# Contents

# 1 Topological Spaces and Continuous Maps

## 1.1 Elementary Topology

Given an inner product on an $\mathbb{R}$-vector space $\langle \cdot, \cdot \rangle$, one can define a norm $\|x\| = \sqrt{\langle x, x \rangle}$. Given a norm, one can define a metric $d(x, y) = \|x - y\|$. Given a metric $d$ on a set $X$, one can define open sets in $X$:

given $a \in X$ and $r > 0$, $B(a, r) := \{x \in X \mid d(x, a) < r\}$. Then for $A \subseteq X$, we say $A$ is open in $X$ when $\forall a \in A \exists r > 0$ such that $B(a, r) \subseteq A$. Equivalently, for all $a \in A$, there is $b \in X$, $r > 0$ such that $a \in B(b, r) \subseteq A$.

**Remark**

The set of open sets on a metric space is called the *metric topology* on $X$.

Open sets in a metric space satisfy the following:

1. $\emptyset$ and $X$ are open
2. arbitrary unions of open sets are open
3. finite intersections of open sets are open

**Notation**

For a set of sets $S$, the union of $S$ is

$$\bigcup S := \{x \mid \exists A \in S, x \in A\} = \bigcup_{A \in S} A$$

In the case that $S \neq \emptyset$, the intersection of $S$ is

$$\bigcap S := \{x \mid \forall A \in S, x \in A\} = \bigcap_{A \in S} A$$

**Note**

$\bigcap S$ would contain all elements as the condition $\forall A \in \emptyset$ would be vacuously satisfied. If we are given a universal set $X$, and $S$ is known to be a set of subsets of $X$, then $\bigcap \emptyset = X$.

**Definition 1.1.1**

Let $X$ be a set. $\mathcal{T} \subseteq \mathcal{P}(X)$ is called a *topology* on $X$ if

1. $\emptyset, X \in \mathcal{T}$
2. If $S \subseteq \mathcal{T}$ is nonempty, then $\bigcup S \in \mathcal{T}$
3. If $S \subseteq \mathcal{T}$ is nonempty and finite, then $\bigcap S \in \mathcal{T}$

The elements of $\mathcal{T}$ are called the open sets of $X$. The closed sets are the compliments of the open sets.

**Remark**

To show 3 holds, it suffices to show the intersection of 2 open sets is open (by induction)

**Definition 1.1.2**

If $X$ is a set, and $\mathcal{T}$ is a topology on $X$, then $(X, \mathcal{T})$ is called a *topological* space

**Remark**

When $f : X \to Y$ is a map between metric spaces, $f$ is continuous iff $f^{-1}(V)$ is open in $X$ for every open set $V \subseteq Y$.

**Definition 1.1.3**

For a map $f : X \to Y$ between topological spaces, we say that $f$ is continuous when $f^{-1}(V)$ is open in $X$ for every open set $V \subseteq Y$.

**Example 1.1.1**

if $f : A \subseteq \mathbb{R}^n \longrightarrow B \subseteq \mathbb{R}^m$ is an elementary function, then $f$ is continuous.

**Definition 1.1.4**

When $S, T$ are topologies on $X$ with $S \subseteq T$, we say that $S$ is coarser than $T$ and $T$ is finer than $S$. When $S \subsetneq T$, we use strictly coarser/finer.

**Example 1.1.2**

$\{\emptyset, X\}$ is a topology on $X$ called the *trivial topology*

**Example 1.1.3**

$\mathcal{P}(X)$ is a topology on $X$ called the *discrete topology*

**Example 1.1.4**

When $X = \emptyset, \mathcal{T} \subseteq \mathcal{P}(X) \Rightarrow \mathcal{T} \subseteq \{\emptyset\} \Rightarrow \mathcal{T} = \emptyset \vee \mathcal{T} = \{\emptyset\}$. Thus the only topology on $\emptyset$ is $\{\emptyset\}$.

**Example 1.1.5**

When $X = \{a\}$ the only topology is $\mathcal{T} = \{\emptyset, \{a\}\}$

**Exercise 1.1.1**

Find all topologies on the 2 and 3 element sets.

**Definition 1.1.5**

Let $X$ be a topological space. Let $A \subseteq X$.

1. The *interior* of $A$ (in $X$) denoted by $\operatorname{int}(A)$ is the union of all open sets in $X$ which are contained in $A$.
2. The *closure* of $A$ denoted $\overline{A}$ is the intersection of all closed sets in $X$ which contain $A$.
3. The *boundary* of $A$, denoted by $\partial A$, given by $\partial A = \overline{A} \setminus \operatorname{int}(A)$

**Note**

The set of closed sets in a topological space is closed under arbitrary intersections and under finite unions. In particular $\emptyset, X$ are closed

**Theorem 1.1.1**

Let $X$ be a topological space, $A \subseteq X$.

1. $\operatorname{int}(A)$ is open, and is the largest open set which is contained in $A$
2. $\overline{A}$ is closed, and is the smallest closed set which contains $A$
3. $A$ is open iff $A = \operatorname{int}(A)$
4. A is closed iff $A = \overline{A}$
5. $\operatorname{int}(\operatorname{int}(A)) = \operatorname{int}(A)$
6. $\overline{\overline{A}} = \overline{A}$

**Definition 1.1.6**

Let $X$ be a topological space, let $A \subseteq X$, let $a \in X$.

1. We say that $a$ is an *interior point* of $A$ when $a \in A$ and there is an open set $U$ such that $a \in U \subseteq A$
2. We say that $a$ is a *limit point* of $A$ when for every open set $U \ni a$ we have $U \cap (A \setminus \{a\}) \neq \emptyset$. The set of limit points of $A$ is denoted by $A'$
3. We say that $a$ is a boundary point of $A$ when every open set $U \ni a$, we have $U \cap A \neq \emptyset$ and $U \cap A^c \neq \emptyset$

**Theorem 1.1.2**

Let $X$ be a topological space and let $A \subseteq X$.
1. $\text{int}(A)$ is equal to the set of all interior points
2. For $a \in X$,
$$a \in A' \Longleftrightarrow a \in \overline{A \setminus \{a\}}$$
3. $A$ is closed iff $A' \subseteq A$
4. $\overline{A} = A \cup A'$
5. $\overline{A}$ is the disjoint union
$$\overline{A} = \text{int}(A) \sqcup \partial A$$
6. $\partial A$ is equal to the set of boundary points of $A$

## 1.2 Topological Bases

**Theorem 1.2.1**

Let $X$ be a set. Then the intersection of any set of topologies on $X$ is also a topology on $X$.

***Proof:*** Let $\{\mathcal{T}_\alpha\}_{\alpha \in I}$ be a collection of topologies on $X$. Let $\mathcal{T} = \bigcap_\alpha \mathcal{T}_\alpha$
1. Since $X, \emptyset \in \mathcal{T}_\alpha$ for all $\alpha \in I$. We have $X, \emptyset \in \mathcal{T}$
2. Let $\{U_i\} \subseteq \mathcal{T}$. For all $\alpha \in I$, we have each $U_i \in \mathcal{T}_\alpha$. Thus $\bigcup_i U_i \in \mathcal{T}_\alpha \Longrightarrow \bigcup_i U_i \in \mathcal{T}$ as desired.
3. Let $U_1, ..., U_n \in \mathcal{T}$. Then again for all $\alpha \in I$, we have each $U_i \in \mathcal{T}_\alpha$. Thus $\bigcap_{i=1}^n U_i \in \mathcal{T}_\alpha \Longrightarrow \bigcap_{i=1}^n U_i \in \mathcal{T}$

□

**Corollary 1.2.2**

When $X$ is a set and $\mathcal{S}$ is any set of subsets of $X$ (that is $S \subseteq \mathcal{P}(X)$), there is a unique smallest (coarsest) topology $\mathcal{T}$ on $X$ which contains $\mathcal{S}$. Indeed $\mathcal{T}$ is the intersection of (the set of) all topologies on $X$ containing $\mathcal{S}$.
This topology $\mathcal{T}$ is called the topology on $X$ *generated by* $\mathcal{S}$

**Definition 1.2.1**

Let $X$ be a set. A *basis of sets* on $X$ is a set $\mathcal{B}$ of subsets of $X$ (So $\mathcal{B} \subseteq \mathcal{P}(X)$) such that
1. $\mathcal{B}$ covers $X$, that is $\bigcup \mathcal{B} = X$
2. For every $C, D \in \mathcal{B}$ and $a \in C \cap D$. There is $B \in \mathcal{B}$ such that $a \in B \subseteq C \cap D$.

When $\mathcal{B}$ is a basis of sets in $X$ and $\mathcal{T}$ is the topology on $X$ generated by $\mathcal{B}$, we say that $\mathcal{B}$ is a *basis for* $\mathcal{T}$. The elements in $\mathcal{B}$ are called *basic open sets* in $X$.

**Theorem 1.2.3** **Characterization of Open Sets in Terms of Basic Open Sets**

Let $X$ be a topological space, Let $\mathcal{B}$ be a basis for the topology on $X$.

1. For $A \subseteq X$, $A$ is open iff for every $a \in A$, there is $B \in \mathcal{B}$ such that $a \in B \subseteq A$ *
2. The open sets in $X$ are the unions of (sets of) elements in $\mathcal{B}$

Equivalently,

1. $\mathcal{T} = \{A \subseteq X \mid \forall a \in A, \exists B \in \mathcal{B}\ a \in B \subseteq A\}$
2. $\mathcal{T} = \{\bigcup C \mid C \subseteq \mathcal{B}\}$

***Proof:*** Let $\mathcal{T}$ be the topology on $X$ (generated by $\mathcal{B}$). Let $\mathcal{S}$ be the set of all sets $A \subseteq X$ with property * ($\forall a \in A \exists B \in \mathcal{B} : a \in B \subseteq A$). And let $\mathcal{R}$ be the set of (arbitrary) unions of (sets of) elements in $\mathcal{B}$. Recall that $\mathcal{T}$ is the intersection of the set of all topologies on $X$ which contain $\mathcal{B}$. Note that $\mathcal{S}$ contains $\mathcal{B}$ (obviously). Let us show that $\mathcal{S}$ is a topology on $X$. We have $\emptyset \in \mathcal{S}$ vacuously and $X \in \mathcal{S}$ because $\mathcal{B}$ covers $X$ (given $a \in X$, we can choose $B \in \mathcal{B}$ with $a \in B$). When $U_k \in S$ for every $k \in K$ (where $K$ is any index set). Let $a \in \bigcup_k U_k$. Choose $\ell \in K$ so that $a \in U_\ell$. Since $U_\ell \in \mathcal{S}$, we can choose $B \in \mathcal{B}$ so that $a \in B \subseteq U_\ell$. Since $U_\ell \subseteq \bigcup_k U_k$, we have $a \in B \subseteq \bigcup_k U_k$. Thus $\bigcup_k U_k$ satisfies *, hence $\bigcup_k U_k \in \mathcal{S}$ as required. Suppose $U, V \in \mathcal{S}$ Let $a \in U \cap V$. Since $U \in \mathcal{S}$ we can choose $C \in \mathcal{B}$ with $a \in C \subseteq U$. Since $V \in \mathcal{S}$, we can choose $D \in \mathcal{B}$ with $a \in D \subseteq V$. Since $\mathcal{B}$ is a basis, $C, D \in \mathcal{B}$ and $a \in C \cap D$, we can choose $B \in \mathcal{B}$ with $a \in B \subseteq C \cap D$. Then we have

$$a \in B \subseteq C \cap D \subseteq U \cap V$$

Thus $U \cap V$ satisfies * so that $U \cap V \in \mathcal{S}$ as required. Thus $\mathcal{S}$ is a topology on $X$ containing $\mathcal{B}$, hence $\mathcal{T} \subseteq \mathcal{S}$. Let us show that $\mathcal{S} \subseteq \mathcal{R}$ let $U \in \mathcal{S}$. For each $a \in U$, choose $B_a \in \mathcal{B}$ with $a \in B_a \subseteq U$. Then we have

$$U = \bigcup_{a \in U} B_a \in \mathcal{R}$$

Thus $\mathcal{S} \subseteq \mathcal{R}$. Finally note that $\mathcal{R} \subseteq \mathcal{T}$ because if $U = \bigcup_k B_k$ with $B_k \in \mathcal{B}$, then each $B_k \in \mathcal{T}$, and $\mathcal{T}$ is a topology, so

$$U = \bigcup_{k \in K} B_k \in \mathcal{T}$$

□

**Theorem 1.2.4** **Characterization of a Basis in terms of the Open Sets**

Let $X$ be a topological space with topology $\mathcal{T}$. Let $\mathcal{B} \subseteq \mathcal{T}$. Then $\mathcal{B}$ is a basis for $\mathcal{T}$ iff $\forall U \in \mathcal{T} \forall a \in U \exists B \in \mathcal{B} \quad a \in B \subseteq U$. *

***Proof:*** If $\mathcal{B}$ is a basis for $\mathcal{T}$, then * holds by part 1 of the previous theorem. Suppose * holds. Let us show that $\mathcal{B}$ is a basis of sets in $X$. Note that $\mathcal{B}$ covers $X$ since, taking $U = X$ in * we have $\forall a \in X \exists B \in \mathcal{B} \quad a \in B \subseteq X$. Also note that given $C, D \in \mathcal{B}$ and $a \in C \cap D$, then by taking $U = C \cap D$ in * (noting that $C, D \in \mathcal{B} \subseteq \mathcal{T}$ so that $U = C \cap D \in \mathcal{T}$) we can choose $B \in \mathcal{B}$ with $a \in B \subseteq C \cap D$. Thus $\mathcal{B}$ is a basis of sets in $X$. It remains to show that $\mathcal{T}$ is the topology generated by $\mathcal{B}$. Let $\mathcal{S}$ be the topology generated by $\mathcal{B}$. By part 1 of the previous theorem, $S$ is the set of all unions of elements in $\mathcal{B}$. Also $\mathcal{S}$ is the smallest topology which contains $\mathcal{B}$. Since $\mathcal{B} \subseteq \mathcal{T}$ and $\mathcal{T}$ is a topology, we have $\mathcal{S} \subseteq \mathcal{T}$. Also we have $\mathcal{T} \subseteq \mathcal{S}$ because given $U \in \mathcal{T}$, by property *, for each $a \in U$, we can choose $B_a \in \mathcal{B}$ with $a \in B_a \subseteq U$, and then we have $U = \bigcup_{a \in U} B_a \in \mathcal{S}$ since it is a union of elements in $\mathcal{B}$ $\square$

**Example 1.2.1**

When $X$ is a metric space, the set $\mathcal{B}$ of all open balls in $X$ is a basis for the metric topology on $X$.

**Remark**

We can use a basis for testing various topological properties:
When $X$ is a topological space, and $\mathcal{B}$ is a basis for the topology on $X$, and $A \subseteq X$ and $a \in X$. Then

$$\begin{aligned} a \in \operatorname{int}(A) &\iff \exists B \in \mathcal{B} \text{ with } a \in B \subseteq A \\ a \in \overline{A} &\iff \forall B \in \mathcal{B} \text{ with } a \in B \quad B \cap A \neq \emptyset \\ a \in A' &\iff \forall B \in \mathcal{B} \text{ with } a \in B \quad (B \setminus \{a\}) \cap A \neq \emptyset \\ a \in \partial A &\iff \forall B \in \mathcal{B} \text{ with } a \in B \quad B \cap A \neq \emptyset \text{ and } B \cap (X \setminus A) \neq \emptyset \end{aligned}$$

**Definition 1.2.2**

A topological space $X$ is called *Hausdorff* when for all $a, b \in X$ with $a \neq b$, there exist disjoint open sets $U$ and $V$ in $X$ with $a \in U$ and $b \in V$.

**Example 1.2.2**

Metric spaces are Hausdorff

## 1.3 Subspaces

**Definition 1.3.1** **Subspace Topology**

Let $Y$ be a topological space with topology $\mathcal{S}$, and $X \subseteq Y$ be a subset. Let

$$\mathcal{T} := \{V \cap X \mid V \in \mathcal{S}\}$$

Then $\mathcal{T}$ is a topology on $X$:
Indeed $\emptyset \in \mathcal{S}$ so $\emptyset \cap X = \emptyset \in \mathcal{T}$ and $Y \in \mathcal{S}$ so $Y \cap X = X \in \mathcal{T}$. If $K$ is any index set and $U_k \in \mathcal{T}$ for each $k \in K$, then for each $k \in K$ we can choose $V_k \in \mathcal{S}$ such that $U_k = V_k \cap X$ and then we have

$$\begin{aligned}\bigcup_{k \in K} U_k &= \bigcup_{k \in K} (V_k \cap X) \\ &= \left(\bigcup_{k \in K} V_k\right) \cap X \in \mathcal{T}\end{aligned}$$

since $\bigcup_{k \in K} V_k \in \mathcal{S}$. Similarly, when $K$ is finite and $U_k \in \mathcal{T}$ for each $k \in K$ we have $\bigcap_{k \in K} U_k \in \mathcal{T}$ The topology $\mathcal{T}$ on $X$ is called the *subspace topology* on $X$ (inherited from the topology on $Y$).

**Theorem 1.3.1**

Let $Y$ be a topological space, let $\mathcal{C}$ be a basis for the topology on $Y$. Let $X \subseteq Y$ be a subset. Then the set

$$\mathcal{B} = \{C \cap X \mid C \in \mathcal{C}\}$$

is a basis for the subspace topology on $X$.

***Proof:*** Exercise □

**Theorem 1.3.2**

Let $Z$ be a topological space, let $Y \subseteq Z$ be a subspace and $X \subseteq Y$ be a subset. Then the subspace topology on $X$ inherited from $Y$ is equal to the subspace topology on $X$ inherited from $Z$.

***Proof:*** Exercise □

**Theorem 1.3.3**

Let $Y$ be a metric space, (using the metric topology) and let $X \subseteq Y$. Then the subspace topology on $X$ (inherited from the topology on $Y$) is equal to the metric topology on $X$ using the metric on $X$ obtained by restricting the metric on $Y$.

***Proof:*** Exercise □

## 1.4 Continuous Maps

**Definition 1.4.1**

Let $X, Y$ be topological spaces.

1. For $f : X \to Y$ and $a \in X$, we say that $f$ is *continuous at* $a$ when for every open set $V \subseteq Y$ with $f(a) \in V$, there exists an open set $U \subseteq X$ with $a \in U \subseteq f^{-1}(V)$.
2. We say that $f$ is *continuous* (in or on $X$) when for every open set $V \subseteq Y$, $f^{-1}(V)$ is open in $X$.
3. A *homeomorphism* from $X$ to $Y$ is is a bijective map $f : X \to Y$ such that both $f$ and its inverse $f^{-1} : Y \to X$ are continuous. We say that $X$ and $Y$ are *homeomorphic*, and we write $X \cong Y$, when there exists a homeomorphism $f : X \to Y$. (and we remark that $f^{-1} : Y \to X$ is also a homeomorphism).

**Theorem 1.4.1**

Constant maps and inclusion maps are continuous.

***Proof:*** For $f : X \to Y$ given by $f(x) = c \in Y$ for all $x \in X$. When $V$ is open in $Y$,

$$f^{-1}(V) = \begin{cases} X \text{ if } c \in V \\ \emptyset \text{ if } c \notin V \end{cases}$$

When $X \subseteq Y$ is a subspace and $f : X \to Y$ is given by $f(x) = x$ for all $x \in X$, when $V$ is open in $Y$.

$$\begin{aligned} f^{-1}(V) &= \{x \in X \mid f(x) \in V\} \\ &= \{x \in X \mid x \in V\} \\ &= V \cap X \end{aligned}$$

which is open in $X$. (when $X$ uses the subspace topology) □

**Remark**

When $Y$ is a topological space and $X \subseteq Y$ we shall assume, unless otherwise noted, that $X$ uses the subspace topology.

**Theorem 1.4.2** **Equivalent Definitions of Continuity**

Let $f : X \to Y$ be a map between topological spaces

1. $f$ is continuous iff $f$ is continuous at every $a \in X$
2. $f$ is continuous iff for every closed set $K \subseteq Y$, $f^{-1}(K)$ is closed in $X$.
3. If $\mathcal{C}$ is a basis for the topology on $Y$ then $f$ is continuous iff for every $C \in \mathcal{C}$, $f^{-1}(C)$ is open in $X$.

***Proof of 1:*** Suppose $f$ is continuous on $X$. Let $a \in X$. Let $V$ be an open set in $Y$ with $f(a) \in V$. Let $U = f^{-1}(V)$, then $f^{-1}(V)$ is open, since $f$ is continuous and $a \in U \subseteq f^{-1}(V)$. Suppose, conversely, that $f$ is continuous at every $a \in X$. Let $V$ be an open set in $Y$. For each $a \in f^{-1}(V)$ since $f$ is continuous at $a$ with $f(a) \in V$, we can choose an open set $U_a$ in $X$ with $a \in U_a \subseteq f^{-1}(V)$. Then

$$f^{-1}(V) = \bigcup_{a \in f^{-1}(V)} U_a$$

which is open in $X$, since it is a union in open sets in $X$. □

**Theorem 1.4.3**

Let $f : X \to Y, g : Y \to Z$ be continuous maps between topological spaces, then the composite map $h = g \circ f : X \to Z$ is continuous.

***Proof:*** Show that $h^{-1}(W) = f^{-1}(g^{-1}(W))$ □

**Remark**

Homeomorphism of topological spaces behaves like an equivalence relation on the class of all topological spaces. For topological spaces $X, Y, Z$

1. $X \cong X$ (since $\mathrm{id}_X$ is a homeomorphism – a special case of the inclusion map)
2. If $X \cong Y$ then $Y \cong X$ (when $f : X \to Y$ is a homeomorphism, so is $f^{-1} : Y \to X$)
3. If $X \cong Y \cong Z$ then $X \cong Z$ (if $f : X \to Y, g : Y \to Z$ are homeomorphisms then so is $g \circ f$)

**Theorem 1.4.4** **Restriction of Domain and Restriction or Expansion of Codomain**

Let $X, Y, Z$ be topological spaces. Suppose $f : X \to Y$ is continuous.

1. For any subspace $A \subseteq X$, the restriction $f|_A : A \to Y$ is continuous.
2. If $Y \subseteq Z$ is a subspace then $f : Y \to Z$ is continuous and if $B \subseteq Y$ with $f(X) \subseteq B$, then $f : X \to B$ is continuous.

***Proof:*** Exercise □

**Lemma 1.4.5** **Glueing/Pasting Lemma**

Let $f : X \to Y$ be a map between topological spaces

1. If $X = \bigcup_{k \in K} U_k$ where each $U_k$ is open in $X$ and if each restriction map $f|_{U_k} : U_k \to Y$ is continuous (where $U_k$ is using the subspace topology), then $f$ is continuous.
2. If $X = C_1 \cup \cdots \cup C_n$ where each $C_k$ is closed in $X$, and if each restriction $f|_{C_k} : C_k \to Y$ is continuous, then $f$ is continuous.

***Proof of 1:*** Suppose $X = \bigcup_{k \in K} U_k$ where each $U_k$ is open in $X$ and suppose each restriction $f|_{U_k}$ is continuous. Let $V \subseteq Y$ be open. Note that

$$\begin{aligned} f^{-1}(V) &= \{x \in X \mid f(x) \in V\} \\ &= \bigcup_{k \in K} \{x \in U_k \mid f(x) \in V\} \\ &= \bigcup_{k \in K} \left\{x \in U_k \,\middle|\, f|_{U_k}(x) \in V\right\} \\ &= \bigcup_{k \in K} f|_{U_k}^{-1}(V) \end{aligned}$$

For each $k \in K$, since $f|_{U_k}$ is continuous, we know that $f|_{U_k}^{-1}(V)$ is open in $U_k$. Since $U_k$ is using the subspace topology, we can choose an open $W_k$ in $X$ such that $f|_{U_k}^{-1}(V) = W_k \cap U_k$. This is open in $X$ since $W_k$ and $U_k$ are both open in $X$. Since $f^{-1}(V) = \bigcup_{k \in K} f|_{U_k}^{-1}(V)$ it is a union of open sets in $X$, so it is open in $X$. Thus $f$ is continuous. □

***Proof of 2:*** Exercise. First show that for $f : X \to Y$, $f$ is continuous iff $f^{-1}(C)$ is closed in $X$ for every closed set $C$ in $Y$. And, show that when $A \subseteq X \subseteq Y$, $A$ is closed in $X$ (using the subspace topology from $Y$) iff $A = B \cap X$ for some closed set $B$ in $Y$. □

**Example 1.4.1**

The map $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = \begin{cases} 2x & x \leq 0 \\ x^2 & x > 0 \end{cases}$ is continuous.

## 1.5 Examples of Homeomorphisms

**Example 1.5.1**

The circle

$$\{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1\}$$

in $\mathbb{R}^2$ is homeomorphic to the ellipse

$$\left\{(x, y) \in \mathbb{R}^2 \,\middle|\, \frac{(x-a)^2}{A^2} + \frac{(y-b)^2}{B^2} = 1\right\}$$

in $\mathbb{R}^2$

**Example 1.5.2**

$\mathbb{R} \cong (-1, 1) \subseteq \mathbb{R}$

**Example 1.5.3**

The standard unit $n$-sphere in $\mathbb{R}^{n+1}$ is the set

$$\mathbb{S}^n = \{x \in \mathbb{R}^{n+1} \mid \|x\| = 1\}$$

Where $p$ is the north pole

$$p = e_{n+1} = (0, ..., 0, 1) \in \mathbb{S}^n$$

We have $\mathbb{S}^n \setminus \{p\} \cong \mathbb{R}^n$

# 2 Examples of Topological Spaces

**Definition 2.0.1**

Let $X$ be a set. We sometimes write $X_t$ to indicate that $X$ is using the trivial topology $\mathcal{T}_t = \{\emptyset, X\}$. We sometimes write $X_d$ to indicate $X$ is using the discrete topology $\mathcal{T}_d = \mathcal{P}(X)$. We sometimes write $X_c$ to indicate $X$ is using the co-finite topology $\mathcal{T}_c = \{A \subseteq X \mid A = \emptyset \text{ or } X \setminus A \text{ is finite}\}$. Note the closed sets in $X_c$ are exactly the finite ones and $X$.

**Definition 2.0.2**

When $X$ is a metric space, we assume, unless otherwise indicated, that $X$ uses the metric topology. Sometimes, we might write $X_m$ to indicate that $X$ is using the metric topology $\mathcal{T}_m$.

**Definition 2.0.3**

When $Y$ is a topological space, and $X \subseteq Y$, we assume, unless otherwise indicated, that $X$ uses the subspace topology. Sometimes, we might write $X_s$ to indicate that $X$ is using the subspace topology $\mathcal{T}_s$. When $X \subseteq \mathbb{R}^n$, we shall assume, unless otherwise indicated, that $X$ is using $\mathcal{T}_m = \mathcal{T}_s$

**Definition 2.0.4**

Let $X$ be a set. A (strict, linear or total) *order* on $X$ is a binary relation $<$ on $X$ such that

1. For all $x, y \in X$ exactly one of the following holds:
   a. $x < y$
   b. $x = y$
   c. $y < x$
2. For all $x, y, z \in X$, if $x < y$ and $y < z$ then $x < z$

An *ordered set* is a set $X$ with an order $<$. When $X$ is an ordered set, we also define $\leq, >, \geq$ by stipulating that for all $x, y \in X$

$$x \leq y \iff (x < y \lor x = y)$$
$$x > y \iff y < x$$
$$x \geq y \iff y \leq x$$

**Remark**

In an ordered set $X$ we can define an *upper bound*, a *lower bound*, the *supremum*, the *infimum*, the *maximum*, and the *minimum* of a subset $A \subseteq X$.

**Example 2.0.1**

Let $X$ be an ordered set and $A \subseteq X$, $M = \max(A)$ when $M \in A$ with $M \geq x$ for all $x \in A$. Similarly, $m$ for minimum.

**Definition 2.0.5**

When $X$ is an ordered set, we have the following subsets which are called *intervals* in $X$. For $a, b \in X$ with $a < b$ we have

$$(a, b) := \{x \in X \mid a < x < b\}$$
$$(a, b] := \{x \in X \mid a < x \leq b\}$$
$$[a, b) := \{x \in X \mid a \leq x < b\}$$
$$[a, b] := \{x \in X \mid a \leq x \leq b\}$$

**Definition 2.0.6**

Let $X$ be an ordered set. The *order topology* on $X$ is the topology $\mathcal{T}_o$ which is generated by the basis $\mathcal{B}_o$ of sets in $X$ which consist of the following intervals:

- $(a, b)$ where $a, b \in X$, $a < b$
- $(a, M]$ where $M = \max X$ and $a \in X$ with $a \neq M$ (in the case that $X$ has a maximum)
- $[m, b)$ where $m = \min X$ and $b \in X$ with $b \neq m$ (in the case that $X$ has a minimum)

We sometimes write $X_o$ to indicate that $X$ is using the order topology $\mathcal{T}_o$

**Exercise 2.0.1**

Verify $\mathcal{B}_o$ is a basis.

**Example 2.0.2**

$\mathbb{R} = \mathbb{R}_o = \mathbb{R}_m$

**Definition 2.0.7**

Let $X$ be an ordered set the *lower limit topology* on $X$ is the topology $\mathcal{T}_\ell$ generated by the basis $\mathcal{B}_\ell$ which consists of intervals of the form $[a, b)$ where $a, b \in X$ with $a < b$ we sometimes write $X_\ell$ to indicate that $X$ is using the lower limit topology.

**Note**

on $\mathbb{R}$, $\mathcal{T}_\ell$ is not equal to $\mathcal{T}_m$. Note that when $a, b \in \mathbb{R}$ with $a < b$,

$$(a, b) = \bigcup_{n=m}^{\infty} \left[a + \frac{1}{n}, b\right) \text{ where } \frac{1}{m} < b - a$$

which is open in $\mathbb{R}_\ell$. So we have $\mathcal{T}_o \subseteq \mathcal{T}_\ell$

**Example 2.0.3**

Let $X = (0, 1) \cup \{2\} \subseteq \mathbb{R}$. Note that $\mathcal{T}_o \neq \mathcal{T}_m = \mathcal{T}_s$ on $X$. (Where $X$ uses the standard order inherited from $\mathbb{R}$). For example $\{2\}$ is open in $X_m$. But is not open in $X_o$ because any open set in $X_o$ which contains 2, must contain a basic open set $B$ with $2 \in B$. So it must contain a set of the form

$$B = (a, 2]_X = (a, 1) \cup \{2\} \text{ where } a \in (0, 1)$$

So they include elements other than 2

**Example 2.0.4**

When $X$ is an ordered set, the *dictionary* (or *lexicographic*) order on $X^2$ is given by

$$(a, b) < (c, d) \iff (a = c \text{ and } b < d) \text{ or } a < c$$

Note that on $\mathbb{R}^2$, the order topology $\mathcal{T}_o$ is not equal to the standard metric topology $\mathcal{T}_m$

## 2.1 Products of Topological Spaces

**Definition 2.1.1**

Let $X, Y$ be sets, then the Cartesian product of $X$ and $Y$ is

$$X \times Y = \{(x, y) \mid x \in X, y \in Y\}$$

**Definition 2.1.2**

Let $K$ be a non-empty index set and let $X_k$ be a set for each $k \in K$. Then the Cartesian product of the (indexed set of) sets $X_k$, $k \in K$

$$\prod_{k \in K} X_k = \left\{ x : K \to \bigcup_{k \in K} X_k \;\middle|\; x(k) \in X_k \text{ for all } k \in K \right\}$$

and we write $x(k)$ as $x_k$. In the case that $K = \{1, ..., n\}$ we write

$$\prod_{k \in K} X_k = \prod_{k=1}^{n} X_k = X_1 \times \cdots \times X_n$$

In the case that $K = \mathbb{Z}^+$ we write

$$\prod_{k \in K} X_k = \prod_{k=1}^{\infty} X_k = X_1 \times X_2 \times \cdots$$

In the case that $K = \{1, ..., n\}$ and $X_k = X$ for all $k \in K$, we also write

$$\prod_{k \in K} X_k = \prod_{k \in K} X = \underbrace{X \times X \times \cdots \times X}_{n \text{ times}} = X^n$$

In the case that $K = \mathbb{Z}^+$, and $X_k = X$ for all $k \in K$, we also write

$$\prod_{k \in K} X_k = \prod_{k=1}^{\infty} = X \times X \times \cdots = X^\omega$$

In the case that $X$ is a vector space, we write

$$X^\infty = \{x = (x_1, x_2 ...) \in X^\omega \mid x_k = 0 \text{ for all but finitely many } k \in \mathbb{Z}^+\}$$

In this case $X^\infty$ and $X^\omega$ are both vector spaces.
When $X_k$ is a set for each $k \in K$, for each $\ell \in K$ we have the projection map

$$p_\ell : \prod_{k \in K} X_k \to x_\ell$$

given by $p_\ell(x) = x_\ell = x(\ell)$. For any set $Y$, a function $f : Y \to \prod_{k \in K} X_k$ determines, and is determined by, its component functions

$$f_\ell : Y \to X_\ell$$

where $f_\ell = p_\ell \circ f$ so $f_\ell(y) = f(y)_\ell = f(y)(\ell)$

**Definition 2.1.3**

When $X_k$ is a topological space for each $k \in K$, there are two commonly used topologies on $\prod_{k \in K} X_k$.

1. The *box topology* on $\prod_{k \in K} X_k$ is the topology generated by the basis of sets of the form
$$\prod_{k \in K} U_k \subseteq \prod_{k \in K} X_k$$
Where each $U_k$ is open in $X_k$
2. The *product topology* on $\prod_{k \in K} X_k$ is the topology generated by the basis of sets consisting of the sets of the form $\prod_{k \in K} U_k$ where each $U_k$ is open in $X_k$ with $U_k = X_k$ for all but finitely many $k \in K$.

**Note**

The above two proposed bases are indeed bases of sets because
$$\left(\prod_{k \in K} U_k\right) \cap \left(\prod_{k \in K} V_k\right) = \prod_{k \in K} (U_k \cap V_k)$$
Also note that when $K$ is finite, these two topologies are equal. When $K$ is infinite, the box topology is finer than the product topology.

**Theorem 2.1.1**

Let $\mathcal{B}_k$ be a basis for $X_k$ for each $k \in K$. Then the set of sets of the form
$$\prod_{k \in K} B_k \text{ where } B_k \in \mathcal{B}_k \text{ for all } k \in K$$
is a basis for the box topology on $\prod_{k \in K} X_k$, and the set of sets of the form
$$\prod_{k \in K} B_k \text{ where } B_k \in \mathcal{B}_k \cup \{X_k\} \text{ for all } k \in K$$
with $B_k = X_k$ for all but finitely many $k \in K$ is a basis for the product topology on $\prod_{k \in K} X_k$.

***Proof:*** Exercise □

**Theorem 2.1.2**

For each $k \in K$, let $X_k$ be a subspace of $Y_k$ (using the subspace topology). Then the box topology on $\prod_{k \in K} X_k$ is equal to the subspace topology on $\prod_{k \in K} X_k$ as a subspace of $\prod_{k \in K} Y_k$ using the box topology, and the product topology on $\prod_{k \in K} X_k$ is equal to the subspace topology on $\prod_{k \in K} X_k$ as a subspace of $\prod_{k \in K} Y_k$ using the product topology.

**Theorem 2.1.3**

Let $Y$ be a topological space, and let $X_k$ be a topological space for each $k \in K$, and let $f : Y \to \prod_{k\in K} X_k$. Then when $\prod_{k\in K} X_k$ uses the product topology, $f$ is continuous if and only if each component map $f_\ell : Y \to X_\ell$ is continuous.

***Proof:*** Suppose that $f$ is continuous, then (using either the box or product topologies on $\prod_{k\in K} X_k$) each projection map $p_\ell : \prod_{k\in K} X_k \to X_\ell$ is continuous because when $U \subseteq X_\ell$ is open,

$$\begin{aligned} p_\ell^{-1}(U) &= \left\{ x \in \prod_{k\in K} X_k \;\middle|\; x_\ell = p_\ell(x) \in U \right\} \\ &= \prod_{k\in K} U_k \end{aligned}$$

where

$$U_k = \begin{cases} U & \text{if } k = \ell \\ X_k & \text{if } k \neq \ell \end{cases}$$

which is open in $\prod_{k\in K} X_k$ (using either the box or product topology) It follows that each component function $f_\ell$ is continuous because

$$f_\ell = p_\ell \circ f$$

Suppose, conversely, that each component map

$$f = p_\ell \circ f : Y \to \prod_{k\in K} X_k$$

is continuous, and that $\prod_{k\in K} X_k$ is using the product topology. To show that $f$ is continuous, it suffices to show that $f^{-1}(B)$ is open in $Y$ for every basic open set $B$ in $\prod_{k\in K} X_k$. Let $B$ be a basic open set (for the product topology) on $\prod_{k\in K} X_k$. Say $B = \prod_{k\in K} U_k$ where each $U_k$ is open in $X_k$ with $U_k = X_k$ for all but finitely many indices $k \in K$. Let $L \subseteq K$ be the finite set of all indices $k \in K$ for which $U_k \neq X_k$. We have

$$\begin{aligned} f^{-1}(B) &= \left\{ y \in Y \;\middle|\; f(y) \in \prod_{k\in K} U_k \right\} \\ &= \{ y \in Y \mid f_k(y) = f(y)_k \in U_k \text{ for all } k \in K \} \\ &= \{ y \in Y \mid f_k(y) \in U_k \text{ for all } k \in L \} \\ &= \bigcap_{k\in L} f_k^{-1}(U_k) \end{aligned}$$

Which is open in $Y$ since it is a finite intersection of open sets in $Y$ (with $f_k^{-1}(U_k)$) is open in $Y$ because $U_k$ is open in $X_k$ and $f_k : Y \to X_k$ is continuous. $\square$

**Remark**

$$\mathbb{R}^\infty \subseteq \ell_1 \subseteq \ell_p \subseteq \ell_q \subseteq \ell_\infty \subseteq \mathbb{R}^\omega$$

for $1 \leq p \leq q \leq \infty$. Recall that these norms induce different topologies.

Question: do any of the $p$-norms induce the box or product topology on $\mathbb{R}^\infty \subseteq \mathbb{R}^\omega$?
Question: is there a norm or metric on $\mathbb{R}^\omega$ which induces the box or product topology?

**Remark**

Also, we have the $p$-norms on $\mathbb{R}^n$. They all give the same topology on $\mathbb{R}^n$. More generally, when $X$ is a finite dimensional vector space, all norms on $X$ induce the same topology on $X$. When $L : X \to Y$ is a linear map between normed linear spaces, $L$ is continuous iff $\|L\|_{\text{op}} < \infty$ iff $L\left(\overline{B_X}(0,1)\right)$ is bounded in $Y$. And when $X$ is finite dimensional, $\overline{B_X}(0,1)$ is compact and $L\left(\overline{B_X}(0,1)\right)$ is bounded, so $L$ is continuous. In particular, when $X$ is finite dimensional and $\|\cdot\|_1, \|\cdot\|_2$ are two norms on $X$,

$$\mathrm{id}_X : (X, \|\cdot\|_1) \longrightarrow (X, \|\cdot\|_2)$$

is continuous, and it is equal to its own inverse which is continuous, so $\mathrm{id}_X$ is a homeomorphism, so for a set $U \subseteq X$, $U$ is open in $(X, \|\cdot\|_1)$ if and only if $U$ is open in $(X, \|\cdot\|_2)$. Consequently, every finite dimensional vector space $X$ has a *standard* topology. (Pick a basis $\{u_1, \ldots, u_n\}$, define

$$\left\langle \sum x_k u_k, \sum y_k u_k \right\rangle = \sum x_k y_k = x \cdot y$$

So the map $L : X \to \mathbb{R}^n$ given by

$$L\left(\sum x_k u_k\right) = \sum x_k e_k = x$$

is an inner product space isomorphism.) Then use the inner product to define a norm, a metric, and a topology. The resulting topology doesn't depend on the choice of basis.

## 2.2 Quotient Spaces

**Definition 2.2.1**

Let $X$ be a set. Let $\sim$ be an equivalence relation on $X$. For $a \in X$, the *equivalence class* of $a$ is

$$[a] = \{x \in X \mid a \sim x\}$$

Recall distinct equivalence classes are disjoint, and $X$ is the disjoint union of distinct equivalence classes. The set of all equivalence classes is denoted by $X/{\sim}$ , is called the quotient set of $X$ by $\sim$.

$$X/{\sim} = \{[a] \mid a \in X\}$$

The map $q : X \to X/{\sim}$ given by $x \mapsto [x]$ is called the quotient map.

**Definition 2.2.2**

When $X$ is a topological space, the *quotient topology* on $X/\sim$ is the topology obtained by stipulating that for $V \subseteq X/\sim$, $V$ is open in $X/\sim$ if and only if $q^{-1}(V)$ is open in $X$.

**Note**

When $V \subseteq X/\sim$ so $V$ is a set of equivalence classes.

$$\begin{aligned} q^{-1}(V) &= \{x \in X \mid q(x) \in V\} \\ &= \{x \in X \mid [x] \in V\} \\ &= \bigcup_{[x]\in V} [x] \\ &= \bigcup V \end{aligned}$$

**Remark**

For sets $X$ and $Y$,

1. When $Y$ is a topological space and $X \subseteq Y$ is a subset, the subspace topology is the coarsest topology on $X$ for which the inclusion map $i : X \to Y$ is continuous
$$i^{-1}(V) = \{x \in X \mid i(x) \in V\} = \{x \in X \mid x \in V\} = V \cap X$$
2. When $X$ and $Y$ are both topological spaces, the product topology on $X \times Y$ is the coarsest topology for which the two projection maps $p_X : X \times Y \to X$, $p_Y : X \times Y \to Y$ are both continuous
$$p_X^{-1}(U) = U \times Y \quad p_Y^{-1}(V) = V \times X$$
3. When $X$ is a topological space and $\sim$ an equivalence relation on $X$, the quotient topology on $X/\sim$ is the finest topology on $X/\sim$ for which the quotient map $q : X \to X/\sim$ is continuous

**Note**

Let $X$ be a set and $\sim$ an equivalence relation on $X$. Note that any function $g : X/\sim \to Y$ (where $Y$ is any set) determines and is determined by a function $f : X \to Y$ which is constant on equivalence classes (meaning that for $x_1, x_2 \in X$ if $x_1 \sim x_2$ then $f(x_1) = f(x_2)$) with $g$ given by $g([x]) = f(x)$ and with $f$ given by $f = g \circ q$. So $f(x) = g(q(x)) = g([x])$

**Theorem 2.2.1**

Let $X, Y$ be topological spaces. Let $\sim$ be an equivalence relation on $X$. Let $f : X/\sim \to Y$. Let $g : X \to Y$ be the map given by $g(x) = f([x])$, that is $g = f \circ q$. Then $f$ is continuous if and only if $g$ is continuous.

***Proof:*** If $f$ is continuous, then $g$ is continuous because $g = f \circ q$ which is the composite of two continuous maps. Suppose that $g$ is continuous. Let $V \subseteq Y$, be open. We need to show that $f^{-1}(V)$ is open in $X/\sim$ . By definition of the quotient topology

$$f^{-1}(V) \text{ is open in } X/\sim \iff q^{-1}(f^{-1}(V)) \text{ is open in } X$$

But

$$q^{-1}(f^{-1}(V)) = (f \circ q)^{-1}(V) = g^{-1}(V)$$

Which is open in $X$ since $g$ is continuous. □

**Definition 2.2.3**

For a group $G$ and a set $X$, a *group action* of $G$ on $X$ is a function $* : G \times X \to X$, where we write $*\,(a, x)$ as $a * x$ or $ax$, such that

1. When $e \in G$ is the identity element we have $e * x = x$ for all $x \in X$.
2. For all $a, b \in G$ and all $x \in X$, we have

$$a * (b * x) = \underbrace{(ab)}_{\text{group op}} * x$$

We say that $G$ *acts on* $X$ (by using the group action).

**Remark**

A group action of $G$ on $X$ determines and is determined by a group homomorphism $\rho : G \to \mathrm{Perm}(X)$ where $\rho(a)(x) = a * x$ (the homomorphism $\rho$ is called a *representation* of $G$)

**Remark**

Given an action of $G$ on $X$, we can define an equivalence relation on $X$ by

$$x \sim y \iff y = a * x \text{ for some } a \in G.$$

In this case, the equivalence class of $x$ is called the *orbit of* $x$ (we might write $[x]$ as $\mathrm{Orb}(x)$) and we write the quotient $X/\sim$ as $X/G$. So

$$\begin{aligned} X/G &= \{[x] \mid x \in X\} \\ &= \{\mathrm{Orb}(x) \mid x \in X\} \end{aligned}$$

**Example 2.2.1**

For $\mathbb{S}^1 = \{u \in \mathbb{R}^2 \mid \|u\| = 1\}$, we have $\mathbb{S}^1 \times \mathbb{R} \cong \mathbb{R}^2 \setminus \{0\}$. Define

$$\begin{aligned} f : \mathbb{S}^1 \times \mathbb{R} &\longrightarrow \mathbb{R}^2 \setminus \{0\} \\ (u, t) &\longmapsto e^t u \end{aligned}$$

and define

$$\begin{aligned} g : \mathbb{R}^2 \setminus \{0\} &\longrightarrow \mathbb{S}^1 \times \mathbb{R} \\ x &\longmapsto \left(\frac{x}{\|x\|}, \ln\|x\|\right) \end{aligned}$$

These maps are continuous (they are elementary functions) and they are inverses of each other.

**Example 2.2.2**

$\mathbb{S}^1$ acts on $\mathbb{R}^2 = \mathbb{C}$ by complex multiplication. For $a \in \mathbb{R}^2 = \mathbb{C}$,

$$\mathrm{Orb}(a) = [a] = \{ua \mid u \in \mathbb{S}^1\}$$

which is equal to the circle centered at 0 of radius $\|a\|$ (with $[0] = \{0\}$).
Show that $\mathbb{R}^2/\mathbb{S}^1 \cong [0, \infty) \subseteq \mathbb{R}$ we define

$$\begin{aligned} f : \mathbb{R}^2/\mathbb{S}^1 &\longrightarrow [0, \infty) \\ [x] &\longmapsto \|x\| \end{aligned}$$

and define

$$\begin{aligned} h : [0, \infty) &\longrightarrow \mathbb{R}^2/\mathbb{S}^1 \\ r &\longmapsto [r] = [(r, 0)] = \{re^{i\theta} \mid \theta \in \mathbb{R}\} \end{aligned}$$

Note that $f$ is continuous because for the map $g : \mathbb{R}^2 \to [0, \infty) \subseteq \mathbb{R}$ given by $g(x) = \|x\|$. We have $g = f \circ q$. Since $g$ is continuous, it follows that $f$ is continuous. Also $h$ is continuous because $h = q \circ i$ where $i : [0, \infty) \longrightarrow \mathbb{R}^2$ is the inclusion map $i(r) = (r, 0)$. Finally, note that $f$ and $h$ are inverses.

**Example 2.2.3**

$\mathbb{R}^+ = (0, \infty)$ acts on $\mathbb{R}^2$ be multiplication that is by $t * x = tx$. The orbits are for $o \neq x \in \mathbb{R}^2$, $[x] = \{tx \mid 0 < t \in \mathbb{R}\}$ which is the (open) ray from 0 through $x$ and $[0] = \{0\}$. Each of the rays $[x]$ for $0 \neq x \in \mathbb{R}^2$ intersects a unique point on $\mathbb{S}^1$. Which gives a fairly natural bijective map

$$f : \mathbb{R}^2/\mathbb{R}^+ \longrightarrow \mathbb{S}^1 \cup \{0\}$$
$$[x] \longmapsto \begin{cases} \frac{x}{\|x\|} & \text{if } 0 \neq x \in \mathbb{R}^2 \\ 0 & \text{if } x = 0 \in \mathbb{R}^2 \end{cases}$$

The inverse $g : \mathbb{S}^1 \cup \{0\} \to \mathbb{R}^2/\mathbb{R}^+$ is given by $u \mapsto [u]$. Note that $g$ is continuous ($g = q \circ i$ where $i$ is the inclusion map $i : \mathbb{S}^1 \cup \{0\} \to \mathbb{R}^2$). But $f$ is not continuous, for example the set $\{0\}$ is open in $\mathbb{S}^1 \cup \{0\}$ (it is an open ball) but $f^{-1}(\{0\}) = \{[0]\} \subseteq \mathbb{R}^2/\mathbb{R}^+$ and $q^{-1}(\{[0]\}) = \{0\}$ is not open in $\mathbb{R}^2$. In fact, $\mathbb{R}^2/\mathbb{R}^+ \ncong \mathbb{S}^1 \cup \{0\}$. One way to show this is to note that $\mathbb{S}^1 \cup \{0\}$ has a singleton which is open ($\{0\}$), but $\mathbb{R}^2/\mathbb{R}^+$ has no singleton which is open.

**Remark**

$\mathbb{R}^2/\mathbb{R}^+$ is not Hausdorff, so it is not metrizable (there is no metric we can define on $\mathbb{R}^2/\mathbb{R}^+$ for which that quotient topology is equal to the metric topology)

**Example 2.2.4**

$\mathbb{Z}$ acts by addition on $\mathbb{R}$ (by $n * x = x + n$). The orbits are the sets $[x] = \{x + n \mid n \in \mathbb{Z}\} = x + \mathbb{Z}$ Show that $\mathbb{R}/\mathbb{Z} \cong \mathbb{S}^1$. Define

$$\begin{aligned} f : \mathbb{R}/\mathbb{Z} &\longrightarrow \mathbb{S}^1 \\ [t] &\longmapsto e^{i2\pi t} \end{aligned}$$

(and note that when $[s] = [t]$ say $s = t + n$ where $n \in \mathbb{Z}$ we have

$$e^{i2\pi s} = e^{i2\pi(t+n)} = e^{i2\pi t}$$

) Note that $f$ is continuous because the map $f : \mathbb{R} \to \mathbb{S}^1$ given by $g(t) = e^{i2\pi t}$ is continuous with $g = f \circ q$. The inverse map

$$\begin{aligned} h : \mathbb{S}^1 &\longrightarrow \mathbb{R}/\mathbb{Z} \\ e^{i\theta} &\longmapsto \left[\frac{\theta}{2\pi}\right] \end{aligned}$$

To see that $h$ is continuous, we can express $h$ in Cartesian coordinates. We remark that there is an angle map

$$\begin{aligned} \theta : \mathbb{R}^2 \setminus \{0\} &\longrightarrow [0, 2\pi) \\ (x, y) &\longmapsto \begin{cases} \arccos \frac{x}{\sqrt{x^2+y^2}} & \text{if } y > 0 \text{ or } (y = 0 \text{ and } x \neq 0) \\ 2\pi - \arccos \frac{x}{\sqrt{x^2+y^2}} & \text{if } y < 0 \text{ or } (y = 0 \text{ and } x < 0) \end{cases} \end{aligned}$$

This map is not continuous along the positive $x$-axis. In Cartesian coordinates, $h : \mathbb{S}^1 \to \mathbb{R}/\mathbb{Z}$ is given by

$$h(x, y) = \begin{cases} \left[\frac{1}{2\pi} \arccos(x)\right] & \text{if } y \geq 0 \\ \left[1 - \frac{1}{2\pi} \arccos(x)\right] & \text{if } y \leq 0 \end{cases}$$

that is by

$$h(x, y) = \begin{cases} h_1(x, y) \text{ if } (x, y) \in A \\ h_2(x, y) \text{ if } (x, y) \in B \end{cases}$$

Where

$$A = \left\{(x, y) \in \mathbb{S}^1 \,\middle|\, y \geq 0\right\}$$
$$B = \left\{(x, y) \in \mathbb{S}^1 \,\middle|\, y \leq 0\right\}$$

and

$$h_1(x, y) = \frac{1}{2\pi} \arccos x$$
$$h_2(x, y) = 1 - \frac{1}{2\pi} \arccos x$$

# 3 Connected, Path-Connected and Compact Spaces

**Definition 3.0.1**

Let $X$ be a topological space. For subsets $A, B \subseteq X$, we say that $A$ and $B$ *separate* $X$ when $A \neq \emptyset$, $B \neq \emptyset$, $A \cap B = \emptyset$ and $A \cup B = X$. We say that $X$ is *disconnected* when there exist (nonempty disjoint) open sets $U, V \subseteq X$ which separate $X$. Otherwise, we say that $X$ is *connected*.

**Proposition 3.0.1**

$X$ is connected if and only if the only clopen sets are $X$ and $\emptyset$.

***Proof:*** If $X$ is disconnected, we can find open sets $U, V \subseteq X$ which separate $X$ then the sets $\emptyset, U, V, X$ are clopen. On the other hand, if $\emptyset \neq U \subsetneq X$ with both $U$ both open and closed in $X$, then $U$ and $V = X \setminus U$ are open sets in $X$ which separate $X$. □

**Exercise 3.0.1**

When $X$ is a metric space and $A \subseteq X$ is a subspace, then $A$ is connected if and only if there do not exist open sets $U, V$ in $X$ such that $U \cap A \neq \emptyset$, $V \cap A \neq \emptyset$, $U \cap V = \emptyset$ and $A \subseteq U \cup V$.

**Example 3.0.1**

The connected sets in $\mathbb{R}$ are the intervals (including $\emptyset, \{a\}, \mathbb{R}$)

**Example 3.0.2**

The (non-empty) connected subsets of $\mathbb{Q}$ are the singletons (by using the density of the irrationals)

**Theorem 3.0.2**

If $f : X \to Y$ is a continuous map between topological spaces, and if $X$ is connected, then $f(X)$ is connected.

***Proof:*** Suppose $X$ is connected and $f : X \to Y$ is continuous. By restricting the codomain, the map $f : X \to f(X)$ is also continuous. Suppose, for a contradiction that $f(X)$ is disconnected. Let $U, V$ be open sets in $f(X)$ which separate $f(X)$. Then $f^{-1}(U)$ and $f^{-1}(V)$ are open sets in $X$ which separate $X$, so that $X$ is disconnected, giving the desired contradiction. □

**Lemma 3.0.3**

Let $X$ be a subspace of $Y$. Suppose $Y$ is disconnected. Let $U, V$ be open sets in $Y$ that separate $Y$. If $X$ is connected, then $X \subseteq U$ or $X \subseteq V$.

***Proof:*** Suppose $X \nsubseteq U$ and $X \nsubseteq V$. Since $U \cup V = Y$, it follows that $X \cap U \neq \emptyset$ and $X \cap V \neq \emptyset$. And these two sets are open sets in $X$ which separate $X$. □

**Theorem 3.0.4**

Let $X = \bigcup_{k \in K} A_k$ where each subspace $A_k$ is connected. With $\bigcap_k A_k \neq \emptyset$. Then $X$ is connected.

***Proof:*** Suppose, for a contradiction, that $X$ is disconnected. Let $U, V$ be open sets in $X$ which separate $X$. Let $p \in \bigcap_{k \in K} A_k \subseteq X = U \cup V$. Either $p \in U$ or $p \in V$ (but not both) say $p \in U$. For each index $k$, since $A_k$ is connected either $A_k \subseteq U$ or $A_k \subseteq V$ and since $p \in A_k$, $p \notin V$, we must have $A_k \subseteq U$. Since $A_K \subseteq U$ for every $k \in K$, we have $X = \bigcup_{k \in K} A_k \subseteq U$. This is not possible since $U$ and $V$ separate $X$. □

**Theorem 3.0.5**

The product of two connected spaces is connected.

***Proof:*** Let $X$ and $Y$ be connected spaces. Suppose both $X$ and $Y$ are nonempty (since if either one was, $\emptyset$ is connected). Choose $a \in X$ and $b \in Y$ so $(a, b) \in X \times Y$. Since $X \times \{b\} \cong X$ and $X$ is connected, it follows that $X \times \{b\}$ is connected. For each $x \in X$, since $\{x\} \times Y \cong Y$ and $Y$ is connected, it follows that $\{x\} \times Y$ is connected. Since $X \times \{b\}$ and $\{x\} \times Y$ are connected and $(X \times \{b\}) \cap (\{x\} \times Y) \neq \emptyset$ (since $(x, b)$ is in both), it follows from the previous theorem that the set $A_x = (X \times \{b\}) \cup (\{x\} \times Y)$ is connected. Since each $A_x$ is connected and $\bigcap_{x \in X} A_x \neq \emptyset$ (indeed $(a, b)$ is in the intersection) it follows that $\bigcup_{x \in X} A_x = X \times Y$ is connected. □

**Lemma 3.0.6**

Let $X$ be a subspace of $Y$. Let $U, V$ be subsets of $X$ which separate $X$ (not necessarily open). Then $U$ is open in $X$ if and only if $U \cap \overline{V} = \emptyset$. Symmetrically, $V$ is open in $X$ if and only if $V \cap \overline{U} = \emptyset$ where $\overline{U} = \mathrm{Cl}_Y(U), \overline{V} = \mathrm{Cl}_Y(V)$

***Proof:***

$$\begin{aligned}
& U \text{ is open in } X \\
\Longrightarrow\ & V \text{ is closed in } X \\
\Longrightarrow\ & V = \mathrm{Cl}_X(V) = \bigcap\{K \mid K \subseteq X \text{ closed in } X \text{ with } V \subseteq K\}
\end{aligned}$$

□

**Theorem 3.0.7**

Let $X$ be a topological space, let $A, B$ be subspaces with $A \subseteq B \subseteq \overline{A}$. If $A$ is connected, then so is $B$. In particular, if $A$ is connected, then so is $\overline{A}$.

***Proof:*** Suppose $A$ is connected. Suppose for a contradiction that $B$ is not connected. Let $U, V \subseteq B$ be open sets in $B$ which separate $B$. Since $A$ is connected and $U, V$ are open sets in $B$, which separate $B$, by previous lemma, either $A \subseteq U$ or $A \subseteq V$. Say $A \subseteq U$. Since $A \subseteq U$ we have $\overline{A} \subseteq \overline{U}$ so that $B \subseteq \overline{A} \subseteq \overline{U}$. By the previous lemma, $V \cap \overline{U} = \emptyset$ hence $V \cap B = \emptyset$, but $V \subseteq B$ so $V = \emptyset$ which contradicts the fact that $U$ and $V$ separate $B$. □

**Theorem 3.0.8**

Let $X_k$ be a connected topological space for each $k \in K$. Then $\prod X_k$ is connected using the product topology.

***Proof:*** If $X_k = \emptyset$ for some $k \in K$ then $\prod X_k = \emptyset$ (which is connected). Suppose that $X_k \neq \emptyset$ for all $k \in K$. For each $k \in K$, choose $a_k \in X_k$. Let $a \in \prod X_k$ be given by $a(k) = a_k$ for all $k \in K$. Let $\mathcal{F}$ be the set of all finite subsets of $K$. For each $J \in \mathcal{F}$, let $Y_J = \{y \in \prod X_k \mid y_k = a_k \ \forall k \notin J\} \subseteq \prod X_k$. We claim that $Y_J \cong \prod_{j \in J} X_j$ (using the product topology). There is a fairly natural map

$$f : Y_j \to \prod_{j \in J} X_j$$

given by

$$f(y)(j) = y_j$$

with inverse $\square$