + bi \mid a, b \in \mathbb{Z} \text{ and } i^2 = -1\} \subseteq \mathbb{C}.$$

Then one can show that  $\mathbb{Z}[i]$  is a subring of  $\mathbb{C}$ , called the *ring of Gaussian integers*.

## 8.3 Ideals

**Note**

Let  $R$  be a ring and  $A$  an additive subgroup of  $R$ . Since  $(R, +)$  is abelian, we have  $A \triangleleft R$ . Thus we have the additive quotient group

$$R/A = \{r + A \mid r \in R\} \text{ with } r + A = \{r + a \mid a \in A\}$$

Using the known properties about cosets and quotient groups, we have

**Proposition 8.3**

Let  $R$  be a ring and  $A$  an additive subgroup of  $R$ . For  $r, s \in R$ , we have

1.  $r + A = s + A$  iff  $(r - s) \in A$
2.  $(r + A) + (s + A) = (r + s) + A$
3.  $0 + A = A$  is the (additive) identity of  $R/A$
4.  $-(r + A) = (-r) + A$  is the (additive) inverse of  $r + A$
5.  $k(r + A) = kr + A$  for all  $k \in \mathbb{Z}$

**Remark**

Since  $R$  is a ring, it is natural to ask if we could make  $R/A$  a ring. A natural way to define multiplication in  $R/A$  is that

$$(r + A)(s + A) = (rs + A) \quad \forall r, s \in R \quad (*)$$

Note that we could have  $r + A = r_1 + A$  and  $s + A = s_1 + A$  with  $r \neq r_1$  and  $s \neq s_1$ . Thus in order for  $(*)$  to make sense, a necessary condition is

$$r + A = r_1 + A \text{ and } s + A = s_1 + A \implies rs + A = r_1s_1 + A$$

In this case, we say that multiplication  $(r + A)(s + A)$  is *well-defined*.

### Proposition 8.4

Let  $A$  be an additive subgroup of a ring  $R$ . For  $a \in A$  define

$$Ra = \{ra \mid r \in R\} \text{ and } aR = \{ar \mid r \in R\}$$

Then the following are equivalent:

1.  $Ra \subseteq A$  and  $aR \subseteq A \quad \forall a \in A$
2. For  $r, s \in R$ , the multiplication  $(r + A)(s + A)$  is well-defined in  $R/A$ .

**Proof of (1)  $\Rightarrow$  (2):** If  $r + A = r_1 + A$  and  $s + A = s_1 + A$ , we need to show that  $rs + A = r_1s_1 + A$ . Since  $(r - r_1) \in A$  and  $(s - s_1) \in A$ , by (1), we have

$$rs - r_1s_1 = rs - r_1s + r_1s - r_1s_1 = (r - r_1)s + r_1(s - s_1) \in A$$

By proposition 8.3,  $rs + A = r_1s_1 + A$ . □

**Proof of (2)  $\Rightarrow$  (1):** Let  $r \in R$  and  $a \in A$ . By prop 8.1, we have

$$ra + A = (r + A)(a + A) = (r + A)(0 + A) = r0 + A = 0 + A = A$$

Thus  $ra \in A$  and we have  $Ra \subseteq A$ . Similarly, we can show  $aR \subseteq A$ . □

### Definition 8.3.1

An additive subgroup  $A$  of a ring  $R$  is an *ideal* of  $R$  if  $Ra \subseteq R$  and  $aR \subseteq R$ .

#### Ideal Test:

1.  $0 \in A$
2. For  $a, b \in A$  and  $r \in R$ , we have  $a - b \in A$  and  $ra, ar \in A$

### Example 8.3.1

If  $R$  is a ring, then  $\{0\}$  and  $R$  are ideals of  $R$ .

### Example 8.3.2

Let  $R$  be a commutative ring and  $a_1, \dots, a_n \in R$ . Consider the set  $I$  generated by  $a_1, \dots, a_n$  i.e.

$$I = \langle a_1, \dots, a_n \rangle = \{r_1a_1 + \dots + r_na_n \mid r_i \in R\}$$

Then one can show that  $I$  is an ideal.

### Proposition 8.5

Let  $A$  be an ideal of a ring  $R$ . If  $1_R \in A$ , then  $A = R$ .

**Proof:** For every  $r \in R$ , since  $A$  is an ideal and  $1_R \in A$ , we have  $r = r1_R \in A$ . It follows that  $R \subseteq A \subseteq R$  and hence  $R = A$ . □

From the above discussion, we have

**Proposition 8.6**

Let  $A$  be an ideal of a ring  $R$ . Then the additive quotient group  $R/A$  is a ring with multiplication  $(r+A)(s+A) = rs+A$ . The unity of  $R/A$  is  $1+A$ .

**Definition 8.3.2**

Let  $A$  be an ideal of a ring  $R$ . The ring  $R/A$  is called a *quotient ring of  $R$  by  $A$* .

**Definition 8.3.3**

Let  $R$  be a commutative ring and  $A$  an ideal of  $R$ . If  $A = aR = Ra$  for some  $a \in R$ , we say  $A$  is a *principal ideal generated by  $a$*  and is denoted by  $A = \langle a \rangle$ .

**Example 8.3.3**

If  $n \in \mathbb{Z}$ , then  $\langle n \rangle = n\mathbb{Z}$  is an ideal of  $\mathbb{Z}$ .

**Proposition 8.7**

All ideals of  $\mathbb{Z}$  are of the form  $\langle z \rangle$  for some  $z \in \mathbb{Z}$ . If  $\langle n \rangle \neq \{0\}$  and  $n \in \mathbb{N}$ , then the generator is uniquely determined.

**Proof:** Let  $A$  be an ideal of  $\mathbb{Z}$ . If  $A = \{0\}$ , then  $A = \langle 0 \rangle$ . Otherwise, choose  $a \in A$  with  $a \neq 0$  and  $|a|$  minimum. Clearly,  $\langle a \rangle \subseteq A$ . To prove the other inclusion, let  $b \in A$ . By the division algorithm, we have  $b = qa + r$  with  $q, r \in \mathbb{Z}$  and  $0 \leq r < |a|$ . If  $r \neq 0$ , since  $A$  is an ideal, and  $a, b \in A$ , we have  $r = b - qa \in A$  with  $|r| < |a|$ , a contradiction. Thus  $r = 0$  and  $b = qa$ , i.e.  $b \in \langle a \rangle$ . It follows that  $A = \langle a \rangle$ .  $\square$

## 8.4 Isomorphism Theorems

**Definition 8.4.1**

Let  $R, S$  be rings. A mapping  $\theta : R \rightarrow S$  is a *ring homomorphism* if for all  $a, b \in R$

1.  $\theta(a+b) = \theta(a) + \theta(b)$
2.  $\theta(ab) = \theta(a)\theta(b)$
3.  $\theta(1_R) = 1_S$

**Example 8.4.1**

The mapping  $k \mapsto [k]$  from  $\mathbb{Z}$  to  $\mathbb{Z}_n$  is a surjective ring homomorphism.

**Example 8.4.2**

If  $R_1, R_2$  are rings, the projection  $\pi_1 : R_1 \times R_2 \rightarrow R_1$  defined by  $\pi_1(r_1, r_2) = r_1$  is a surjective ring homomorphism. Similarly for  $\pi_2$ .

### Proposition 8.8

Let  $\theta : R \rightarrow S$  be a ring homomorphism.

1.  $\theta(0_R) = 0_S$
2.  $\theta(-r) = -\theta(r)$
3.  $\theta(kr) = k\theta(r)$  for all  $k \in \mathbb{Z}$
4.  $\theta(r^n) = \theta(r)^n$  for all  $n \in \mathbb{N} \cup \{0\}$
5. If  $a \in R^*$  (the set of elements in  $R$  which have multiplicative inverses, such  $a$  is called a *unit* of  $R$ ) then  $\theta(a^k) = \theta(a)^k$  for all  $k \in \mathbb{Z}$ .

### Definition 8.4.2

A *ring isomorphism* is a bijective homomorphism. If there exists an isomorphism between rings  $R$  and  $S$ , we say  $R$  and  $S$  are isomorphic, denoted  $R \cong S$ .

### Exercise 8.4.1

Let  $\theta : R \rightarrow S$  be a bijection of rings with  $\theta(rr') = \theta(r)\theta(r')$  for all  $r, r' \in R$ . Write  $\theta(1_R) = e$ . Prove that  $se = es = s$  for all  $s \in S$  (hence condition 3 for a ring homomorphism can be omitted in this case).

### Definition 8.4.3

Let  $\theta : R \rightarrow S$  be a ring homomorphism. The *kernel* of  $\theta$  is defined by

$$\ker \theta = \{r \in R \mid \theta(r) = 0\} \subseteq R$$

and the *image* of  $\theta$  is defined by

$$\text{im } \theta = \theta(R) = \{\theta(r) \mid r \in R\} \subseteq S$$

We have seen earlier that  $\ker \theta$  and  $\text{im } \theta$  are additive subgroups of  $R$  and  $S$  respectively.

### Proposition 8.9

Let  $\theta : R \rightarrow S$  be a ring homomorphism. Then

1.  $\text{im } \theta$  is a subring of  $S$
2.  $\ker \theta$  is an ideal of  $R$

**Proof of 1:** Since  $\text{im } \theta$  is an additive subgroup of  $S$ , it suffices to show that  $\theta(R)$  is closed under multiplication, and  $1_S \in \theta(R)$ . Note that  $1_S = \theta(1_R) \in \theta(R)$ . Also if  $s_1 = \theta(r_1)$  and  $s_2 = \theta(r_2)$ , then

$$s_1 s_2 = \theta(r_1)\theta(r_2) = \theta(r_1 r_2) \in \theta(R)$$

By the subring test,  $\text{im } \theta$  is a subring of  $S$ . □

**Proof of 2:** Since  $\ker \theta$  is an additive subgroup of  $R$ , it suffices to show that  $ra, ar \in \ker \theta$  for all  $r \in R$ ,  $a \in \ker \theta$ . If  $r \in R$  and  $a \in \ker \theta$ , then

$$\theta(ra) = \theta(r)\theta(a) = \theta(r) \cdot 0 = 0$$

Thus  $ra \in \ker \theta$ . Similarly, one can show  $ar \in \ker \theta$ . Thus  $\ker \theta$  is an ideal of  $R$ .  $\square$

### Theorem 8.10

### First Isomorphism Theorem

Let  $\theta : R \rightarrow S$  be a ring homomorphism. We have  $R/\ker \theta \cong \text{im } \theta$ .

**Proof:** Let  $A = \ker \theta$ . Since  $A$  is an ideal of  $R$ ,  $R/A$  is a ring. Define the map

$$\begin{aligned}\bar{\theta} : R/A &\longrightarrow \text{im } \theta \\ r + A &\longmapsto \theta(r)\end{aligned}$$

Note that  $r + A = s + A \iff r - s \in A \iff \theta(r - s) = 0 \iff \theta(r) = \theta(s)$ . Thus  $\bar{\theta}$  is well defined and injective. Also,  $\bar{\theta}$  is clearly surjective. One can show that  $\bar{\theta}$  is a homomorphism. It follows that  $\bar{\theta}$  is a ring isomorphism and  $\text{im } \theta \cong R/\ker \theta$   $\square$

### Remark

Let  $A, B$  be subsets of a ring  $R$ . If  $A$  and  $B$  are both subrings, then  $A \cap B$  is the largest subring of  $R$  contained in both  $A$  and  $B$ .

### Notation

To consider the smallest subring of  $R$  containing both  $A$  and  $B$  ( $A, B$  not necessarily subrings), we define the *sum*  $A + B$  to be

$$A + B = \{a + b \mid a \in A \text{ and } b \in B\}$$

One can show

### Proposition 8.11

If  $R$  is a ring, then we have

1. If  $A, B$  are subrings of  $R$  (with  $1_A = 1_B = 1_R$ ) then  $A \cap B$  is a subring of  $R$ .
2. If  $A$  is a subring and  $B$  is an ideal of  $R$ , then  $A + B$  is a subring of  $R$
3. If  $A$  and  $B$  are ideals of  $R$ , then  $A + B$  is an ideal of  $R$ .

Using the first isomorphism theorem, one can show (see A8)

### Theorem 8.12

### Second Isomorphism Theorem

Let  $A$  be a subring and  $B$  an ideal of a ring  $R$ . Then  $A + B$  is a subring of  $R$ ,  $B$  is an ideal of  $A + B$ ,  $A \cap B$  is an ideal of  $A$  and

$$(A + B)/B \cong A/(A \cap B)$$

**Theorem 8.13****Third Isomorphism Theorem**

Let  $A$  and  $B$  be ideals of a ring  $R$  with  $A \subseteq B$ . Then  $B/A$  is an ideal in  $R/A$  and

$$(R/A)/(B/A) \cong R/B$$

**Corollary 8.14****Correspondence Theorem / Fourth Isomorphism Theorem**

Let  $R$  be a ring and  $A$  an ideal. There exists a bijection between the set of ideals  $B$  of  $R$  that contains  $A$  and the set of ideals of  $R/A$ .

**Example 8.4.3**

Combining the third isomorphism theorem and the fact that all ideals of  $\mathbb{Z}$  are principal, all ideals of  $\mathbb{Z}_n$  are principal.

**Theorem 8.15****Chinese Remainder Theorem**

Let  $A, B$  be ideals of  $R$

1. If  $A + B = R$  then  $R/(A \cap B) \cong R/A \times R/B$
2. If  $A + B = R$  and  $A \cap B = \{0\}$ , then  $R \cong R/A \times R/B$

**Proof:** (2) obviously follows from (1), so we prove (1). Define  $\theta : R \rightarrow R/A \times R/B$  by  $\theta(r) = (r + A, r + B)$ . Then  $\theta$  is a ring homomorphism with  $\ker \theta = A \cap B$ . To show  $\theta$  is surjective, let  $(s + A, t + B) \in R/A \times R/B$  with  $s, t \in R$ . Since  $A + B = R$ , there exists  $a \in A$  and  $b \in B$  such that  $1 = a + b$ . Let  $r = sb + ta$ . Then

$$s - r = s - sb - ta = s(1 - b) - ta = sa - ta = (s - t)a \in A$$

Thus  $s + A = r + A$ . Similarly, we have  $t + B = r + B$ . Thus  $\theta(r) = (r + A, r + B) = (s + A, t + B)$ . Thus  $\text{im } \theta = R/A \times R/B$ . By the first isomorphism theorem, we have

$$R/(A \cap B) \cong R/A \times R/B$$

□

**Example 8.4.4**

Let  $m, n \in \mathbb{N}$  with  $\gcd(m, n) = 1$ . By Bézout's Lemma, we have  $1 = mr + ns$  for some  $r, s \in \mathbb{Z}$ . Thus  $1 \in m\mathbb{Z} + n\mathbb{Z}$  and hence  $m\mathbb{Z} + n\mathbb{Z} = \mathbb{Z}$ . Also, since  $\gcd(m, n) = 1$ , we have  $m\mathbb{Z} \cap n\mathbb{Z} = mn\mathbb{Z}$ . By CRT,

**Corollary 8.16**

1. If  $m, n \in \mathbb{N}$  with  $\gcd(m, n) = 1$ , then

$$\mathbb{Z}_{mn} \cong \mathbb{Z}_m \times \mathbb{Z}_n$$

2. If  $m, n \in \mathbb{N}$  with  $m, n \geq 2$  and  $\gcd(m, n) \neq 1$ ,  $\varphi(mn) = \varphi(m)\varphi(n)$ , where  $\varphi(m) = |\mathbb{Z}_m^*|$  is the Euler  $\varphi$ -function

**Proof of 2:** From (1), we have

$$(\mathbb{Z}_{mn})^* \cong (\mathbb{Z}_m \times \mathbb{Z}_n)^* \cong \mathbb{Z}_m^* \times \mathbb{Z}_n^*$$

Since  $|\mathbb{Z}_m^*| = \varphi(m)$ , we have  $\varphi(mn) = \varphi(m)\varphi(n)$  □

**Remark**

Let  $m, n \in \mathbb{Z}$  with  $\gcd(m, n) = 1$ . For  $a, b \in \mathbb{Z}$ , by Cor 8.16 and the proof of Thm 8.15, for  $[a] \in \mathbb{Z}_m$  and  $[b] \in \mathbb{Z}_n$ , there exists a unique  $[c] \in \mathbb{Z}_{mn}$  such that  $[c] = [a]$  in  $\mathbb{Z}_m$  and  $[c] = [b]$  in  $\mathbb{Z}_n$ . In other words, the simultaneous congruences  $x \equiv a \pmod{m}$  and  $x \equiv b \pmod{n}$  has a unique solution  $x \equiv c \pmod{mn}$ , which is CRT in Math 135.

**Proposition 8.17**

If  $R$  is a ring with  $|R| = p$  a prime, then  $R \cong \mathbb{Z}_p$ .

**Proof:** Define  $\theta : \mathbb{Z}_p \rightarrow R$  by  $\theta[k] = k1_R$ . Note that since  $R$  is an additive group and  $|R| = p$ , by Lagrange,  $o(1_R) = 1$  or  $p$ . Since  $1_R \neq 0$ , we have  $o(1_R) = p$ . Thus

$$[k] = [m] \iff p \mid (k - m) \iff (k - m)1_R = 0 \iff k1_R = m1_R \text{ in } R$$

Thus  $\theta$  is well-defined and injective. Since  $|\mathbb{Z}_p| = p = |R|$  and  $\theta$  is injective,  $\theta$  is also surjective. Finally, one can prove that  $\theta$  is a ring homomorphism. It follows that  $\theta$  is a ring isomorphism and  $R \cong \mathbb{Z}_p$ . □

**Exercise 8.4.2**

What are the possible rings  $R$  with  $|R| = p^2$  where  $p$  is a prime.

## 9 Commutative Rings

### 9.1 Integral Domains and Fields

**Definition 9.1.1**

Let  $R$  be a ring. We say  $u \in R$  is a *unit* if  $u$  has a multiplicative inverse in  $R$ . Denoted by  $u^{-1}$ . We have  $uu^{-1} = 1 = u^{-1}u$ . Note that if  $u$  is a unit in  $R$ , and  $r, s \in R$  we have

$$ur = us \implies s = s \quad \text{and} \quad ru = su \implies r = s$$

Let  $R^*$  denote the set of all units in  $R$ . One can show that  $(R, \cdot)$  is group called the *group of units* of  $R$ .

**Example 9.1.1**

Note that 2 is a unit in  $\mathbb{Q}$ , but not a unit in  $\mathbb{Z}$ . We have  $\mathbb{Q}^* = \mathbb{Q} \setminus \{0\}$  and  $\mathbb{Z}^* = \{\pm 1\}$ .

**Exercise 9.1.1**

Consider the ring of Gaussian integers  $\mathbb{Z}[i]$ . One can show  $\mathbb{Z}[i]^* = \{\pm 1, \pm i\}$

Hint: Prove that  $|xy| = |x||y|$  and  $|x| = 1 \iff x$  is a unit.

**Definition 9.1.2**

A ring  $R \neq \{0\}$  is a *division ring* if  $R^* = R \setminus \{0\}$  i.e. every nonzero element of  $R$  is a unit of  $R$ . A commutative division ring is called a *field*.

**Example 9.1.2**

$\mathbb{Q}, \mathbb{R}, \mathbb{C}$  are fields, but  $\mathbb{Z}$  is not a field.

**Example 9.1.3**

We recall that the equation  $[a][x] = [1]$  in  $\mathbb{Z}_n$  has a solution iff  $\gcd(a, n) = 1$  for all  $a \in \{1, 2, \dots, p-1\}$ . Thus  $\mathbb{Z}_p^* = \mathbb{Z}_p \setminus \{0\}$  and  $\mathbb{Z}_p$  is a field. However, if  $n$  is not prime, say  $n = ab$  with  $1 < a, b < n$ . Then the nonzero congruence classes  $[a], [b]$  are not units in  $\mathbb{Z}_n$  as there is no solution for  $[a][x] = [1]$  and hence  $\mathbb{Z}_n^* \neq \mathbb{Z}_n \setminus \{0\}$ . Thus  $\mathbb{Z}_n$  is a field iff  $n$  is a prime.

**Remark**

If  $R$  is a division ring or a field, then its only ideals are  $\{0\}$  or  $R$  since if  $A \neq \{0\}$  is an ideal of  $R$ , then  $0 \neq a \in A$  implies that  $1 = aa^{-1} \in A$ . By prop 8.5,  $A = R$ . As a consequence, if we have a ring homomorphism a field  $F$  to a ring  $S$ , since  $\ker \theta$  is an ideal,  $\ker \theta = \{0\}$  or  $F$ . Hence  $\theta$  is either injective or the zero map.

**Exercise 9.1.2**

(This is quite hard) Prove that every finite division ring is a field.

**Note**

For  $r, s \in \mathbb{R}$ , we have  $rs = 0$  implies that  $r = 0$  or  $s = 0$ . This property is useful in solving equations, say if  $x^2 - x - 6 = 0$  i.e.  $(x-3)(x-2) = 0$ , then  $x = 3$  or  $x = 2$ . However, such property is not always true. For example,  $[2][3] = [6] = [0]$  in  $\mathbb{Z}_6$ , but  $[2] \neq [0]$  and  $[3] \neq [0]$ .

**Exercise 9.1.3**

Solve  $[(x-2)(x-3)] = [0]$  in  $\mathbb{Z}_6$ .

**Definition 9.1.3**

Let  $R \neq \{0\}$  be a ring. For  $0 \neq a \in R$ , we say  $a$  is a *zero divisor* if there exists  $0 \neq b \in R$  such that  $ab = 0$ .

**Example 9.1.4**

In  $\mathbb{Z}_6$ ,  $[2], [3], [4]$  are zero divisors since  $[2][3] = [0] = [4][3]$ .

**Example 9.1.5**

Note that in  $M_2(\mathbb{R})$ , we have

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Thus  $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$  is a zero divisor.

**Proposition 9.1**

Given a ring  $R$ , for all  $a, b, c \in R$ , the following are equivalent:

1. If  $ab = 0$ , then  $a = 0$  or  $b = 0$
2. If  $ab = ac$  and  $a \neq 0$ , then  $b = c$
3. If  $ba = ca$  and  $a \neq 0$ , then  $b = c$

**Proof:** We prove  $(1) \iff (2)$  and the proof of  $(1) \iff (3)$  is similar.

$(1) \implies (2)$ : Let  $ab = ac$  with  $a \neq 0$ . Then  $a(b - c) = 0$ . By (1), since  $a \neq 0$ , we have  $b - c = 0$  i.e.  $b = c$ .

$(2) \implies (1)$ : Let  $ab = 0$  in  $R$ . Two cases:

1. If  $a = 0$ , then we are done
2. If  $a \neq 0$ , then  $ab = 0 = a0$ . By (2), since  $a \neq 0$ , we have  $b = 0$ .

□

**Definition 9.1.4**

A commutative ring  $R \neq \{0\}$  is an *integral domain* if it has no zero divisors i.e. if  $ab = 0$  in  $R$ , then  $a = 0$  or  $b = 0$ .

**Example 9.1.6**

$\mathbb{Z}$  is an integral domain since for  $a, b \in \mathbb{Z}$ ,  $ab = 0$  implies  $a = 0$  or  $b = 0$ .

**Example 9.1.7**

Note that if  $p$  is a prime, if  $p \mid ab$  then  $p \mid a$  or  $p \mid b$  i.e.  $[a][b] = [0]$  in  $\mathbb{Z}_p$  implies that  $[a] = [0]$  or  $[b] = [0]$ . Thus  $\mathbb{Z}_p$  is an integral domain. However, if  $n = ab$  with  $1 < a, b < n$ , then  $[a][b] = [0]$  with  $[a] \neq [0]$  and  $[b] \neq [0]$ . Thus  $\mathbb{Z}_n$  is an integral domain iff  $n$  is a prime.

**Proposition 9.2**

Every field is an integral domain.

**Proof:** Let  $ab = 0$  in a field  $R$ . We need to show that  $a = 0$  or  $b = 0$ . Two cases:

1. If  $a = 0$ , then we are done
2. If  $a \neq 0$ , since  $R$  is a field,  $a \in R^*$  and  $a^{-1} \in R$  exists. Then

$$b = 1 \cdot b = (a^{-1}ab) = a^{-1}(ab) = a^{-1}0 = 0$$

Thus  $R$  is an integral domain. □

**Remark**

Using the above proof, one can show that every subring of a field is an integral domain.

**Remark**

The converse of Prop 9.2 is not true, for example,  $\mathbb{Z}$  is an integral domain but not a field.

**Example 9.1.8**

The Gaussian integers  $\mathbb{Z}[i]$  is an integral domain, but not a field.

**Proposition 9.3**

Every finite integral domain is a field.

**Proof:** Let  $R$  be a finite integral domain and  $0 \neq a \in R$ . Consider the map

$$\begin{aligned}\theta : R &\longrightarrow R \\ r &\longmapsto ar\end{aligned}$$

Since  $R$  is an integral domain,  $ar = as$  and  $a \neq 0$  implies that  $r = s$ . Hence  $\theta$  is injective. Since  $R$  is finite,  $\theta$  is surjective. In particular, there is  $b \in R$  such that  $ab = 1$ . Since  $R$  is commutative, we have  $ab = ba$ , i.e.  $a$  is a unit. Hence  $R^* = R \setminus \{0\}$  and  $R$  is a field. □

**Recall**

The characteristic of a ring  $R$ , denoted by  $\text{ch}(R)$  is the order of  $1_R$  in  $(R, +)$ . We write  $\text{ch}(R) = 0$  if  $o(1_R) = \infty$  and  $\text{ch}(R) = n$  if  $o(1_R) = n \in \mathbb{N}$ .

### Proposition 9.4

The characteristic of any integral domain is either 0 or a prime  $p$ .

**Proof:** Let  $R$  be an integral domain. Two cases:

1. If  $\text{ch}(R) = \infty$ , then we are done.
2. Note that since  $R \neq \{0\}$ , we have  $n \neq 1$ . If  $\text{ch}(R) = n \in \mathbb{N} \setminus \{1\}$ , suppose that  $n$  is not prime, say  $n = ab$  with  $1 < a, b < n$ . If 1 is the unity of  $R$ , then by Prop 8.1, we have

$$(a \cdot 1)(b \cdot 1) = (ab)(1 \cdot 1) = n \cdot 1 = 0$$

Since  $R$  is an integral domain, we have  $a \cdot 1 = 0$  or  $b \cdot 1 = 0$ , which leads to a contradiction since  $o(1) = n$ . Thus  $n$  is prime.

□

### Remark

Let  $R$  be an integral domain with  $\text{ch}(R) = p$ , a prime. For  $a, b \in R$ , we have

$$(a + b)^p = a^p + \binom{p}{1}a^{p-1}b + \binom{p}{2}a^{p-2}b^2 + \dots + \binom{p}{p-1}ab^{p-1} + b^p$$

Since  $p$  is a prime,  $p \mid \binom{p}{i}$  for all  $1 \leq i \leq (p-1)$ . Since  $\text{ch}(R) = p$ , we have

$$(a + b)^p = a^p + b^p$$

## 9.2 Prime Ideals and Maximal Ideals

Let  $p$  be a prime and  $a, b \in \mathbb{Z}$ . We recall from Math 135 that  $p \mid ab$  implies  $p \mid a$  or  $p \mid b$ . In other words, if  $ab \in p\mathbb{Z}$ , then  $a \in p\mathbb{Z}$  or  $b \in p\mathbb{Z}$ .

### Definition 9.2.1

Let  $R$  be a commutative ring. An ideal  $P \neq R$  of  $R$  is a *prime ideal* if whenever  $r, s \in R$  satisfy  $rs \in P$ , then  $r \in P$  or  $s \in P$ .

### Example 9.2.1

$\{0\}$  is prime ideal of  $\mathbb{Z}$

### Example 9.2.2

For  $n \in \mathbb{N}$  with  $n \geq 2$ ,  $n\mathbb{Z}$  is a prime ideal of  $\mathbb{Z}$  if and only if  $n$  is prime.

### Proposition 9.5

If  $R$  is a commutative ring, then an ideal  $P$  of  $R$  is a prime ideal if and only if  $R/P$  is an integral domain.

**Proof:** Since  $R$  is a commutative ring, so is  $R/P$ . Note that

$$R/P \neq \{0\} \iff 0 + P \neq 1 + P \iff 1 \notin P \iff P \neq R.$$

Also, for  $r, s \in R$ , we have

$$\begin{aligned} P \text{ is a prime ideal} &\iff rs \in P \text{ implies that } r \in P \text{ or } s \in P \\ &\iff (r + P)(s + P) = 0 + P \text{ implies that} \\ &\quad r + P = 0 + P \text{ or } s + P = 0 + P \\ &\iff R/P \text{ is an integral domain} \end{aligned}$$

□

### Definition 9.2.2

Let  $R$  be a commutative ring. An ideal  $M \neq R$  of  $R$  is a *maximal ideal* if whenever  $A$  is an ideal such that  $M \subseteq A \subseteq R$ , then  $A = M$  or  $A = R$ .

### Remark

Let  $M$  be a maximal ideal of  $R$  and  $r \notin M$ . Then  $M \subseteq \langle r \rangle + M \subseteq R$ . Since  $M \neq \langle r \rangle + M$ , we have  $\langle r \rangle + M = R$ .

### Proposition 9.6

If  $R$  is a commutative ring, then an ideal  $M$  of  $R$  is a maximal ideal if and only if  $R/M$  is a field.

**Proof:** Since  $R$  is a commutative ring, so is  $R/M$ . Note that

$$R/M \neq \{0\} \iff 0 + M \neq 1 + M \iff 1 \notin M \iff M \neq R$$

Also, for  $r \in R$ , note that  $r \notin M$  iff  $r + M \neq 0 + M$ . Thus we have

$$\begin{aligned} M \text{ is a maximal ideal} &\iff \langle r \rangle + M = R \text{ for any } r \notin M \\ &\iff 1 \in \langle r \rangle + M \text{ for all } r \notin M \\ &\iff \forall r \notin M, \text{ there exists } s \in R \text{ s.t. } 1 + M = rs + M \\ &\iff \forall r + M \neq 0 + M, \text{ there exists } s + M \in R/M \text{ s.t. } (r + M)(s + M) = 1 + M \\ &\iff R/M \text{ is a field} \end{aligned}$$

□

Combining Prop 9.2, 9.5 and 9.6, we have

### Corollary 9.7

Every maximal ideal of a commutative ring is a prime ideal.

**Remark**

The converse of Cor 9.7 is not true. For example, in  $\mathbb{Z}$ ,  $\{0\}$  is a prime ideal, but not a max ideal.

**Example 9.2.3**

Consider the ideal  $\langle x^2 + 1 \rangle$  in the ring  $\mathbb{Z}[x]$ . The map  $\theta : \mathbb{Z}[x] \rightarrow \mathbb{Z}[i]$  defined by  $\theta(f(x)) = f(i)$  is surjective since  $\theta(a + bx) = a + bi$ . Also, one can check that the kernel of the map is  $\langle x^2 + 1 \rangle$  (see Piazza). By the first isomorphism theorem, we have  $\mathbb{Z}[x]/\langle x^2 + 1 \rangle \cong \mathbb{Z}[i]$ . Since  $\mathbb{Z}[i]$  is an integral domain, but not a field, we conclude that the ideal  $\langle x^2 + 1 \rangle$  is prime, but not maximal. Note that  $\langle x^2 + 1 \rangle \subsetneq \langle x^2 + 1, 3 \rangle \subsetneq \mathbb{Z}[x]$ . We have  $\mathbb{Z}[x]/\langle x^2 + 1, 3 \rangle \cong \mathbb{Z}_3[x]/\langle x^2 + 1 \rangle$  and  $x^2 + 1$  is irreducible in  $\mathbb{Z}_3[x]$ .

### 9.3 Fields of Fractions

Let  $R$  be an integral domain. We now construct a field  $F$  of all fractions  $\frac{r}{s}$  from  $R$ .

Let  $R$  be an integral domain and let  $D = R \setminus \{0\}$ . Consider the set  $X = R \times D$ . We say  $(r, s) \equiv (r_1, s_1)$  on  $X$  iff  $rs_1 = r_1s$ . One can show that  $\equiv$  is an equivalence relation on  $X$ . Motivated by the case  $R = \mathbb{Z}$ , we now define the fraction  $\frac{r}{s}$  to be the equivalence class  $[(r, s)]$  of the pair  $(r, s)$  on  $X$ . Let  $F$  denote the set of all these fractions, i.e.

$$F = \left\{ \frac{r}{s} \mid r \in R, s \in D \right\} = \left\{ \frac{r}{s} \mid r, s \in R, s \neq 0 \right\}$$

The addition and multiplication of  $F$  are defined by

$$\frac{r}{s} + \frac{r_1}{s_1} = \frac{rs_1 + r_1s}{ss_1} \quad \text{and} \quad \frac{r}{s} \cdot \frac{r_1}{s_1} = \frac{rr_1}{ss_1}$$

Note that  $ss_1 \neq 0$  since  $R$  is an integral domain and thus these operations are well-defined. Then one can show that with the above defined addition and multiplication,  $F$  becomes a field with the zero being  $\frac{0}{1}$ , the unity  $\frac{1}{1}$ , the negative of  $\frac{r}{s}$  is  $\frac{-r}{s}$ . Moreover, if  $\frac{r}{s} \neq 0$  in  $F$ , then  $r \neq 0$  and thus  $\frac{s}{r} \in F$  and we have

$$\frac{r}{s} \cdot \frac{s}{r} = \frac{rs}{sr} = \frac{rs}{rs} = \frac{1}{1} \in F$$

In addition, we have  $R \cong R'$  where  $R' = \left\{ \frac{r}{1} \mid r \in R \right\} \subseteq F$ . Thus we have

**Theorem 9.8**

Let  $R$  be an integral domain. Then there exists a field  $F$  consisting of fractions  $\frac{r}{s}$  with  $r, s \in R$  and  $s \neq 0$ . By identifying  $r = \frac{r}{1}$  for all  $r \in R$ , we can view  $R$  as a subring of  $F$ . The field  $F$  is called the *field of fractions of  $R$* .

## 10 Polynomial Rings

### 10.1 Polynomials

Let  $R$  be a ring and  $x$  a variable. Let

$$R[x] = \{f(x) = a_0 + a_1x + \cdots + a_mx^m \mid m \in \mathbb{N} \cup \{0\} \text{ and } a_i \in R \ (0 \leq i \leq m)\}$$

Such  $f(x)$  is called a *polynomial in  $x$  over  $R$* . If  $a_m \neq 0$ , we say  $f(x)$  has degree  $m$ , denoted by  $\deg f = m$ , and we say the  $a_m$  is *leading coefficient* of  $f(x)$ . If the leading coefficient  $a_m = 1$ , we say  $f(x)$  is *monic*. If  $\deg f = 0$ , then  $f(x) = a_0 \in R \setminus \{0\}$ . In this case, we say  $f(x)$  is a *constant polynomial*. Note that

$$f(x) = 0 \iff a_0 = 0, a_1 = 0, \dots$$

0 is also a constant polynomial and we define  $\deg 0 = -\infty$ . Let  $f(x) = a_0 + a_1x + \cdots + a_mx^m \in R[x]$  and  $g(x) = b_0 + b_1x + \cdots + b_nx^n \in R[x]$  with  $m \leq n$ . Then we write  $a_i = 0$  for all  $m+1 \leq i \leq n$ . We can define addition and multiplication on  $R[x]$  as follows:

$$f(x) + g(x) = (a_0 + b_0) + (a_1 + b_1)x + \cdots + (a_n + b_n)x^n$$

and

$$\begin{aligned} f(x)g(x) &= (a_0 + a_1x + \cdots + a_mx^m)(b_0 + b_1x + \cdots + b_nx^n) \\ &= a_0b_0 + (a_0b_1 + a_1b_0)x + (a_2b_0 + a_1b_1 + a_0b_2)x^2 + \cdots + (a_mb_n)x^{m+n} \\ &= c_0 + c_1x + c_2x^2 + \cdots + c_{m+n}x^{m+n} \end{aligned}$$

where

$$c_i = a_0b_i + a_1b_{i-1} + \cdots + a_{i-1}b_1 + a_ib_0$$

### Proposition 10.1

Let  $R$  be a ring and  $x$  a variable

1.  $R[x]$  is a ring
2.  $R$  is a subring of  $R[x]$
3. If  $Z = Z(R)$  is the center of  $R$ , then  $Z(R[x]) = Z[x]$

**Proof:** (1) and (2) are left as exercises. □

**Proof of 3:** Let  $f(x) = a_0 + a_1x + \cdots + a_mx^m \in Z[x]$  and  $g(x) = b_0 + b_1x + \cdots + b_nx^n \in R[x]$ . We have

$$f(x)g(x) = c_0 + c_1x + c_2x^2 + \cdots + c_{m+n}x^{m+n}$$

with

$$c_i = a_0b_i + a_1b_{i-1} + \cdots + a_{i-1}b_1 + a_ib_0$$

Since  $a_i \in Z$ , we have  $a_i b_j = a_j b_i$  for all  $i, j$ . Thus we get  $f(x)g(x) = g(x)f(x)$  for all  $g(x) \in R[x]$  and hence  $Z[x] \subseteq Z(R[x])$ . To show the other inclusion, if  $h(x) = c_0 + c_1x + \cdots + c_sx^s \in Z(R[x])$ , then for all  $r \in R$ , we have  $h(x)r = rh(x)$ . Thus,  $c_i r = r c_i$  for all  $r \in R$  and  $0 \leq i \leq s$ . Thus,  $c_i \in Z$  and  $Z(R[x]) \subseteq Z[x]$ . It follows that  $Z(R[x]) = Z[x]$ . □

**Warning**

Although  $f(x) \in R[x]$  can be used to define a function from  $R$  to  $R$ , the polynomial is not the same as the function it defines. For example, if  $R = \mathbb{Z}_2$ , there are only 4 different functions from  $\mathbb{Z}_2$  to  $\mathbb{Z}_2$ . However, the polynomial ring  $\mathbb{Z}_2[x]$  is an infinite set.

**Proposition 10.2**

Let  $R$  be an integral domain. Then

1.  $R[x]$  is an integral domain.
2. If  $f \neq 0$  and  $g \neq 0$  in  $R[x]$ , then

$$\deg(fg) = \deg(f) + \deg(g) \quad (\text{product formula})$$

3. The units in  $R[x]$  are  $R^*$ , the units in  $R$ .

**Proof of 1,2:** Suppose that  $f(x) \neq 0$  and  $g(x) \neq 0$  are polynomials in  $R[x]$ , say

$f(x) = a_0 + a_1x + \dots + a_mx^m$  and  $g(x) = b_0 + b_1x + \dots + b_nx^n$  with  $a_m \neq 0$  and  $a_n \neq 0$ . Then

$$f(x)g(x) = (a_m b_n)x^{m+n} + \dots + a_0 b_0$$

Since  $R$  is an integral domain,  $a_m b_n \neq 0$  and thus  $f(x)g(x) \neq 0$ . It follows that  $R[x]$  is an integral domain. Moreover,

$$\deg(fg) = \deg(f) + \deg(g)$$

Thus (1) and (2) follow. □

**Proof of 3:** Let  $u(x) \in R[x]$  be a unit with inverse  $v(x)$ . Since  $u(x)v(x) = 1$ , by (2) we have

$$\deg(u) + \deg(v) = \deg 1 = 0$$

Since  $u(x)v(x) = 1$ , we have  $u(x) \neq 0$  and  $v(x) \neq 0$ . Since  $\deg u \geq 0$  and  $\deg v \geq 0$ , the above equation implies that  $\deg u = 0 = \deg v$ . Thus  $u(x), v(x)$  are units in  $R$  and hence  $R[x]^* \subseteq R^*$ . Since  $R^* \subseteq R[x]^*$ , we have  $R[x]^* = R^*$ . □

**Remark**

Note that in  $\mathbb{Z}_4[x]$ , we have  $2x \cdot 2x = 4x^2 = 0$  thus the product formula doesn't hold here since  $\mathbb{Z}_4$  is not an integral domain.

**Remark**

To extend the product formula in Prop 10.2 to 0, we define  $\deg 0 = \pm\infty$ .

**10.2 Polynomials Over a Field****Definition 10.2.1**

Let  $F$  be a field and  $f(x), g(x) \in F[x]$ . We say  $f(x)$  divides  $g(x)$ , denoted by  $f(x) | g(x)$ , if there exists  $q(x) \in F[x]$  such that  $g(x) = q(x)f(x)$ .

### Proposition 10.3

Let  $F$  be a field.  $f(x), g(x), h(x) \in F[x]$ .

1. If  $f(x) | g(x)$  and  $g(x) | h(x)$ , then  $f(x) | h(x)$ . (transitivity of divisibility)
2. If  $f(x) | g(x)$  and  $f(x) | g(x)$ , then  $f(x) | (g(x)u(x) + h(x)v(x))$  for any  $u(x), v(x) \in F[x]$  (divisibility of integer combinations)

### Recall

For  $a, b \in \mathbb{Z}$  if  $a | b$  and  $b | a$  and  $a, b > 0$ , then  $a = b$ . The following is its analogue in  $F[x]$

### Proposition 10.4

Let  $F$  be a field and  $f(x), g(x) \in F[x]$  be monic polynomials. If  $f(x) | g(x)$  and  $g(x) | f(x)$ , then  $f(x) = g(x)$ .

**Proof:** Since  $f(x) | g(x)$  and  $g(x) | f(x)$ , we have  $g(x) = r(x)f(x)$  and  $f(x) = s(x)g(x)$  for some  $r(x), s(x) \in F[x]$ . Then  $f(x) = s(x)r(x)f(x)$ . By Prop 10.2, we have  $\deg f = \deg s + \deg r + \deg f$ , which implies that  $\deg s = \deg r = 0$ . Thus  $f(x) = sg(x)$  for some  $s \in F$ . Since both  $f(x)$  and  $g(x)$  are monic, we have  $s = 1$  and hence  $f(x) = g(x)$ .  $\square$

### Proposition 10.5

### Division Algorithm

Let  $F$  be a field and  $f(x), g(x) \in F[x]$  with  $f(x) \neq 0$ . Then there exist unique  $q(x), r(x) \in F[x]$  such that

$$g(x) = q(x)f(x) + r(x) \quad \text{with } \deg r < \deg f$$

Note that this includes the case for  $r = 0$  (this explains why we define  $\deg 0 = -\infty$ ).

**Proof:** We first prove by induction that such  $q(x)$  and  $r(x)$  exist. Write  $m = \deg f$  and  $n = \deg g$ . If  $n < m$ , then  $g(x) = 0 \cdot f(x) + g(x)$ . Suppose  $n \geq m$  and the result holds for all  $g(x) \in F[x]$  with  $\deg g < n$ . Write  $f(x) = a_0 + a_1x + \dots + a_mx^m$  with  $a_m \neq 0$  and  $g(x) = b_0 + b_1x + \dots + b_nx^n$ . Since  $F$  is a field,  $a_m^{-1}$  exists. Consider

$$\begin{aligned} g_1(x) &= g(x) - b_n a_m^{-1} x^{n-m} f(x) \\ &= (b_n x^n + b_{n-1} x^{n-1} + \dots + b_0) - b_n a_m^{-1} x^{n-m} (a_m x^m + \dots + a_1 x + a_0) \\ &= 0 \cdot x^n + (b_{n-1} - b_n a_m^{-1} a_{m-1}) x^{n-1} + \dots \end{aligned}$$

Since  $\deg g_1 < n$ , by induction, there exist  $q_1(x), r_1(x) \in F[x]$  such that  $g_1(x) = q_1(x)f(x) + r_1(x)$  with  $\deg r_1 < \deg f$ . It follows that

$$\begin{aligned} g(x) &= q_1(x) + b_n a_m^{-1} x^{n-m} f(x) \\ &= (q_1(x)f(x) + r_1(x)) + b_n a_m^{-1} x^{n-m} f(x) \\ &= (q_1(x) + b_n a_m^{-1} x^{n-m})f(x) + r_1(x) \end{aligned}$$

By taking  $q(x) = q_1(x) + b_n a_m^{-1} x^{n-m}$  and  $r(x) = r_1(x)$ , we have

$$g(x) = q(x)f(x) + r(x) \quad \text{with } \deg r < \deg f$$

To prove uniqueness, suppose that we have  $g(x) = q_1(x)f(x) + r_1(x)$  with  $\deg r_1 < \deg f$ . Then

$$r(x) - r_1(x) = (q_1(x) - q(x))f(x).$$

If  $q_1(x) - q(x) \neq 0$ , we get

$$\deg(r - r_1) = \deg((q_1 - q)f) = \deg(q_1 - q) + \deg f \geq \deg f$$

which leads to a contradiction since  $\deg(r - r_1) < \deg f$ . Thus  $q_1(x) - q(x) = 0$  and hence  $r(x) - r_1(x) = 0$ . It follows that  $q_1(x) = q(x)$  and  $r_1(x) = r(x)$ .  $\square$

### Note

For  $a, b \in \mathbb{Z} \setminus \{0\}$ , the Bézout Lemma states that  $\gcd(a, b) = ax + by$  for some  $x, y \in \mathbb{Z}$ .

### Proposition 10.6

Let  $F$  be a field and  $f(x), g(x) \in F[x] \setminus \{0\}$ . Then there exists  $d(x) \in F[x]$  which satisfies the following conditions:

1.  $d(x)$  is monic
2.  $d(x) \mid f(x)$  and  $d(x) \mid g(x)$
3. If  $e(x) \mid f(x)$  and  $e(x) \mid g(x)$ , then  $e(x) \mid d(x)$
4.  $d(x) = u(x)f(x) + v(x)g(x)$  for some  $u(x), v(x) \in F[x]$

Note that if both  $d(x)$  and  $d_1(x)$  satisfy the above conditions, since  $d(x) \mid d_1(x)$  and  $d_1(x) \mid d(x)$  and both of them are monic, By prop 10.4, we have  $d(x) = d_1(x)$ . We call such  $d(x)$  the greatest common divisor of  $f(x)$  and  $g(x)$  denote it by  $d(x) = \gcd(f(x), g(x))$

**Proof:** Consider the set  $X = \{u(x)f(x) + v(x)g(x), u(x), v(x) \in F[x]\}$ . Since  $f(x) \in X$ , the set  $X$  contains nonzero polynomials and thus monic polynomials (since  $F$  is a field, if  $h(x) \in X$  with leading coefficient  $a$ , then  $a^{-1}h(x) \in X$  is monic). Among all monic polynomials in  $X$ , choose  $d(x) = u(x)f(x) + v(x)g(x)$  of minimal degree. Then (1) and (4) are satisfied. For (3), if  $e(x) \mid f(x)$  and  $e(x) \mid g(x)$ , since  $d(x) = u(x)f(x) + v(x)g(x)$  by prop 10.3,  $e(x) \mid d(x)$ . It remains to prove (2). By the division algorithm, write  $f(x) = q(x)d(x) + r(x)$  with  $\deg r < \deg d$ . Then

$$\begin{aligned} r(x) &= f(x) - q(x)d(x) \\ &= f(x) - q(x)(u(x)f(x) + v(x)g(x)) \\ &= (1 - q(x)u(x))f(x) - (q(x)v(x))g(x) \end{aligned}$$

Note that if  $r(x) \neq 0$ , write  $c \neq 0$  be the leading coefficient of  $r(x)$ . Since  $F$  is a field,  $c^{-1}$  exists. The above expression shows that  $c^{-1}r(x)$  is a monic polynomial of  $X$  with  $\deg(c^{-1}r) = \deg r < \deg d$ , which contradicts the choice of  $d(x)$ . Thus  $r(x) = 0$  and we have  $d(x) \mid f(x)$ . Similarly, we can show  $d(x) \mid g(x)$ . Thus (2) follows.  $\square$

**Recall**

$p \in \mathbb{Z}$  is a prime if  $p \geq 2$  and whenever  $p = ab$  with  $a, b \in \mathbb{Z}$ , then  $a = \pm 1$  or  $b = \pm 1$  (note that  $\pm 1$  are the units of  $\mathbb{Z}$ ).

**Definition 10.2.2**

If  $F$  is a field, a polynomial  $\ell(x) \neq 0$  in  $F[x]$  is *irreducible* if  $\deg \ell \geq 1$  and whenever  $\ell(x) = \ell_1(x)\ell_2(x)$  in  $F[x]$ , we have  $\deg \ell_1 = 0$  or  $\deg \ell_2 = 0$  ( $\deg 0$  polynomials are the units in  $F[x]$ ). Polynomials that are not irreducible are *reducible*.

**Example 10.2.1**

If  $\ell(x) \in F[x]$  satisfies  $\deg \ell = 1$ , then  $\ell(x)$  is irreducible.

**Exercise 10.2.1**

One can show that if  $\deg f \in \{2, 3\}$ , then  $f$  is irreducible iff  $f(d) \neq 0$  for all  $d \in F$ .

**Example 10.2.2**

Let  $\ell(x), f(x) \in F[x]$ . If  $\ell(x)$  is irreducible and  $\ell(x) \nmid f(x)$  then  $\gcd(\ell(x), f(x)) = 1$

**Recall**

Given a prime  $p \in \mathbb{Z}$  and  $a, b \in \mathbb{Z}$ , Euclid's Lemma states that if  $p \mid ab$  then  $p \mid a$  or  $p \mid b$ .

**Proposition 10.7**

Let  $F$  be a field and  $f(x), g(x) \in F[x]$ . If  $\ell(x) \in F[x]$  is irreducible and  $\ell \mid f(x)g(x)$ , then  $\ell \mid f(x)$  or  $\ell \mid g(x)$ .

**Proof:** Suppose  $\ell(x) \mid f(x)g(x)$ . Two cases:

1. If  $\ell(x) \mid f(x)$  then we are done.
2. If  $\ell(x) \nmid f(x)$ , then  $d(x) = \gcd(\ell(x), f(x)) = 1$ . By Prop 10.6, we have

$$1 = \ell(x)u(x) + f(x)v(x) \quad \text{for some } u(x), v(x) \in F[x]$$

Then

$$g(x) = g(x)\ell(x)u(x) + g(x)f(x)v(x)$$

Since  $\ell(x) \mid \ell(x)$  and  $\ell(x) \mid f(x)g(x)$ , By prop 10.3, we have  $\ell(x) \mid g(x)$ .

□

**Remark**

Let  $f_1(x), \dots, f_n(x) \in F[x]$  and let  $\ell(x) \in F[x]$  be irreducible. If  $\ell(x) \mid f_1(x) \cdots f_n(x)$ , by applying Prop 10.7 repeatedly, we get  $\ell(x) \mid f_i(x)$  for some  $i$ .

**Recall**

For an integer  $n \in \mathbb{Z}$  with  $|n| \geq 2$ , up to  $\pm 1$  sign,  $n$  can be written uniquely as a product of primes. By induction and Prop 10.7, we have the following analogous result in  $F[x]$ .

**Theorem 10.8****Unique Factorization Theorem**

Let  $F$  be a field and  $f(x) \in F[x]$  with  $\deg f \geq 1$ . Then we can write  $f(x) = c\ell_1(x) \cdots \ell_m(x)$  where  $c \in F^*$  and  $\ell_i(x)$  are monic, irreducible polynomials (not necessarily distinct.) The formulation is unique up to the order of  $\ell_i$ .

**Exercise 10.2.2**

Use Theorem 10.8 to prove there are infinitely many irreducible polynomials in  $F[x]$ .

**Recall**

In  $\mathbb{Z}$ , all ideals are of the form  $\langle n \rangle = n\mathbb{Z}$  and if  $n \in \mathbb{N}$ , then  $n$  is uniquely determined.

**Proposition 10.9**

Let  $F$  be a field. Then all ideals of  $F[x]$  are of the form  $\langle h(x) \rangle = h(x)F[x]$  for some  $h(x) \in F[x]$ . If  $\langle h(x) \rangle \neq 0$  and  $h(x)$  is monic, then the generator is uniquely determined.

**Proof:** Let  $A$  be an ideal of  $F[x]$ . If  $A = \{0\}$ , then  $A = \langle 0 \rangle$ . If  $A \neq \{0\}$ , then it contains a monic polynomial (since  $F$  is a field, if  $f \in A$  with leading coefficient  $a$ , then  $a^{-1}f \in A$ ). Among all monic polynomials in  $A$ , choose  $h(x) \in A$  of minimum degree. Then  $\langle h(x) \rangle \subseteq A$ . To prove the other inclusion, let  $f(x) \in A$ . By the division algorithm, we have  $f(x) = q(x)h(x) + r(x)$  with  $q(x), r(x) \in F[x]$  and  $\deg r < \deg h$ . If  $r(x) \neq 0$ , let  $u \neq 0$  be its leading coefficient. Since  $A$  is an ideal and  $f(x), h(x) \in A$  we have  $u^{-1}r(x) = u^{-1}(f(x) - q(x)h(x)) = u^{-1}f(x) - u^{-1}q(x)h(x) \in A$  which is a monic polynomial in  $A$  with  $\deg(u^{-1}r) < \deg h$ . This contradicts the minimum property of  $\deg h$ . Thus  $r(x) = 0$  and  $f(x) = q(x)h(x)$ . It follows that  $f(x) \in \langle h(x) \rangle$  and hence  $A = \langle h(x) \rangle$ . To prove uniqueness, suppose  $A = \langle h(x) \rangle = \langle h_1(x) \rangle$ . Since  $h(x) \mid h_1(x)$  and  $h_1(x) \mid h(x)$  and both of them are monic, by Prop 10.4, we have  $h(x) = h_1(x)$ . □

**Recall**

We have seen in  $\mathbb{Z}$  that all ideals are of the form  $\langle n \rangle$  for some  $n \in \mathbb{Z}$ . For  $n \geq 2$ , if we divide an integer by  $n$ , the remainder  $r \in \{0, 1, \dots, n-1\}$ . Write  $\langle n \rangle = n\mathbb{Z}$ . Then we have

$$\mathbb{Z}_n = \mathbb{Z}/\langle n \rangle = \{0 + \langle n \rangle, \dots, (n-1) + \langle n \rangle\} = \{[0], \dots, [n-1]\}$$

We now consider its analogue in  $F[x]$ . Let  $F$  be a field. By Prop 10.9, all ideals of  $F[x]$  are of the form  $\langle h(x) \rangle$ . Suppose that  $h(x)$  is monic and  $\deg h = m \geq 1$ . Consider the quotient ring  $R = F[x]/\langle h(x) \rangle$  and thus

$$R = \left\{ \overline{f(x)} := f(x) + \langle h(x) \rangle \mid f(x) \in F[x] \right\}$$

Write  $t = \bar{x} = x + \langle h(x) \rangle$ . We have  $h(t) = 0$  in  $R$  (exercise). By the division algorithm, we can write  $f(x) = q(x)h(x) + r(x)$  with  $\deg r < \deg h = m$ . Thus one can show that

$$R = \{ \overline{a_0} + \overline{a_1}t + \cdots + \overline{a_{m-1}}t^{m-1} \mid a_i \in F \text{ and } h(t) = 0 \}$$

Consider the map  $\theta : F \rightarrow R$  given by  $\theta(a) = \bar{a}$ . Since  $\theta$  is not the zero map and  $\ker \theta$  is an ideal of  $F$ , we have  $\ker \theta = \{0\}$ . Thus  $\theta$  is an injective ring homomorphism. Since  $F \cong \theta(F)$ , by identifying  $F$  with  $\theta(F)$ , we have

$$R = \{ a_0 + a_1t + \cdots + a_{m-1}t^{m-1} \mid a_i \in F \text{ and } h(t) = 0 \}$$

Note that in  $R$  we have

$$a_0 + a_1t + \cdots + a_{m-1}t^{m-1} = b_0 + b_1t + \cdots + b_{m-1}t^{m-1} \iff a_i = b_i \ (\forall 0 \leq i \leq m-1)$$

Hence this representation of the elements in  $R$  is unique.

### Proposition 10.10

Let  $F$  be a field and  $h(x) \in F[x]$  be monic with  $\deg h = m \geq 1$ . Then the quotient ring  $R = F[x]/\langle h(x) \rangle$  is given by

$$R = \{ a_0 + a_1t + \cdots + a_{m-1}t^{m-1} \mid a_i \in F \text{ and } h(t) = 0 \}$$

in which an element of  $R$  can be uniquely represented by the above form.

### Example 10.2.3

Consider the ring  $\mathbb{R}[x]$ . Let  $h(x) = x^2 + 1 \in \mathbb{R}[x]$ . By Prop 10.10, we have

$$\mathbb{R}[x]/\langle x^2 + 1 \rangle = \{ a + bt \mid a, b \in \mathbb{R} \text{ and } t^2 + 1 = 0 \} \cong \mathbb{C}$$

### Proposition 10.11

Let  $F$  be a field and  $h(x) \in F[x]$  with  $\deg h \geq 1$ . The following are equivalent:

1.  $F[x]/\langle h(x) \rangle$  is a field.
2.  $F[x]/\langle h(x) \rangle$  is an integral domain.
3.  $h(x)$  is irreducible in  $F[x]$

**Proof:** Write  $A = \langle h(x) \rangle$

(1  $\implies$  2) Every field is an integral domain.

(2  $\implies$  3) If  $h(x) = f(x)g(x)$  with  $f(x), g(x) \in F[x]$ , then

$$(f(x) + A)(g(x) + A) = f(x)g(x) + A = h(x) + A = 0 + A \text{ in } F[x]/A$$

By (2), either  $f(x) + A = 0 + A$  or  $g(x) + A = 0 + A$ , i.e. either  $f(x) \in A$  or  $g(x) \in A$ . If  $f(x) \in A = \langle h(x) \rangle$ , then  $f(x) = q(x)h(x)$  for some  $q(x) \in F[x]$ . Thus  $h(x) = f(x)g(x) = q(x)h(x)g(x)$ . Since  $F[x]$  is an integral domain, this implies that  $q(x)h(x) = 1$ , which implies that  $\deg g = 0$ . Similarly, if  $g(x) \in A$ , then  $\deg f = 0$ . Thus  $h(x)$  is irreducible in  $F[x]$ . (3  $\Rightarrow$  1) Note that  $F[x]/A$  is a commutative ring. Thus to show that it is a field, it suffices to show that every nonzero element of  $F[x]/A$  has an inverse. Let  $f(x) + A \neq 0 + A$  with  $f(x) \in F[x]$ . Then  $f(x) \notin A$  and hence  $h(x) \nmid f(x)$ . Since  $h(x)$  is irreducible and  $h(x) \nmid f(x)$ , we have  $\gcd(f(x), h(x)) = 1$ . By Prop 10.6, there exist  $u(x), v(x) \in F[x]$  such that  $1 = u(x)h(x) + v(x)f(x)$ . Thus  $(v(x) + A)(f(x) + A) = 1 + A$  (since  $h(x) \in A$ ). It follows that  $f(x) + A$  has an inverse in  $F[x]/A$  and hence  $F[x]/A$  is a field.  $\square$

### Example 10.2.4

Since  $\mathbb{R}[x]/\langle x^2 + 1 \rangle \cong \mathbb{C}$ , which is a field, the polynomial  $x^2 + 1$  is irreducible in  $\mathbb{R}[x]$ .

### Example 10.2.5

Since  $x^2 + x + 1$  has no root in  $\mathbb{Z}_2$  (since 0 and 1 are not roots), it is irreducible in  $\mathbb{Z}_2[x]$ . Thus  $\mathbb{Z}_2[x]/\langle x^2 + x + 1 \rangle = \{a + bt \mid a, b \in \mathbb{Z}_2 \text{ and } t^2 + t + 1 = 0\}$  is a field of 4 elements.

### Remark

Before the previous example, the only finite fields we know are of the form  $\mathbb{Z}_p$  where  $p$  is a prime. We have seen before that there are infinitely many irreducible polynomials in  $\mathbb{Z}_p[x]$ . One can show that for any  $n \in \mathbb{N}$ , there exists at least one irreducible polynomial of degree  $n$  in  $\mathbb{Z}_p[x]$ , say  $f_n(x)$ . Since  $f_n(x)$  is irreducible,  $\mathbb{Z}_p[x]/\langle f_n(x) \rangle$  is a field of order  $p^n$ . Note that  $\mathbb{Z}_{p^n}$  is NOT a field if  $n \geq 2$ .

### Analogies Between $\mathbb{Z}$ and $F[x]$

	$\mathbb{Z}$	$F[x]$
elements	$m$	$f(x)$
size	$ m  = \text{absolute value}$	$\deg f$
units	$\pm 1; \mathbb{Z} \setminus \{0\}/\{\pm 1\} = \mathbb{N}$	$F^*; F[x] \setminus \{0\}/F^* = \{\text{monic polynomials}\}$
unique factorization	$m = \pm 1 p_1^{\alpha_1} \cdots p_n^{\alpha_n}, p_i \text{ prime}$	$f = c \ell_1^{\alpha_1} \cdots \ell_r^{\alpha_r}, c \in F^*, \ell_i = \ell_i(t) = \text{monic, irreducible}$
ideals	$\langle n \rangle$ (unique if $n \in \mathbb{N}$ )	$\langle h(x) \rangle$ (unique if $h(x)$ monic)
ideals	$\mathbb{Z}/\langle n \rangle$ is a field iff $n$ is a prime	$F[x]/\langle h(x) \rangle$ is a field iff $h(x)$ is irreducible

### 10.3 Fermat's Last Theorem in $F[x]$

#### The Pythagorean Theorem

We have  $x^2 + y^2 = z^2$ . The triples  $(x, y, z) = (3, 4, 5)$  and  $(5, 12, 13)$  are positive integer solutions.

#### Proposition 10.12

#### Euclid's Formula

For  $a, b \in \mathbb{N}$  with  $a > b$ , we can take

$$x = a^2 - b^2, \quad y = 2ab, \quad z = a^2 + b^2$$

Then the triple  $(x, y, z)$  satisfies  $x^2 + y^2 + z^2$  and we have infinitely many positive integer solutions.

Question: How about positive integer solutions for

$$x^n + y^n + z^n \quad \text{with } n \in \mathbb{N} \text{ and } n \geq 3$$

#### Theorem 10.13

#### Fermat's Last Theorem

For  $n \in \mathbb{N}$  with  $n \geq 3$ , the equation  $x^n + y^n = z^n$  has no solution with  $x, y, z \in \mathbb{N}$ .

- The result was first claimed by Fermat in the mid 17<sup>th</sup> century without proof
- The theorem was proved by Wiles and others in 1994
- The proof involves the use of elliptic curves, modular forms and Galois representations

Let  $F$  be a field and  $n \in \mathbb{N}$ . Consider

$$f(x)^n + g(x)^n = h(x)^n$$

with  $f(x), g(x), h(x) \in F[x]$ . We say a solution  $(f, g, h)$  is *non-trivial* if  $\deg(f), \deg(g), \deg(h) \geq 1$ .

#### Theorem 10.14

Let  $F$  be a field with  $\text{ch}(F) = 0$ . For  $n \in \mathbb{N}$  with  $n \geq 3$ , there is no non-trivial solutions to the equation

$$f(x)^n + g(x)^n = h(x)^n$$

with  $f(x), g(x), h(x) \in F[x]$ .

**Proof:** Suppose we have a non-trivial solution  $f^n + g^n = h^n$ . By dividing all common divisors of  $f, g, h$ , we can assume  $(f, g, h)$  is “coprime”, i.e.  $\gcd(f, g) = \gcd(f, h) = \gcd(g, h) = 1$ . Without loss of generality, suppose

$$\deg f = \deg h \geq \deg g$$

Write  $f' = df/dx$ . Since  $f^n + g^n = h^n$ , by the chain rule, we have

$$nf^{n-1}f' + ng^{n-1}g' = nh^{n-1}h'.$$

Since  $\text{ch}(F) = 0$ , we have  $n \cdot 1 \neq 0$ . By multiplying both sides by  $h$  we get

$$\begin{aligned} f^{n-1}f'h + g^{n-1}g'h &= f^n h' + g^n h' \\ \implies f^{n-1}(f'h - fh') &= g^{n-1}(gh' - g'h) \end{aligned}$$

Since  $f$  and  $g$  are coprime, the above equation implies that  $f^{n-1} \mid (gh' - g'h)$ . Thus

$$(n-1) \deg f \leq \deg g + \deg h - 1$$

Since  $\deg f = \deg h \geq \deg g$ , we have

$$(n-2) \deg g \leq (n-2) \deg f \leq \deg g - 1$$

which is a contradiction if  $n \geq 3$ .  $\square$

### Note

Let  $p$  be a prime and consider  $\mathbb{Z}_p[x]$ . We have

$$(f+g)^p = f^p + g^p$$

Let  $h = f + g$ . Then the equation  $f^p + g^p = h^p$  has infinitely many non-trivial solutions.

### Theorem 10.15

Let  $F$  be a field with  $\text{ch}(F) = p$ . For  $n \in \mathbb{N}$  with  $n \geq 3$  and  $\gcd(n, p) = 1$ , there is no non-trivial solutions for

$$f^n + g^n = h^n$$

with  $f, g, h \in F[x]$ .

## 10.4 Prime Number Theorem and Riemann Hypothesis in $\mathbb{Z}_p[t]$

Define  $\pi(x) = \#\{p \leq x \mid p \text{ prime}\}$

Conjecture:

$$\pi(x) \sim \text{li } x = \int_2^x \frac{dt}{\log t} \sim \frac{x}{\log x}$$

Hence the probability of primes  $\leq x$  is  $\frac{1}{\log x}$ . For example, about 1% of  $n \in \mathbb{N}$  with  $n < e^{100}$  are primes.

### Definition 10.4.1

For  $s \in \mathbb{C}$ , the *Riemann zeta function* is defined to be

$$\zeta(s) = \sum_{n \in \mathbb{N}} \frac{1}{n^s} = \prod_{p \text{ prime}} \left(1 + \frac{1}{p^s} + \frac{1}{p^{2s}} + \dots\right) = \prod_{p \text{ prime}} \left(1 - \frac{1}{p^s}\right)^{-1}$$

- Absolutely convergent for  $\text{Re}(s) > 1$
- Functional equation: relate  $\zeta(s)$  with  $\zeta(1-s)$
- Analytic continuation: extend  $\zeta(s)$  to  $s \in \mathbb{C}$

Riemann Hypothesis:

All “non-trivial zeros” of  $\zeta(s)$  lie on  $\operatorname{Re}(s) = \frac{1}{2}$

**Theorem 10.16**

For any  $n \in \mathbb{N}$ , we have

$$\pi(x) = \operatorname{li}(x) + O\left(\frac{x}{(\log x)^n}\right)$$

**Theorem 10.17**

For any  $\varepsilon > 0$ , assuming RH,

$$\pi(x) = \operatorname{li}(x) + O\left(x^{\frac{1}{2}+\varepsilon}\right)$$

Let's consider the Prime Number Theorem in  $\mathbb{Z}_p[t]$ . For  $f \in \mathbb{Z}_p[t]$ , define  $|f| = p^{\deg f}$ . For  $s \in \mathbb{C}$ , the *zeta function* in  $\mathbb{Z}_p[t]$  is defined to be

$$\zeta_p(s) = \sum_{f \text{ monic}} \frac{1}{|f|^s} = \prod_{\substack{\ell \text{ monic} \\ \text{irreducible}}} \left(1 - \frac{1}{|\ell|^s}\right)^{-1}$$

Note that

$$\#\{f \mid f \text{ monic, } \deg f = d\} = p^d$$

So

$$\zeta_p(s) = \sum_{d=0}^{\infty} \frac{p^d}{(p^d)^s} = \frac{1}{1 - p^{1-s}}$$

- Functional equation: relate  $\zeta_p(s)$  with  $\zeta_p(1-s)$
- Analytic continuation: extend  $\zeta_p(s)$  to  $s \in \mathbb{C}$

Define  $a_d = \#\{\ell \mid \ell \text{ monic, irreducible, } \deg \ell = d\}$ . Then we have

$$\zeta_p(s) = \prod_{d=1}^{\infty} \left(1 - \frac{1}{(p^d)^s}\right)^{-a_d}$$

Write  $T = p^{-s}$ . Then

$$\zeta_p(s) = \frac{1}{1 - pT} = \prod_{d=1}^{\infty} (1 - T^d)^{-a_d}$$

Note that by taking logarithmic derivative,

$$\left(\log\left(\frac{1}{1 - pT}\right)\right)' = \frac{p}{1 - pT}$$

By taking logarithmic derivatives and multiplying by  $T$ , we have

$$\frac{pT}{1-pT} = \sum_{d=1}^{\infty} \frac{da_d T^d}{1-T^d}$$

By expanding both sides into power series and comparing the coefficients of  $T^n$ , we have,

$$p^n = \sum_{d|n} da_d \quad (*)$$

### Definition 10.4.2

Now define the *Möbius function* on  $d \in \mathbb{N}$

$$\mu(d) = \begin{cases} 1 & \text{if } d = 1 \\ (-1)^r & \text{if } d \text{ is a product of distinct } r \text{ primes} \\ 0 & \text{otherwise} \end{cases}$$

### Proposition 10.18

### Möbius inversion formula

$$f(n) = \sum_{d|n} g(d) \iff g(n) = \sum_{d|n} \mu(d) f\left(\frac{n}{d}\right)$$

By  $(*)$  and the Möbius inversion formula, we have

$$da_d = p^d + O\left(p^{\frac{d}{2}}\right)$$

i.e.

$$a_d = \frac{p^d}{d} + O\left(\frac{p^{\frac{d}{2}}}{d}\right)$$

Define  $\pi_p(x) = \#\{\ell \mid \ell \text{ monic, irreducible, } |\ell| \leq x\}$  By the estimates of  $a_d$ ,

$$\pi_p(x) = \frac{p}{p-1} \cdot \frac{x}{\log x} + O\left(x^{\frac{1}{2}+\varepsilon}\right)$$

Which is RH in  $\mathbb{Z}_p[t]$