

Real Analysis I

Jake Thakur

Section 1.1: The Completeness Axiom

Definition

$S \subseteq \mathbb{R}$ is inductive if

- (i) $1 \in S$
- (ii) $x \in S \Rightarrow x + 1 \in S$

Definition

\mathbb{N} is the intersection of all inductive subsets of \mathbb{R}

Principle of Mathematical Induction

For each $n \in \mathbb{N}$ let $S(n)$ be some mathematical assertion. Suppose also that

- (i) $S(1)$ is true
- (ii) Whenever $S(n)$ is true, then $S(n + 1)$ is true

Then $S(n)$ is true $\forall n \in \mathbb{N}$

Definition

$$\mathbb{Z} = \mathbb{N} \cup \{0\} \cup \{-n | n \in \mathbb{N}\}$$

Fact

$m, n \in \mathbb{Z} \Rightarrow$

- (i) $m + n \in \mathbb{Z}$
- (ii) $m - n \in \mathbb{Z}$
- (iii) $mn \in \mathbb{Z}$

Definition

$$\mathbb{Q} = \{\frac{m}{n} | m, n \in \mathbb{Z}, n \neq 0\}$$

Fact

- (i) Each $x \in \mathbb{Q} \Rightarrow x = \frac{m}{n}, m$ or n is odd
- (ii) n^2 is even $\Rightarrow n$ is even

Proposition 1.2

$$\exists \text{ No } x \in \mathbb{Q} \ni x^2 = 2$$

Definition

$S \subset \mathbb{R}, S \neq \emptyset$ is Bounded Above if $\exists c \in \mathbb{R} \ni x \leq c \forall x \in S \Rightarrow c$ is an Upper Bound for S

Completeness Axiom

If $S \subset \mathbb{R}, S \neq \emptyset$, and S is Bounded Above, set $U_S = \{c \in \mathbb{R} | c \text{ is an upper bound for } S\}$

Then $\exists a \in U_S \ni a \leq c \forall c \in U_S$

$a = \sup S = \text{supremum of } S \text{ (least upper bound)}$

("Given a bounded, nonempty set S , and the set of all upper bounds of S , U_S , then there exists a least element in U_S that is the least upper bound for S (its supremum)")

Proposition 1.3

$$\text{If } c > 0, \text{ then } \exists! x > 0 \ni x^2 = c$$

Theorem 1.4

$S \subset \mathbb{R}, S \neq \emptyset$, and S is Bounded Below, set $L_S = \{b \in \mathbb{R} | b \text{ is a lower bound for } S\}$

Then $\exists d \in L_S \ni d \geq b \forall b \in L_S$

$d = \inf S = \text{infimum of } S \text{ (greatest lower bound)}$

("Given a bounded, nonempty set S , and the set of all lower bounds of S , L_S , then there exists a greatest element in L_S that is the greatest lower bound for S (its infimum)")

Section 1.2: The Distribution of \mathbb{Z} & \mathbb{Q}

Theorem 1.5 (Archimedian Property)

- (i) $c > 0 \Rightarrow \exists n \in \mathbb{N} \ni n > c$
- (ii) $\epsilon > 0 \Rightarrow \exists n \in \mathbb{N} \ni \frac{1}{n} < \epsilon$

Proposition 1.6

Let $n \in \mathbb{Z}$, then \exists No $k \in \mathbb{Z} \ni k \in (n, n+1)$

Proposition 1.7

Suppose $S \neq \emptyset$, $S \subset \mathbb{Z}$, and S is Bounded Above, then S has a Maximum $m \in S$
Note: $m \in S \Rightarrow m = \sup S$

Theorem 1.8

For any $c \in \mathbb{R} \exists! k \in \mathbb{Z} \ni k \in [c, c+1)$

Definition

$S \subset \mathbb{R}$ is Dense in \mathbb{R} if for any $I = (a, b)$, $a < b$, $S \cap I \neq \emptyset$

Theorem 1.9

\mathbb{Q} is Dense in \mathbb{R}

Corollary 1.10

$\mathbb{R} \setminus \mathbb{Q}$ is Dense in \mathbb{R}

Section 1.3: Inequalities and Identities

Definition

$$x \in \mathbb{R}, |x| = \begin{cases} x, & x \geq 0 \\ -x, & x < 0 \end{cases}$$

Fact 1

$$d > 0, |c| \leq d \Leftrightarrow -d \leq c \leq d$$

Fact 2

$$x \in \mathbb{R}, -|x| \leq x \leq |x|$$

Theorem 1.11 (Triangle Inequality)

If $a, b \in \mathbb{R}$, Then $|a + b| \leq |a| + |b|$

Proposition 1.12

$a, r \in \mathbb{R}, r > 0$, TFAE:

- (i) $|x - a| < r$
- (ii) $a - r < x < a + r$
- (iii) $x \in (a - r, a + r)$

Difference of Powers Formula

$$n \in \mathbb{N} \text{ and } a, b \in \mathbb{R}, \\ a^n - b^n = (a - b) \sum_{k=0}^{n-1} a^{n-1-k} b^k$$

Geometric Series Formula

$$r \in \mathbb{R}, r \neq 1, \text{ then } \frac{1-r^{n+1}}{1-r} = 1 + r + \cdots + r^n = \sum_{k=0}^n r^k$$

Definition

$$n! = \begin{cases} 1, & n = 0, 1 \\ n * (n-1) * \cdots * 2 * 1, & n > 1 \end{cases}$$

Definition

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}$$

Binomial Formula

$$n \in \mathbb{N} \text{ and } a, b \in \mathbb{R}, \\ (a+b)^n = \sum_{k=0}^n \binom{n}{k} a^{n-k} b^k$$

Section 2.1: Convergence of Sequences

Definition

A sequence of real numbers is a function $f : \mathbb{N} \rightarrow \mathbb{R}$
set $a_n = f(n)$, then characterize f by $\{a_1, a_2, \dots\} = \{a_n\}_{n=1}^{\infty} = \{a_n\}$

Definition

$\{a_n\}$ Converges to $a \in \mathbb{R}$ provided that
for each $\epsilon > 0 \exists N \in \mathbb{N} \ni |a_n - a| < \epsilon \forall n \geq N$

Proposition 2.6

$\{\frac{1}{n}\}$ converges to 0

Fact

$\{1 + \frac{1}{2} + \dots + \frac{1}{2^n}\}$ converges to 2

Fact

$a_n \rightarrow a, a_n \rightarrow b \Rightarrow a = b$
(limits are unique)

Fact

$\{(-1)^n\}$ does not converge

Lemma 2.9 (Comparison Lemma)

Suppose we have $\{a_n\}, \{b_n\}$ with $a_n \rightarrow a$. Then $b_n \rightarrow b$ if
 $\exists c \geq 0$ and $N_1 \in \mathbb{N} \ni |b_n - b| \leq c|a_n - a| \forall n \geq N_1$

Theorem 2.10 (Sum Property)

$$a_n \rightarrow a, b_n \rightarrow b \Rightarrow a_n + b_n \rightarrow a + b$$

Lemma 2.11

$$a_n \rightarrow a, \alpha \in \mathbb{R} \Rightarrow (\alpha)a_n \rightarrow (\alpha)a$$

Theorem 2.13 (Product property)

$$a_n \rightarrow a, b_n \rightarrow b \Rightarrow a_n b_n \rightarrow ab$$

Fact 1

$$a_n \rightarrow a \Rightarrow |a_n| \rightarrow |a|$$

Proposition 2.14

$$b_n \rightarrow b \neq 0 \Rightarrow \frac{1}{b_n} \rightarrow \frac{1}{b}$$

Theorem 2.15 (Quotient property)

$$a_n \rightarrow a, b_n \rightarrow b \neq 0 \Rightarrow \frac{a_n}{b_n} \rightarrow \frac{a}{b}$$

Proposition 2.16 (Linear property)

$$a_n \rightarrow a, b_n \rightarrow b, \alpha, \beta \in \mathbb{R} \Rightarrow (\alpha)a_n + (\beta)b_n \rightarrow (\alpha)a + (\beta)b$$

Fact 2

$$a_n = c \forall n \Rightarrow a_n \rightarrow c$$

Proposition 2.17

$$P: \mathbb{R} \rightarrow \mathbb{R}, a_n \rightarrow a \Rightarrow P(a_n) \rightarrow P(a)$$

Section 2.2: Sequences & Sets

Theorem 2.18

$\{a_n\}$ converges $\Rightarrow \{a_n\}$ is bounded

Proposition 2.19

S is dense in $\mathbb{R} \Leftrightarrow$ each $x \in \mathbb{R}$ is a limit of a sequence in S

Theorem 2.20 (Sequential Density of \mathbb{Q})

Every $x \in \mathbb{R}$ is the limit of a sequence of rational numbers

Lemma 2.21

$d_n \rightarrow d, d_n \geq 0 \Rightarrow d \geq 0$

Theorem 2.22

$\{c_n\} \subset [a, b], c_n \rightarrow c \Rightarrow c \in [a, b]$

Definition

$S \subset \mathbb{R}$ is closed if whenever $\{a_n\} \subset S$ and $a_n \rightarrow a$ then $a \in S$

Fact

$[a, b]$ is closed

Section 2.3: The Monotone Convergence Theorem

Definition

$\{a_n\}$ is monotonically increasing if $a_{n+1} \geq a_n$ for each n

Definition

$\{a_n\}$ is monotonically decreasing if $a_{n+1} \leq a_n$ for each n

Definition

$\{a_n\}$ is monotone if it is either monotonically increasing or decreasing

Theorem 2.25 (Monotone Convergence Theorem)

If $\{a_n\}$ is monotone, then

$\{a_n\}$ converges $\Leftrightarrow \{a_n\}$ is bounded

Note: if $\{a_n\}$ is monotonically increasing, $a_n \rightarrow \sup\{a_n\}$

Note: if $\{a_n\}$ is monotonically decreasing, $a_n \rightarrow \inf\{a_n\}$

Proposition 2.28

Let $c \in \mathbb{R}$, $|c| < 1 \Rightarrow \lim_{n \rightarrow \infty} c^n = 0$

Theorem 2.29 (Nested Interval Theorem)

Let $\{a_n\}$ and $\{b_n\}$ be such that $a_n < b_n$ and set $I_n = [a_n, b_n]$.

Assume that $I_{n+1} \subset I_n$ and that $\lim_{n \rightarrow \infty} [b_n - a_n] = 0$. Then $\exists! x \in \bigcap_{n=1}^{\infty} I_n$

Section 2.4: The Sequential Compactness Theorem

Definition

For a given $\{a_n\}$ let $\{n_k\}$ be a sequence of natural numbers that is strictly increasing. Then the sequence $\{b_k\}$ defined by $b_k = a_{n_k}$, with $k = 1, 2, \dots$ is called a subsequence of $\{a_n\}$, denoted $\{b_k\} = \{a_{n_k}\}_{k=1}^{\infty} = \{a_{n_k}\}$

Fact

Given a sequence $\{n_k\}_{k=1}^{\infty}$ of natural numbers that is strictly increasing, we have that $n_k \geq k$ for every $k \in \mathbb{N}$

Proposition 2.30

Let $\{a_n\}$ converge to a , i.e., $a_n \rightarrow a$

Then $\lim_{n \rightarrow \infty} a_{n_k} = a$ for every subsequence $\{a_{n_k}\}$ of $\{a_n\}$

Theorem 2.32

For every $\{a_n\}$ $\exists \{n_k\} \ni \{a_{n_k}\}$ is monotone

Theorem 2.33

Every bounded sequence has a convergent subsequence

Definition

$S \subseteq \mathbb{R}$ is sequentially compact if every sequence $\{a_n\} \subset S$ has a convergent subsequence whose limit is in S

Theorem 2.36 (Sequential Compactness Theorem)

$a, b \in \mathbb{R}$ with $a < b \Rightarrow [a, b]$ is sequentially compact

Section 3.1: Continuity

Definition

For $D \subseteq \mathbb{R}$, $f : D \rightarrow \mathbb{R}$ is continuous at $x_0 \in D$ if whenever $\{x_n\} \subset D$ and $x_n \rightarrow x_0$ then $f(x_n) \rightarrow f(x_0)$

($f : D \rightarrow \mathbb{R}$ is continuous if it is continuous $\forall x_0 \in D$)

Fact

$P : \mathbb{R} \rightarrow \mathbb{R} \Rightarrow x_0 \in \mathbb{R}, \{x_n\} \subset \mathbb{R}, x_n \rightarrow x_0 \Rightarrow P(x_n) \rightarrow P(x_0) \Rightarrow P$ is continuous

Theorem 3.4

Suppose $f : D \rightarrow \mathbb{R}, g : D \rightarrow \mathbb{R}$ are continuous at $x_0 \in D$, then

$f + g : D \rightarrow \mathbb{R}$ and $fg : D \rightarrow \mathbb{R}$ are continuous at $x_0 \in D$

and if $g(x) \neq 0 \forall x \in D$ then $\frac{f}{g} : D \rightarrow \mathbb{R}$ is continuous at $x_0 \in D$

Corollary 3.5

Let P and Q be polynomials, then $\frac{P}{Q} : D \rightarrow \mathbb{R}$ is continuous where $D = \{x \in \mathbb{R} \mid Q(x) \neq 0\}$

Theorem 3.6

$f : D \rightarrow \mathbb{R}, g : U \rightarrow \mathbb{R}, f(D) \subseteq U$ and suppose that f is continuous at $x_0 \in D$ and g is continuous at $f(x_0) \in U$ then $g \circ f$ is continuous at x_0 ; namely, $g \circ f : D \rightarrow \mathbb{R}$

Fact

$f(x) = \sqrt{x}$ is continuous on $D = [0, +\infty)$

Section 3.1 (Sup): Trigonometric Continuity

Fact 1

if $\theta_n \rightarrow 0$, then $\sin \theta_n \rightarrow 0$

Fact 2

if $\theta_n \rightarrow 0$, then $\cos \theta_n \rightarrow 1$

Fact

$\sin \theta$ is continuous,
 $\cos \theta$ is continuous,
 $\tan \theta$ is continuous at $\cos \theta \neq 0$ ($\theta \neq (2n+1) * \frac{\pi}{2}$),
 $\csc \theta$ is continuous at $\sin \theta \neq 0$ ($\theta \neq n\pi$),
 $\sec \theta$ is continuous at $\cos \theta \neq 0$ ($\theta \neq (2n+1) * \frac{\pi}{2}$),
 $\cot \theta$ is continuous at $\sin \theta \neq 0$ ($\theta \neq n\pi$)

Section 3.2: Extreme Value Theorem

Definition

For $f : D \rightarrow \mathbb{R}$ we define $f(D) = \{y \in \mathbb{R} \mid y = f(x) \text{ for some } x \in D\}$
 $f(D)$ is the image of f

Definition

$f : D \rightarrow \mathbb{R}$ attains a maximum (max value) if $\exists x_0 \in D \ni f(x) \leq f(x_0) \forall x \in D$
Such a point x_0 is a maximizer of f

$f : D \rightarrow \mathbb{R}$ attains a minimum (min value) if $\exists x'_0 \in D \ni f(x'_0) \leq f(x) \forall x \in D$
Such a point x'_0 is a minimizer of f

Lemma 3.10

If $f : [a, b] \rightarrow \mathbb{R}$ is continuous, then $f([a, b])$ is bounded above ($\exists m \ni f(x) \leq m \forall x \in [a, b]$)

Theorem 3.9 (Extreme Value Theorem)

If $f : [a, b] \rightarrow \mathbb{R}$ is continuous, then f attains both a max and a min

Fact

Let $S \subset [a, b]$, then $\inf S \in [a, b]$, and $\sup S \in [a, b]$

Section 3.3: Intermediate Value Theorem

Theorem 3.11 (Intermediate Value Theorem)

Suppose $f : [a, b] \rightarrow \mathbb{R}$ is continuous and let $c \in \mathbb{R}$ be any number strictly between $f(a)$ and $f(b)$; i.e., $f(a) < c < f(b)$ or $f(b) < c < f(a)$, then $\exists x_0 \in (a, b) \ni f(x_0) = c$

Fact

Suppose $f : D \rightarrow \mathbb{R}$ is continuous. If $\exists [a, b] \subset D \ni f(a) < 0$
and $f(b) > 0$ (or vice-versa), then $\exists x_0 \in (a, b) \ni f(x_0) = 0$
"A real, continuous function that is positive on one side and negative on the other contains a root"

Definition

$D \subseteq \mathbb{R}$ is convex if $u, v \in D, (u < v) \Rightarrow [u, v] \subset D$

Fact

If $D \subset \mathbb{R}$ is convex then D is an interval

Theorem 3.14

If I is an interval and $f : I \rightarrow \mathbb{R}$ is continuous then $f(I)$ is an interval

Section 3.4: Uniform Continuity

Definition

$f : D \rightarrow \mathbb{R}$ is uniformly continuous on D if whenever $\{u_n\}, \{v_n\} \subset D \ni u_n - v_n \rightarrow 0$, then $f(u_n) - f(v_n) \rightarrow 0$

Note: if $v_n = x_0 \forall n$, then $u_n - v_n \rightarrow 0 \Rightarrow u_n \rightarrow x_0$, so uniform continuity \Rightarrow continuity at each $x_0 \in D$

Fact

$f(x) = x$ is uniformly continuous but $f(x) = x^2$ is not

Theorem 3.17

$f : [a, b] \rightarrow \mathbb{R}$ is continuous $\Rightarrow f$ is uniformly continuous on $[a, b]$

Fact

If $f : D \rightarrow \mathbb{R}$ satisfies Lipschitz Continuity: $|f(u) - f(v)| \leq c|u - v|, \forall u, v \in D$ and for some $c \geq 0$, then f is uniformly continuous.

Section 3.5: The $\epsilon - \delta$ Criteria for Continuity

Definition

"The $\epsilon - \delta$ Criterion At a Point" - $f : D \rightarrow \mathbb{R}$ satisfies the $\epsilon - \delta$ criterion at a point $x_0 \in D$, if for each $\epsilon > 0 \exists \delta > 0 \ni$ for $x \in D$, $|x - x_0| < \delta \Rightarrow |f(x) - f(x_0)| < \epsilon$

Theorem 3.20

For $f : D \rightarrow \mathbb{R}$ and $x_0 \in D$, TFAE:

- (i) f is continuous at x_0
- (ii) The $\epsilon - \delta$ criterion at x_0 holds

Definition

"The $\epsilon - \delta$ Criterion On the Domain of a Function" - $f : D \rightarrow \mathbb{R}$ satisfies the $\epsilon - \delta$ criterion on D , if for each $\epsilon > 0 \exists \delta > 0 \ni u, v \in D$, $|u - v| < \delta \Rightarrow |f(u) - f(v)| < \epsilon$

Theorem 3.22

For $f : D \rightarrow \mathbb{R}$, TFAE:

- (i) $f : D \rightarrow \mathbb{R}$ is uniformly continuous
- (ii) $f : D \rightarrow \mathbb{R}$ satisfies the $\epsilon - \delta$ criterion on D

Section 3.6: Images and Inverses; Monotone Functions

Definition

- (i) $f : D \rightarrow \mathbb{R}$ is monotonically increasing if $u, v \in D$ and $u < v \Rightarrow f(u) \leq f(v)$
- (ii) $f : D \rightarrow \mathbb{R}$ is monotonically decreasing if $u, v \in D$ and $u < v \Rightarrow f(u) \geq f(v)$
- (iii) $f : D \rightarrow \mathbb{R}$ is monotone if it is monotonically increasing or decreasing

Theorem 3.23

Suppose $f : D \rightarrow \mathbb{R}$ is monotone. If $f(D)$ is an interval, then f is continuous

Corollary 3.25

Suppose $f : I \rightarrow \mathbb{R}$ is monotone, then f is continuous $\Leftrightarrow f(I)$ is an interval

Definition

- (i) $f : D \rightarrow \mathbb{R}$ is strictly increasing if $u, v \in D$ and $u < v \Rightarrow f(u) < f(v)$
- (ii) $f : D \rightarrow \mathbb{R}$ is strictly decreasing if $u, v \in D$ and $u < v \Rightarrow f(u) > f(v)$

Definition

$f : D \rightarrow \mathbb{R}$ is one-to-one (injective) if for each $y \in f(D)$ $\exists! x \in D \ni f(x) = y$

"No element in the image may have more than one element in the domain mapping to it"

Fact

If f is strictly increasing or decreasing, then f is one-to-one

Definition

Suppose $f : D \rightarrow \mathbb{R}$ is one-to-one. If $y \in f(D)$, let $x \in D \ni f(x) = y$
Define $f^{-1} : f(D) \rightarrow D$ by $f^{-1}(y) = x$, so f^{-1} is well-defined since x is unique

Note:

- (i) $f^{-1}(f(x)) = x$, where $x \in D$
- (ii) $f(f^{-1}(y)) = y$, where $y \in f(D)$

Theorem 3.29

$f : I \rightarrow \mathbb{R}$ is continuous and strictly increasing or decreasing \Rightarrow
 $f^{-1} : f(I) \rightarrow \mathbb{R}$ is continuous

Section 3.7: Limits

Definition

$h(x) = \frac{f(x)-f(x_0)}{x-x_0}$ gives the slope of the line at point x_0
and $h(x)$ is continuous on $[a, b] \setminus \{x_0\}$

Definition

$D \subset \mathbb{R}$, $x_0 \in \mathbb{R}$ is a limit point of D if $\exists \{x_n\} \subset D \setminus \{x_0\} \ni x_n \rightarrow x_0$

Definition

If $f : D \rightarrow \mathbb{R}$ and x_0 is a limit point of D , then we denote $\lim_{x \rightarrow x_0} f(x) = l$
If whenever $\{x_n\} \subset D \setminus \{x_0\}$ and $x_n \rightarrow x_0$ we have that $\lim_{n \rightarrow \infty} f(x_n) = l$
(x_0 may or may not be in D)

Example

$D = \mathbb{R} \setminus \{x_0\}$, $f(x) = x^2 \Rightarrow h(x) = \frac{x^2-(x_0)^2}{x-x_0}$ and suppose
 $\{x_n\} \subset D$, $x_n \rightarrow x_0 \Rightarrow h(x_n) = \frac{(x_n)^2-(x_0)^2}{x_n-x_0} = \frac{(x_n+x_0)(x_n-x_0)}{x_n-x_0} = x_n + x_0$
So $\lim_{n \rightarrow \infty} h(x_n) = \lim_{n \rightarrow \infty} (x_n + x_0) = x_0 + x_0 = 2x_0$

Theorem 3.36

Suppose $f : D \rightarrow \mathbb{R}$, $g : D \rightarrow \mathbb{R}$, and x_0 is a limit point of D , so that

$$\lim_{x \rightarrow x_0} f(x) = A, \quad \lim_{x \rightarrow x_0} g(x) = B \Rightarrow$$

- (i) $\lim_{x \rightarrow x_0} [f(x) + g(x)] = A + B$
- (ii) $\lim_{x \rightarrow x_0} [f(x)g(x)] = AB$
- (ii)(a) $\alpha \in \mathbb{R}, \quad \lim_{x \rightarrow x_0} [\alpha f(x)] = \alpha A$
- (iii) $B \neq 0, \quad g(x) \neq 0 \quad \forall x \in D, \quad \lim_{x \rightarrow x_0} \left[\frac{f(x)}{g(x)} \right] = \frac{A}{B}$

Theorem 3.37

$f : D \rightarrow \mathbb{R}$, $g : U \rightarrow \mathbb{R}$ and
 x_0 is a limit point of $D \ni \lim_{x \rightarrow x_0} f(x) = y_0$,
 y_0 is a limit point of $U \ni \lim_{y \rightarrow y_0} g(y) = e$,
and suppose that $f(D \setminus \{x_0\}) \subset U \setminus \{y_0\}$, then
 $\lim_{x \rightarrow x_0} (g \circ f)(x) = e$

Definition

$x_0 \in D$ is an isolated point if $\exists r > 0 \ni (x_0 - r, x_0 + r) \cap D = \{x_0\}$

Limits and Continuity Theorem

For $f : D \rightarrow \mathbb{R}$, $x_0 \in D$, then

f is continuous at $x_0 \Leftrightarrow x_0$ is an isolated point of D or $\lim_{x \rightarrow x_0} f(x) = f(x_0)$

So f is continuous at $x_0 \Leftrightarrow \lim_{x \rightarrow x_0} f(x) = f(x_0)$

Section 4.1: The Algebra of Derivatives

Definition

$x_0 \in \mathbb{R}, I \subset \mathbb{R} \ni I = (a, b)$ and $x_0 \in I \Rightarrow I$ is a neighborhood of x_0

Definition

$x_0 \in \mathbb{R}$ and I is a neighborhood of $x_0 \Rightarrow f : I \rightarrow \mathbb{R}$ is differentiable at x_0 IF $\lim_{x \rightarrow x_0} \frac{f(x) - f(x_0)}{x - x_0}$ exists.

We say $f'(x_0) := \lim_{x \rightarrow x_0} \frac{f(x) - f(x_0)}{x - x_0}$ and is called the derivative of f at x_0

Definition

If $f : I \rightarrow \mathbb{R}$ is differentiable at each $x_0 \in I$ then f is differentiable and $f' : I \rightarrow \mathbb{R}$ is the derivative of f

Definition

The line determined by $y = f(x_0) + f'(x_0)(x - x_0)$ is the tangent line to the graph of f at $(x_0, f(x_0))$

For $y_0 = f(x_0)$, $y - y_0 = f'(x_0)(x - x_0)$

Proposition 4.4

$n \in \mathbb{N}, f(x) = x^n \forall x \in I = \mathbb{R} \Rightarrow f : \mathbb{R} \rightarrow \mathbb{R}$ is differentiable and $f'(x) = nx^{n-1}$

Proposition 4.5

$x_0 \in \mathbb{R}, I$ is a neighborhood of $x_0, I = (a, b)$ and $f : I \rightarrow \mathbb{R}$ is differentiable at $x_0 \Rightarrow f$ is continuous at x_0

Theorem 4.6

$x_0 \in \mathbb{R}$, I is a neighborhood of x_0 , $f : I \rightarrow \mathbb{R}$ and $g : I \rightarrow \mathbb{R}$ is differentiable at x_0 , then

(i) $f + g : I \rightarrow \mathbb{R}$ is differentiable at x_0 and
 $(f + g)'(x_0) = f'(x_0) + g'(x_0)$

(ii) $fg : I \rightarrow \mathbb{R}$ is differentiable at x_0 and
 $(fg)'(x_0) = f(x_0)g'(x_0) + f'(x_0)g(x_0)$

(iii) If $g(x) \neq 0 \forall x \in I$ then $\frac{1}{g} : I \rightarrow \mathbb{R}$ is differentiable at x_0 and
 $(\frac{1}{g})'(x_0) = \frac{-g'(x_0)}{[g(x_0)]^2}$

(iv) If $g(x) \neq 0 \forall x \in I$ then $\frac{f}{g} : I \rightarrow \mathbb{R}$ is differentiable at x_0 and
 $(\frac{f}{g})'(x_0) = \frac{f'(x_0)g(x_0) - f(x_0)g'(x_0)}{[g(x_0)]^2}$

Proposition 4.7

$n \in \mathbb{Z}$, $D = \mathbb{R}$ if $n \geq 0$ and $D = \mathbb{R} \setminus \{0\}$ if $n < 0$, then for $f : D \rightarrow \mathbb{R}$ defined by $f(x) = x^n$, f is differentiable and $f'(x) = nx^{n-1}$

Corollary 4.8

$p, q : \mathbb{R} \rightarrow \mathbb{R}$ are polynomials, $D = \mathbb{R} \setminus \{x \mid q(x) = 0\}$, then
 $\frac{p}{q} : D \rightarrow \mathbb{R}$ is differentiable