Real Analysis I

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Section 1.1: The Completeness Axiom

Definition

 $S\subseteq\mathbb{R} \text{ is inductive if} \\ \text{(i) } 1\in S$

(ii) $x \in S \Rightarrow x + 1 \in S$

Definition

 $\mathbb N$ is the intersection of all inductive subsets of $\mathbb R$

Principle of Mathematical Induction

For each $n \in N$ let S(n) be some mathematical assertion. Suppose also that

- (i) S(1) is true
- (ii) Whenever S(n) is true, then S(n+1) is true

Then S(n) is true $\forall n \in N$

Definition

$$\mathbb{Z} = \mathbb{N} \cup \{0\} \cup \{-n | n \in \mathbb{N}\}\$$

Fact

 $m, n \in \mathbb{Z} \Rightarrow$

- (i) $m+n \in \mathbb{Z}$
- (ii) $m n \in \mathbb{Z}$
- (iii) $mn \in \mathbb{Z}$

Definition

$$\mathbb{Q} = \{ \frac{m}{n} | m, n \in \mathbb{Z}, n \neq 0 \}$$

Fact

- (i) Each $x \in \mathbb{Q} \Rightarrow x = \frac{m}{n}, m$ or n is odd
- (ii) n^2 is even $\Rightarrow n$ is even

Proposition 1.2

$$\exists \text{ No } x \in \mathbb{Q} \ni x^2 = 2$$

Definition

 $S \subset \mathbb{R}, S \neq \emptyset$ is Bounded Above if $\exists c \in \mathbb{R} \ \ni x \leq c \ \forall x \in S \Rightarrow c$ is an Upper Bound for S

Completeness Axiom

If $S \subset \mathbb{R}, S \neq \emptyset$, and S is Bounded Above, set $U_S = \{c \in \mathbb{R} | c \text{ is an upper bound for } S\}$

Then $\exists a \in U_S \ni a \leq c \ \forall c \in U_S$ $a = \sup S = \text{supremum of S (least upper bound)}$

("Given a bounded, nonempty set S, and the set of all upper bounds of S, U_S , then there exists a least element in U_S that is the least upper bound for S (its supremum)")

Proposition 1.3

If c > 0, then $\exists ! \ x > 0 \ \ni x^2 = c$

Theorem 1.4

 $S \subset \mathbb{R}, S \neq \emptyset$, and S is Bounded Below, set $L_S = \{b \in \mathbb{R} | b \text{ is an lower bound for } S\}$

Then $\exists d \in L_S \ni d \geq b \ \forall b \in U_S$ d = infS = infimum of S (greatest lower bound)

("Given a bounded, nonempty set S, and the set of all lower bounds of S, L_S , then there exists a greatest element in L_S that is the greatest lower bound for S (its infimum)")

Section 1.2: The Distribution of $\mathbb{Z} \ \& \ \mathbb{Q}$

Theorem 1.5 (Archimedian Property)

 $\begin{array}{l} \text{(i) } c>0 \Rightarrow \exists n \in \mathbb{N} \text{ } \ni n>c \\ \text{(ii) } \epsilon>0 \Rightarrow \exists n \in \mathbb{N} \text{ } \ni \frac{1}{n}<\epsilon \end{array}$

Proposition 1.6

Let $n \in \mathbb{Z}$, then \exists No $k \in \mathbb{Z}$ $\ni k \in (n, n+1)$

Proposition 1.7

Suppose $S \neq \emptyset, S \subset \mathbb{Z},$ and S is Bounded Above, then S has a Maximum $m \in S$ Note: $m \in S \Rightarrow m = \sup S$

Theorem 1.8

For any $c \in \mathbb{R} \exists ! \ k \in \mathbb{Z} \ni k \in [c, c+1)$

Definition

 $S \subset \mathbb{R}$ is Dense in \mathbb{R} if for any $I = (a, b), a < b, S \cap I \neq \emptyset$

Theorem 1.9

 $\mathbb Q$ is Dense in $\mathbb R$

Corollary 1.10

 $\mathbb{R}\setminus\mathbb{Q}$ is Dense in \mathbb{R}

Section 1.3: Inequalities and Identities

Definition

$$x \in \mathbb{R}, |x| = \begin{cases} x, & x \ge 0 \\ -x, & x < 0 \end{cases}$$

Fact 1

$$d > 0, |c| \le d \Leftrightarrow -d \le c \le d$$

Fact 2

$$x \in \mathbb{R}, -|x| \le x \le |x|$$

Theorem 1.11 (Triangle Inequality)

If $a, b \in \mathbb{R}$, Then $|a + b| \le |a| + |b|$

Proposition 1.12

 $a, r \in \mathbb{R}, r > 0$, TFAE:

- (i) |x a| < r
- (ii) a r < x < a + r
- (iii) $x \in (a-r, a+r)$

Difference of Powers Formula

$$n\in\mathbb{N}$$
 and $a,b\in\mathbb{R},$ $a^n-b^n=(a-b)\sum_{k=0}^{n-1}a^{n-1-k}b^k$

Geometric Series Formula

$$r \in \mathbb{R}, r \neq 1$$
, then $\frac{1-r^{n+1}}{1-r} = 1 + r + \dots + r^n = \sum_{k=0}^n r^k$

Definition

$$n! = \begin{cases} 1, & n = 0, 1\\ n * (n-1) * \cdots * 2 * 1, & n > 1 \end{cases}$$

Definition

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}$$

Binomial Formula

$$n \in \mathbb{N}$$
 and $a, b \in \mathbb{R}$,
 $(a+b)^n = \sum_{k=0}^n \binom{n}{k} a^{n-k} b^k$

Section 2.1: Convergence of Sequences

Definition

A sequence of real numbers is a function $f: \mathbb{N} \to \mathbb{R}$ set $a_n = f(n)$, then characterize f by $\{a_1, a_2, \dots\} = \{a_n\}_{n=1}^{\infty} = \{a_n\}$

Definition

 $\{a_n\}$ Converges to $a \in \mathbb{R}$ provided that for each $\epsilon > 0 \ \exists N \in \mathbb{N} \ \ni |a_n - a| < \epsilon \ \forall n \ge N$

Proposition 2.6

 $\{\frac{1}{n}\}$ converges to 0

Fact

 $\left\{1+\frac{1}{2}+\cdots+\frac{1}{2^n}\right\}$ converges to 2

Fact

 $a_n \to a, a_n \to b \Rightarrow a = b$ (limits are unique)

Fact

 $\{(-1)^n\}$ does not converge

Lemma 2.9 (Comparison Lemma)

Suppose we have $\{a_n\}, \{b_n\}$ with $a_n \to a$. Then $b_n \to b$ if $\exists c \ge 0$ and $N_1 \in \mathbb{N} \ni |b_n - b| \le c|a_n - a| \ \forall n \ge N_1$

Theorem 2.10 (Sum Property)

$$a_n \to a, \ b_n \to b \Rightarrow a_n + b_n \to a + b$$

Lemma 2.11

$$a_n \to a, \ \alpha \in \mathbb{R} \Rightarrow (\alpha)a_n \to (\alpha)a$$

Theorem 2.13 (Product property)

$$a_n \to a, \ b_n \to b \Rightarrow a_n b_n \to ab$$

Fact 1

$$a_n \to a \Rightarrow |a_n| \to |a|$$

Proposition 2.14

$$b_n \to b \neq 0 \Rightarrow \frac{1}{b_n} \to \frac{1}{b}$$

Theorem 2.15 (Quotient property)

$$a_n \to a, \ b_n \to b \neq 0 \Rightarrow \frac{a_n}{b_n} \to \frac{a}{b}$$

Proposition 2.16 (Linear property)

$$a_n \to a, \ b_n \to b, \ \alpha, \beta \in \mathbb{R} \Rightarrow (\alpha)a_n + (\beta)b_n \to (\alpha)a + (\beta)b$$

Fact 2

$$a_n = c \ \forall n \Rightarrow a_n \to c$$

Proposition 2.17

$$P: \mathbb{R} \to \mathbb{R}, \ a_n \to a \Rightarrow P(a_n) \to P(a)$$

Section 2.2: Sequences & Sets

Theorem 2.18

 $\{a_n\}$ converges $\Rightarrow \{a_n\}$ is bounded

Proposition 2.19

S is dense in $\mathbb{R} \Leftrightarrow \text{each } x \in \mathbb{R}$ is a limit of a sequence in S

Theorem 2.20 (Sequential Density of \mathbb{Q})

Every $x \in \mathbb{R}$ is the limit of a sequence of rational numbers

Lemma 2.21

 $d_n \to d, \ d_n \ge 0 \Rightarrow d \ge 0$

Theorem 2.22

 $\{c_n\} \subset [a,b], \ c_n \to c \Rightarrow c \in [a,b]$

Definition

 $S \subset \mathbb{R}$ is closed if whenever $\{a_n\} \subset S$ and $a_n \to a$ then $a \in S$

Fact

[a, b] is closed

Section 2.3: The Monotone Convergence Theorem

Definition

 $\{a_n\}$ is monotonically increasing if $a_{n+1} \ge a_n$ for each n

Definition

 $\{a_n\}$ is monotonically decreasing if $a_{n+1} \leq a_n$ for each n

Definition

 $\{a_n\}$ is monotone if it is either monotonically increasing or decreasing

Theorem 2.25 (Monotone Convergence Theorem)

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If \{a_n\} is monotone, then \{a_n\} converges \Leftrightarrow \{a_n\} is bounded

Note: if \{a_n\} is monotonically increasing, a_n \to \sup\{a_n\}

Note: if \{a_n\} is monotonically decreasing, a_n \to \inf\{a_n\}
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Proposition 2.28

Let $c \in \mathbb{R}$, $|c| < 1 \Rightarrow \lim_{n \to \infty} c^n = 0$

Theorem 2.29 (Nested Interval Theorem)

Let $\{a_n\}$ and $\{b_n\}$ be such that $a_n < b_n$ and set $I_n = [a_n, b_n]$. Assume that $I_{n+1} \subset I_n$ and that $\lim_{n \to \infty} [b_n - a_n] = 0$. Then $\exists ! \ x \in \bigcap_{n=1}^{\infty} I_n$

Section 2.4: The Sequential Compactness Theorem

Definition

For a given $\{a_n\}$ let $\{n_k\}$ be a sequence of natural numbers that is strictly increasing. Then the sequence $\{b_k\}$ defined by $b_k = a_{n_k}$, with $k = 1, 2, \cdots$ is called a subsequence of $\{a_n\}$, denoted $\{b_k\} = \{a_{n_k}\}_{k=1}^{\infty} = \{a_{n_k}\}$

Fact

Given a sequence $\{n_k\}_{k=1}^{\infty}$ of natural numbers that is strictly increasing, we have that $n_k \geq k$ for every $k \in \mathbb{N}$

Proposition 2.30

Let $\{a_n\}$ converge to a, i.e., $a_n \to a$ Then $\lim_{n\to\infty} a_{n_k} = a$ for every subsequence $\{a_{n_k}\}$ of $\{a_n\}$

Theorem 2.32

For every $\{a_n\} \exists \{n_k\} \ni \{a_{n_k}\}$ is monotone

Theorem 2.33

Every bounded sequence has a convergent subsequence

Definition

 $S \subseteq \mathbb{R}$ is sequentially compact if every sequence $\{a_n\} \subset S$ has a convergent subsequence whose limit is in S

Theorem 2.36 (Sequential Compactness Theorem)

 $a, b \in \mathbb{R}$ with $a < b \Rightarrow [a, b]$ is sequentially compact

Section 3.1: Continuity

Definition

For $D \subseteq \mathbb{R}$, $f: D \to \mathbb{R}$ is continuous at $x_0 \in D$ if whenever $\{x_n\} \subset D$ and $x_n \to x_0$ then $f(x_n) \to f(x_0)$

 $(f: D \to \mathbb{R} \text{ is continuous if it is continuous } \forall x_0 \in D)$

Fact

 $P: \mathbb{R} \to \mathbb{R} \Rightarrow x_0 \in \mathbb{R}, \{x_n\} \subset \mathbb{R}, x_n \to x_0 \Rightarrow P(x_n) \to P(x_0) \Rightarrow P$ is continuous

Theorem 3.4

Suppose $f: D \to \mathbb{R}, g: D \to \mathbb{R}$ are continuous at $x_0 \in D$, then $f+g: D \to \mathbb{R}$ and $fg: D \to \mathbb{R}$ are continuous at $x_0 \in D$ and if $g(x) \neq 0 \ \forall \ x \in D$ then $\frac{f}{g}: D \to \mathbb{R}$ is continuous at $x_0 \in D$

Corollary 3.5

Let P and Q be polynomials, then $\frac{P}{Q}:D\to\mathbb{R}$ is continuous where $D=\{x\in\mathbb{R}\mid Q(x_0)\neq 0\}$

Theorem 3.6

 $f: D \to \mathbb{R}, g: U \to \mathbb{R}, f(D) \subseteq U$ and suppose that f is continuous at $x_0 \in D$ and g is continuous at $f(x_0) \in U$ then $g \circ f$ is continuous at x_0 ; namely, $g \circ f: D \to \mathbb{R}$

Fact

 $f(x) = \sqrt{x}$ is continuous on $D = [0, +\infty)$

Section 3.1 (Sup): Trigonometric Continuity

Fact 1

if $\theta_n \to 0$, then $\sin \theta_n \to 0$

Fact 2

if $\theta_n \to 0$, then $\cos \theta_n \to 1$

Fact

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\begin{array}{l} \sin\theta \text{ is continuous,} \\ \cos\theta \text{ is continuous,} \\ \tan\theta \text{ is continuous at } \cos\theta \neq 0 \text{ } (\theta \neq (2n+1)*\frac{\pi}{2}), \\ \csc\theta \text{ is continuous at } \sin\theta \neq 0 \text{ } (\theta \neq n\pi), \\ \sec\theta \text{ is continuous at } \cos\theta \neq 0 \text{ } (\theta \neq ((2n+1)*\frac{\pi}{2}), \\ \cot\theta \text{ is continuous at } \sin\theta \neq 0 \text{ } (\theta \neq n\pi) \end{array}
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