

Real Analysis I

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Section 1.1: The Completeness Axiom

Definition

$S \subseteq \mathbb{R}$ is inductive if

- (i) $1 \in S$
- (ii) $x \in S \Rightarrow x + 1 \in S$

Definition

\mathbb{N} is the intersection of all inductive subsets of \mathbb{R}

Principle of Mathematical Induction

For each $n \in \mathbb{N}$ let $S(n)$ be some mathematical assertion. Suppose also that

- (i) $S(1)$ is true
- (ii) Whenever $S(n)$ is true, then $S(n + 1)$ is true

Then $S(n)$ is true $\forall n \in \mathbb{N}$

Definition

$$\mathbb{Z} = \mathbb{N} \cup \{0\} \cup \{-n | n \in \mathbb{N}\}$$

Fact

$m, n \in \mathbb{Z} \Rightarrow$

- (i) $m + n \in \mathbb{Z}$
- (ii) $m - n \in \mathbb{Z}$
- (iii) $mn \in \mathbb{Z}$

Definition

$$\mathbb{Q} = \left\{ \frac{m}{n} \mid m, n \in \mathbb{Z}, n \neq 0 \right\}$$

Fact

- (i) Each $x \in \mathbb{Q} \Rightarrow x = \frac{m}{n}, m$ or n is odd
- (ii) n^2 is even $\Rightarrow n$ is even

Proposition 1.2

$$\exists \text{ No } x \in \mathbb{Q} \ni x^2 = 2$$

Definition

$S \subset \mathbb{R}, S \neq \emptyset$ is Bounded Above if $\exists c \in \mathbb{R} \ni x \leq c \forall x \in S \Rightarrow c$ is an Upper Bound for S

Completeness Axiom

If $S \subset \mathbb{R}, S \neq \emptyset$, and S is Bounded Above, set $U_S = \{c \in \mathbb{R} \mid c \text{ is an upper bound for } S\}$

Then $\exists a \in U_S \ni a \leq c \forall c \in U_S$

$a = \sup S =$ supremum of S (least upper bound)

("Given a bounded, nonempty set S , and the set of all upper bounds of S , U_S , then there exists a least element in U_S that is the least upper bound for S (its supremum)")

Proposition 1.3

$$\text{If } c > 0, \text{ then } \exists! x > 0 \ni x^2 = c$$

Theorem 1.4

$S \subset \mathbb{R}, S \neq \emptyset$, and S is Bounded Below, set $L_S = \{b \in \mathbb{R} \mid b \text{ is a lower bound for } S\}$

Then $\exists d \in L_S \ni d \geq b \forall b \in L_S$

$d = \inf S =$ infimum of S (greatest lower bound)

("Given a bounded, nonempty set S , and the set of all lower bounds of S , L_S , then there exists a greatest element in L_S that is the greatest lower bound for S (its infimum)")

Section 1.2: The Distribution of \mathbb{Z} & \mathbb{Q}

Theorem 1.5 (Archimedian Property)

- (i) $c > 0 \Rightarrow \exists n \in \mathbb{N} \ni n > c$
- (ii) $\epsilon > 0 \Rightarrow \exists n \in \mathbb{N} \ni \frac{1}{n} < \epsilon$

Proposition 1.6

Let $n \in \mathbb{Z}$, then \exists No $k \in \mathbb{Z} \ni k \in (n, n+1)$

Proposition 1.7

Suppose $S \neq \emptyset$, $S \subset \mathbb{Z}$, and S is Bounded Above, then S has a Maximum $m \in S$
Note: $m \in S \Rightarrow m = \sup S$

Theorem 1.8

For any $c \in \mathbb{R} \exists! k \in \mathbb{Z} \ni k \in [c, c+1)$

Definition

$S \subset \mathbb{R}$ is Dense in \mathbb{R} if for any $I = (a, b)$, $a < b$, $S \cap I \neq \emptyset$

Theorem 1.9

\mathbb{Q} is Dense in \mathbb{R}

Corollary 1.10

$\mathbb{R} \setminus \mathbb{Q}$ is Dense in \mathbb{R}

Section 1.3: Inequalities and Identities

Definition

$$x \in \mathbb{R}, |x| = \begin{cases} x, & x \geq 0 \\ -x, & x < 0 \end{cases}$$

Fact 1

$$d > 0, |c| \leq d \Leftrightarrow -d \leq c \leq d$$

Fact 2

$$x \in \mathbb{R}, -|x| \leq x \leq |x|$$

Theorem 1.11 (Triangle Inequality)

If $a, b \in \mathbb{R}$, Then $|a + b| \leq |a| + |b|$

Proposition 1.12

$a, r \in \mathbb{R}, r > 0$, TFAE:

- (i) $|x - a| < r$
- (ii) $a - r < x < a + r$
- (iii) $x \in (a - r, a + r)$

Difference of Powers Formula

$$n \in \mathbb{N} \text{ and } a, b \in \mathbb{R}, \\ a^n - b^n = (a - b) \sum_{k=0}^{n-1} a^{n-1-k} b^k$$

Geometric Series Formula

$$r \in \mathbb{R}, r \neq 1, \text{ then } \frac{1-r^{n+1}}{1-r} = 1 + r + \cdots + r^n = \sum_{k=0}^n r^k$$

Definition

$$n! = \begin{cases} 1, & n = 0, 1 \\ n * (n-1) * \cdots * 2 * 1, & n > 1 \end{cases}$$

Definition

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}$$

Binomial Formula

$$n \in \mathbb{N} \text{ and } a, b \in \mathbb{R},$$
$$(a + b)^n = \sum_{k=0}^n \binom{n}{k} a^{n-k} b^k$$

Section 2.1: Convergence of Sequences

Definition

A sequence of real numbers is a function $f : \mathbb{N} \rightarrow \mathbb{R}$
set $a_n = f(n)$, then characterize f by $\{a_1, a_2, \dots\} = \{a_n\}_{n=1}^{\infty} = \{a_n\}$

Definition

$\{a_n\}$ Converges to $a \in \mathbb{R}$ provided that
for each $\epsilon > 0 \exists N \in \mathbb{N} \ni |a_n - a| < \epsilon \forall n \geq N$

Proposition 2.6

$\{\frac{1}{n}\}$ converges to 0

Fact

$\{1 + \frac{1}{2} + \dots + \frac{1}{2^n}\}$ converges to 2

Fact

$a_n \rightarrow a, a_n \rightarrow b \Rightarrow a = b$
(limits are unique)

Fact

$\{(-1)^n\}$ does not converge

Lemma 2.9 (Comparison Lemma)

Suppose we have $\{a_n\}, \{b_n\}$ with $a_n \rightarrow a$. Then $b_n \rightarrow b$ if
 $\exists c \geq 0$ and $N_1 \in \mathbb{N} \ni |b_n - b| \leq c|a_n - a| \forall n \geq N_1$

Theorem 2.10 (Sum Property)

$$a_n \rightarrow a, b_n \rightarrow b \Rightarrow a_n + b_n \rightarrow a + b$$

Lemma 2.11

$$a_n \rightarrow a, \alpha \in \mathbb{R} \Rightarrow (\alpha)a_n \rightarrow (\alpha)a$$

Theorem 2.13 (Product property)

$$a_n \rightarrow a, b_n \rightarrow b \Rightarrow a_n b_n \rightarrow ab$$

Fact 1

$$a_n \rightarrow a \Rightarrow |a_n| \rightarrow |a|$$

Proposition 2.14

$$b_n \rightarrow b \neq 0 \Rightarrow \frac{1}{b_n} \rightarrow \frac{1}{b}$$

Theorem 2.15 (Quotient property)

$$a_n \rightarrow a, b_n \rightarrow b \neq 0 \Rightarrow \frac{a_n}{b_n} \rightarrow \frac{a}{b}$$

Proposition 2.16 (Linear property)

$$a_n \rightarrow a, b_n \rightarrow b, \alpha, \beta \in \mathbb{R} \Rightarrow (\alpha)a_n + (\beta)b_n \rightarrow (\alpha)a + (\beta)b$$

Fact 2

$$a_n = c \ \forall n \Rightarrow a_n \rightarrow c$$

Proposition 2.17

$$P : \mathbb{R} \rightarrow \mathbb{R}, a_n \rightarrow a \Rightarrow P(a_n) \rightarrow P(a)$$

Section 2.2: Sequences & Sets

Theorem 2.18

$\{a_n\}$ converges $\Rightarrow \{a_n\}$ is bounded

Proposition 2.19

S is dense in $\mathbb{R} \Leftrightarrow$ each $x \in \mathbb{R}$ is a limit of a sequence in S

Theorem 2.20 (Sequential Density of \mathbb{Q})

Every $x \in \mathbb{R}$ is the limit of a sequence of rational numbers

Lemma 2.21

$d_n \rightarrow d, d_n \geq 0 \Rightarrow d \geq 0$

Theorem 2.22

$\{c_n\} \subset [a, b], c_n \rightarrow c \Rightarrow c \in [a, b]$

Definition

$S \subset \mathbb{R}$ is closed if whenever $\{a_n\} \subset S$ and $a_n \rightarrow a$ then $a \in S$

Fact

$[a, b]$ is closed

Section 2.3: The Monotone Convergence Theorem

Definition

$\{a_n\}$ is monotonically increasing if $a_{n+1} \geq a_n$ for each n

Definition

$\{a_n\}$ is monotonically decreasing if $a_{n+1} \leq a_n$ for each n

Definition

$\{a_n\}$ is monotone if it is either monotonically increasing or decreasing

Theorem 2.25 (Monotone Convergence Theorem)

If $\{a_n\}$ is monotone, then
 $\{a_n\}$ converges $\Leftrightarrow \{a_n\}$ is bounded

Note: if $\{a_n\}$ is monotonically increasing, $a_n \rightarrow \sup\{a_n\}$

Note: if $\{a_n\}$ is monotonically decreasing, $a_n \rightarrow \inf\{a_n\}$

Proposition 2.28

Let $c \in \mathbb{R}$, $|c| < 1 \Rightarrow \lim_{n \rightarrow \infty} c^n = 0$

Theorem 2.29 (Nested Interval Theorem)

Let $\{a_n\}$ and $\{b_n\}$ be such that $a_n < b_n$ and set $I_n = [a_n, b_n]$.
Assume that $I_{n+1} \subset I_n$ and that $\lim_{n \rightarrow \infty} [b_n - a_n] = 0$. Then $\exists! x \in \cap_{n=1}^{\infty} I_n$

Section 2.4: The Sequential Compactness Theorem

Definition

For a given $\{a_n\}$ let $\{n_k\}$ be a sequence of natural numbers that is strictly increasing. Then the sequence $\{b_k\}$ defined by $b_k = a_{n_k}$, with $k = 1, 2, \dots$ is called a subsequence of $\{a_n\}$, denoted $\{b_k\} = \{a_{n_k}\}_{k=1}^{\infty} = \{a_{n_k}\}$

Fact

Given a sequence $\{n_k\}_{k=1}^{\infty}$ of natural numbers that is strictly increasing, we have that $n_k \geq k$ for every $k \in \mathbb{N}$

Proposition 2.30

Let $\{a_n\}$ converge to a , i.e., $a_n \rightarrow a$
Then $\lim_{n \rightarrow \infty} a_{n_k} = a$ for every subsequence $\{a_{n_k}\}$ of $\{a_n\}$

Theorem 2.32

For every $\{a_n\}$ $\exists \{n_k\} \ni \{a_{n_k}\}$ is monotone

Theorem 2.33

Every bounded sequence has a convergent subsequence

Definition

$S \subseteq \mathbb{R}$ is sequentially compact if every sequence $\{a_n\} \subset S$ has a convergent subsequence whose limit is in S

Theorem 2.36 (Sequential Compactness Theorem)

$a, b \in \mathbb{R}$ with $a < b \Rightarrow [a, b]$ is sequentially compact

Section 3.1: Continuity

Definition

For $D \subseteq \mathbb{R}$, $f : D \rightarrow \mathbb{R}$ is continuous at $x_0 \in D$ if whenever $\{x_n\} \subset D$
and $x_n \rightarrow x_0$ then $f(x_n) \rightarrow f(x_0)$

($f : D \rightarrow \mathbb{R}$ is continuous if it is continuous $\forall x_0 \in D$)

Fact

$P : \mathbb{R} \rightarrow \mathbb{R} \Rightarrow x_0 \in \mathbb{R}, \{x_n\} \subset \mathbb{R}, x_n \rightarrow x_0 \Rightarrow P(x_n) \rightarrow P(x_0) \Rightarrow P$ is continuous

Theorem 3.4

Suppose $f : D \rightarrow \mathbb{R}, g : D \rightarrow \mathbb{R}$ are continuous at $x_0 \in D$, then
 $f + g : D \rightarrow \mathbb{R}$ and $fg : D \rightarrow \mathbb{R}$ are continuous at $x_0 \in D$
and if $g(x) \neq 0 \forall x \in D$ then $\frac{f}{g} : D \rightarrow \mathbb{R}$ is continuous at $x_0 \in D$

Corollary 3.5

Let P and Q be polynomials, then $\frac{P}{Q} : D \rightarrow \mathbb{R}$ is continuous where $D = \{x \in \mathbb{R} \mid Q(x) \neq 0\}$

Theorem 3.6

$f : D \rightarrow \mathbb{R}, g : U \rightarrow \mathbb{R}, f(D) \subseteq U$ and suppose that f is continuous at $x_0 \in D$ and g is continuous at $f(x_0) \in U$ then $g \circ f$ is continuous at x_0 ; namely, $g \circ f : D \rightarrow \mathbb{R}$

Fact

$f(x) = \sqrt{x}$ is continuous on $D = [0, +\infty)$

Section 3.1 (Sup): Trigonometric Continuity

Fact 1

if $\theta_n \rightarrow 0$, then $\sin \theta_n \rightarrow 0$

Fact 2

if $\theta_n \rightarrow 0$, then $\cos \theta_n \rightarrow 1$

Fact

$\sin \theta$ is continuous,

$\cos \theta$ is continuous,

$\tan \theta$ is continuous at $\cos \theta \neq 0$ ($\theta \neq (2n+1) * \frac{\pi}{2}$),

$\csc \theta$ is continuous at $\sin \theta \neq 0$ ($\theta \neq n\pi$),

$\sec \theta$ is continuous at $\cos \theta \neq 0$ ($\theta \neq (2n+1) * \frac{\pi}{2}$),

$\cot \theta$ is continuous at $\sin \theta \neq 0$ ($\theta \neq n\pi$)

Section 3.2: Extreme Value Theorem

Definition

For $f : D \rightarrow \mathbb{R}$ we define $f(D) = \{y \in \mathbb{R} \mid y = f(x) \text{ for some } x \in D\}$
 $f(D)$ is the image of f

Definition

$f : D \rightarrow \mathbb{R}$ attains a maximum (max value) if $\exists x_0 \in D \ni f(x) \leq f(x_0) \forall x \in D$
Such a point x_0 is a maximizer of f

$f : D \rightarrow \mathbb{R}$ attains a minimum (min value) if $\exists x'_0 \in D \ni f(x'_0) \leq f(x) \forall x \in D$
Such a point x'_0 is a minimizer of f

Lemma 3.10

If $f : [a, b] \rightarrow \mathbb{R}$ is continuous, then $f([a, b])$ is bounded above ($\exists m \ni f(x) \leq m \forall x \in [a, b]$)

Theorem 3.9 (Extreme Value Theorem)

If $f : [a, b] \rightarrow \mathbb{R}$ is continuous, then f attains both a max and a min

Fact

Let $S \subset [a, b]$, then $\inf S \in [a, b]$, and $\sup S \in [a, b]$

Section 3.3: Intermediate Value Theorem

Theorem 3.11 (Intermediate Value Theorem)

Suppose $f : [a, b] \rightarrow \mathbb{R}$ is continuous and let $c \in \mathbb{R}$ be any number strictly between $f(a)$ and $f(b)$; i.e., $f(a) < c < f(b)$ or $f(b) < c < f(a)$, then $\exists x_0 \in (a, b) \ni f(x_0) = c$

Definition

$D \subseteq \mathbb{R}$ is convex if $u, v \in D$, $(u < v) \Rightarrow [u, v] \subset D$

Fact

If $D \subset \mathbb{R}$ is convex then D is an interval

Theorem 3.14

If I is an interval and $f : I \rightarrow \mathbb{R}$ is continuous then $f(I)$ is an interval

Section 3.4: Uniform Continuity