Contents

1 Topological Spaces and Continuous Maps

1.1 Elementary Topology

Given an inner product on an $\mathbb{R}$-vector space $\langle \cdot, \cdot \rangle$, one can define a norm $\|x\| = \sqrt{\langle x, x \rangle}$. Given a norm, one can define a metric $d(x, y) = \|x - y\|$. Given a metric d on a set X, one can define open sets in X:

given $a \in X$ and $r > 0$, $B(a, r) := \{x \in X \mid d(x, a) < r\}$. Then for $A \subseteq X$, we say A is open in X when $\forall a \in A \exists r > 0$ such that $B(a, r) \subseteq A$. Equivalently, for all $a \in A$, there is $b \in X$, $r > 0$ such that $a \in B(b, r) \subseteq A$.

Remark

The set of open sets on a metric space is called the *metric topology* on X.

Open sets in a metric space satisfy the following:

1. $\emptyset$ and X are open
2. arbitrary unions of open sets are open
3. finite intersections of open sets are open

Notation

For a set of sets S, the union of S is

$$\bigcup S := \{x \mid \exists A \in S, x \in A\} = \bigcup_{A \in S} A$$

In the case that $S \neq \emptyset$, the intersection of S is

$$\bigcap S := \{x \mid \forall A \in S, x \in A\} = \bigcap_{A \in S} A$$

Note

$\bigcap S$ would contain all elements as the condition $\forall A \in \emptyset$ would be vacuously satisfied. If we are given a universal set X, and S is known to be a set of subsets of X, then $\bigcap \emptyset = X$.

Definition 1.1.1

Let X be a set. $\mathcal{T} \subseteq \mathcal{P}(X)$ is called a *topology* on X if

1. $\emptyset, X \in \mathcal{T}$
2. If $S \subseteq \mathcal{T}$ is nonempty, then $\bigcup S \in \mathcal{T}$
3. If $S \subseteq \mathcal{T}$ is nonempty and finite, then $\bigcap S \in \mathcal{T}$

The elements of $\mathcal{T}$ are called the open sets of X. The closed sets are the compliments of the open sets.

Remark

To show 3 holds, it suffices to show the intersection of 2 open sets is open (by induction)

Definition 1.1.2

If X is a set, and $\mathcal{T}$ is a topology on X, then $(X, \mathcal{T})$ is called a *topological* space

Remark

When $f : X \to Y$ is a map between metric spaces, f is continuous iff $f^{-1}(V)$ is open in X for every open set $V \subseteq Y$.

Definition 1.1.3

For a map $f : X \to Y$ between topological spaces, we say that f is continuous when $f^{-1}(V)$ is open in X for every open set $V \subseteq Y$.

Example 1.1.1

if $f : A \subseteq \mathbb{R}^n \longrightarrow B \subseteq \mathbb{R}^m$ is an elementary function, then f is continuous.

Definition 1.1.4

When S, T are topologies on X with $S \subseteq T$, we say that S is coarser than T and T is finer than S. When $S \subsetneq T$, we use strictly coarser/finer.

Example 1.1.2

$\{\emptyset, X\}$ is a topology on X called the *trivial topology*

Example 1.1.3

$\mathcal{P}(X)$ is a topology on X called the *discrete topology*

Example 1.1.4

When $X = \emptyset, \mathcal{T} \subseteq \mathcal{P}(X) \Rightarrow \mathcal{T} \subseteq \{\emptyset\} \Rightarrow \mathcal{T} = \emptyset \vee \mathcal{T} = \{\emptyset\}$. Thus the only topology on $\emptyset$ is $\{\emptyset\}$.

Example 1.1.5

When $X = \{a\}$ the only topology is $\mathcal{T} = \{\emptyset, \{a\}\}$

Exercise 1.1.1

Find all topologies on the 2 and 3 element sets.

Definition 1.1.5

Let X be a topological space. Let $A \subseteq X$.

1. The *interior* of A (in X) denoted by A° is the union of all open sets in X which are contained in A.
2. The *closure* of A denoted $\overline{A}$ is the intersection of all closed sets in X which contain A.
3. The *boundary* of A, denoted by ∂A, given by $\partial A = \overline{A} \setminus A^\circ$

Note

The set of closed sets in a topological space is closed under arbitrary intersections and under finite unions. In particular $\emptyset, X$ are closed

Theorem 1.1.1

Let X be a topological space, $A \subseteq X$.

1. A° is open, and is the largest open set which is contained in A
2. $\overline{A}$ is closed, and is the smallest closed set which contains A
3. A is open iff $A = A^\circ$
4. A is closed iff $A = \overline{A}$
5. $A^{\circ\circ} = A^\circ$
6. $\overline{\overline{A}} = \overline{A}$

Definition 1.1.6

Let X be a topological space, let $A \subseteq X$, let $a \in X$.

1. We say that a is an *interior point* of A when $a \in A$ and there is an open set U such that $a \in U \subseteq A$
2. We say that a is a *limit point* of A when for every open set $U \ni a$ we have $U \cap (A \setminus \{a\}) \neq \emptyset$. The set of limit points of A is denoted by A'
3. We say that a is a boundary point of A when every open set $U \ni a$, we have $U \cap A \neq \emptyset$ and $U \cap A^c \neq \emptyset$

Theorem 1.1.2

Let X be a topological space and let $A \subseteq X$.

1. A° is equal to the set of all interior points
2. For $a \in X$,
$$a \in A' \Longleftrightarrow a \in \overline{A \setminus \{a\}}$$
3. A is closed iff $A' \subseteq A$
4. $\overline{A} = A \cup A'$
5. $\overline{A}$ is the disjoint union
$$\overline{A} = A^\circ \sqcup \partial A$$
6. ∂A is equal to the set of boundary points of A

1.2 Topological Bases

Theorem 1.2.1

Let X be a set. Then the intersection of any set of topologies on X is also a topology on X.

Proof: Let $\{\mathcal{T}_\alpha\}_{\alpha \in I}$ be a collection of topologies on X. Let $\mathcal{T} = \bigcap_\alpha \mathcal{T}_\alpha$

1. Since $X, \emptyset \in \mathcal{T}_\alpha$ for all $\alpha \in I$. We have $X, \emptyset \in \mathcal{T}$
2. Let $\{U_i\} \subseteq \mathcal{T}$. For all $\alpha \in I$, we have each $U_i \in \mathcal{T}_\alpha$. Thus $\bigcup_i U_i \in \mathcal{T}_\alpha \Longrightarrow \bigcup_i U_i \in \mathcal{T}$ as desired.
3. Let $U_1, \ldots, U_n \in \mathcal{T}$. Then again for all $\alpha \in I$, we have each $U_i \in \mathcal{T}_\alpha$. Thus $\bigcap_{i=1}^n U_i \in \mathcal{T}_\alpha \Longrightarrow \bigcap_{i=1}^n U_i \in \mathcal{T}$

□

Corollary 1.2.2

When X is a set and $\mathcal{S}$ is any set of subsets of X (that is $S \subseteq \mathcal{P}(X)$), there is a unique smallest (coarsest) topology $\mathcal{T}$ on X which contains $\mathcal{S}$. Indeed $\mathcal{T}$ is the intersection of (the set of) all topologies on X containing $\mathcal{S}$.
This topology $\mathcal{T}$ is called the topology on X *generated by* $\mathcal{S}$

Definition 1.2.1

Let X be a set. A *basis of sets* on X is a set $\mathcal{B}$ of subsets of X (So $\mathcal{B} \subseteq \mathcal{P}(X)$) such that

1. $\mathcal{B}$ covers X, that is $\bigcup \mathcal{B} = X$
2. For every $C, D \in \mathcal{B}$ and $a \in C \cap D$. There is $B \in \mathcal{B}$ such that $a \in B \subseteq C \cap D$.

When $\mathcal{B}$ is a basis of sets in X and $\mathcal{T}$ is the topology on X generated by $\mathcal{B}$, we say that $\mathcal{B}$ is a *basis for* $\mathcal{T}$. The elements in $\mathcal{B}$ are called *basic open sets* in X.

Theorem 1.2.3 **Characterization of Open Sets in Terms of Basic Open Sets**

Let X be a topological space, Let $\mathcal{B}$ be a basis for the topology on X.

1. For $A \subseteq X$, A is open iff for every $a \in A$, there is $B \in \mathcal{B}$ such that $a \in B \subseteq A$ *
2. The open sets in X are the unions of (sets of) elements in $\mathcal{B}$

Equivalently,

1. $\mathcal{T} = \{A \subseteq X \mid \forall a \in A, \exists B \in \mathcal{B}\ a \in B \subseteq A\}$
2. $\mathcal{T} = \{\bigcup C \mid C \subseteq \mathcal{B}\}$

Proof: Let $\mathcal{T}$ be the topology on X (generated by $\mathcal{B}$). Let $\mathcal{S}$ be the set of all sets $A \subseteq X$ with property * ($\forall a \in A \exists B \in \mathcal{B} : a \in B \subseteq A$). And let $\mathcal{R}$ be the set of (arbitrary) unions of (sets of) elements in $\mathcal{B}$. Recall that $\mathcal{T}$ is the intersection of the set of all topologies on X which contain $\mathcal{B}$. Note that $\mathcal{S}$ contains $\mathcal{B}$ (obviously). Let us show that $\mathcal{S}$ is a topology on X. We have $\emptyset \in \mathcal{S}$ vacuously and $X \in \mathcal{S}$ because $\mathcal{B}$ covers X (given $a \in X$, we can choose $B \in \mathcal{B}$ with $a \in B$). When $U_k \in S$ for every $k \in K$ (where K is any index set). Let $a \in \cup_k U_k$. Choose $\ell \in K$ so that $a \in U_\ell$. Since $U_\ell \in \mathcal{S}$, we can choose $B \in \mathcal{B}$ so that $a \in B \subseteq U_\ell$. Since $U_\ell \subseteq \bigcup_k U_k$, we have $a \in B \subseteq \bigcup_k U_k$. Thus $\bigcup_k U_k$ satisfies *, hence $\bigcup_k U_k \in \mathcal{S}$ as required. Suppose $U, V \in \mathcal{S}$ Let $a \in U \cap V$. Since $U \in \mathcal{S}$ we can choose $C \in \mathcal{B}$ with $a \in C \subseteq U$. Since $V \in \mathcal{S}$, we can choose $D \in \mathcal{B}$ with $a \in D \subseteq V$. Since $\mathcal{B}$ is a basis, $C, D \in \mathcal{B}$ and $a \in C \cap D$, we can choose $B \in \mathcal{B}$ with $a \in B \subseteq C \cap D$. Then we have

$$a \in B \subseteq C \cap D \subseteq U \cap V$$

Thus $U \cap V$ satisfies * so that $U \cap V \in \mathcal{S}$ as required. Thus $\mathcal{S}$ is a topology on X containing $\mathcal{B}$, hence $\mathcal{T} \subseteq \mathcal{S}$. Let us show that $\mathcal{S} \subseteq \mathcal{R}$ let $U \in \mathcal{S}$. For each $a \in U$, choose $B_a \in \mathcal{B}$ with $a \in B_a \subseteq U$. Then we have

$$U = \bigcup_{a \in U} B_a \in \mathcal{R}$$

Thus $\mathcal{S} \subseteq \mathcal{R}$. Finally note that $\mathcal{R} \subseteq \mathcal{T}$ because if $U = \bigcup_k B_k$ with $B_k \in \mathcal{B}$, then each $B_k \in \mathcal{T}$, and $\mathcal{T}$ is a topology, so

$$U = \bigcup_{k \in K} B_k \in \mathcal{T}$$

□

Theorem 1.2.4 **Characterization of a Basis in terms of the Open Sets**

Let X be a topological space with topology $\mathcal{T}$. Let $\mathcal{B} \subseteq \mathcal{T}$. Then $\mathcal{B}$ is a basis for $\mathcal{T}$ iff
$\forall U \in \mathcal{T} \forall a \in U \exists B \in \mathcal{B} \quad a \in B \subseteq U$. *

Proof: If $\mathcal{B}$ is a basis for $\mathcal{T}$, then * holds by part 1 of the previous theorem. Suppose * holds. Let us show that $\mathcal{B}$ is a basis of sets in X. Note that $\mathcal{B}$ covers X since, taking $U = X$ in * we have
$\forall a \in X \exists B \in \mathcal{B} \quad a \in B \subseteq X$. Also note that given $C, D \in \mathcal{B}$ and $a \in C \cap D$, then by taking
$U = C \cap D$ in * (noting that $C, D \in \mathcal{B} \subseteq \mathcal{T}$ so that $U = C \cap D \in \mathcal{T}$) we can choose $B \in \mathcal{B}$ with
$a \in B \subseteq C \cap D$. Thus $\mathcal{B}$ is a basis of sets in X. It remains to show that $\mathcal{T}$ is the topology generated by $\mathcal{B}$. Let $\mathcal{S}$ be the topology generated by $\mathcal{B}$. By part 1 of the previous theorem, S is the set of all unions of

elements in $\mathcal{B}$. Also $\mathcal{S}$ is the smallest topology which contains $\mathcal{B}$. Since $\mathcal{B} \subseteq \mathcal{T}$ and $\mathcal{T}$ is a topology, we have $\mathcal{S} \subseteq \mathcal{T}$. Also we have $\mathcal{T} \subseteq \mathcal{S}$ because given $U \in \mathcal{T}$, by property *, for each $a \in U$, we can choose $B_a \in \mathcal{B}$ with $a \in B_a \subseteq U$, and then we have $U = \bigcup_{a \in U} B_a \in \mathcal{S}$ since it is a union of elements in $\mathcal{B}$ $\square$

Example 1.2.1

When X is a metric space, the set $\mathcal{B}$ of all open balls in X is a basis for the metric topology on X.

Remark

We can use a basis for testing various topological properties:
When X is a topological space, and $\mathcal{B}$ is a basis for the topology on X, and $A \subseteq X$ and $a \in X$. Then

$$\begin{aligned}
a \in A^\circ &\iff \exists B \in \mathcal{B} \text{ with } a \in B \subseteq A \\
a \in \overline{A} &\iff \forall B \in \mathcal{B} \text{ with } a \in B \quad B \cap A \neq \emptyset \\
a \in A' &\iff \forall B \in \mathcal{B} \text{ with } a \in B \quad (B \setminus \{a\}) \cap A \neq \emptyset \\
a \in \partial A &\iff \forall B \in \mathcal{B} \text{ with } a \in B \quad B \cap A \neq \emptyset \text{ and } B \cap (X \setminus A) \neq \emptyset
\end{aligned}$$

Definition 1.2.2

A topological space X is called *Hausdorff* when for all $a, b \in X$ with $a \neq b$, there exist disjoint open sets U and V in X with $a \in U$ and $b \in V$.

Example 1.2.2

Metric spaces are Hausdorff

1.3 Subspaces

Definition 1.3.1 **Subspace Topology**

Let Y be a topological space with topology $\mathcal{S}$, and $X \subseteq Y$ be a subset. Let

$$\mathcal{T} := \{V \cap X \mid V \in \mathcal{S}\}$$

Then $\mathcal{T}$ is a topology on X:
Indeed $\emptyset \in \mathcal{S}$ so $\emptyset \cap X = \emptyset \in \mathcal{T}$ and $Y \in \mathcal{S}$ so $Y \cap X = X \in \mathcal{T}$. If K is any index set and $U_k \in \mathcal{T}$ for each $k \in K$, then for each $k \in K$ we can choose $V_k \in \mathcal{S}$ such that $U_k = V_k \cap X$ and then we have

$$\begin{aligned} \bigcup_{k \in K} U_k &= \bigcup_{k \in K} (V_k \cap X) \\ &= \left(\bigcup_{k \in K} V_k \right) \cap X \in \mathcal{T} \end{aligned}$$

since $\bigcup_{k \in K} V_k \in \mathcal{S}$. Similarly, when K is finite and $U_k \in \mathcal{T}$ for each $k \in K$ we have $\bigcap_{k \in K} U_k \in \mathcal{T}$ The topology $\mathcal{T}$ on X is called the *subspace topology* on X (inherited from the topology on Y).

Theorem 1.3.1

Let Y be a topological space, let $\mathcal{C}$ be a basis for the topology on Y. Let $X \subseteq Y$ be a subset. Then the set

$$\mathcal{B} = \{C \cap X \mid C \in \mathcal{C}\}$$

is a basis for the subspace topology on X.

Proof: Exercise $\square$

Theorem 1.3.2

Let Z be a topological space, let $Y \subseteq Z$ be a subspace and $X \subseteq Y$ be a subset. Then the subspace topology on X inherited from Y is equal to the subspace topology on X inherited from Z.

Proof: Exercise $\square$

Theorem 1.3.3

Let Y be a metric space, (using the metric topology) and let $X \subseteq Y$. Then the subspace topology on X (inherited from the topology on Y) is equal to the metric topology on X using the metric on X obtained by restricting the metric on Y.

Proof: Exercise $\square$

1.4 Continuous Maps

Definition 1.4.1

Let X, Y be topological spaces.

1. For $f : X \to Y$ and $a \in X$, we say that f is *continuous at* a when for every open set $V \subseteq Y$ with $f(a) \in V$, there exists an open set $U \subseteq X$ with $a \in U \subseteq f^{-1}(V)$.
2. We say that f is *continuous* (in or on X) when for every open set $V \subseteq Y$, $f^{-1}(V)$ is open in X.
3. A *homeomorphism* from X to Y is is a bijective map $f : X \to Y$ such that both f and its inverse $f^{-1} : Y \to X$ are continuous. We say that X and Y are *homeomorphic*, and we write $X \cong Y$, when there exists a homeomorphism $f : X \to Y$. (and we remark that $f^{-1} : Y \to X$ is also a homeomorphism).

Theorem 1.4.1

Constant maps and inclusion maps are continuous.

Proof: For $f : X \to Y$ given by $f(x) = c \in Y$ for all $x \in X$. When V is open in Y,

$$f^{-1}(V) = \begin{cases} X \text{ if } c \in V \\ \emptyset \text{ if } c \notin V \end{cases}$$

When $X \subseteq Y$ is a subspace and $f : X \to Y$ is given by $f(x) = x$ for all $x \in X$, when V is open in Y.

$$\begin{aligned} f^{-1}(V) &= \{x \in X \mid f(x) \in V\} \\ &= \{x \in X \mid x \in V\} \\ &= V \cap X \end{aligned}$$

which is open in X. (when X uses the subspace topology) □

Remark

When Y is a topological space and $X \subseteq Y$ we shall assume, unless otherwise noted, that X uses the subspace topology.

Theorem 1.4.2 **Equivalent Definitions of Continuity**

Let $f : X \to Y$ be a map between topological spaces

1. f is continuous iff f is continuous at every $a \in X$
2. f is continuous iff for every closed set $K \subseteq Y$, $f^{-1}(K)$ is closed in X.
3. If $\mathcal{C}$ is a basis for the topology on Y then f is continuous iff for every $C \in \mathcal{C}$, $f^{-1}(C)$ is open in X.

Proof of 1: Suppose f is continuous on X. Let $a \in X$. Let V be an open set in Y with $f(a) \in V$. Let $U = f^{-1}(V)$, then $f^{-1}(V)$ is open, since f is continuous and $a \in U \subseteq f^{-1}(V)$. Suppose, conversely, that f is continuous at every $a \in X$. Let V be an open set in Y. For each $a \in f^{-1}(V)$ since f is continuous at a with $f(a) \in V$, we can choose an open set U_a in X with $a \in U_a \subseteq f^{-1}(V)$. Then

$$f^{-1}(V) = \bigcup_{a \in f^{-1}(V)} U_a$$

which is open in X, since it is a union in open sets in X. □

Theorem 1.4.3

Let $f : X \to Y$, $g : Y \to Z$ be continuous maps between topological spaces, then the composite map $h = g \circ f : X \to Z$ is continuous.

Proof: Show that $h^{-1}(W) = f^{-1}(g^{-1}(W))$ □

Remark

Homeomorphism of topological spaces behaves like an equivalence relation on the class of all topological spaces. For topological spaces X, Y, Z

1. $X \cong X$ (since id_X is a homeomorphism – a special case of the inclusion map)
2. If $X \cong Y$ then $Y \cong X$ (when $f : X \to Y$ is a homeomorphism, so is $f^{-1} : Y \to X$)
3. If $X \cong Y \cong Z$ then $X \cong Z$ (if $f : X \to Y, g : Y \to Z$ are homeomorphisms then so is $g \circ f$)

Theorem 1.4.4 **Restriction of Domain and Restriction or Expansion of Codomain**

Let X, Y, Z be topological spaces. Suppose $f : X \to Y$ is continuous.

1. For any subspace $A \subseteq X$, the restriction $f|_A : A \to Y$ is continuous.
2. If $Y \subseteq Z$ is a subspace then $f : Y \to Z$ is continuous and if $B \subseteq Y$ with $f(X) \subseteq B$, then $f : X \to B$ is continuous.

Proof: Exercise □

Lemma 1.4.5 **Glueing/Pasting Lemma**

Let $f : X \to Y$ be a map between topological spaces

1. If $X = \bigcup_{k \in K} U_k$ where each U_k is open in X and if each restriction map $f|_{U_k} : U_k \to y$ is continuous (where U_k is using the subspace topology), then f is continuous.
2. If $X = C_1 \cup \cdots \cup C_n$ where each C_k is closed in X, and if each restriction $f|_{C_k} : C_k \to Y$ is continuous, then f is continuous.