Contents

1 Topological Spaces and Continuous Maps

1.1 Elementary Topology

Given an inner product on an $\mathbb{R}$-vector space $\langle \cdot, \cdot \rangle$, one can define a norm $\|x\| = \sqrt{\langle x, x \rangle}$. Given a norm, one can define a metric $d(x, y) = \|x - y\|$. Given a metric d on a set X, one can define open sets in X:

given $a \in X$ and $r > 0$, $B(a, r) := \{x \in X \mid d(x, a) < r\}$. Then for $A \subseteq X$, we say A is open in X when $\forall a \in A \exists r > 0$ such that $B(a, r) \subseteq A$. Equivalently, for all $a \in A$, there is $b \in X$, $r > 0$ such that $a \in B(b, r) \subseteq A$.

Remark

The set of open sets on a metric space is called the *metric topology* on X.

Open sets in a metric space satisfy the following:

1. $\emptyset$ and X are open
2. arbitrary unions of open sets are open
3. finite intersections of open sets are open

Notation

For a set of sets S, the union of S is

$$\bigcup S := \{x \mid \exists A \in S, x \in A\} = \bigcup_{A \in S} A$$

In the case that $S \neq \emptyset$, the intersection of S is

$$\bigcap S := \{x \mid \forall A \in S, x \in A\} = \bigcap_{A \in S} A$$

Note

$\bigcap S$ would contain all elements as the condition $\forall A \in \emptyset$ would be vacuously satisfied. If we are given a universal set X, and S is known to be a set of subsets of X, then $\bigcap \emptyset = X$.

Definition 1.1.1

Let X be a set. $\mathcal{T} \subseteq \mathcal{P}(X)$ is called a *topology* on X if

1. $\emptyset, X \in \mathcal{T}$
2. If $S \subseteq \mathcal{T}$ is nonempty, then $\bigcup S \in \mathcal{T}$
3. If $S \subseteq \mathcal{T}$ is nonempty and finite, then $\bigcap S \in \mathcal{T}$

The elements of $\mathcal{T}$ are called the open sets of X. The closed sets are the compliments of the open sets.

Remark

To show 3 holds, it suffices to show the intersection of 2 open sets is open (by induction)

Definition 1.1.2

If X is a set, and $\mathcal{T}$ is a topology on X, then $(X, \mathcal{T})$ is called a *topological* space

Remark

When $f : X \to Y$ is a map between metric spaces, f is continuous iff $f^{-1}(V)$ is open in X for every open set $V \subseteq Y$.

Definition 1.1.3

For a map $f : X \to Y$ between topological spaces, we say that f is continuous when $f^{-1}(V)$ is open in X for every open set $V \subseteq Y$.

Example 1.1.1

if $f : A \subseteq \mathbb{R}^n \longrightarrow B \subseteq \mathbb{R}^m$ is an elementary function, then f is continuous.

Definition 1.1.4

When S, T are topologies on X with $S \subseteq T$, we say that S is coarser than T and T is finer than S. When $S \subsetneq T$, we use strictly coarser/finer.

Example 1.1.2

$\{\emptyset, X\}$ is a topology on X called the *trivial topology*

Example 1.1.3

$\mathcal{P}(X)$ is a topology on X called the *discrete topology*

Example 1.1.4

When $X = \emptyset, \mathcal{T} \subseteq \mathcal{P}(X) \Rightarrow \mathcal{T} \subseteq \{\emptyset\} \Rightarrow \mathcal{T} = \emptyset \vee \mathcal{T} = \{\emptyset\}$. Thus the only topology on $\emptyset$ is $\{\emptyset\}$.

Example 1.1.5

When $X = \{a\}$ the only topology is $\mathcal{T} = \{\emptyset, \{a\}\}$

Exercise 1.1.1

Find all topologies on the 2 and 3 element sets.

Definition 1.1.5

Let X be a topological space. Let $A \subseteq X$.

1. The *interior* of A (in X) denoted by $\text{int}(A)$ is the union of all open sets in X which are contained in A.
2. The *closure* of A denoted $\overline{A}$ is the intersection of all closed sets in X which contain A.
3. The *boundary* of A, denoted by ∂A, given by $\partial A = \overline{A} \setminus \text{int}(A)$

Note

The set of closed sets in a topological space is closed under arbitrary intersections and under finite unions. In particular $\emptyset, X$ are closed

Theorem 1.1.1

Let X be a topological space, $A \subseteq X$.

1. $\text{int}(A)$ is open, and is the largest open set which is contained in A
2. $\overline{A}$ is closed, and is the smallest closed set which contains A
3. A is open iff $A = \text{int}(A)$
4. A is closed iff $A = \overline{A}$
5. $\text{int}(\text{int}(A)) = \text{int}(A)$
6. $\overline{\overline{A}} = \overline{A}$

Definition 1.1.6

Let X be a topological space, let $A \subseteq X$, let $a \in X$.

1. We say that a is an *interior point* of A when $a \in A$ and there is an open set U such that $a \in U \subseteq A$
2. We say that a is a *limit point* of A when for every open set $U \ni a$ we have $U \cap (A \setminus \{a\}) \neq \emptyset$. The set of limit points of A is denoted by A'
3. We say that a is a boundary point of A when every open set $U \ni a$, we have $U \cap A \neq \emptyset$ and $U \cap A^c \neq \emptyset$

Theorem 1.1.2

Let X be a topological space and let $A \subseteq X$.

1. $\operatorname{int}(A)$ is equal to the set of all interior points
2. For $a \in X$,
$$a \in A' \iff a \in \overline{A \setminus \{a\}}$$
3. A is closed iff $A' \subseteq A$
4. $\overline{A} = A \cup A'$
5. $\overline{A}$ is the disjoint union
$$\overline{A} = \operatorname{int}(A) \sqcup \partial A$$
6. ∂A is equal to the set of boundary points of A

1.2 Topological Bases

Theorem 1.2.1

Let X be a set. Then the intersection of any set of topologies on X is also a topology on X.

Proof: Let $\{\mathcal{T}_\alpha\}_{\alpha \in I}$ be a collection of topologies on X. Let $\mathcal{T} = \bigcap_\alpha \mathcal{T}_\alpha$

1. Since $X, \emptyset \in \mathcal{T}_\alpha$ for all $\alpha \in I$. We have $X, \emptyset \in \mathcal{T}$
2. Let $\{U_i\} \subseteq \mathcal{T}$. For all $\alpha \in I$, we have each $U_i \in \mathcal{T}_\alpha$. Thus $\bigcup_i U_i \in \mathcal{T}_\alpha \implies \bigcup_i U_i \in \mathcal{T}$ as desired.
3. Let $U_1, ..., U_n \in \mathcal{T}$. Then again for all $\alpha \in I$, we have each $U_i \in \mathcal{T}_\alpha$. Thus $\bigcap_{i=1}^n U_i \in \mathcal{T}_\alpha \implies \bigcap_{i=1}^n U_i \in \mathcal{T}$

$\square$

Corollary 1.2.2

When X is a set and $\mathcal{S}$ is any set of subsets of X (that is $S \subseteq \mathcal{P}(X)$), there is a unique smallest (coarsest) topology $\mathcal{T}$ on X which contains $\mathcal{S}$. Indeed $\mathcal{T}$ is the intersection of (the set of) all topologies on X containing $\mathcal{S}$.
This topology $\mathcal{T}$ is called the topology on X *generated by* $\mathcal{S}$

Definition 1.2.1

Let X be a set. A *basis of sets* on X is a set $\mathcal{B}$ of subsets of X (So $\mathcal{B} \subseteq \mathcal{P}(X)$) such that

1. $\mathcal{B}$ covers X, that is $\bigcup \mathcal{B} = X$
2. For every $C, D \in \mathcal{B}$ and $a \in C \cap D$. There is $B \in \mathcal{B}$ such that $a \in B \subseteq C \cap D$.

When $\mathcal{B}$ is a basis of sets in X and $\mathcal{T}$ is the topology on X generated by $\mathcal{B}$, we say that $\mathcal{B}$ is a *basis for* $\mathcal{T}$. The elements in $\mathcal{B}$ are called *basic open sets* in X.

Theorem 1.2.3 **Characterization of Open Sets in Terms of Basic Open Sets**

Let X be a topological space, Let $\mathcal{B}$ be a basis for the topology on X.
1. For $A \subseteq X$, A is open iff for every $a \in A$, there is $B \in \mathcal{B}$ such that $a \in B \subseteq A$ *
2. The open sets in X are the unions of (sets of) elements in $\mathcal{B}$

Equivalently,
1. $\mathcal{T} = \{A \subseteq X \mid \forall a \in A, \exists B \in \mathcal{B}\ a \in B \subseteq A\}$
2. $\mathcal{T} = \{\bigcup C \mid C \subseteq \mathcal{B}\}$

Proof: Let $\mathcal{T}$ be the topology on X (generated by $\mathcal{B}$). Let $\mathcal{S}$ be the set of all sets $A \subseteq X$ with property * ($\forall a \in A \exists B \in \mathcal{B} : a \in B \subseteq A$). And let $\mathcal{R}$ be the set of (arbitrary) unions of (sets of) elements in $\mathcal{B}$. Recall that $\mathcal{T}$ is the intersection of the set of all topologies on X which contain $\mathcal{B}$. Note that $\mathcal{S}$ contains $\mathcal{B}$ (obviously). Let us show that $\mathcal{S}$ is a topology on X. We have $\emptyset \in \mathcal{S}$ vacuously and $X \in \mathcal{S}$ because $\mathcal{B}$ covers X (given $a \in X$, we can choose $B \in \mathcal{B}$ with $a \in B$). When $U_k \in S$ for every $k \in K$ (where K is any index set). Let $a \in \bigcup_k U_k$. Choose $\ell \in K$ so that $a \in U_\ell$. Since $U_\ell \in \mathcal{S}$, we can choose $B \in \mathcal{B}$ so that $a \in B \subseteq U_\ell$. Since $U_\ell \subseteq \bigcup_k U_k$, we have $a \in B \subseteq \bigcup_k U_k$. Thus $\bigcup_k U_k$ satisfies *, hence $\bigcup_k U_k \in \mathcal{S}$ as required. Suppose $U, V \in \mathcal{S}$ Let $a \in U \cap V$. Since $U \in \mathcal{S}$ we can choose $C \in \mathcal{B}$ with $a \in C \subseteq U$. Since $V \in \mathcal{S}$, we can choose $D \in \mathcal{B}$ with $a \in D \subseteq V$. Since $\mathcal{B}$ is a basis, $C, D \in \mathcal{B}$ and $a \in C \cap D$, we can choose $B \in \mathcal{B}$ with $a \in B \subseteq C \cap D$. Then we have

$$a \in B \subseteq C \cap D \subseteq U \cap V$$

Thus $U \cap V$ satisfies * so that $U \cap V \in \mathcal{S}$ as required. Thus $\mathcal{S}$ is a topology on X containing $\mathcal{B}$, hence $\mathcal{T} \subseteq \mathcal{S}$. Let us show that $\mathcal{S} \subseteq \mathcal{R}$ let $U \in \mathcal{S}$. For each $a \in U$, choose $B_a \in \mathcal{B}$ with $a \in B_a \subseteq U$. Then we have

$$U = \bigcup_{a \in U} B_a \in \mathcal{R}$$

Thus $\mathcal{S} \subseteq \mathcal{R}$. Finally note that $\mathcal{R} \subseteq \mathcal{T}$ because if $U = \bigcup_k B_k$ with $B_k \in \mathcal{B}$, then each $B_k \in \mathcal{T}$, and $\mathcal{T}$ is a topology, so

$$U = \bigcup_{k \in K} B_k \in \mathcal{T}$$

□

Theorem 1.2.4 **Characterization of a Basis in terms of the Open Sets**

Let X be a topological space with topology $\mathcal{T}$. Let $\mathcal{B} \subseteq \mathcal{T}$. Then $\mathcal{B}$ is a basis for $\mathcal{T}$ iff $\forall U \in \mathcal{T} \forall a \in U \exists B \in \mathcal{B} \quad a \in B \subseteq U.$ *

Proof: If $\mathcal{B}$ is a basis for $\mathcal{T}$, then * holds by part 1 of the previous theorem. Suppose * holds. Let us show that $\mathcal{B}$ is a basis of sets in X. Note that $\mathcal{B}$ covers X since, taking $U = X$ in * we have $\forall a \in X \exists B \in \mathcal{B} \quad a \in B \subseteq X$. Also note that given $C, D \in \mathcal{B}$ and $a \in C \cap D$, then by taking $U = C \cap D$ in * (noting that $C, D \in \mathcal{B} \subseteq \mathcal{T}$ so that $U = C \cap D \in \mathcal{T}$) we can choose $B \in \mathcal{B}$ with $a \in B \subseteq C \cap D$. Thus $\mathcal{B}$ is a basis of sets in X. It remains to show that $\mathcal{T}$ is the topology generated by $\mathcal{B}$. Let $\mathcal{S}$ be the topology generated by $\mathcal{B}$. By part 1 of the previous theorem, S is the set of all unions of elements in $\mathcal{B}$. Also $\mathcal{S}$ is the smallest topology which contains $\mathcal{B}$. Since $\mathcal{B} \subseteq \mathcal{T}$ and $\mathcal{T}$ is a topology, we have $\mathcal{S} \subseteq \mathcal{T}$. Also we have $\mathcal{T} \subseteq \mathcal{S}$ because given $U \in \mathcal{T}$, by property *, for each $a \in U$, we can choose $B_a \in \mathcal{B}$ with $a \in B_a \subseteq U$, and then we have $U = \bigcup_{a \in U} B_a \in \mathcal{S}$ since it is a union of elements in $\mathcal{B}$ $\square$

Example 1.2.1

When X is a metric space, the set $\mathcal{B}$ of all open balls in X is a basis for the metric topology on X.

Remark

We can use a basis for testing various topological properties:
When X is a topological space, and $\mathcal{B}$ is a basis for the topology on X, and $A \subseteq X$ and $a \in X$. Then

$$\begin{aligned} a \in \operatorname{int}(A) &\Longleftrightarrow \exists B \in \mathcal{B} \text{ with } a \in B \subseteq A \\ a \in \overline{A} &\Longleftrightarrow \forall B \in \mathcal{B} \text{ with } a \in B \quad B \cap A \neq \emptyset \\ a \in A' &\Longleftrightarrow \forall B \in \mathcal{B} \text{ with } a \in B \quad (B \setminus \{a\}) \cap A \neq \emptyset \\ a \in \partial A &\Longleftrightarrow \forall B \in \mathcal{B} \text{ with } a \in B \quad B \cap A \neq \emptyset \text{ and } B \cap (X \setminus A) \neq \emptyset \end{aligned}$$

Definition 1.2.2

A topological space X is called *Hausdorff* when for all $a, b \in X$ with $a \neq b$, there exist disjoint open sets U and V in X with $a \in U$ and $b \in V$.

Example 1.2.2

Metric spaces are Hausdorff

1.3 Subspaces

Definition 1.3.1 **Subspace Topology**

Let Y be a topological space with topology $\mathcal{S}$, and $X \subseteq Y$ be a subset. Let

$$\mathcal{T} := \{V \cap X \mid V \in \mathcal{S}\}$$

Then $\mathcal{T}$ is a topology on X:
Indeed $\emptyset \in \mathcal{S}$ so $\emptyset \cap X = \emptyset \in \mathcal{T}$ and $Y \in \mathcal{S}$ so $Y \cap X = X \in \mathcal{T}$. If K is any index set and $U_k \in \mathcal{T}$ for each $k \in K$, then for each $k \in K$ we can choose $V_k \in \mathcal{S}$ such that $U_k = V_k \cap X$ and then we have

$$\begin{aligned} \bigcup_{k \in K} U_k &= \bigcup_{k \in K} (V_k \cap X) \\ &= \left(\bigcup_{k \in K} V_k \right) \cap X \in \mathcal{T} \end{aligned}$$

since $\bigcup_{k \in K} V_k \in \mathcal{S}$. Similarly, when K is finite and $U_k \in \mathcal{T}$ for each $k \in K$ we have $\bigcap_{k \in K} U_k \in \mathcal{T}$ The topology $\mathcal{T}$ on X is called the *subspace topology* on X (inherited from the topology on Y).

Theorem 1.3.1

Let Y be a topological space, let $\mathcal{C}$ be a basis for the topology on Y. Let $X \subseteq Y$ be a subset. Then the set

$$\mathcal{B} = \{C \cap X \mid C \in \mathcal{C}\}$$

is a basis for the subspace topology on X.

Proof: Exercise □

Theorem 1.3.2

Let Z be a topological space, let $Y \subseteq Z$ be a subspace and $X \subseteq Y$ be a subset. Then the subspace topology on X inherited from Y is equal to the subspace topology on X inherited from Z.

Proof: Exercise □

Theorem 1.3.3

Let Y be a metric space, (using the metric topology) and let $X \subseteq Y$. Then the subspace topology on X (inherited from the topology on Y) is equal to the metric topology on X using the metric on X obtained by restricting the metric on Y.

Proof: Exercise □

1.4 Continuous Maps

Definition 1.4.1

Let X, Y be topological spaces.

1. For $f : X \to Y$ and $a \in X$, we say that f is *continuous at* a when for every open set $V \subseteq Y$ with $f(a) \in V$, there exists an open set $U \subseteq X$ with $a \in U \subseteq f^{-1}(V)$.
2. We say that f is *continuous* (in or on X) when for every open set $V \subseteq Y$, $f^{-1}(V)$ is open in X.
3. A *homeomorphism* from X to Y is is a bijective map $f : X \to Y$ such that both f and its inverse $f^{-1} : Y \to X$ are continuous. We say that X and Y are *homeomorphic*, and we write $X \cong Y$, when there exists a homeomorphism $f : X \to Y$. (and we remark that $f^{-1} : Y \to X$ is also a homeomorphism).

Theorem 1.4.1

Constant maps and inclusion maps are continuous.

Proof: For $f : X \to Y$ given by $f(x) = c \in Y$ for all $x \in X$. When V is open in Y,

$$f^{-1}(V) = \begin{cases} X \text{ if } c \in V \\ \emptyset \text{ if } c \notin V \end{cases}$$

When $X \subseteq Y$ is a subspace and $f : X \to Y$ is given by $f(x) = x$ for all $x \in X$, when V is open in Y.

$$\begin{aligned} f^{-1}(V) &= \{x \in X \mid f(x) \in V\} \\ &= \{x \in X \mid x \in V\} \\ &= V \cap X \end{aligned}$$

which is open in X. (when X uses the subspace topology) □

Remark

When Y is a topological space and $X \subseteq Y$ we shall assume, unless otherwise noted, that X uses the subspace topology.

Theorem 1.4.2 **Equivalent Definitions of Continuity**

Let $f : X \to Y$ be a map between topological spaces

1. f is continuous iff f is continuous at every $a \in X$
2. f is continuous iff for every closed set $K \subseteq Y$, $f^{-1}(K)$ is closed in X.
3. If $\mathcal{C}$ is a basis for the topology on Y then f is continuous iff for every $C \in \mathcal{C}$, $f^{-1}(C)$ is open in X.

Proof of 1: Suppose f is continuous on X. Let $a \in X$. Let V be an open set in Y with $f(a) \in V$. Let $U = f^{-1}(V)$, then $f^{-1}(V)$ is open, since f is continuous and $a \in U \subseteq f^{-1}(V)$. Suppose, conversely, that f is continuous at every $a \in X$. Let V be an open set in Y. For each $a \in f^{-1}(V)$ since f is continuous at a with $f(a) \in V$, we can choose an open set U_a in X with $a \in U_a \subseteq f^{-1}(V)$. Then

$$f^{-1}(V) = \bigcup_{a \in f^{-1}(V)} U_a$$

which is open in X, since it is a union in open sets in X. □

Theorem 1.4.3

Let $f : X \to Y$, $g : Y \to Z$ be continuous maps between topological spaces, then the composite map $h = g \circ f : X \to Z$ is continuous.

Proof: Show that $h^{-1}(W) = f^{-1}(g^{-1}(W))$ □

Remark

Homeomorphism of topological spaces behaves like an equivalence relation on the class of all topological spaces. For topological spaces X, Y, Z

1. $X \cong X$ (since id_X is a homeomorphism – a special case of the inclusion map)
2. If $X \cong Y$ then $Y \cong X$ (when $f : X \to Y$ is a homeomorphism, so is $f^{-1} : Y \to X$)
3. If $X \cong Y \cong Z$ then $X \cong Z$ (if $f : X \to Y, g : Y \to Z$ are homeomorphisms then so is $g \circ f$)

Theorem 1.4.4 **Restriction of Domain and Restriction or Expansion of Codomain**

Let X, Y, Z be topological spaces. Suppose $f : X \to Y$ is continuous.

1. For any subspace $A \subseteq X$, the restriction $f|_A : A \to Y$ is continuous.
2. If $Y \subseteq Z$ is a subspace then $f : Y \to Z$ is continuous and if $B \subseteq Y$ with $f(X) \subseteq B$, then $f : X \to B$ is continuous.

Proof: Exercise □

Lemma 1.4.5 **Glueing/Pasting Lemma**

Let $f : X \to Y$ be a map between topological spaces

1. If $X = \bigcup_{k \in K} U_k$ where each U_k is open in X and if each restriction map $f|_{U_k} : U_k \to Y$ is continuous (where U_k is using the subspace topology), then f is continuous.
2. If $X = C_1 \cup \cdots \cup C_n$ where each C_k is closed in X, and if each restriction $f|_{C_k} : C_k \to Y$ is continuous, then f is continuous.

Proof of 1: Suppose $X = \bigcup_{k \in K} U_k$ where each U_k is open in X and suppose each restriction $f|_{U_k}$ is continuous. Let $V \subseteq Y$ be open. Note that

$$\begin{aligned} f^{-1}(V) &= \{x \in X \mid f(x) \in V\} \\ &= \bigcup_{k \in K} \{x \in U_k \mid f(x) \in V\} \\ &= \bigcup_{k \in K} \left\{x \in U_k \,\middle|\, f|_{U_k}(x) \in V\right\} \\ &= \bigcup_{k \in K} f|_{U_k}^{-1}(V) \end{aligned}$$

For each $k \in K$, since $f|_{U_k}$ is continuous, we know that $f|_{U_k}^{-1}(V)$ is open in U_k. Since U_k is using the subspace topology, we can choose an open W_k in X such that $f|_{U_k}^{-1}(V) = W_k \cap U_k$. This is open in X since W_k and U_k are both open in X. Since $f^{-1}(V) = \bigcup_{k \in K} f|_{U_k}^{-1}(V)$ it is a union of open sets in X, so it is open in X. Thus f is continuous. □

Proof of 2: Exercise. First show that for $f : X \to Y$, f is continuous iff $f^{-1}(C)$ is closed in X for every closed set C in Y. And, show that when $A \subseteq X \subseteq Y$, A is closed in X (using the subspace topology from Y) iff $A = B \cap X$ for some closed set B in Y. □

Example 1.4.1

The map $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = \begin{cases} 2x & x \leq 0 \\ x^2 & x > 0 \end{cases}$ is continuous.

1.5 Examples of Homeomorphisms

Example 1.5.1

The circle

$$\{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1\}$$

in $\mathbb{R}^2$ is homeomorphic to the ellipse

$$\left\{(x, y) \in \mathbb{R}^2 \,\middle|\, \frac{(x-a)^2}{A^2} + \frac{(y-b)^2}{B^2} = 1\right\}$$

in $\mathbb{R}^2$

Example 1.5.2

$\mathbb{R} \cong (-1, 1) \subseteq \mathbb{R}$

Example 1.5.3

The standard unit n-sphere in $\mathbb{R}^{n+1}$ is the set

$$\mathbb{S}^n = \{x \in \mathbb{R}^{n+1} \mid \|x\| = 1\}$$

Where p is the north pole

$$p = e_{n+1} = (0, ..., 0, 1) \in \mathbb{S}^n$$

We have $\mathbb{S}^n \setminus \{p\} \cong \mathbb{R}^n$

2 Examples of Topological Spaces

Definition 2.0.1

Let X be a set. We sometimes write X_t to indicate that X is using the trivial topology $\mathcal{T}_t = \{\emptyset, X\}$. We sometimes write X_d to indicate X is using the discrete topology $\mathcal{T}_d = \mathcal{P}(X)$. We sometimes write X_c to indicate X is using the co-finite topology $\mathcal{T}_c = \{A \subseteq X \mid A = \emptyset \text{ or } X \setminus A \text{ is finite}\}$. Note the closed sets in X_c are exactly the finite ones and X.

Definition 2.0.2

When X is a metric space, we assume, unless otherwise indicated, that X uses the metric topology. Sometimes, we might write X_m to indicate that X is using the metric topology $\mathcal{T}_m$.

Definition 2.0.3

When Y is a topological space, and $X \subseteq Y$, we assume, unless otherwise indicated, that X uses the subspace topology. Sometimes, we might write X_s to indicate that X is using the subspace topology $\mathcal{T}_s$. When $X \subseteq \mathbb{R}^n$, we shall assume, unless otherwise indicated, that X is using $\mathcal{T}_m = \mathcal{T}_s$

Definition 2.0.4

Let X be a set. A (strict, linear or total) *order* on X is a binary relation $<$ on X such that

1. For all $x, y \in X$ exactly one of the following holds:
   a. $x < y$
   b. $x = y$
   c. $y < x$
2. For all $x, y, z \in X$, if $x < y$ and $y < z$ then $x < z$

An *ordered set* is a set X with an order $<$. When X is an ordered set, we also define $\leq, >, \geq$ by stipulating that for all $x, y \in X$

$$x \leq y \iff (x < y \lor x = y)$$
$$x > y \iff y < x$$
$$x \geq y \iff y \leq x$$

Remark

In an ordered set X we can define an *upper bound*, a *lower bound*, the *supremum*, the *infimum*, the *maximum*, and the *minimum* of a subset $A \subseteq X$.

Example 2.0.1

Let X be an ordered set and $A \subseteq X$, $M = \max(A)$ when $M \in A$ with $M \geq x$ for all $x \in A$. Similarly, m for minimum.

Definition 2.0.5

When X is an ordered set, we have the following subsets which are called *intervals* in X. For $a, b \in X$ with $a < b$ we have

$$(a, b) := \{x \in X \mid a < x < b\}$$
$$(a, b] := \{x \in X \mid a < x \leq b\}$$
$$[a, b) := \{x \in X \mid a \leq x < b\}$$
$$[a, b] := \{x \in X \mid a \leq x \leq b\}$$

Definition 2.0.6

Let X be an ordered set. The *order topology* on X is the topology $\mathcal{T}_o$ which is generated by the basis $\mathcal{B}_o$ of sets in X which consist of the following intervals:

- (a, b) where $a, b \in X$, $a < b$
- $(a, M]$ where $M = \max X$ and $a \in X$ with $a \neq M$ (in the case that X has a maximum)
- $[m, b)$ where $m = \min X$ and $b \in X$ with $b \neq m$ (in the case that X has a minimum)

We sometimes write X_o to indicate that X is using the order topology $\mathcal{T}_o$

Exercise 2.0.1

Verify $\mathcal{B}_o$ is a basis.

Example 2.0.2

$\mathbb{R} = \mathbb{R}_o = \mathbb{R}_m$

Definition 2.0.7

Let X be an ordered set the *lower limit topology* on X is the topology $\mathcal{T}_\ell$ generated by the basis $\mathcal{B}_\ell$ which consists of intervals of the form $[a, b)$ where $a, b \in X$ with $a < b$ we sometimes write X_ℓ to indicate that X is using the lower limit topology.

Note

on $\mathbb{R}$, $\mathcal{T}_\ell$ is not equal to $\mathcal{T}_m$. Note that when $a, b \in \mathbb{R}$ with $a < b$,

$$(a, b) = \bigcup_{n=m}^{\infty} \left[a + \frac{1}{n}, b\right) \text{ where } \frac{1}{m} < b - a$$

which is open in $\mathbb{R}_\ell$. So we have $\mathcal{T}_o \subseteq \mathcal{T}_\ell$

Example 2.0.3

Let $X = (0, 1) \cup \{2\} \subseteq \mathbb{R}$. Note that $\mathcal{T}_o \neq \mathcal{T}_m = \mathcal{T}_s$ on X. (Where X uses the standard order inherited from $\mathbb{R}$). For example $\{2\}$ is open in X_m. But is not open in X_o because any open set in X_o which contains 2, must contain a basic open set B with $2 \in B$. So it must contain a set of the form

$$B = (a, 2]_X = (a, 1) \cup \{2\} \text{ where } a \in (0, 1)$$

So they include elements other than 2

Example 2.0.4

When X is an ordered set, the *dictionary* (or *lexicographic*) order on X^2 is given by

$$(a, b) < (c, d) \iff (a = c \text{ and } b < d) \text{ or } a < c$$

Note that on $\mathbb{R}^2$, the order topology $\mathcal{T}_o$ is not equal to the standard metric topology $\mathcal{T}_m$

2.1 Products of Topological Spaces

Definition 2.1.1

Let X, Y be sets, then the Cartesian product of X and Y is

$$X \times Y = \{(x, y) \mid x \in X, y \in Y\}$$

Definition 2.1.2

Let K be a non-empty index set and let X_k be a set for each $k \in K$. Then the Cartesian product of the (indexed set of) sets X_k, $k \in K$

$$\prod_{k \in K} X_k = \left\{ x : K \to \bigcup_{k \in K} X_k \,\middle|\, x(k) \in X_k \text{ for all } k \in K \right\}$$

and we write $x(k)$ as x_k. In the case that $K = \{1, ..., n\}$ we write

$$\prod_{k \in K} X_k = \prod_{k=1}^{n} X_k = X_1 \times \cdots \times X_n$$

In the case that $K = \mathbb{Z}^+$ we write

$$\prod_{k \in K} X_k = \prod_{k=1}^{\infty} X_k = X_1 \times X_2 \times \cdots$$

In the case that $K = \{1, ..., n\}$ and $X_k = X$ for all $k \in K$, we also write

$$\prod_{k \in K} X_k = \prod_{k \in K} X = \underbrace{X \times X \times \cdots \times X}_{n \text{ times}} = X^n$$

In the case that $K = \mathbb{Z}^+$, and $X_k = X$ for all $k \in K$, we also write

$$\prod_{k \in K} X_k = \prod_{k=1}^{\infty} = X \times X \times \cdots = X^\omega$$

In the case that X is a vector space, we write

$$X^\infty = \{x = (x_1, x_2 ...) \in X^\omega \mid x_k = 0 \text{ for all but finitely many } k \in \mathbb{Z}^+\}$$

In this case X^∞ and X^ω are both vector spaces.
When X_k is a set for each $k \in K$, for each $\ell \in K$ we have the projection map

$$p_\ell : \prod_{k \in K} X_k \to X_\ell$$

given by $p_\ell(x) = x_\ell = x(\ell)$. For any set Y, a function $f : Y \to \prod_{k \in K} X_k$ determines, and is determined by, its component functions

$$f_\ell : Y \to X_\ell$$

where $f_\ell = p_\ell \circ f$ so $f_\ell(y) = f(y)_\ell = f(y)(\ell)$

Definition 2.1.3

When X_k is a topological space for each $k \in K$, there are two commonly used topologies on $\prod_{k \in K} X_k$.

1. The *box topology* on $\prod_{k \in K} X_k$ is the topology generated by the basis of sets of the form
$$\prod_{k \in K} U_k \subseteq \prod_{k \in K} X_k$$
Where each U_k is open in X_k
2. The *product topology* on $\prod_{k \in K} X_k$ is the topology generated by the basis of sets consisting of the sets of the form $\prod_{k \in K} U_k$ where each U_k is open in X_k with $U_k = X_k$ for all but finitely many $k \in K$.

Note

The above two proposed bases are indeed bases of sets because
$$\left(\prod_{k \in K} U_k\right) \cap \left(\prod_{k \in K} V_k\right) = \prod_{k \in K} (U_k \cap V_k)$$
Also note that when K is finite, these two topologies are equal. When K is infinite, the box topology is finer than the product topology.

Theorem 2.1.1

Let $\mathcal{B}_k$ be a basis for X_k for each $k \in K$. Then the set of sets of the form
$$\prod_{k \in K} B_k \text{ where } B_k \in \mathcal{B}_k \text{ for all } k \in K$$
is a basis for the box topology on $\prod_{k \in K} X_k$, and the set of sets of the form
$$\prod_{k \in K} B_k \text{ where } B_k \in \mathcal{B}_k \cup \{X_k\} \text{ for all } k \in K$$
with $B_k = X_k$ for all but finitely many $k \in K$ is a basis for the product topology on $\prod_{k \in K} X_k$.

Proof: Exercise □

Theorem 2.1.2

For each $k \in K$, let X_k be a subspace of Y_k (using the subspace topology). Then the box topology on $\prod_{k \in K} X_k$ is equal to the subspace topology on $\prod_{k \in K} X_k$ as a subspace of $\prod_{k \in K} Y_k$ using the box topology, and the product topology on $\prod_{k \in K} X_k$ is equal to the subspace topology on $\prod_{k \in K} X_k$ as a subspace of $\prod_{k \in K} Y_k$ using the product topology.

Theorem 2.1.3

Let Y be a topological space, and let X_k be a topological space for each $k \in K$, and let $f : Y \to \prod_{k\in K} X_k$. Then when $\prod_{k\in K} X_k$ uses the product topology, f is continuous if and only if each component map $f_\ell : Y \to X_\ell$ is continuous.

Proof: Suppose that f is continuous, then (using either the box or product topologies on $\prod_{k\in K} X_k$) each projection map $p_\ell : \prod_{k\in K} X_k \to X_\ell$ is continuous because when $U \subseteq X_\ell$ is open,

$$\begin{aligned} p_\ell^{-1}(U) &= \left\{ x \in \prod_{k\in K} X_k \;\middle|\; x_\ell = p_\ell(x) \in U \right\} \\ &= \prod_{k\in K} U_k \end{aligned}$$

where

$$U_k = \begin{cases} U & \text{if } k = \ell \\ X_k & \text{if } k \neq \ell \end{cases}$$

which is open in $\prod_{k\in K} X_k$ (using either the box or product topology) It follows that each component function f_ℓ is continuous because

$$f_\ell = p_\ell \circ f$$

Suppose, conversely, that each component map

$$f = p_\ell \circ f : Y \to \prod_{k\in K} X_k$$

is continuous, and that $\prod_{k\in K} X_k$ is using the product topology. To show that f is continuous, it suffices to show that $f^{-1}(B)$ is open in Y for every basic open set B in $\prod_{k\in K} X_k$. Let B be a basic open set (for the product topology) on $\prod_{k\in K} X_k$. Say $B = \prod_{k\in K} U_k$ where each U_k is open in X_k with $U_k = X_k$ for all but finitely many indices $k \in K$. Let $L \subseteq K$ be the finite set of all indices $k \in K$ for which $U_k \neq X_k$. We have

$$\begin{aligned} f^{-1}(B) &= \left\{ y \in Y \;\middle|\; f(y) \in \prod_{k\in K} U_k \right\} \\ &= \{ y \in Y \mid f_k(y) = f(y)_k \in U_k \text{ for all } k \in K \} \\ &= \{ y \in Y \mid f_k(y) \in U_k \text{ for all } k \in L \} \\ &= \bigcap_{k\in L} f_k^{-1}(U_k) \end{aligned}$$

Which is open in Y since it is a finite intersection of open sets in Y (with $f_k^{-1}(U_k)$) is open in Y because U_k is open in X_k and $f_k : Y \to X_k$ is continuous. $\square$

Remark

$$\mathbb{R}^\infty \subseteq \ell_1 \subseteq \ell_p \subseteq \ell_q \subseteq \ell_\infty \subseteq \mathbb{R}^\omega$$

for $1 \leq p \leq q \leq \infty$. Recall that these norms induce different topologies.

Question: do any of the p-norms induce the box or product topology on $\mathbb{R}^\infty \subseteq \mathbb{R}^\omega$?
Question: is there a norm or metric on $\mathbb{R}^\omega$ which induces the box or product topology?

Remark

Also, we have the p-norms on $\mathbb{R}^n$. They all give the same topology on $\mathbb{R}^n$. More generally, when X is a finite dimensional vector space, all norms on X induce the same topology on X. When $L : X \to Y$ is a linear map between normed linear spaces, L is continuous iff $\|L\|_{\text{op}} < \infty$ iff $L\left(\overline{B_X}(0,1)\right)$ is bounded in Y. And when X is finite dimensional, $\overline{B_X}(0,1)$ is compact and $L\left(\overline{B_X}(0,1)\right)$ is bounded, so L is continuous. In particular, when X is finite dimensional and $\|\cdot\|_1, \|\cdot\|_2$ are two norms on X,

$$\text{id}_X : (X, \|\cdot\|_1) \longrightarrow (X, \|\cdot\|_2)$$

is continuous, and it is equal to its own inverse which is continuous, so id_X is a homeomorphism, so for a set $U \subseteq X$, U is open in $(X, \|\cdot\|_1)$ if and only if U is open in $(X, \|\cdot\|_2)$. Consequently, every finite dimensional vector space X has a *standard* topology. (Pick a basis $\{u_1, ..., u_n\}$, define

$$\left\langle \sum x_k u_k, \sum y_k u_k \right\rangle = \sum x_k y_k = x \cdot y$$

So the map $L : X \to \mathbb{R}^n$ given by

$$L\left(\sum x_k u_k\right) = \sum x_k e_k = x$$

is an inner product space isomorphism.) Then use the inner product to define a norm, a metric, and a topology. The resulting topology doesn't depend on the choice of basis.

2.2 Quotient Spaces

Definition 2.2.1

Let X be a set. Let $\sim$ be an equivalence relation on X. For $a \in X$, the *equivalence class* of a is

$$[a] = \{x \in X \mid a \sim x\}$$

Recall distinct equivalence classes are disjoint, and X is the disjoint union of distinct equivalence classes. The set of all equivalence classes is denoted by $X/\sim$, is called the quotient set of X by $\sim$.

$$X/\sim = \{[a] \mid a \in X\}$$

The map $q : X \to X/\sim$ given by $x \mapsto [x]$ is called the quotient map.

Definition 2.2.2

When X is a topological space, the *quotient topology* on $X/\sim$ is the topology obtained by stipulating that for $V \subseteq X/\sim$, V is open in $X/\sim$ if and only if $q^{-1}(V)$ is open in X.

Note

When $V \subseteq X/\sim$ so V is a set of equivalence classes.

$$\begin{aligned} q^{-1}(V) &= \{x \in X \mid q(x) \in V\} \\ &= \{x \in X \mid [x] \in V\} \\ &= \bigcup_{[x] \in V} [x] \\ &= \bigcup V \end{aligned}$$

Remark

For sets X and Y,

1. When Y is a topological space and $X \subseteq Y$ is a subset, the subspace topology is the coarsest topology on X for which the inclusion map $i : X \to Y$ is continuous
$$i^{-1}(V) = \{x \in X \mid i(x) \in V\} = \{x \in X \mid x \in V\} = V \cap X$$
2. When X and Y are both topological spaces, the product topology on $X \times Y$ is the coarsest topology for which the two projection maps $p_X : X \times Y \to X$, $p_Y : X \times Y \to Y$ are both continuous
$$p_X^{-1}(U) = U \times Y \quad p_Y^{-1}(V) = V \times X$$
3. When X is a topological space and $\sim$ an equivalence relation on X, the quotient topology on $X/\sim$ is the finest topology on $X/\sim$ for which the quotient map $q : X \to X/\sim$ is continuous

Note

Let X be a set and $\sim$ an equivalence relation on X. Note that any function $g : X/\sim \to Y$ (where Y is any set) determines and is determined by a function $f : X \to Y$ which is constant on equivalence classes (meaning that for $x_1, x_2 \in X$ if $x_1 \sim x_2$ then $f(x_1) = f(x_2)$) with g given by $g([x]) = f(x)$ and with f given by $f = g \circ q$. So $f(x) = g(q(x)) = g([x])$

Theorem 2.2.1

Let X, Y be topological spaces. Let $\sim$ be an equivalence relation on X. Let $f : X/\sim \to Y$. Let $g : X \to Y$ be the map given by $g(x) = f([x])$, that is $g = f \circ q$. Then f is continuous if and only if g is continuous.

Proof: If f is continuous, then g is continuous because $g = f \circ q$ which is the composite of two continuous maps. Suppose that g is continuous. Let $V \subseteq Y$, be open. We need to show that $f^{-1}(V)$ is open in $X/\sim$. By definition of the quotient topology

$$f^{-1}(V) \text{ is open in } X/\sim \Longleftrightarrow q^{-1}(f^{-1}(V)) \text{ is open in } X$$

But

$$q^{-1}(f^{-1}(V)) = (f \circ q)^{-1}(V) = g^{-1}(V)$$

Which is open in X since g is continuous. □

Definition 2.2.3

For a group G and a set X, a *group action* of G on X is a function $* : G \times X \to X$, where we write $*\,(a, x)$ as $a * x$ or ax, such that

1. When $e \in G$ is the identity element we have $e * x = x$ for all $x \in X$.
2. For all $a, b \in G$ and all $x \in X$, we have

$$a * (b * x) = \underbrace{(ab)}_{\text{group op}} * x$$

We say that G *acts on* X (by using the group action).

Remark

A group action of G on X determines and is determined by a group homomorphism $\rho : G \to \mathrm{Perm}(X)$ where $\rho(a)(x) = a * x$ (the homomorphism ρ is called a *representation* of G)

Remark

Given an action of G on X, we can define an equivalence relation on X by

$$x \sim y \Longleftrightarrow y = a * x \text{ for some } a \in G.$$

In this case, the equivalence class of x is called the *orbit of* x (we might write $[x]$ as $\mathrm{Orb}(x)$) and we write the quotient $X/\sim$ as X/G. So

$$\begin{aligned} X/G &= \{[x] \mid x \in X\} \\ &= \{\mathrm{Orb}(x) \mid x \in X\} \end{aligned}$$

Example 2.2.1

For $\mathbb{S}^1 = \{u \in \mathbb{R}^2 \mid \|u\| = 1\}$, we have $\mathbb{S}^1 \times \mathbb{R} \cong \mathbb{R}^2 \setminus \{0\}$. Define

$$\begin{aligned} f : \mathbb{S}^1 \times \mathbb{R} &\longrightarrow \mathbb{R}^2 \setminus \{0\} \\ (u,t) &\longmapsto e^t u \end{aligned}$$

and define

$$\begin{aligned} g : \mathbb{R}^2 \setminus \{0\} &\longrightarrow \mathbb{S}^1 \times \mathbb{R} \\ x &\longmapsto \left(\frac{x}{\|x\|}, \ln\|x\| \right) \end{aligned}$$

These maps are continuous (they are elementary functions) and they are inverses of each other.

Example 2.2.2

$\mathbb{S}^1$ acts on $\mathbb{R}^2 = \mathbb{C}$ by complex multiplication. For $a \in \mathbb{R}^2 = \mathbb{C}$,

$$\operatorname{Orb}(a) = [a] = \{ua \mid u \in \mathbb{S}^1\}$$

which is equal to the circle centered at 0 of radius $\|a\|$ (with $[0] = \{0\}$).
Show that $\mathbb{R}^2/\mathbb{S}^1 \cong [0,\infty) \subseteq \mathbb{R}$ we define

$$\begin{aligned} f : \mathbb{R}^2/\mathbb{S}^1 &\longrightarrow [0,\infty) \\ [x] &\longmapsto \|x\| \end{aligned}$$

and define

$$\begin{aligned} h : [0,\infty) &\longrightarrow \mathbb{R}^2/\mathbb{S}^1 \\ r &\longmapsto [r] = [(r,0)] = \{re^{i\theta} \mid \theta \in \mathbb{R}\} \end{aligned}$$

Note that f is continuous because for the map $g : \mathbb{R}^2 \to [0,\infty) \subseteq \mathbb{R}$ given by $g(x) = \|x\|$. We have $g = f \circ q$. Since g is continuous, it follows that f is continuous. Also h is continuous because $h = q \circ i$ where $i : [0,\infty) \longrightarrow \mathbb{R}^2$ is the inclusion map $i(r) = (r,0)$. Finally, note that f and h are inverses.

Example 2.2.3

$\mathbb{R}^+ = (0, \infty)$ acts on $\mathbb{R}^2$ be multiplication that is by $t * x = tx$. The orbits are for $o \neq x \in \mathbb{R}^2$, $[x] = \{tx \mid 0 < t \in \mathbb{R}\}$ which is the (open) ray from 0 through x and $[0] = \{0\}$. Each of the rays $[x]$ for $0 \neq x \in \mathbb{R}^2$ intersects a unique point on $\mathbb{S}^1$. Which gives a fairly natural bijective map

$$f : \mathbb{R}^2/\mathbb{R}^+ \longrightarrow \mathbb{S}^1 \cup \{0\}$$
$$[x] \longmapsto \begin{cases} \frac{x}{\|x\|} & \text{if } 0 \neq x \in \mathbb{R}^2 \\ 0 & \text{if } x = 0 \in \mathbb{R}^2 \end{cases}$$

The inverse $g : \mathbb{S}^1 \cup \{0\} \to \mathbb{R}^2/\mathbb{R}^+$ is given by $u \mapsto [u]$. Note that g is continuous ($g = q \circ i$ where i is the inclusion map $i : \mathbb{S}^1 \cup \{0\} \to \mathbb{R}^2$). But f is not continuous, for example the set $\{0\}$ is open in $\mathbb{S}^1 \cup \{0\}$ (it is an open ball) but $f^{-1}(\{0\}) = \{[0]\} \subseteq \mathbb{R}^2/\mathbb{R}^+$ and $q^{-1}(\{[0]\}) = \{0\}$ is not open in $\mathbb{R}^2$. In fact, $\mathbb{R}^2/\mathbb{R}^+ \not\cong \mathbb{S}^1 \cup \{0\}$. One way to show this is to note that $\mathbb{S}^1 \cup \{0\}$ has a singleton which is open ($\{0\}$), but $\mathbb{R}^2/\mathbb{R}^+$ has no singleton which is open.

Remark

$\mathbb{R}^2/\mathbb{R}^+$ is not Hausdorff, so it is not metrizable (there is no metric we can define on $\mathbb{R}^2/\mathbb{R}^+$ for which that quotient topology is equal to the metric topology)

Example 2.2.4

$\mathbb{Z}$ acts by addition on $\mathbb{R}$ (by $n * x = x + n$). The orbits are the sets $[x] = \{x + n \mid n \in \mathbb{Z}\} = x + \mathbb{Z}$ Show that $\mathbb{R}/\mathbb{Z} \cong \mathbb{S}^1$. Define

$$\begin{aligned} f : \mathbb{R}/\mathbb{Z} &\longrightarrow \mathbb{S}^1 \\ [t] &\longmapsto e^{i2\pi t} \end{aligned}$$

(and note that when $[s] = [t]$ say $s = t + n$ where $n \in \mathbb{Z}$ we have

$$e^{i2\pi s} = e^{i2\pi(t+n)} = e^{i2\pi t}$$

) Note that f is continuous because the map $f : \mathbb{R} \to \mathbb{S}^1$ given by $g(t) = e^{i2\pi t}$ is continuous with $g = f \circ q$. The inverse map

$$\begin{aligned} h : \mathbb{S}^1 &\longrightarrow \mathbb{R}/\mathbb{Z} \\ e^{i\theta} &\longmapsto \left[\frac{\theta}{2\pi}\right] \end{aligned}$$

To see that h is continuous, we can express h in Cartesian coordinates. We remark that there is an angle map

$$\begin{aligned} \theta : \mathbb{R}^2 \setminus \{0\} &\longrightarrow [0, 2\pi) \\ (x, y) &\longmapsto \begin{cases} \arccos \frac{x}{\sqrt{x^2+y^2}} & \text{if } y > 0 \text{ or } (y = 0 \text{ and } x \neq 0) \\ 2\pi - \arccos \frac{x}{\sqrt{x^2+y^2}} & \text{if } y < 0 \text{ or } (y = 0 \text{ and } x < 0) \end{cases} \end{aligned}$$

This map is not continuous along the positive x-axis. In Cartesian coordinates, $h : \mathbb{S}^1 \to \mathbb{R}/\mathbb{Z}$ is given by

$$h(x, y) = \begin{cases} \left[\frac{1}{2\pi} \arccos(x)\right] & \text{if } y \geq 0 \\ \left[1 - \frac{1}{2\pi} \arccos(x)\right] & \text{if } y \leq 0 \end{cases}$$

that is by

$$h(x, y) = \begin{cases} h_1(x, y) \text{ if } (x, y) \in A \\ h_2(x, y) \text{ if } (x, y) \in B \end{cases}$$

Where

$$A = \left\{(x, y) \in \mathbb{S}^1 \,\middle|\, y \geq 0\right\}$$
$$B = \left\{(x, y) \in \mathbb{S}^1 \,\middle|\, y \leq 0\right\}$$

and

$$h_1(x, y) = \frac{1}{2\pi} \arccos x$$
$$h_2(x, y) = 1 - \frac{1}{2\pi} \arccos x$$

3 Connected, Path-Connected and Compact Spaces

Definition 3.0.1

Let X be a topological space. For subsets $A, B \subseteq X$, we say that A and B *separate* X when $A \neq \emptyset$, $B \neq \emptyset$, $A \cap B = \emptyset$ and $A \cup B = X$. We say that X is *disconnected* when there exist (nonempty disjoint) open sets $U, V \subseteq X$ which separate X. Otherwise, we say that X is *connected*.

Proposition 3.0.1

X is connected if and only if the only clopen sets are X and $\emptyset$.

Proof: If X is disconnected, we can find open sets $U, V \subseteq X$ which separate X then the sets $\emptyset, U, V, X$ are clopen. On the other hand, if $\emptyset \neq U \subsetneq X$ with both U both open and closed in X, then U and $V = X \setminus U$ are open sets in X which separate X. □

Exercise 3.0.1

When X is a metric space and $A \subseteq X$ is a subspace, then A is connected if and only if there do not exist open sets U, V in X such that $U \cap A \neq \emptyset$, $V \cap A \neq \emptyset$, $U \cap V = \emptyset$ and $A \subseteq U \cup V$.

Example 3.0.1

The connected sets in $\mathbb{R}$ are the intervals (including $\emptyset$, $\{a\}$, $\mathbb{R}$)

Example 3.0.2

The (non-empty) connected subsets of $\mathbb{Q}$ are the singletons (by using the density of the irrationals)

Theorem 3.0.2

If $f : X \to Y$ is a continuous map between topological spaces, and if X is connected, then $f(X)$ is connected.

Proof: Suppose X is connected and $f : X \to Y$ is continuous. By restricting the codomain, the map $f : X \to f(X)$ is also continuous. Suppose, for a contradiction that $f(X)$ is disconnected. Let U, V be open sets in $f(X)$ which separate $f(X)$. Then $f^{-1}(U)$ and $f^{-1}(V)$ are open sets in X which separate X, so that X is disconnected, giving the desired contradiction. □

Lemma 3.0.3

Let X be a subspace of Y. Suppose Y is disconnected. Let U, V be open sets in Y that separate Y. If X is connected, then $X \subseteq U$ or $X \subseteq V$.

Proof: Suppose $X \nsubseteq U$ and $X \nsubseteq V$. Since $U \cup V = Y$, it follows that $X \cap U \neq \emptyset$ and $X \cap V \neq \emptyset$. And these two sets are open sets in X which separate X. □

Theorem 3.0.4

Let $X = \bigcup_{k \in K} A_k$ where each subspace A_k is connected. With $\bigcap_k A_k \neq \emptyset$. Then X is connected.

Proof: Suppose, for a contradiction, that X is disconnected. Let U, V be open sets in X which separate X. Let $p \in \bigcap_{k \in K} A_k \subseteq X = U \cup V$. Either $p \in U$ or $p \in V$ (but not both) say $p \in U$. For each index k, since A_k is connected either $A_k \subseteq U$ or $A_k \subseteq V$ and since $p \in A_k$, $p \notin V$, we must have $A_k \subseteq U$. Since $A_k \subseteq U$ for every $k \in K$, we have $X = \bigcup_{k \in K} A_k \subseteq U$. This is not possible since U and V separate X. □

Theorem 3.0.5

The product of two connected spaces is connected.

Proof: Let X and Y be connected spaces. Suppose both X and Y are nonempty (since if either one was, $\emptyset$ is connected). Choose $a \in X$ and $b \in Y$ so $(a, b) \in X \times Y$. Since $X \times \{b\} \cong X$ and X is connected, it follows that $X \times \{b\}$ is connected. For each $x \in X$, since $\{x\} \times Y \cong Y$ and Y is connected, it follows that $\{x\} \times Y$ is connected. Since $X \times \{b\}$ and $\{x\} \times Y$ are connected and $(X \times \{b\}) \cap (\{x\} \times Y) \neq \emptyset$ (since (x, b) is in both), it follows from the previous theorem that the set $A_x = (X \times \{b\}) \cup (\{x\} \times Y)$ is connected. Since each A_x is connected and $\bigcap_{x \in X} A_x \neq \emptyset$ (indeed (a, b) is in the intersection) it follows that $\bigcup_{x \in X} A_x = X \times Y$ is connected. □

Lemma 3.0.6

Let X be a subspace of Y. Let U, V be subsets of X which separate X (not necessarily open). Then U is open in X if and only if $U \cap \overline{V} = \emptyset$. Symmetrically, V is open in X if and only if $V \cap \overline{U} = \emptyset$ where $\overline{U} = \mathrm{Cl}_Y(U), \overline{V} = \mathrm{Cl}_Y(V)$

Theorem 3.0.7

Let X be a topological space, let A, B be subspaces with $A \subseteq B \subseteq \overline{A}$. If A is connected, then so is B. In particular, if A is connected, then so is $\overline{A}$.

Proof: Suppose A is connected. Suppose for a contradiction that B is not connected. Let $U, V \subseteq B$ be open sets in B which separate B. Since A is connected and U, V are open sets in B, which separate B, by previous lemma, either $A \subseteq U$ or $A \subseteq V$. Say $A \subseteq U$. Since $A \subseteq U$ we have $\overline{A} \subseteq \overline{U}$ so that $B \subseteq \overline{A} \subseteq \overline{U}$. By the previous lemma, $V \cap \overline{U} = \emptyset$ hence $V \cap B = \emptyset$, but $V \subseteq B$ so $V = \emptyset$ which contradicts the fact that U and V separate B. □

Theorem 3.0.8

Let X_k be a connected topological space for each $k \in K$. Then $\prod X_k$ is connected using the product topology.

Definition 3.0.2

When X is a topological space, and $A \subseteq X$, we say that A is *dense* in X when $\overline{A} = X$. Note that

$$\begin{aligned}\overline{A} = X &\iff \text{the only closed set } K \subseteq X \text{ with } A \subseteq K \text{ is } K = X \\ &\iff \text{the only open set } U \subseteq X \text{ with } A \cap U = \emptyset \text{ is } U = \emptyset \\ &\iff \text{for every nonempty open set } U \subseteq X \text{ we have } A \cap U \neq \emptyset\end{aligned}$$

When $\mathcal{B}$ is a basis for the topology on X, verify that $\overline{A} = X$ if and only if for all $\emptyset \neq B \in \mathcal{B}$ we have $A \cap B \neq \emptyset$.

Example 3.0.3

$\mathbb{R}^\omega = \prod_{k=1}^{\infty} \mathbb{R}$ using the box topology is <u>not</u> connected. Indeed verify that the sets

$$\begin{aligned}U &= \{x \in \mathbb{R}^\omega \mid \|x\|_\infty < \infty\} \\ &= \text{the set of all bounded sequences in } \mathbb{R}\end{aligned}$$

and

$$\begin{aligned}V &= \{x \in \mathbb{R}^\omega \mid \|x\|_\infty = \infty\} \\ &= \text{the set of all unbounded sequences in } \mathbb{R}\end{aligned}$$

are open in $\mathbb{R}^\omega$ (with the box topology) and they cover $\mathbb{R}^\omega$.

3.1 Connected Components

Definition 3.1.1

Let X be a topological space. Define a binary relation $\sim$ on X by stipulating that for $a, b \in X$

$$a \sim b \iff \text{there exists a connected subspace } A \subseteq X \text{ with } a, b \in A$$

Note that $\sim$ is an equivalence relation. Indeed $a \sim a$ since $\{a\}$ is connected. If $a \sim b$ then obviously $b \sim a$. If $a \sim b$ and $b \sim c$ then we can choose connected subspaces $A, B \subseteq X$ with $a, b \in A$, $b, c \in B$, then by a previous lemma, since $b \in A \cap B$, we have $A \cup B$ is connected, and $a, c \in A \cup B$, so that $a \sim c$. The equivalence classes in X under $\sim$ are called the *connected components* of X. (Note that the connected components are disjoint and they cover X).

Theorem 3.1.1

Let X be a topological space. The connected components of X are the maximal connected subspaces of X. Indeed, each connected component of X is connected, and every non-empty connected subspace of X is contained inside exactly one of the connected components.

Proof: $\square$

3.2 Path-Connectedness

Definition 3.2.1

Let X be a topological space. For $a, b \in X$, a (continuous) *path* from a to b in X is a continuous map $\alpha : [0,1] \subseteq \mathbb{R} \to X$ with $\alpha(0) = a$ and $\alpha(1) = b$. We say that X is *path connected* when for every $a, b \in X$ there exists a path from a to b in X.

Theorem 3.2.1

Every path-connected space is connected.

Proof: Suppose X is path-connected. Suppose, for a contradiction, that X is not connected. Choose open sets $U, V \subseteq X$ which separate X. Choose $a \in U$ and $b \in V$. Since X is path-connected we can choose a path $\alpha : [0,1] \subseteq \mathbb{R} \to X$ with $\alpha(0) = a$ $\alpha(1) = b$. Then the sets $\alpha^{-1}(U)$ and $\alpha^{-1}(V)$ are open and separate $[0,1]$, contradiction. □

Theorem 3.2.2

The image of a path connected space under a continuous map is path connected. In particular, for topological spaces X and Y. If $X \cong Y$, then X is path connected if and only if Y is path connected.

Proof: Let $f : X \to Y$ be continuous and suppose X is path connected. Let $c, d \in f(X)$. Choose $a, b \in X$ with $f(a) = c$, $f(b) = d$. Since X is path connected, we can choose a path α in X from a to b. Then $\beta = f \circ \alpha$ is path in Y from c to d. □

Note

Convex sets are path connected (in normed linear spaces). More generally, the image of a convex set (in a normed linear spaces) under a continuous map is path connected, hence connected.

Example 3.2.1

$A = \{x \in \mathbb{R}^2 \mid 1 \leq \|x\| \leq 2\}$ is the image of $[1,2] \times [0, 2\pi]$ under the polar coordinates map $p : \mathbb{R}^2 \to \mathbb{R}^2$ given by $p(r, \theta) = (r\cos\theta, r\sin\theta)$ and thus path connected. (Using the fact that rectangles (also balls) are convex and hence connected).

Proposition 3.2.3

Using the product topology, a product of path-connected spaces is path connected.

Proof: Let X_k be path connected for each $k \in K$. Let $a, b \in \prod X_k$. For each $k \in K$, choose a path α_k in X_k from a_k to b_k. Then the map $\alpha : [0,1] \to \prod X_k$ given by

$$\alpha(t)(k) = \alpha(t)_k = \alpha_k(t)$$

is a (continuous) path in $\prod X_k$ from a to b. □

Remark

Using the box topology, this isn't true.

Definition 3.2.2

Let X be a topological space. Define a binary relation $\sim$ on X by stipulating that for $a, b \in X$

$$a \sim b \Longleftrightarrow \text{there exists a path in } X \text{ from } a \text{ to } b$$

Note that this is an equivalence relation on X, indeed for $a, b, c \in X$:

1. $a \sim a$ since the constant path κ_a is a path from a to a in X.
2. If $a \sim b$ then there is a path α from a to b. Then $\beta(t) = \alpha(1 - t)$
3. If $a \sim b$ and $b \sim c$ with paths α, β then $\gamma : [0, 1] \to X$ given by

$$\gamma(t) = \begin{cases} \alpha(2t) & \text{for } 0 \leq t \leq \frac{1}{2} \\ \beta(2t - 1) & \text{for } \frac{1}{2} \leq t \leq 1 \end{cases}$$

   is a (continuous) path in X from a to c (by the glueing lemma).

The equivalence classes in X under $\sim$ are called the *path components of* X

Theorem 3.2.4

Let X be a topological space. The path components of X are the maximal path connected subspaces of X. Indeed, each path component of X is path connected, and every path connected subspace of X is contained in exactly one of the path components of X.

Proof: path components are path connected by the definition of $\sim$. Let A be any path connected subspace of X. Let P, Q be any path components for which $A \cap P \neq \emptyset$ and $A \cap Q \neq \emptyset$. Choose $p \in A \cap P$ and $q \in A \cap Q$. Since $p, q \in A$ and A is path connected, we have $p \sim q$ and hence $P = [p] = [q] = Q$ since the path components cover X and A intersects with a unique path component P, we have $A \subseteq P$. □

Note

In a topological space X, since each connected subspace of X is contained in a unique connected component of X, and since each path component of X is path connected, hence connected, it follows that each connected component of X is a (disjoint) union of some of the path components of X.

Example 3.2.2

Let $A = \left\{\left(x, \sin\frac{1}{x}\right) \mid o < x \leq \frac{1}{\pi}\right\}$. Let $B = \{(0, y) \mid -1 \leq y \leq 1\}$. Let $X = A \cup B$. We see that $\overline{A} = A \cup B = X$. Note that A is path connected because it is the image of the convex set $\left(0, \frac{1}{\pi}\right]$ under the continuous map $g : \left(0, \frac{1}{p}\right) \to \mathbb{R}^2$ given by $g(x) = \left(s, \sin\frac{1}{x}\right)$. Also B is convex hence path connected. Note that X is connected since it is the closure of a connected set A. We claim that X is not path connected, indeed there is no path in X from a point in A to a point in B. Since A and B are path connected with $\left(\frac{1}{\pi}, 0\right) \in A$ and $(0, 0) \in B$, it suffices to show that there is no path in $X = A \cup B$ from $\left(\frac{1}{\pi}, 0\right)$ to $(0, 0)$. Suppose for a contradiction that there is such a path $\alpha : [0, 1] \to A \cup B$ from $\left(\frac{1}{\pi}, 0\right)$ to $(0, 0)$ in $X = A \cup B$. Note that the map $\alpha : [0, 1] \to \mathbb{R}^2$ is continuous, say α is given by $\alpha(t) = (x(t), y(t))$ where $x, y : [0, 1] \to \mathbb{R}$ are both continuous with $(x(t), y(t)) \in X = A \cup B$ for all $t \in [0, 1]$ and with $x(0) = \frac{1}{\pi}, x(1) = 0, y(0) = y(1) = 0$. Also recall that when $(x, y) \in X = A \cup B$ with $x > 0$ we have $(x, y) \in A$ so that $y = \sin\frac{1}{x}$. Since $x : [0, 1] \to \mathbb{R}$ is continuous with $x(0) = \frac{1}{\pi}$ and $x(1) = 0$. By IVT, we can choose $0 < t_1 < t_2 < \cdots < 1$ so that $x(t_n) = \frac{2}{(2n+1)\pi}$ and hence $y(t_n) = \sin\frac{1}{x(t_n)} = \sin\frac{(2n+1)\pi}{2} = (-1)^n$. Since $(t_n)_{n \geq 1}$ is increasing and bounded above (by 1) it converges with $\lim_{n\to\infty} t_n = s = \sup\{t_n \mid n \in \mathbb{N}\} \leq 1$ and we have $0 < t_n < s \leq 1$ for all $n \in \mathbb{N}$. Since $t_n \to s$ and since α is continuous at s, we have

$$(x(s), y(s)) = \alpha(s) = \lim_{n\to\infty} \alpha(t_n) = \left(\lim_{n\to\infty} x(t_n), \lim_{n\to\infty} y(t_n)\right)$$

so we have

$$\lim_{n\to\infty} x(t_n) = \lim_{n\to\infty} \frac{2}{(2n+1)\pi} = 0$$

but

$$\lim_{n\to\infty} y(t_n) = \lim_{n\to\infty} (-1)^n$$

which does not exist. In conclusion, $X = A \cup B$ is connected, but not path connected. Since X is connected, it only has one connected component, namely X. Since X is not path connected, it has at least 2 path components so, since A and B are path connected with $X = A \cup B$, A and B are the two path components of X.

3.3 Compactness

Definition 3.3.1

Let X be a topological space. For a set $\mathcal{S}$ of subsets of X, we say that $\mathcal{S}$ *covers* X or that $\mathcal{S}$ is *a cover* of X when $X = \bigcup \mathcal{S}$. When $\mathcal{S}$ is a cover of X, a *subcover* is a subset $\mathcal{R} \subseteq \mathcal{S}$ such that $X = \bigcup \mathcal{R}$. An *open cover* of X is a set of open sets which covers X. We say that X is *compact* when every open cover of X has a finite subcover.

Theorem 3.3.1

The image of a compact space under a continuous map is compact.

Proof: Let $f : X \to Y$ be a map between topological spaces. Suppose that X is compact and f is continuous. Note the map $f : X \to f(X)$ (by restricting codomain) is continuous. We claim that $f(X)$ is compact. Let $\mathcal{T}$ be an open cover of $f(X)$. Let $\mathcal{S} = \{f^{-1}(V) \mid V \in \mathcal{T}\}$. Then $\mathcal{S}$ is an open cover of X. Since X is compact, $\mathcal{S}$ has a finite subcover, $V_1, ..., V_n \in \mathcal{T}$ so that $X = \bigcup_{k=1}^{n} f^{-1}(V_k)$. Then $f(X) = \bigcup_{k=1}^{n} V_k$ so that $\{V_1, ..., V_n\}$ is a finite subcover of $\mathcal{T}$. Thus $f(X)$ is compact, as claimed. □

Theorem 3.3.2 **Heine-Borel**

For $A \subseteq \mathbb{R}^n$, A is compact iff A is closed and bounded.

Definition 3.3.2

Let X be a subspace of Y. For a set $\mathcal{T}$ of subsets of Y, we say $\mathcal{T}$ covers X in Y or $\mathcal{T}$ is a cover of X in Y, when $X \subseteq \bigcup \mathcal{T}$. When $\mathcal{T}$ is a cover of X in Y, a subcover of $\mathcal{T}$ (of X in Y) is a subset $\mathcal{R} \subseteq \mathcal{T}$ such that $X \subseteq \bigcup \mathcal{R}$. An open cover of X in Y is a set $\mathcal{T}$ of open sets in Y with $X \subseteq \bigcup \mathcal{T}$. We say that X is compact in Y when every open cover of X in Y has a finite subcover (of X in Y).

Theorem 3.3.3

Let X be a subspace of Y. Then X is compact (in itself) iff X is compact in Y.

Proof: Suppose X is compact (in X) let $\mathcal{T}$ be an open cover of X in Y. Let $\mathcal{S} = \{V \cap X \mid V \in \mathcal{T}\}$. Note that $\mathcal{S}$ is an open cover of X. Since X is compact in itself, we can choose $V_1, ..., V_n \in \mathcal{T}$ such that $X = \bigcup_{k=1}^{n} (V_k \cap X) = \bigcup_{k=1}^{n} V_k \cap X$. Then $X \subseteq \bigcup_{k=1}^{n} V_k$ so that $\{V_1, ..., V_n\}$ is a finite subcover of $\mathcal{T}$ (for X in Y). Suppose, conversely, that X is compact in Y. Let $\mathcal{S}$ be an open cover of X (in X). For each $U \in \mathcal{S}$ we can choose V_U open in Y such that $U = V_U \cap X$. Then $\mathcal{T} = \{V_U \mid U \in \mathcal{S}\}$ is an open cover of X in Y. Since X is compact in Y, we can choose $U_1, ..., U_n \in \mathcal{S}$ such that $X \subseteq \bigcup_{k=1}^{n} V_{U_k}$. Then

$$X = \bigcup_{k=1}^{n} V_{U_k} \cap X = \bigcup_{k=1}^{n} \left(V_{U_k} \cap X\right) = \bigcup_{k=1}^{n} U_k$$

so that $\{U_1, ..., U_n\}$ is a finite subcover of $\mathcal{S}$ (of X in X). □

Remark

When X is a subspace of a metric space Y (but not in general when X is a subspace of a topological space Y), we have an analogous result for the connectedness of X in Y: X is connected in Y when there do not exist open sets U, V in Y which separate X in Y, meaning that $U \cap X \neq \emptyset, V \cap X \neq \emptyset, U \cap V = \emptyset, X \subseteq U \cup V$. Verify that (when Y is a metric space) X is connected (in itself) iff X is connected in Y.

Theorem 3.3.4

Every closed subspace of a compact topological space is compact.

Proof: Let X be a subspace of Y. Suppose Y is compact (in Y) and that X is closed in Y. Let $\mathcal{S}$ be an open cover of X in Y. Since X is closed in Y, $X^c = Y \setminus X$ is open in Y. Note that $\mathcal{S} \cup \{X^c\}$ is an open cover of Y. Since Y is compact, we can choose a finite subcover of $\mathcal{S} \cup \{X^c\}$ so we can choose a finite subset $\mathcal{R} \subseteq \mathcal{S}$ such that $\mathcal{R} \cup \{X^c\}$ covers Y. Then $\mathcal{R}$ is a finite subcover of $\mathcal{S}$ (of X in Y). □

Theorem 3.3.5

Every compact subspace of a Hausdorff space is closed.

Proof: Let $X \subseteq Y$ be a subspace. Suppose that X is compact and Y is Hausdorff. We shall show $X^c = Y \setminus X$ is open in Y. Let $b \in X^c$. For each $a \in X$, since Y is Hausdorff we can choose disjoint open sets U_a and V_a in Y with $a \in U_a$ and $b \in V_a$. Note that $\mathcal{S} = \{U_a \mid a \in X\}$ is an open cover of X in Y. Since X is compact, we can choose $a_1, \ldots, a_n \in X$ such that $X \subseteq \bigcup_{k=1}^n U_{a_k}$. Let $U = \bigcup_{k=1}^n U_{a_k}$ and $V = \bigcap_{k=1}^n V_{a_k}$. Note that $X \subseteq U$, $b \in V$ and $U \cap V = \emptyset$. Since $X \subseteq U$ and $U \cap V = \emptyset$, we also have $X \cap V = \emptyset$ so that $V \subseteq X^c$. Hence X^c is open in Y, and X is closed. □