Contents

1 Groups

1.1 Notation

1. $\mathbb{N} = \{1, 2, ...\}$
2. $\mathbb{Z} = \{..., -1, 0, 1, ...\}$
3. $\mathbb{Q} = \left\{\frac{a}{b} : a \in \mathbb{Z}, b \in \mathbb{N}\right\}$
4. $\mathbb{R} =$ real numbers
5. $\mathbb{C} = \{a + bi : a, b \in \mathbb{R}, i^2 = -1\}$

For $n \in \mathbb{N}$, $\mathbb{Z}_n =$ integers modulo $n = \{[0], ..., [n-1]\}$ where
$[r] = \{z \in \mathbb{Z} : Z \equiv r \bmod n\}$
We note that the set $S = \mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{R}, \mathbb{C}, \mathbb{Z}_n$ has 2 operations $+, \cdot$.
For $n \in \mathbb{N}$, an $n \times n$ matrix over $\mathbb{R}$ (or $\mathbb{Q}$ or $\mathbb{C}$) is an $n \times n$ array

$$A = [a_{ij}] = \begin{bmatrix} a_{11} & \dots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \dots & a_{nn} \end{bmatrix}$$

with $a_{ij} \in \mathbb{R}$.
Note we can also do $+, \cdot$. For $A, B \in M_n(\mathbb{R})$

$$A + B := [a_{ij} + b_{ij}] \quad A \cdot B := \left[\sum_{k=1}^{n} a_{ik} b_{kj}\right]$$

1.2 Groups

Definition 1.2.1

Let G be a set and $* : G \times G \to G$. We say G is a *group* if the following are satisfied:

1. Associativity: if $a, b, c \in G$, then $a * (b * c) = (a * b) * c$
2. Identity: there is $e \in G$ such that $a * e = e * a = a$ for all $a \in G$
3. Inverses: for all $a \in G$, there is $a^{-1} \in G$ such that $a * a^{-1} = a^{-1} * a = e$

Definition 1.2.2

A group is called *abelian* if $a * b = b * a$ for all $a, b \in G$

Exercise 1.2.1

Prove in the definition of a group, 1-sided identity and inverses are enough to have 2-sided identity and inverses

Proposition 1.2.1 **previous exercise**

Suppose G is a set, $* : G \times G \to G$ is associative. Suppose there is $e \in G$ such that $e * a = a$ for all $a \in G$. Further suppose that for every $a \in G$, there is $a^{-1} \in G$ such that $a^{-1} * a = e$. Then for all $a \in G$,

1. $a * e = a$
2. $a * a^{-1} = e$

Proof of 1: Let $a \in G$, then

$$a^{-1} * a * e = e * e = e = a^{-1} * a$$

Multiplying on the left by ${a^{-1}}^{-1}$ gives

$$\begin{aligned} & {a^{-1}}^{-1} * a^{-1} * a * e = {a^{-1}}^{-1} * a^{-1} * a \\ \implies & e * a * e = e * a \\ \implies & a * e = a \end{aligned}$$

□

Proof of 2: Let $a \in G$, then

$$a^{-1} * a * a^{-1} = e * a^{-1} = a^{-1}$$

Again multiplying on the left by ${a^{-1}}^{-1}$ gives

$$a * a^{-1} = e$$

□

Proposition 1.2.2

Let G be a group, let $a \in G$. Then

1. The group identity is unique
2. The inverse of a is unique

Proof of 1: Suppose e_1, e_2 are both identities. Then

$$e_1 = e_1 * e_2 = e_2$$

□

Proof of 2: Suppose b_1, b_2 are inverses of a. Then

$$b_1 = b_1 * e = b_1 * (a * b_2) = (b_1 * a) * b_2 = e * b_2 = b_2$$

□

Example 1.2.1

$(\mathbb{Z}, +), (\mathbb{Q}, +), (\mathbb{R}, +), (\mathbb{C}, +)$ are all abelian groups

Example 1.2.2

$(\mathbb{Z}, \cdot), (\mathbb{Q}, \cdot), (\mathbb{R}, \cdot), (\mathbb{C}, \cdot)$ are not groups as 0 has no inverse

Example 1.2.3

but $(\mathbb{Q} \setminus \{0\}, \cdot), (\mathbb{R} \setminus \{0\}, \cdot), (\mathbb{C} \setminus \{0\}, \cdot)$ are abelian groups

Definition 1.2.3

For a set $(S, \cdot)$ let $S^* \subseteq S$ denote the set of all elements with inverses.

Exercise 1.2.2

what is $\mathbb{Z}_n^*$?

Example 1.2.4

$(M_n(\mathbb{R}), +)$ is an abelian group.

Example 1.2.5

Consider $\left(M_{n(\mathbb{R})}, \cdot\right)$ The identity matrix is $\begin{bmatrix} 1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 1 \end{bmatrix} \in M_n(\mathbb{R})$ However, since not all $M \in M_n(\mathbb{R})$ have multiplicative inverses, $(M_n(\mathbb{R}), \cdot)$ is not a group.

Notation

$\mathrm{GL}_n(\mathbb{R}) = \{M \in M_n(\mathbb{R}) : \det(M) \neq 0\}$

Note

If $A, B \in \mathrm{GL}_n(\mathbb{R})$, then $\det(AB) = \det(A)\det(B) \neq 0$ Thus $AB \in \mathrm{GL}_n(\mathbb{R})$. The associativity of $\mathrm{GL}_n(\mathbb{R})$ inherits from $M_n(\mathbb{R})$. Also the identity matrix satisfies $\det(I) = 1 \neq 0$ and thus $I \in \mathrm{GL}_n(\mathbb{R})$. Finally, for $M \in \mathrm{GL}_n(\mathbb{R})$, there exists $M^{-1} \in M_n(\mathbb{R})$ such that $MM^{-1} = I = M^{-1}M$ since $\det(M^{-1}) = \frac{1}{\det(M)} \neq 0$, we have $M^{-1} \in \mathrm{GL}_n(\mathbb{R})$. Thus $(\mathrm{GL}_n(\mathbb{R}), \cdot)$ is a group, called the *general linear group of degree n over $\mathbb{R}$*

Note

if $n \geq 2$, then $\mathrm{GL}_n(\mathbb{R})$ is not abelian.

Exercise 1.2.3

What is $(\mathrm{GL}_1(\mathbb{R}), \cdot)$?

Example 1.2.6

Let G, H be groups. The *direct product* is the set $G \times H$ with the component wise operation defined by

$$(g_1, h_1) * (g_2, h_2) = (g_1 *_G g_2, h_1 *_H h_2)$$

One can check that $G \times H$ is a group with identity (e_G, e_H) and the inverse of (g, h) is (g^{-1}, h^{-1})

Note

One can show by induction that if $G_1, \ldots, G_n$ are groups, then $G_1 \times \cdots \times G_n$ is also a group.

Notation

Given a group G and $g_1, g_2 \in G$, we often denote $g_1 * g_2$ by $g_1 g_2$ and its identity by 1. Also the unique inverse of an element $g \in G$ is denoted by g^{-1}. Also for $n \in \mathbb{N}$, we define $g^n = g * g * \cdots * g$ (n-times) and $g^{-n} = \left(g^{-1}\right)^n$. Finally, we denote $g^0 = 1$.

Proposition 1.2.3

Let G be a group and $g, h \in G$ we have

1. $g^{-1^{-1}} = g$
2. $(gh)^{-1} = h^{-1}g^{-1}$
3. $g^n g^m = g^{n+m}$ for all $n, m \in \mathbb{Z}$
4. $(g^n)^m = g^{nm}$ for all $n, m \in \mathbb{Z}$

Proof of 1: Since

$$g^{-1}g = 1 = gg^{-1}$$

so $g^{-1^{-1}} = g$ □

Proof of 2:

$$(gh)(h^{-1}g^{-1}) = g(hh^{-1})g^{-1} = g1g^{-1} = 1$$

Similarly,

$$(h^{-1}g^{-1})(gh) = 1$$

Thus $(gh)^{-1} = h^{-1}g^{-1}$ □

Proof of 3: We proceed by considering cases:

1. if $n = 0$ then
$$g^n g^m = g^0 g^m = 1g^m = g^m = g^{0+m} = g^{n+m}$$
2. if $n > 0$, we will proceed by induction on n. Case 1 establishes the base case. Let $m \in \mathbb{Z}$, $n \in \mathbb{Z}_{\geq 0}$. Suppose that $g^n g^m = g^{n+m}$ Then
$$g^{n+1}g^m = gg^n g^m = gg^{n+m} = g^{n+m+1}$$
3. if $n < 0$, then $n = -k$ for some $k \in \mathbb{N}$. We have
$$g^k g^n g^m = g^{k+n} g^m = g^0 g^m = g^m$$
also
$$g^k g^{n+m} = g^{k+m+n} = g^m$$
Thus
$$g^k g^n g^m = g^k g^{n+m}$$
So

$$g^n g^m = g^{n+m}$$

as desired.

□

Proof of 4: We proceed by considering cases:

1. if $m = 0$, then $(g^n)^m = (g^n)^0 = 1 = g^0 = g^{n0} = g^{nm}$
2. if $m > 0$, then

$$(g^n)^m = \underbrace{g^n g^n \cdots g^n}_{m \text{ times}} = g^{nm}$$

3. if $m < 0$, then $m = -k$ for some $k \in \mathbb{N}$. We will induct on k. For $k = 1$ we see that $(g^n)^{-1} = g^{-n}$ since

$$g^n g^{-n} = g^{n-n} = g^0 = 1$$

Suppose $(g^n)^{-\ell} = g^{-n\ell}$ for all $1 \leq \ell \leq k$ Then

$$(g^n)^{-k-1} = (g^n)^{-k}(g^n)^{-1} = g^{-nk} g^{-n} = g^{-nk-n} = g^{-n(k+1)}$$

□

Exercise 1.2.4

prove 3,4

Warning

In general, it is not the case that if $g, h \in G$ then $(gh)^n = g^n h^n$, this is not true unless G is abelian

Proposition 1.2.4

Let G be a group and $g, h, f \in G$ Then

1. They satisfy the left and right cancellation. More precisely,
   a. if $gh = gf$ then $h = f$
   b. if $hg = fg$ then $h = f$
2. Given $a, b \in G$ the equations $ax = b$ and $ya = b$ have unique solutions for $x, y \in G$

Proof of 1-a: By left-multiplying by g^{-1}, we have

$$gh = gf \iff g^{-1}gh = g^{-1}gf \iff h = f$$

□

Proof of 1-b: similar to 1-a □

Proof of 2: Let $x = a^{-1}b$ then

$$ax = aa^{-1}b = b$$

If u is another solution, then $au = b = ax$. By 1-a, $u = x$. Similarly, $y = ba^{-1}$ is the unique solution of $ya = b$ □

1.3 Symmetric Groups

Definition 1.3.1

Given a non-empty set L, *a permutation* of L is a bijection from L to L. The set of all permutations of L is denoted by S_L

Example 1.3.1

Consider the set $L = \{1, 2, 3\}$ which has the following different permutations

$$\begin{pmatrix}123\\123\end{pmatrix}, \begin{pmatrix}123\\132\end{pmatrix}, \begin{pmatrix}123\\213\end{pmatrix}, \begin{pmatrix}123\\231\end{pmatrix}, \begin{pmatrix}123\\312\end{pmatrix}, \begin{pmatrix}123\\321\end{pmatrix}$$

Where $\begin{pmatrix}123\\123\end{pmatrix}$ denotes the bijection

$$\sigma : \{1, 2, 3\} \longrightarrow \{1, 2, 3\}$$

$$\sigma(1) = 1, \sigma(2) = 2, \sigma(3) = 3$$

Notation

For $n \in \mathbb{N}$ we denote by $S_n = S_{\{1,2,\ldots,n\}}$ the set of all permutations of $\{1, 2, \ldots, n\}$. We have seen that the order of $S_3 = 3! = 6$. To consider the general S_n, we note that for a permutation $\sigma \in S_n$, there are n choices for $\sigma(1)$, $n - 1$ choices for $\sigma(2)$,..., 1 choice for $\sigma(n)$ Thus

Proposition 1.3.1

$|S_n| = n!$

Note

For Möbius quizzes, use “9 dots” for permutations.

Remark

Given $\sigma, \tau \in S_n$ we can compose them to get a new element $\sigma\tau$, where $\sigma\tau = \{1, 2, ..., n\} \to \{1, 2, ..., n\}$ given by $x \mapsto \sigma(\tau(x))$ Since both σ, τ are bijections, $\sigma\tau \in S_n$

Example 1.3.2

Compute $\sigma\tau$ and $\tau\sigma$ if

$$\sigma = \begin{pmatrix} 1234 \\ 3412 \end{pmatrix}, \quad \tau = \begin{pmatrix} 1234 \\ 2431 \end{pmatrix}$$

Then $\sigma\tau(1) = \sigma(2) = 4$,... Then $\sigma\tau = \left(\begin{smallmatrix} 1234 \\ 4213 \end{smallmatrix}\right)$, and $\tau\sigma = \left(\begin{smallmatrix} 1234 \\ 3124 \end{smallmatrix}\right)$
We note that $\sigma\tau \neq \tau\sigma$

Note

For any $\sigma, \tau \in S_n$ we have that $\tau\sigma, \sigma\tau \in S_n$ but $\sigma\tau \neq \tau\sigma$ in general on the other hand, for any σ, τ, μ we have $\sigma(\tau\mu) = (\sigma\tau)\mu$. Also note the *identity permutation* $\varepsilon \in S_n$ is defined as

$$\varepsilon = \begin{pmatrix} 12\cdots n \\ 12\cdots n \end{pmatrix}$$

Thus for any $\sigma \in S_n$, we have $\sigma\varepsilon = \varepsilon\sigma = \sigma$
Finally, for $\sigma \in S_n$, since it is a bijection, there is a unique bijection $\sigma^{-1} \in S_n$ called the *inverse permutation* of σ such that for all $x, y \in \{1, 2, ..., n\}$

$$\sigma^{-1}(x) = y \iff \sigma(y) = x$$

It follows that

$$\sigma(\sigma^{-1}(x)) = \sigma(y) = x$$

and

$$\sigma^{-1}(\sigma(y)) = y$$

i.e we have

$$\sigma\sigma^{-1} = \sigma^{-1}\sigma = \varepsilon$$

Example 1.3.3

$$\sigma = \begin{pmatrix} 12345 \\ 45123 \end{pmatrix}$$

Then

$$\sigma^{-1} = \begin{pmatrix} 12345 \\ 34512 \end{pmatrix}$$

From the above we have

Proposition 1.3.2

$(S_n, \circ)$ is a group, called the *symmetric group of degree* n

Exercise 1.3.1

Write down all rotations and reflections that fix an equilateral triangle. Then check why it is the "same" as S_3

Example 1.3.4

Consider

$$\sigma = \begin{pmatrix} 123456789(10) \\ 317694258(10) \end{pmatrix} \in S_{10}$$

We note that $1 \to 3 \to 7 \to 2 \to 1$ and $4 \to 6 \to 4$ and $5 \to 9 \to 8$ and $10 \to 10$ Thus σ can be *decomposed* into one 4-cycle (1372), one 2-cycle (46), and one 3-cycle (598) and one 1-cycle (10) (we usually do not write 1-cycles) Note that these cycles are *pairwise disjoint* and we have

$$\sigma = (1372)(46)(598)$$

We can also write $\sigma = (46)(598)(1372)$, or $\sigma = (64)(985)(7213)$

Theorem 1.3.3 **Cycle Decomposition**

If Given $\sigma \in S_n$ with $\sigma \neq \varepsilon$, then σ is a product of (one or more) disjoint cycles of length at least 2. This factorization is unique up to the order of the factors.

Proof: See bonus 1. □

Convention

Every permutation of S_n can be regarded as a permutation in S_{n+1} by fixing the number $n+1$, thus

$$S_1 \subseteq S_2 \subseteq \cdots \subseteq S_n \subseteq S_{n+1}$$

1.4 Cayley Tables

Definition 1.4.1

For a finite group G, defining its operation by means of a table is sometimes convenient. Given $x, y \in G$, the product xy is the entry of the table in the row corresponding to x and the column corresponding to y, such a table is a *Cayley table.*

Remark

By cancellation, the entries in each row or column of a Cayley table are all distinct

Example 1.4.1

Consider $(\mathbb{Z}_2, +)$ its Cayley table is

$\mathbb{Z}_2$	$[0]$	$[1]$
$[0]$	$[0]$	$[1]$
$[1]$	$[1]$	$[0]$

Example 1.4.2

Consider the group $\mathbb{Z}^* = \{1, -1\}$. Its Cayley table is

$\mathbb{Z}^*$	1	-1
1	1	-1
-1	-1	1

Note

If we replace 1 by $[0]$ and -1 by $[1]$ the Cayley tables of $\mathbb{Z}^*$ and $\mathbb{Z}_2$ become the same. In this case, we say $\mathbb{Z}^*$ and $\mathbb{Z}_2$ are *isomorphic* denoted by

$$\mathbb{Z}^* \cong \mathbb{Z}_2$$

Example 1.4.3

For $n \in \mathbb{N}$, the *cyclic group of order* n is defined by

$$C_n = \{1, a, a^2, ..., a^{n-1}\} \text{ with } a^n = 1 \text{ and } 1, a, ..., a^{n-1} \text{ are distinct}$$

The Cayley table of C_n is as follows

C_n	1	a	a^2	$\cdots$	a^{n-2}	a^{n-1}
1	1	a	a^2	$\cdots$	a^{n-2}	a^{n-1}
a	a	a^2	a^3	$\cdots$	a^{n-1}	1
a^2	a^2	a^3	a^4	$\cdots$	1	a
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$
a^{n-2}	a^{n-2}	a^{n-1}	1	$\cdots$	a^{n-4}	a^{n-3}
a^{n-1}	a^{n-1}	1	a	$\cdots$	a^{n-3}	a^{n-2}