Undergraduate Real Analysis References

Jake Thakur

Section 1.1: The Completeness Axiom

Definition

$S \subseteq \mathbb{R}$ is inductive if
(i) $1 \in S$
(ii) $x \in S \Rightarrow x + 1 \in S$

Definition

$\mathbb{N}$ is the intersection of all inductive subsets of $\mathbb{R}$

Principle of Mathematical Induction

For each $n \in N$ let $S(n)$ be some mathematical assertion. Suppose also that
(i) $S(1)$ is true
(ii) Whenever $S(n)$ is true, then $S(n+1)$ is true
Then $S(n)$ is true $\forall n \in N$

Definition

$\mathbb{Z} = \mathbb{N} \cup \{0\} \cup \{-n | n \in \mathbb{N}\}$

Fact

$m, n \in \mathbb{Z} \Rightarrow$
(i) $m + n \in \mathbb{Z}$
(ii) $m - n \in \mathbb{Z}$
(iii) $mn \in \mathbb{Z}$

Definition

$\mathbb{Q} = \{\frac{m}{n} | m, n \in \mathbb{Z}, n \neq 0\}$

Fact

(i) Each $x \in \mathbb{Q} \Rightarrow x = \frac{m}{n}, m$ or n is odd
(ii) n^2 is even $\Rightarrow n$ is even

Proposition 1.2

$\exists$ No $x \in \mathbb{Q} \ni x^2 = 2$

Definition

$S \subset \mathbb{R}, S \neq \emptyset$ is Bounded Above if $\exists c \in \mathbb{R} \ni x \leq c\ \forall x \in S \Rightarrow c$ is an Upper Bound for S

Completeness Axiom

If $S \subset \mathbb{R}, S \neq \emptyset$, and S is Bounded Above, set $U_S = \{c \in \mathbb{R} | c$ is an upper bound for $S\}$
Then $\exists a \in U_S \ni a \leq c\ \forall c \in U_S$
$a = supS =$ supremum of S (least upper bound)

("Given a bounded, nonempty set S, and the set of all upper bounds of S, U_S, then there exists a least element in U_S that is the least upper bound for S (its supremum)")

Proposition 1.3

If $c > 0$, then $\exists!\ x > 0 \ni x^2 = c$

Theorem 1.4

$S \subset \mathbb{R}, S \neq \emptyset$, and S is Bounded Below, set $L_S = \{b \in \mathbb{R} | b$ is an lower bound for $S\}$
Then $\exists d \in L_S \ni d \geq b\ \forall b \in U_S$
$d = infS =$ infimum of S (greatest lower bound)

("Given a bounded, nonempty set S, and the set of all lower bounds of S, L_S, then there exists a greatest element in L_S that is the greatest lower bound for S (its infimum)")

Section 1.2: The Distribution of $\mathbb{Z}$ & $\mathbb{Q}$

Theorem 1.5 (Archimedian Property)

(i) $c > 0 \Rightarrow \exists n \in \mathbb{N} \ni n > c$
(ii) $\epsilon > 0 \Rightarrow \exists n \in \mathbb{N} \ni \frac{1}{n} < \epsilon$

Proposition 1.6

Let $n \in \mathbb{Z}$, then $\exists$ No $k \in \mathbb{Z} \ni k \in (n, n+1)$

Proposition 1.7

Suppose $S \neq \emptyset, S \subset \mathbb{Z}$, and S is Bounded Above, then S has a Maximum $m \in S$
Note: $m \in S \Rightarrow m = supS$

Theorem 1.8

For any $c \in \mathbb{R}$ $\exists!$ $k \in \mathbb{Z} \ni k \in [c, c+1)$

Definition

$S \subset \mathbb{R}$ is Dense in $\mathbb{R}$ if for any $I = (a, b), a < b, S \cap I \neq \emptyset$

Theorem 1.9

$\mathbb{Q}$ is Dense in $\mathbb{R}$

Corollary 1.10

$\mathbb{R} \setminus \mathbb{Q}$ is Dense in $\mathbb{R}$

Section 1.3: Inequalities and Identities

Definition

$x \in \mathbb{R}$, $|x| = \begin{cases} x, & x \geq 0 \\ -x, & x < 0 \end{cases}$

Fact 1

$d > 0$, $|c| \leq d \Leftrightarrow -d \leq c \leq d$

Fact 2

$x \in \mathbb{R}, -|x| \leq x \leq |x|$

Theorem 1.11 (Triangle Inequality)

If $a, b \in \mathbb{R}$, Then $|a + b| \leq |a| + |b|$

Proposition 1.12

$a, r \in \mathbb{R}, r > 0$, TFAE:
(i) $|x - a| < r$
(ii) $a - r < x < a + r$
(iii) $x \in (a - r, a + r)$

Difference of Powers Formula

$n \in \mathbb{N}$ and $a, b \in \mathbb{R}$,
$a^n - b^n = (a - b) \sum_{k=0}^{n-1} a^{n-1-k} b^k$

Geometric Series Formula

$r \in \mathbb{R}, r \neq 1$, then $\frac{1-r^{n+1}}{1-r} = 1 + r + \cdots + r^n = \sum_{k=0}^{n} r^k$

Definition

$n! = \begin{cases} 1, & n = 0, 1 \\ n * (n-1) * \cdots * 2 * 1, & n > 1 \end{cases}$

Definition

$\binom{n}{k} = \frac{n!}{k!(n-k)!}$

Binomial Formula

$n \in \mathbb{N}$ and $a, b \in \mathbb{R}$,
$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k$

Section 2.1: Convergence of Sequences

Definition

A sequence of real numbers is a function $f : \mathbb{N} \to \mathbb{R}$
set $a_n = f(n)$, then characterize f by $\{a_1, a_2, \cdots\} = \{a_n\}_{n=1}^{\infty} = \{a_n\}$

Definition

$\{a_n\}$ Converges to $a \in \mathbb{R}$ provided that
for each $\epsilon > 0$ $\exists N \in \mathbb{N} \ni |a_n - a| < \epsilon$ $\forall n \geq N$

Proposition 2.6

$\{\frac{1}{n}\}$ converges to 0

Fact

$\{1 + \frac{1}{2} + \cdots + \frac{1}{2^n}\}$ converges to 2

Fact

$a_n \to a$, $a_n \to b \Rightarrow a = b$
(limits are unique)

Fact

$\{(-1)^n\}$ does not converge

Lemma 2.9 (Comparison Lemma)

Suppose we have $\{a_n\}, \{b_n\}$ with $a_n \to a$. Then $b_n \to b$ if
$\exists c \geq 0$ and $N_1 \in \mathbb{N} \ni |b_n - b| \leq c|a_n - a|$ $\forall n \geq N_1$

Theorem 2.10 (Sum Property)

$a_n \to a,\ b_n \to b \Rightarrow a_n + b_n \to a + b$

Lemma 2.11

$a_n \to a,\ \alpha \in \mathbb{R} \Rightarrow (\alpha)a_n \to (\alpha)a$

Theorem 2.13 (Product property)

$a_n \to a,\ b_n \to b \Rightarrow a_n b_n \to ab$

Fact 1

$a_n \to a \Rightarrow |a_n| \to |a|$

Proposition 2.14

$b_n \to b \neq 0 \Rightarrow \frac{1}{b_n} \to \frac{1}{b}$

Theorem 2.15 (Quotient property)

$a_n \to a,\ b_n \to b \neq 0 \Rightarrow \frac{a_n}{b_n} \to \frac{a}{b}$

Proposition 2.16 (Linear property)

$a_n \to a,\ b_n \to b,\ \alpha, \beta \in \mathbb{R} \Rightarrow (\alpha)a_n + (\beta)b_n \to (\alpha)a + (\beta)b$

Fact 2

$a_n = c\ \forall n \Rightarrow a_n \to c$

Proposition 2.17

$P : \mathbb{R} \to \mathbb{R},\ a_n \to a \Rightarrow P(a_n) \to P(a)$

Section 2.2: Sequences & Sets

Theorem 2.18

$\{a_n\}$ converges $\Rightarrow \{a_n\}$ is bounded

Proposition 2.19

S is dense in $\mathbb{R} \Leftrightarrow$ each $x \in \mathbb{R}$ is a limit of a sequence in S

Theorem 2.20 (Sequential Density of $\mathbb{Q}$)

Every $x \in \mathbb{R}$ is the limit of a sequence of rational numbers

Lemma 2.21

$d_n \to d,\ d_n \geq 0 \Rightarrow d \geq 0$

Theorem 2.22

$\{c_n\} \subset [a, b],\ c_n \to c \Rightarrow c \in [a, b]$

Definition

$S \subset \mathbb{R}$ is closed if whenever$\{a_n\} \subset S$ and $a_n \to a$ then $a \in S$

Fact

$[a, b]$ is closed

Section 2.3: The Monotone Convergence Theorem

Definition

$\{a_n\}$ is monotonically increasing if $a_{n+1} \geq a_n$ for each n

Definition

$\{a_n\}$ is monotonically decreasing if $a_{n+1} \leq a_n$ for each n

Definition

$\{a_n\}$ is monotone if it is either monotonically increasing or decreasing

Theorem 2.25 (Monotone Convergence Theorem)

If $\{a_n\}$ is monotone, then
$\{a_n\}$ converges $\Leftrightarrow$ $\{a_n\}$ is bounded

Note: if $\{a_n\}$ is monotonically increasing, $a_n \to sup\{a_n\}$
Note: if $\{a_n\}$ is monotonically decreasing, $a_n \to inf\{a_n\}$

Proposition 2.28

Let $c \in \mathbb{R},\ |c| < 1 \Rightarrow \lim_{n\to\infty} c^n = 0$

Theorem 2.29 (Nested Interval Theorem)

Let $\{a_n\}$ and $\{b_n\}$ be such that $a_n < b_n$ and set $I_n = [a_n, b_n]$.
Assume that $I_{n+1} \subset I_n$ and that $\lim_{n\to\infty}[b_n - a_n] = 0$. Then $\exists!\ x \in \cap_{n=1}^{\infty} I_n$

Section 2.4: The Sequential Compactness Theorem

Definition

For a given $\{a_n\}$ let $\{n_k\}$ be a sequence of natural numbers that is strictly increasing. Then the sequence $\{b_k\}$ defined by $b_k = a_{n_k}$, with $k = 1, 2, \cdots$ is called a subsequence of $\{a_n\}$, denoted $\{b_k\} = \{a_{n_k}\}_{k=1}^{\infty} = \{a_{n_k}\}$

Fact

Given a sequence $\{n_k\}_{k=1}^{\infty}$ of natural numbers that is strictly increasing, we have that $n_k \geq k$ for every $k \in \mathbb{N}$

Proposition 2.30

Let $\{a_n\}$ converge to a, i.e., $a_n \to a$
Then $\lim_{n\to\infty} a_{n_k} = a$ for every subsequence $\{a_{n_k}\}$ of $\{a_n\}$

Theorem 2.32

For every $\{a_n\}$ $\exists \{n_k\} \ni \{a_{n_k}\}$ is monotone

Theorem 2.33

Every bounded sequence has a convergent subsequence

Definition

$S \subseteq \mathbb{R}$ is sequentially compact if every sequence $\{a_n\} \subset S$ has a convergent subsequence whose limit is in S

Theorem 2.36 (Sequential Compactness Theorem)

$a, b \in \mathbb{R}$ with $a < b \Rightarrow [a, b]$ is sequentially compact

Section 3.1: Continuity

Definition

For $D \subseteq \mathbb{R}, f : D \to \mathbb{R}$ is continuous at $x_0 \in D$ if whenever $\{x_n\} \subset D$
and $x_n \to x_0$ then $f(x_n) \to f(x_0)$

($f : D \to \mathbb{R}$ is continuous if it is continuous $\forall\ x_0 \in D$)

Fact

$P : \mathbb{R} \to \mathbb{R} \Rightarrow x_0 \in \mathbb{R}, \{x_n\} \subset \mathbb{R}, x_n \to x_0 \Rightarrow P(x_n) \to P(x_0) \Rightarrow P$ is continuous

Theorem 3.4

Suppose $f : D \to \mathbb{R}, g : D \to \mathbb{R}$ are continuous at $x_0 \in D,$ then
$f + g : D \to \mathbb{R}$ and $fg : D \to \mathbb{R}$ are continuous at $x_0 \in D$
and if $g(x) \neq 0\ \forall\ x \in D$ then $\frac{f}{g} : D \to \mathbb{R}$ is continuous at $x_0 \in D$

Corollary 3.5

Let P and Q be polynomials, then $\frac{P}{Q} : D \to \mathbb{R}$ is continuous where $D = \{x \in \mathbb{R} \mid Q(x_0) \neq 0\}$

Theorem 3.6

$f : D \to \mathbb{R}, g : U \to \mathbb{R}, f(D) \subseteq U$ and suppose that
f is continuous at $x_0 \in D$ and g is continuous at $f(x_0) \in U$
then $g \circ f$ is continuous at x_0; namely, $g \circ f : D \to \mathbb{R}$

Fact

$f(x) = \sqrt{x}$ is continuous on $D = [0, +\infty)$

Section 3.1 (Sup): Trigonometric Continuity

Fact 1

if $\theta_n \to 0,$ then $\sin\theta_n \to 0$

Fact 2

if $\theta_n \to 0,$ then $\cos\theta_n \to 1$

Fact

$\sin\theta$ is continuous,
$\cos\theta$ is continuous,
$\tan\theta$ is continuous at $\cos\theta \neq 0$ $(\theta \neq (2n+1) * \frac{\pi}{2})$,
$\csc\theta$ is continuous at $\sin\theta \neq 0$ $(\theta \neq n\pi)$,
$\sec\theta$ is continuous at $\cos\theta \neq 0$ $(\theta \neq ((2n+1) * \frac{\pi}{2})$,
$\cot\theta$ is continuous at $\sin\theta \neq 0$ $(\theta \neq n\pi)$

Section 3.2: Extreme Value Theorem

Definition

For $f : D \to \mathbb{R}$ we define $f(D) = \{y \in \mathbb{R} \mid y = f(x) \text{ for some } x \in D\}$
$f(D)$ is the image of f

Definition

$f : D \to \mathbb{R}$ attains a maximum (max value) if $\exists\ x_0 \in D \ni f(x) \leq f(x_0)\ \forall x \in D$
Such a point x_0 is a maximizer of f

$f : D \to \mathbb{R}$ attains a minimum (min value) if $\exists\ x_0' \in D \ni f(x_0') \leq f(x)\ \forall x \in D$
Such a point x_0' is a minimizer of f

Lemma 3.10

If $f : [a,b] \to \mathbb{R}$ is continuous, then $f([a,b])$ is bounded above ($\exists m \ni f(x) \leq m\ \forall x \in [a,b]$)

Theorem 3.9 (Extreme Value Theorem)

If $f : [a,b] \to \mathbb{R}$ is continuous, then f attains both a max and a min

$\exists x_0, x_0' \in [a,b] \ni f(x_0) \leq f(x) \leq f(x_0') \forall x \in [a,b]$

Fact

Let $S \subset [a,b]$, then $inf S \in [a,b]$, and $sup S \in [a,b]$

Section 3.3: Intermediate Value Theorem

Theorem 3.11 (Intermediate Value Theorem)

Suppose $f : [a,b] \to \mathbb{R}$ is continuous and let $c \in \mathbb{R}$ be any number strictly between $f(a)$ and $f(b)$; i.e., $f(a) < c < f(b)$ or $f(b) < c < f(a)$, then $\exists x_0 \in (a,b) \ni f(x_0) = c$

Fact

Suppose $f : D \to \mathbb{R}$ is continuous. If $\exists [a,b] \subset D \ni f(a) < 0$
and $f(b) > 0$ (or vice-versa), then $\exists x_0 \in (a,b) \ni f(x_0) = 0$

"A real, continuous function that is positive on one side and negative on the other contains a root"

Definition

$D \subseteq \mathbb{R}$ is convex if $u, v \in D,\ (u < v) \Rightarrow [u, v] \subset D$

Fact

If $D \subset \mathbb{R}$ is convex then D is an interval

Theorem 3.14

If I is an interval and $f : I \to \mathbb{R}$ is continuous then $f(I)$ is an interval

Section 3.4: Uniform Continuity

Definition

$f : D \to \mathbb{R}$ is uniformly continuous on D if whenever $\{u_n\}, \{v_n\} \subset D \ni u_n - v_n \to 0$, then $f(u_n) - f(v_n) \to 0$

Note: if $v_n = x_0\ \forall n$, then $u_n - v_n \to 0 \Rightarrow u_n \to x_0$, so uniform continuity $\Rightarrow$ continuity at each $x_0 \in D$

Fact

$f(x) = x$ is uniformly continuous but $f(x) = x^2$ is not

Theorem 3.17

$f : [a, b] \to \mathbb{R}$ is continuous $\Rightarrow f$ is uniformly continuous on $[a, b]$

Fact

If $f : D \to \mathbb{R}$ satisfies Lipschitz Continuity: $|f(u) - f(v)| \leq c|u - v|,\ \forall u, v \in D$ and for some $c \geq 0$, then f is uniformly continuous.

Fact

Let P be a polynomial. Then on each $[a, b]$, $P : [a, b] \to \mathbb{R}$ is lipschitz continous

Section 3.5: The $\epsilon - \delta$ Criteria for Continuity

Definition

"The $\epsilon - \delta$ Criterion At a Point" - $f : D \to \mathbb{R}$ satisfies the $\epsilon - \delta$ criterion at a point $x_0 \in D$, if for each $\epsilon > 0 \; \exists \; \delta > 0 \ni$ for $x \in D$, $|x - x_0| < \delta \Rightarrow |f(x) - f(x_0)| < \epsilon$

Theorem 3.20

For $f : D \to \mathbb{R}$ and $x_0 \in D$, TFAE:

(i) f is continuous at x_0
(ii) The $\epsilon - \delta$ criterion at x_0 holds

Definition

"The $\epsilon - \delta$ Criterion On the Domain of a Function" - $f : D \to \mathbb{R}$ satisfies the $\epsilon - \delta$ criterion on D, if for each $\epsilon > 0 \; \exists \; \delta > 0 \ni u, v \in D$, $|u - v| < \delta \Rightarrow |f(u) - f(v)| < \epsilon$

Theorem 3.22

For $f : D \to \mathbb{R}$, TFAE:

(i) $f : D \to \mathbb{R}$ is uniformly continuous
(ii) $f : D \to \mathbb{R}$ satisfies the $\epsilon - \delta$ criterion on D

Fact

$I = (a, b)$, $f : I \to \mathbb{R}$ is continuous, then if $x_0 \in (a, b)$ with $f(x_0) > 0$, then $\exists \; I_1 = (a_1, b_1) \subset I \ni f(x) > 0 \; \forall \; x \in I_1$

Section 3.6: Images and Inverses; Monotone Functions

Definition

(i) $f : D \to \mathbb{R}$ is monotonically increasing if $u, v \in D$ and $u < v \Rightarrow f(u) \leq f(v)$
(ii) $f : D \to \mathbb{R}$ is monotonically decreasing if $u, v \in D$ and $u < v \Rightarrow f(u) \geq f(v)$
(iii) $f : D \to \mathbb{R}$ is monotone if it is monotonically increasing or decreasing

Theorem 3.23

Suppose $f : D \to \mathbb{R}$ is monotone. If $f(D)$ is an interval, then f is continuous

Corollary 3.25

Suppose $f : I \to \mathbb{R}$ is monotone, then f is continuous $\Leftrightarrow f(I)$ is an interval

Definition

(i) $f : D \to \mathbb{R}$ is strictly increasing if $u, v \in D$ and $u < v \Rightarrow f(u) < f(v)$
(ii) $f : D \to \mathbb{R}$ is strictly decreasing if $u, v \in D$ and $u < v \Rightarrow f(u) > f(v)$

Definition

$f : D \to \mathbb{R}$ is one-to-one (injective) if for each $y \in f(D)$ $\exists!$ $x \in D \ni f(x) = y$

"No element in the image may have more than one element in the domain mapping to it"

Fact

If f is strictly increasing or decreasing, then f is one-to-one

Fact

If $f : I \to \mathbb{R}$ is continuous and f is one-to-one, then f is strictly monotone

Definition

Suppose $f : D \to \mathbb{R}$ is one-to-one. If $y \in f(D)$, let $x \in D \ni f(x) = y$
Define $f^{-1} : f(D) \to D$ by $f^{-1}(y) = x$, so f^{-1} is well-defined since x is unique

Note:
(i) $f^{-1}(f(x)) = x$, where $x \in D$
(ii) $f(f^{-1}(y)) = y$, where $y \in f(D)$

Theorem 3.29

$f : I \to \mathbb{R}$ is continuous and strictly increasing or decreasing $\Rightarrow$
$f^{-1} : f(I) \to \mathbb{R}$ is continuous

Section 3.7: Limits

Definition

$h(x) = \frac{f(x)-f(x_0)}{x-x_0}$ gives the slope of the line at point x_0
and $h(x)$ is continuous on $[a, b] \setminus \{x_0\}$

Definition

$D \subset \mathbb{R}$, $x_0 \in \mathbb{R}$ is a limit point of D if $\exists \{x_n\} \subset D \setminus \{x_0\} \ni x_n \to x_0$

Definition

If $f : D \to \mathbb{R}$ and x_0 is a limit point of D, then we denote $\lim_{x \to x_0} f(x) = l$
If whenever $\{x_n\} \subset D \setminus \{x_0\}$ and $x_n \to x_0$ we have that $\lim_{n\to\infty} f(x_n) = l$
(x_0 may or may not be in D)

Example

$D = \mathbb{R} \setminus \{x_0\}$, $f(x) = x^2 \Rightarrow h(x) = \frac{x^2-(x_0)^2}{x-x_0}$ and suppose
$\{x_n\} \subset D, x_n \to x_0 \Rightarrow h(x_n) = \frac{(x_n)^2-(x_0)^2}{x_n-x_0} = \frac{(x_n+x_0)(x_n-x_0)}{x_n-x_0} = x_n + x_0$
So $\lim_{n\to\infty} h(x_n) = \lim_{n\to\infty}(x_n + x_0) = x_0 + x_0 = 2x_0$

Theorem 3.36

Suppose $f : D \to \mathbb{R},\ g : D \to \mathbb{R},$ and x_0 is a limit point of D, so that

$$\lim_{x \to x_0} f(x) = A, \quad \lim_{x \to x_0} g(x) = B \Rightarrow$$

(i) $\lim_{x \to x_0}[f(x) + g(x)] = A + B$
(ii) $\lim_{x \to x_0}[f(x)g(x)] = AB$
(ii)(a) $\alpha \in \mathbb{R},\ \lim_{x \to x_0}[\alpha f(x)] = \alpha A$
(iii) $B \neq 0,\ g(x) \neq 0\ \forall x \in D,\ \lim_{x \to x_0}[\frac{f(x)}{g(x)}] = \frac{A}{B}$

Theorem 3.37

$f : D \to \mathbb{R},\ g : U \to \mathbb{R}$ and
x_0 is a limit point of $D \ni \lim_{x \to x_0} f(x) = y_0$,
y_0 is a limit point of $U \ni \lim_{y \to y_0} g(y) = e$,
and suppose that $f(D \setminus \{x_0\}) \subset U \setminus \{y_0\}$, then
$\lim_{x \to x_0}(g \circ f)(x) = e$

Definition

$x_0 \in D$ is an isolated point if $\exists\ r > 0 \ni (x_0 - r, x_0 + r) \cap D = \{x_0\}$

Fact

$x_0 \in D \Rightarrow x_0$ is either a limit point or an isolated point of D

Limits and Continuity Theorem

For $f : D \to \mathbb{R},\ x_0 \in D$, then
f is continuous at $x_0 \Leftrightarrow x_0$ is an isolated point of D or $\lim_{x \to x_0} f(x) = f(x_0)$

So f is continuous at $x_0 \Leftrightarrow \lim_{x \to x_0} f(x) = f(x_0)$

Fact in Review

If $h(x) = g(x)$ on $D \setminus \{x_0\}$ where $g : D \to \mathbb{R}$ is continuous on D, then
$\lim_{x \to x_0} h(x) = \lim_{x \to x_0} g(x) = g(x_0)$

Section 4.1: The Algebra of Derivatives

Definition

$x_0 \in \mathbb{R}, I \subset \mathbb{R} \ni I = (a, b)$ and $x_0 \in I \Rightarrow I$ is a neighborhood of x_0

Definition

$x_0 \in \mathbb{R}$ and I is a neighborhood of $x_0 \Rightarrow f : I \to \mathbb{R}$ is differentiable at x_0 IF $\lim_{x \to x_0} \frac{f(x)-f(x_0)}{x-x_0}$ exists. We say $f'(x_0) \coloneqq \lim_{x \to x_0} \frac{f(x)-f(x_0)}{x-x_0}$ and is called the derivative of f at x_0

Definition

If $f : I \to \mathbb{R}$ is differentiable at each $x_0 \in I$ then f is differentiable and $f' : I \to \mathbb{R}$ is the derivative of f

Definition

The line determined by $y = f(x_0) + f'(x_0)(x - x_0)$ is the tangent line to the graph of f at $(x_0, f(x_0))$

For $y_0 = f(x_0)$, $y - y_0 = f'(x_0)(x - x_0)$

Proposition 4.4

$n \in \mathbb{N}, f(x) = x^n \ \forall x \in I = \mathbb{R} \Rightarrow f : \mathbb{R} \to \mathbb{R}$ is differentiable and $f'(x) = nx^{n-1}$

Proposition 4.5

$x_0 \in \mathbb{R}, I$ is a neighborhood of $x_0, I = (a, b)$ and $f : I \to \mathbb{R}$ is differentiable at $x_0 \Rightarrow$ f is continuous at x_0

Theorem 4.6

$x_0 \in \mathbb{R}, I$ is a neighborhood of $x_0, f : I \to \mathbb{R}$ and $f : I \to \mathbb{R}$ is differentiable at x_0, then

(i) $f + g : I \to \mathbb{R}$ is differentiable at x_0 and
$(f + g)'(x_0) = f'(x_0) + g'(x_0)$

(ii) $fg : I \to \mathbb{R}$ is differentiable at x_0 and
$(fg)'(x_0) = f(x_0)g'(x_0) + f'(x_0)g(x_0)$

(iii) If $g(x) \neq 0 \ \forall x \in I$ then $\frac{1}{g} : I \to \mathbb{R}$ is differentiable at x_0 and

$(\frac{1}{g})'(x_0) = \frac{-g'(x_0)}{[g(x_0)]^2}$

(iv) If $g(x) \neq 0 \ \forall x \in I$ then $\frac{f}{g} : I \to \mathbb{R}$ is differentiable at x_0 and
$(\frac{f}{g})'(x_0) = \frac{f'(x_0)g(x_0) - f(x_0)g'(x_0)}{[g(x_0)]^2}$

Fact

For $x_0, \alpha \in \mathbb{R},\ (\alpha f)'(x_0) = \alpha f'(x_0)$

Fact

$f(x) = c \Rightarrow f'(x) = 0 \ \forall x \in D$

Proposition 4.7

$n \in \mathbb{Z}, D = \mathbb{R}$ if $n \geq 0$ and $D = \mathbb{R} \backslash \{0\}$ if $n < 0,$ then for $f : D \to \mathbb{R}$ defined by $f(x) = x^n,\ f$ is differentiable and $f'(x) = nx^{n-1}$

Corollary 4.8

$p, q : \mathbb{R} \to \mathbb{R}$ are polynomials, $D = \mathbb{R} \setminus \{x \mid q(x) = 0\}$, then
$\frac{p}{q} : D \to \mathbb{R}$ is differentiable

Section 4.2: Differentiating Inverses & Compositions

Theorem 4.11

Suppose $x_0 \in I,$ and $f : I \to \mathbb{R}$ is strictly monotone, continuous, differentiable at x_0, and $f'(x_0) \neq 0$. Let $J = f(I)$ then
$f^{-1} : J \to \mathbb{R}$ is differentiable at $y_0 = f(x_0)$ and $(f^{-1})'(y_0) = \frac{1}{f'(x_0)} = \frac{1}{f'(f^{-1}(y_0))}$

Corollary 4.12

Suppose $f : I \to \mathbb{R}$ is strictly monotone, differentiable, and f' is nonzero on I. Let $J = f(I)$, then $(f^{-1}) : J \to \mathbb{R}$ is differentiable and $\forall x \in J$
$(f^{-1})'(x) = \frac{1}{f'(f^{-1}(x))}$

Proposition 4.13

Let $g(g) = x^{\frac{1}{n}}$ where $n \in \mathbb{N}$ and $x > 0$, then
$g : (0, \infty) \to \mathbb{R}$ is differentiable and $g'(x) = \frac{1}{n} x^{\frac{1}{n}-1}$ $\forall x > 0$

Theorem 4.14 (Chain Rule)

Suppose $x_0 \in I$ with $f : I \to \mathbb{R}$ is differentiable. Say $f(I) \subseteq J$ and suppose $g : J \to \mathbb{R}$ is differentiable at $f(x_0)$, then $g \circ f : I \to \mathbb{R}$ is differentiable at x_0 and $(g \circ f)'(x_0) = g'(f(x_0)) \cdot f'(x_0)$

Proposition 4.15

For $r = \frac{m}{n}$ where $n \neq 0, m \in \mathbb{Z}, n \in \mathbb{N}$, set $h(x) = x^r$, where $x > 0$, then
h is differentiable and $h'(x) = rx^{r-1}$ $\forall x > 0$

Section 4.3: The Mean Value Theorem

Lemma 4.16

Suppose I is a neighborhood of x_0 and $f : I \to \mathbb{R}$ is differentiable at x_0.
If x_0 is a maximizer or a minimizer, then $f'(x_0) = 0$

Theorem 4.17 (Rolle's Theorem)

Suppose $f : [a, b] \to \mathbb{R}$ is continuous and $f : (a, b) \to \mathbb{R}$ is differentiable.
Assume that $f(a) = f(b)$, then $\exists\ x_0 \in (a, b) \ni f'(x_0) = 0$

Theorem 4.18 (Mean Value Theorem)

If $f : [a, b] \to \mathbb{R}$ is continuous and $f : (a, b) \to \mathbb{R}$ is differentiable, then
$\exists\ x_0 \in (a, b) \ni f'(x_0) = \frac{f(b)-f(a)}{b-a}$

Lemma 4.19

Suppose $I = (a, b)$ and $f : I \to \mathbb{R}$ is differentiable. Then
f is constant $\Leftrightarrow f'(x) = 0\ \forall\ x \in I$

Proposition 4.20 (Identity Criterion)

Suppose $g,\ h : I \to \mathbb{R}$ are differentiable. Then
$g = h + c \Leftrightarrow g'(x) = h'(x)\ \forall\ x \in I$

Corollary 4.21

(i) $f : I \to \mathbb{R}$ is differentiable $\ni f'(x) > 0\ \forall\ x \in I \Rightarrow f$ is strictly increasing

(ii) $f : I \to \mathbb{R}$ is differentiable $\ni f'(x) < 0\ \forall\ x \in I \Rightarrow f$ is strictly decreasing

Definition

Suppose $f : D \to \mathbb{R}$, then $x_0 \in D$ is a
(i) local maximizer if $\exists\ \delta > 0 \ni x_0$ is a maximizer for f on $D\ \cap (x_0 - \delta, x_0 + \delta)$
(ii) local minimizer if $\exists\ \delta > 0 \ni x_0$ is a minimizer for f on $D\ \cap (x_0 - \delta, x_0 + \delta)$

Definition

Suppose $f : I \to \mathbb{R}$ is differentiable on I. If $f' : I \to \mathbb{R}$ is differentiable on I, then define $f'' : I \to \mathbb{R}$ by $f''(x) = (f')'(x) = f^{(2)}(x)$ for each $x \in I$
Inductively define $f^{(k)} : I \to \mathbb{R}$, $k \in \mathbb{N}$

Theorem 4.22 (2nd Derivative Test)

Suppose $f, f' : I \to \mathbb{R}$ are differentiable and $x_0 \in I \ni f'(x_0) = 0$. Then
(i) $f''(x_0) > 0 \Rightarrow x_0$ is a local minimizer for f (concave up)
(ii) $f''(x_0) < 0 \Rightarrow x_0$ is a local maximizer for f (concave down)

Fact

If f is continuous on $[a, b]$ and f is differentiable on (a, b),
then f attains its max and min at either
(i) The endpoints a or b
(ii) $x_0 \in (a, b) \ni f'(x_0) = 0$

Section 4.4: Cauchy Mean Value Theorem

Theorem 4.23 (Cauchy Mean Value Theorem)

If $f : [a, b] \to \mathbb{R}$ and $g : [a, b] \to \mathbb{R}$ are continuous on $[a, b]$ and differentiable on (a, b) with $g'(x) \neq 0 \ \forall \ x \in (a, b)$, then $\exists \ x_0 \in (a, b) \ni \frac{f(b)-f(a)}{g(b)-g(a)} = \frac{f'(x_0)}{g'(x_0)}$

Lemma 1

If $h_1(x) = (x - x_0)^n$, then $h_1^{(k)}(x) = \begin{cases} \frac{n!}{(n-k)!} \cdot (x - x_0)^{n-k}, & 0 \leq k \leq n \\ 0, & k > n \end{cases}$

Theorem 4.24

Suppose $f : I \to \mathbb{R}$ has n derivatives on I and suppose at $x_0 \in I$ that $f^{(k)}(x_0) = 0$ for $0 \leq k \leq n-1$, then for each $x \in I$ with $x \neq x_0$ $\exists$ z strictly between x and x_0 $\ni$ $f(x) = \frac{f^{(n)}(z)}{n!} \cdot (x - x_0)^n$

Application

Let $g : I \to \mathbb{R}$ have $n + 1$ derivatives and set for $x_0 \in I$
$h(x) = \sum_{j=0}^{n} \frac{g^{(j)}(x_0)}{j!} \cdot (x - x_0)^j$
Then $g(x) = h(x) + \frac{g^{(n+1)}(z)}{(n+1)!} \cdot (x - x_0)^{n+1}$
(Taylor's Formula with Remainder)

Section 4.4 (sup): Trigonometric Differentiability

Fact 1

(i) $\lim_{\theta \to 0} \frac{\sin \theta}{\theta} = 1$
(ii) $\sin \theta \to 0$ as $\theta \to 0$
(iii) $\cos \theta \to 1$ as $\theta \to 0$
(iv) $\sin(x + y) = \sin x \cos y + \cos x \sin y$
(v) $\cos(x + y) = \cos x \cos y - \sin x \sin y$
(vi) $\frac{d}{dx} \sin x = \cos x$
(vii) $\frac{d}{dx} \cos x = -\sin x$

Section 6.1: Darboux Sums; Upper and Lower Integrals

Definition

The approximation of an area under a curve using rectangles over subintervals with $P = \{x_0, x_1, \cdots, x_n\}$ is a partition of $[a, b]$ with $a = x_0 < x_1 < \cdots < x_n = b$
Then the total area of the rectangles is

$$\sum_{i=1}^{n} M_i(x_i - x_{i-1})$$

Where $x_i - x_{i-1}$ is the width of the ith partition and M_i is the height
$M_i = sup\{f(x) \mid x \in [x_{i-1}, x_i]\}$

If f is continuous then $M_i = f(x_i^*)$ where x_i^* is a maximizer for f on $[x_{i-1}, x_i]$

We can also consider sums of the form

$$\sum_{i=1}^{n} m_i(x_i - x_{i-1})$$

Where $m_i = inf\{f(x) \mid x \in [x_{i-1}, x_i]\}$

Definition

$U(f, P) = \sum_{i=1}^{n} M_i(x_i - x_{i-1})$ is the Upper Darboux Sum

$L(f, P) = \sum_{i=1}^{n} m_i(x_i - x_{i-1})$ is the Lower Darboux Sum

Fact

$L(f, P) \leq U(f, P)$

Lemma 6.1

Suppose $f : [a, b] \to \mathbb{R}$ is bounded and $\exists\ m, M \in \mathbb{R} \ni m \leq f(x) \leq M\ \forall\ x \in [a, b]$

Then for any partition P of $[a, b]$ we have

$$m(b - a) \leq L(f, P) \leq U(f, P) \leq M(b - a)$$

Definition

Given a partition P of $[a, b]$, another partition P^* of $[a, b]$ is a refinement of P if $P \subset P^*$; i.e., if $P = \{x_0, x_1, \cdots, x_n\}$ then each x_i is in P^* also

"A nontrivial refinement takes all the points in the given partition and adds at least one more"

Definition

Let $P_i = P^* \cap [x_{i-1}, x_i]$, then

$U(f, P^*) = \sum_{i=1}^{n} U(f, P_i)$

$L(f, P^*) = \sum_{i=1}^{n} L(f, P_i)$

Lemma 6.2 (Refinement Lemma)

Suppose $f : [a, b] \to \mathbb{R}$ is bounded and P is a partition of $[a, b]$ and P^* is a refinement of P, then

$$L(f, P) \leq L(f, P^*) \leq U(f, P^*) \leq U(f, P)$$

Definition

Let P_1, P_2 be two partitions of $[a, b]$ and set $P^* = P_1 \cup P_2$, then P^* refines both P_1 and P_2 and is called a common refinement of P_1 and P_2

Lemma 6.3

Suppose $f : [a, b] \to \mathbb{R}$ is bounded, P_1, P_2 are partitions of $[a, b]$, then

$$L(f, P_1) \leq U(f, P_2)$$

"Any lower sum is less than or equal to any upper sum"

Definition

Lower Integral

$$\underline{\int_a^b} f \equiv sup\{L(f, P) \mid P \text{ is a partition of } [a, b]\}$$

Upper Integral

$$\overline{\int_a^b} f \equiv inf\{U(f, P) \mid P \text{ is a partition of } [a, b]\}$$

Lemma 6.4

Suppose $f : [a, b] \to \mathbb{R}$ is bounded, then

$$\underline{\int_a^b} f \leq \overline{\int_a^b} f$$

Fact

A telescoping sum is a sum in which subsequent terms cancel each other and only leave the first and last terms

Example

$m(b-a) = m\sum_{i=1}^{n}(x_i - x_{i-1}) =$
$m[(x_1-x_0)+(x_2-x_1)+\cdots+(x_{n-1}-x_{n-2})+(x_n-x_{n-1})] = m(x_n-x_0) = m(b-a)$

Definition

Dirichlet's function: $f : [0, 1] \to \mathbb{R}$

$$\begin{cases} 0, & x \in [0,1] \ni x \in \mathbb{Q} \\ 1, & x \in [0,1] \ni x \in \mathbb{R} \setminus \mathbb{Q} \end{cases}$$

Section 6.2: The Archimedes-Riemman Theorem

Fact (Comparison Lemma for Positive Sequences)

Suppose $\{a_n\}, \{b_n\}$ satisfy $0 \leq a_n \leq b_n \ \forall \ n \in \mathbb{N}$ and $b_n \to 0,$ then $a_n \to 0$

Fact (Order Preserving Property for Sequences)

Suppose $\{a_n\}, \{b_n\}$ with $a_n \to a, \ b_n \to b$ and $\forall \ n \in \mathbb{N} \ a_n \leq b_n,$ then $a \leq b$

Definition

Suppose $f : [a, b] \to \mathbb{R}$ is bounded, then we say f is integrable (on $[a, b]$) if

$$\underline{\int_a^b} f = \overline{\int_a^b} f = I$$

If this is the case then we define

$$\int_a^b f = I = \text{the integral of f on } [a,b]$$

Lemma 6.7

Suppose $f:[a,b]\to\mathbb{R}$ is bounded, and P is a partition of $[a,b]$, then

$$(*)\ L(f,P) \le \underline{\int_a^b} f \le \overline{\int_a^b} f \le U(f,P) \Rightarrow$$

$$\text{(i) } 0 \le \overline{\int_a^b} f - \underline{\int_a^b} f \le U(f,P) - L(f,P)$$

$$\text{(ii) } 0 \le U(f,P) - \overline{\int_a^b} f \le U(f,P) - L(f,P)$$

$$\text{(ii) } 0 \le \underline{\int_a^b} f - L(f,P) \le U(f,P) - L(f,P)$$

Theorem 6.8 (Archimedes-Riemann Theorem)

Suppose $f:[a,b]\to\mathbb{R}$ is bounded, then

f is integrable $\Leftrightarrow$ $\exists$ a sequence of partitions $\{P_n\}$ of $[a,b]$ $\ni \lim_{n\to\infty}[U(f,P_n)-L(f,P_n)] = 0$

Definition

$f:[a,b]\to\mathbb{R}$ is bounded, $\{P_n\}$ is a sequence of partitions of $[a,b]$, then $\{P_n\}$ is an Archimedian Sequence if $U(f,P_n) - L(f,P_n) \to 0$

Definition

A partition of $P_n = \{x_0, x_1, \cdots, x_n\}$ of $[a,b]$ $\ni$ $x_i = a + i(\frac{b-a}{n})$ is a regular partition of $[a,b]$

Definition

Let $P_n = \{x_0, x_1, \cdots, x_n\}$ is a partition of $[a,b]$, then
$\text{gap}(P) = \max_{1\le i\le n}(x_i - x_{i-1})$

Fact 1

$f:[a,b]\to\mathbb{R}$ is bounded and f is monotonically increasing $\Rightarrow$ f is integrable

Definition

$f : [a, b] \to \mathbb{R}$ is a step function if $\exists\ P^* = \{z_0, z_1, \cdots, z_k\}$ of $[a, b]$ and
$c_1, \cdots, c_k \in \mathbb{R} \ni f(x) = c_k, x \in (z_{i-1}, z_i)$
Note: $z_0 = a, z_k = b$

Fact 2

$f : [a, b] \to \mathbb{R}$ is a step function $\Rightarrow f$ is integrable

Fact 3

$f : [a, b] \to \mathbb{R}$ is Lipschitz continuous $\Rightarrow f$ is integrable

Leibniz Notation

$f : [a, b] \to \mathbb{R}$ is integrable

$$\int_a^b f = \int_a^b f(x)dx = \int_a^b f(*)d*$$

Section 6.3: Additivity, Monotonicity, Linearity

Theorem 6.12 (Additivity)

Suppose $f : [a, b] \to \mathbb{R}$ is integrable and $c \in (a, b)$, then f is integrable on $[a, c]$ and $[c, b]$ and

$$\int_a^b f = \int_a^c f + \int_c^b f$$

Theorem 6.13 (Monotonicity)

Suppose $f : [a, b] \to \mathbb{R}$ and $g : [a, b] \to \mathbb{R}$ are integrable and that
$f(x) \leq g(x)\ \forall\ x \in [a, b]$, then

$$\int_a^b f \leq \int_a^b g$$

Lemma 6.14

$f : [a, b] \to \mathbb{R}$ and $g : [a, b] \to \mathbb{R}$ are bounded and P is a partition of $[a, b]$, then

$$L(f, P) + L(g, P) \leq L(f + g, P)$$

$$U(f + g, P) \leq U(f, P) + U(g, P)$$

And for any $\alpha \in \mathbb{R}$

$$\begin{cases} U(\alpha f, P) = \alpha U(f, P) \text{ and } L(\alpha f, P) = \alpha L(f, P), & \alpha \geq 0 \\ U(\alpha f, P) = \alpha L(f, P) \text{ and } L(\alpha f, P) = \alpha U(f, P), & \alpha < 0 \end{cases}$$

Theorem 6.15 (Linearity)

Let $f : [a, b] \to \mathbb{R}$ and $g : [a, b] \to \mathbb{R}$ be integrable. Then for $\alpha, \beta \in \mathbb{R}$
$\alpha f + \beta g$ is integrable and

$$\int_a^b (\alpha f + \beta g) = \alpha \int_a^b f + \beta \int_a^b g$$

Corollary 6.16

Let $f : [a, b] \to \mathbb{R}$ and $|f| : [a, b] \to \mathbb{R}$ be integrable. Then

$$\left| \int_a^b f(x)dx \right| \leq \int_a^b |f(x)|dx$$

Fact A

Suppose a set $S \subseteq \mathbb{R}$ and S is bounded, let $\alpha S = \{\alpha x \mid x \in S\}$, then

$$\begin{cases} \sup\ \alpha S = \alpha \sup\ S, & \alpha \geq 0 \\ \sup\ \alpha S = \alpha \inf\ S, & \alpha < 0 \end{cases}$$

Section 6.4: Continuity and Integrability

Theorem 6.18

$f : [a, b] \to \mathbb{R}$ is continuous $\Rightarrow f$ is integrable on $[a, b]$

Theorem 6.19

$f : [a, b] \to \mathbb{R}$ is bounded and continuous on $(a, b) \Rightarrow f$ is integrable on $[a, b]$
and $\int_a^b f$ does not depend on $f(a), f(b)$

Section 6.4 (sup): Continuity and Integrability

Theorem (6.3, 6)

$f : [a, b] \to \mathbb{R}$ is bounded and $a < c < b$. If f is integrable on $[a, c]$ and $[c, b]$, then f is integrable on $[a, b]$

Definition

$f : [a, b] \to \mathbb{R}$ is bounded and $a < c_1 < c_2 < \cdots < c_k < b$, then f is piecewise integrable if
f is integrable on each of $[a, c_1], [c_1, c_2], \cdots, [c_k, b]$
So f is piecewise integrable $\Rightarrow f$ is integrable

Corollary

Suppose f is bounded and $a < c_1 < c_2 < \cdots < c_k < b$, then
(i) f is continuous on $(a, c_1), (c_1, c_2), \cdots, (c_k, b) \Rightarrow f$ is integrable (f is piecewise continuous)
(ii) f is monotone on $(a, c_1), (c_1, c_2), \cdots, (c_k, b) \Rightarrow f$ is integrable (f is piecewise monotone)

Section 6.5: First Fundamental Theorem of Calculus

Theorem 6.22 (FTC 1)

$f : [a, b] \to \mathbb{R}$ is continuous and is differentiable on (a, b) and
$f' : (a, b) \to \mathbb{R}$ is continuous and bounded, then

$$\int_a^b F'(x)dx = F(b) - F(a)$$

Section 6.6: Second Fundamental Theorem of Calculus

Theorem 6.26 (MVT for Integrals)

Let $f : [a, b] \to \mathbb{R}$ be continuous. Then $\exists\ c \in (a, b) \ni f(c)(b - a) = \int_a^b f$

$$f(c) = \frac{1}{b-a} \int_a^b f$$

Proposition 6.27

For $f : [a, b] \to \mathbb{R}$ is continuous, set $F(x) = \int_a^x f(t)dt$ for $x \in [a, b]$,
then F is continuous on $[a, b]$

Theorem 6.29 (FTC 2)

$f : [a, b] \to \mathbb{R}$ is continuous, $F(x) = \int_a^x f(t)dt$, then F is differentiable on (a, b) and $F'(x) = f(x)$ for $x \in (a, b)$

Corollary 6.30

Suppose that the function $f : [a, b] \to \mathbb{R}$ is continuous. Then

$$\frac{d}{dx}\left[\int_x^b f\right] = -f(x) \text{ for all } x \in (a, b)$$

Definition

Let $f : [a, b] \to \mathbb{R}$ be integrable. Let $c, d \in [a, b] \ni c < d$
Set $\int_d^c f = -\int_c^d f, \int_c^c f = 0$

Fact

For any $x_1, x_2, x_3 \in [a, b]$

$$\int_{x_1}^{x_3} f = \int_{x_1}^{x_2} f + \int_{x_2}^{x_3} f$$

Corollary 6.31

$I = (a, b), f$ is continuous on $I, x_0 \in I \Rightarrow$

$$\frac{d}{dx}\int_{x_0}^{x} f = f(x) \ \forall \ x \in I$$

Corollary 6.32

$I = (a, b), f$ is continuous on $J = (c, d), \phi : J \to I$ is differentiable and $\phi(J) \subseteq I$, then for $x_0 \in I$

$$\frac{d}{dx}\int_{x_0}^{\phi(x)} f = f(\phi(x)) \cdot \phi'(x) \ \forall \ x \in J$$

Section 6.6 (sup): The Logarithm and Exponential Functions

Fact

Divergence of the Harmonic Series:

$S_n = \sum_{k=1}^{n} \frac{1}{k}$ is a harmonic series and diverges (does not converge and grows without bound)

Definitions

$$\ln x = \int_1^x \frac{1}{t} dt, \ x > 0$$
$$\frac{d}{dx} \ln x = \frac{1}{x}$$
$$\ln 1 = \int_1^1 \frac{1}{t} dt = 0$$

Proposition 1

(a) $\ln x$ is strictly increasing
(b) $\ln x \to \infty; \ x \to \infty$
(c) $\ln x \to -\infty; \ x \to 0$

Proposition 2

$a, b > 0,\ r \in \mathbb{Q},\ r > 0$

(i) $\ln ab = \ln a + \ln b$
(ii) $\ln a^r = r \ln a$
(iii) $\ln \frac{1}{b} = -\ln b$

Definition

$f(x) = \ln x \Rightarrow f^{-1}(x) = \exp x = e^x$

$\exp x$ is strictly increasing and

(1) $\ln 1 = 0 \Rightarrow \exp 0 = 1$
(2) $D(f) = (0, +\infty) \Rightarrow R(f^{-1}) = (0, +\infty)$
(3) $R(f) = (-\infty, +\infty) \Rightarrow D(f^{-1}) = (-\infty, +\infty)$

Proposition 3

(i) $\exp(a + b) = \exp a \cdot \exp b$
(ii) $\exp ab = (\exp a)^b,\ b \in \mathbb{Q},\ b > 0$
(iii) $\frac{d}{dx} \exp x = \exp x$
(iv) $\exp(-a) = \frac{1}{\exp a}$

Definition

$$e = \exp 1 = e^1 = \sum_{k=0}^{\infty} \frac{1}{k!}$$

Fact

$x = b^y \Rightarrow y = \log_b(x)$

Section 8.1: Taylor Polynomials

Definition

$I = (a, b),\ x_0 \in I.\ f : I \to \mathbb{R},\ g : I \to \mathbb{R},$ have contact of order n at x_0 provided that f and g have derivatives of order n at x_0 and $f^{(k)}(x_0) = g^{(k)}(x_0)$,

$k = 0, 1 \cdots, n$

Fact

$\frac{d^k}{dx^k}(x - x_0)^l = \begin{cases} k!, & k = l \\ 0, & k \neq l \end{cases}$

Proposition 8.2

$I = (a, b),\ x_0 \in I,\ f : I \to \mathbb{R}$ has n derivatives (at x_0), set:

$$P_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(x_0)(x - x_0)^k}{k!}$$

Then P_n is the unique polynomial of degree $\leq n \ni P_n$ has a contact of order n at x_0 with f (P_n is the Taylor Polynomial for f)

Fact

$$f(x) = e^x \Rightarrow P_n(x) = \sum_{k=0}^{n} \frac{x^k}{k!}$$

Fact

$$f(x) = \sin x \Rightarrow P_n(x) = \sum_{k=0}^{n} \frac{(-1)^k}{(2k+1)!} \cdot x^{2k+1}$$

Section 8.2: Lagrange Remainder Theorem

Lemma 8.7

$I = (a, b),\ x_0 \in I,\ h : I \to \mathbb{R}$ has $n + 1$ derivatives and $h^{(k)}(x_0) = 0$, $k = 0, 1 \cdots, n$. Then for $x \in I,\ x \neq x_0\ \exists\ z$ strictly between x and $x_0 \ni$

$$h(x) = \frac{h^{(n+1)}(z)(x - x_0)^{n+1}}{(n+1)!}$$

Theorem 8.8 (Lagrange Remainder Theorem)

$I = (a, b),\ x_0 \in I,\ f : I \to \mathbb{R}$ has $n + 1$ derivatives. Then for $x \in I$
$x \neq x_0$ $\exists$ c strictly between x and x_0 $\ni$

$$f(x) = P_n(x) + \frac{f^{(n+1)}(c)(x - x_0)^{n+1}}{(n + 1)!} = \sum_{k=0}^{n} \frac{f^{(k)}(x_0)(x - x_0)^k}{k!} + \frac{f^{(n+1)}(c)(x - x_0)^{n+1}}{(n + 1)!}$$

Section 8.3: Convergence of Taylor Polynomials

Lemma 8.20

Suppose $\{C_n\} \subset \mathbb{R} \ni \lim_{n\to\infty} \frac{|C_{n+1}|}{|C_n|} = l$, then

(i) $l < 1 \Rightarrow C_n \to 0$
(ii) $l > 1 \Rightarrow \{C_n\}$ is unbounded

Theorem 8.14

$I = (a, b),\ f : I \to \mathbb{R}$ has derivatives up to all orders,
$x_0 \in I,\ r \ni [x_0 - r, x_0 + r] \subset I$. Suppose also that $\exists$ m $\ni$ for each $n \in \mathbb{N}$,
$x \in [x_0 - r, x_0 + r]$, $|f^{(n)}(x)| \le M^n$, then

$$f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(x_0)}{k!}(x - x_0)^k,\ x \in [x_0 - r, x_0 + r]$$

Section 8.7: The Weirstrass Approximation Theorem

Lemma 8.24

For each $x \in \mathbb{R}$ and each $n \in \mathbb{N},\ n \ge 2$

$$\sum_{k=0}^{n}(x - kn^{-1})^2 \binom{n}{k} x^k(1 - x)^{n-k} - \frac{x(1 - x)}{n}$$

Theorem 8.23 (Weirstrass Approximation)

Let $I = [a, b]$ and suppose $f : I \to \mathbb{R}$ is continuous. Then for each $\epsilon > 0$ there exists a polynomial $P : \mathbb{R} \to \mathbb{R} \ni |f(x) - P(x)| < \epsilon \ \forall \ x \in I$

Where

$$P(x) = \sum_{k=0}^{n} f\left(\tfrac{k}{n}\right) \binom{n}{k} x^k (1-x)^{n-k}$$

Fact

Let f be continuous on $I = [a, b]$, then there exists a sequence of polynomials $\{P_n\} \ni \{P_n\}$ converges uniformly to f

Section 9.1: Sequences and Series

Definition

$\{a_n\}$ us a Cauchy Sequence if given $\epsilon > 0 \ \exists \ N \ni n, \ m \geq N \Rightarrow |a_n - a_m| < \epsilon$

Proposition 9.2

$\{a_n\}$ converges $\Rightarrow \{a_n\}$ is a Cauchy Sequence

Lemma 9.3

$\{a_n\}$ is a Cauchy Sequence $\Rightarrow \{a_n\}$ is bounded

Theorem 9.4

$\{a_n\}$ converges $\Leftrightarrow \{a_n\}$ is a Cauchy Sequence

Definition

For a given $\{a_n\}$ set $S_n = \sum_{k=1}^{n} a_k$

If $\{S_n\}$ converges we write

$$\sum_{k=1}^{\infty} a_k = \lim_{n\to\infty} S_n = \lim_{n\to\infty} \sum_{k=1}^{n} a_k$$

Fact

The Harmonic Series

$$\sum_{k=1}^{\infty} \frac{1}{k} \text{ diverges}$$

Proposition 9.5

$$\sum_{n=1}^{\infty} a_n \text{ converges } \Rightarrow \lim_{n\to\infty} a_n = 0$$

Proposition 9.6

$$|r| < 1 \Rightarrow \sum_{k=0}^{\infty} r^k = \frac{1}{1-r}$$

Theorem 9.7

Suppose $a_k \geq 1$, then

$$\sum_{k=0}^{\infty} a_k \text{ converges } \Leftrightarrow \exists\ M \ni \sum_{k=1}^{n} a_k \leq M$$

Corollary 9.8 (Comparison Test)

$0 \leq a_k \leq b_k$

(i) $\sum_{k=1}^{\infty} b_k$ converges $\Rightarrow \sum_{k=1}^{\infty} a_k$ converges

(ii) $\sum_{k=1}^{\infty} a_k$ diverges $\Rightarrow \sum_{k=1}^{\infty} b_k$ diverges

Corollary 9.11 (Integral Test)

Suppose $a_k \geq 0$ and suppose $\exists\ f : [0, +\infty) \to \mathbb{R},\ , f$ is continuous, $f(k) = a_k,$ and f is monotonically decreasing, then

$$\int_1^n f \leq M \text{ for some } M\ \forall\ n \in \mathbb{N} \Leftrightarrow \sum_{k=1}^{\infty} a_k \text{ converges}$$

Corollary 9.13 (p-Test)

$$\sum_{k=1}^{\infty} \frac{1}{k^p} \text{ converges} \Leftrightarrow p > 1$$

Theorem 9.15 (Alternating Series Test)

Suppose $a_n \geq 0$, $\{a_n\}$ is monotonically decreasing, and $a_n \to 0$, then

$$\sum_{k=1}^{\infty}(-1)^{k+1}a_k \text{ converges}$$

Theorem 9.17 (Cauchy Convergence Criterion for Series)

$$\sum_{k=1}^{\infty} a_k \text{ converges} \Leftrightarrow \text{Given } \epsilon > 0 \ \exists \ N \ni n \geq N \text{ implies for each } k \in \mathbb{N} \ |a_{n+1}+\cdots+a_{n+k}| < \epsilon$$

Definition

The series $\sum_{k=1}^{\infty} a_k$ is said to converge absolutely provided that the series $\sum_{k=1}^{\infty} |a_k|$ converges

Corollary 9.18 (Absolute Convergence Test)

$$\sum_{k=1}^{\infty} |a_k| \text{ converges} \Rightarrow \sum_{k=1}^{\infty} a_k \text{ converges (absolutely)}$$

Theorem 9.20

Suppose for $\sum_{k=1}^{\infty} a_k$ that $\exists \ N$ and $r \in [0,1] \ni \ |a_{n+1}| \leq r|a_n| \ \forall \ n \geq N$, then

$$\sum_{k=1}^{\infty} a_k \text{ converges absolutely}$$

Corollary 9.21 (Ratio Test)

For $\sum_{k=1}^{\infty} a_k$, suppose that

$$\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = l$$

(i) $l < 1 \Rightarrow \sum_{k=1}^{\infty} a_k$ converges (absolutely)

(ii) $l > 1 \Rightarrow \sum_{n=k}^{\infty} a_k$ diverges

Section 9.2: Pointwise Convergence of Sequences of Functions

Definition

$\{f_n\} : D \to \mathbb{R},\ f : D \to \mathbb{R}$, then $\{f_n\}$ converges pointwise to f
($f_n \to f$ pointwise) if $f_n(x) \to f(x)$ for each $x \in D$

Section 9.3: Uniform Convergence

Definition

$\{f_n\} : D \to \mathbb{R},\ f : D \to \mathbb{R}$, then $\{f_n\}$ converges to f uniformly
($f_n \to f$ uniformly) if given $\epsilon > 0\ \exists\ N \ni n \geq N \Rightarrow |f_n(x) - f(x)| < \epsilon\ \forall\ x \in D$

Definition

$\{f_n\} : D \to \mathbb{R}$ is uniformly cauchy if given $\epsilon > 0\ \exists\ N \ni n \geq N$ and $k \in \mathbb{N} \Rightarrow |f_{n+k}(x) - f_n(x)| < \epsilon\ \forall\ x \in D$

Theorem 9.29 (Weirstrass Uniform Convergence Criterion)

$\{f_n\} : D \to \mathbb{R},\ f : D \to \mathbb{R}$, $f_n \to f$ uniformly $\Leftrightarrow \{f_n\}$ is uniformly cauchy

Section 9.4: The Uniform Limit of Functions

Theorem 9.31

$\{f_n : D \to \mathbb{R}\},\ f : D \to \mathbb{R},\ f_n$ is continuous and f_n converges to f uniformly $\Rightarrow f$ is continuous on D

Theorem 9.32

$\{f_n : [a,b] \to \mathbb{R}\},\ f : [a,b] \to \mathbb{R},\ f_n$ is integrable and f_n converges to f uniformly $\Rightarrow f$ is integrable and $\int_a^b f_n \to \int_a^b f$

Theorem 9.33

$I = (a,b),\ \{f_n : I \to \mathbb{R}\}$ is continuously differentiable on I (f_n is continuous on I and f_n' is differentiable on I),
$f : I \to \mathbb{R}$ and (i) $f_n(x) \to f(x)$ for each $x \in I$
(ii) $\exists\ g : I \to \mathbb{R} \ni f_n' \to g$ uniformly on I
then f is continuously differentiable on I and $f'(x) = g(x)$ for each $x \in I$

Theorem 9.34

$I = (a,b),\ \{f_n : I \to \mathbb{R}\}$ is continuously differentiable and (i) $\{f_n\}$ converges uniformly to $f : I \to \mathbb{R}$ and (ii) $\{f_n'\}$ is uniformly Cauchy, then $f : I \to \mathbb{R}$ is continuously differentiable and $f_n' \to f'$ uniformly

Section 9.5: Power Series

$$\text{Define } f : D \to \mathbb{R} \text{ by } f(x) = \lim_{n\to\infty}\left[\sum_{k=0}^{\infty} c_k x^k\right] = \sum_{k=0}^{\infty} c_k x^k \text{ for each } x \in D$$

Section 10.1: The Linear Structure of $\mathbb{R}^n$ and the Scalar Product

Properties

$\mathbf{v} \in \mathbb{R}^n,\ \mathbf{V} = (v_1, \cdots, v_n)$
(i) $\mathbf{u} = \mathbf{v} \Leftrightarrow u_i = v_i,\ i = 1, \cdots, n$
(ii) $\mathbf{u} + \mathbf{v} = (u_1 + v_1, \cdots, u_n + v_n)$
(iii) $\alpha \in \mathbb{R},\ \alpha\mathbf{u} = (\alpha u_1, \cdots, \alpha u_n)$
(iv) $\mathbf{u} - \mathbf{v} = \mathbf{u} + (-1)\mathbf{v}$

Proposition 10.1

$\mathbf{u},\ \mathbf{v},\ \mathbf{w} \in \mathbb{R}^n$, then

$(\mathbf{u}+\mathbf{v})+\mathbf{w} = \mathbf{u}+(\mathbf{v}+\mathbf{w})$
$\mathbf{u}+\mathbf{0} = \mathbf{u},\ \mathbf{0} = (0,\cdots,0)$
$\mathbf{u}-\mathbf{u} = \mathbf{0}$
$\mathbf{u}+\mathbf{v} = \mathbf{v}+\mathbf{u}$

Definition (Scalar Product)

$\mathbf{u},\ \mathbf{v} \in \mathbb{R}^n$, then
$\langle \mathbf{u},\mathbf{v}\rangle = u_1 \cdot v_1 + \cdots + u_n \cdot v_n = \sum_{i=1}^n u_i v_i$

Proposition 10.2

$\mathbf{u},\ \mathbf{v},\ \mathbf{w} \in \mathbb{R}^n,\ \alpha,\ \beta \in \mathbb{R}$, then

(i) $\langle \mathbf{u},\mathbf{u}\rangle \geq 0$ and $\langle \mathbf{u},\mathbf{u}\rangle = 0 \Leftrightarrow \mathbf{u} = \mathbf{0}$
(ii) $\langle \mathbf{u},\mathbf{v}\rangle = \langle \mathbf{v},\mathbf{u}\rangle$
(iii) $\langle \alpha\mathbf{u}+\beta\mathbf{w},\mathbf{v}\rangle = \alpha\langle \mathbf{u},\mathbf{v}\rangle + \beta\langle \mathbf{w},\mathbf{v}\rangle$

Definition (Norm)

$\mathbf{w} \in \mathbb{R}^n,\ ||\mathbf{w}|| = \langle \mathbf{w},\mathbf{w}\rangle^{\frac{1}{2}} = \sqrt{w_1^2+\cdots+w_n^2}$

Definition (Distance)

$\mathbf{u},\ \mathbf{v} \in \mathbb{R}^n,\ \mathrm{dist}(\mathbf{u},\mathbf{v}) = ||\mathbf{u}-\mathbf{v}|| = \sqrt{(u_1-v_1)^2+\cdots+(u_n-v_n)^2}$

Proposition 10.3

$\mathbf{u},\ \mathbf{v} \in \mathbb{R}^2$, then $\langle \mathbf{u},\mathbf{v}\rangle = ||\mathbf{u}|| \cdot ||\mathbf{v}|| \cos\theta$

Definition (Orthogonality)

$\mathbf{u},\mathbf{v} \in \mathbb{R}^n$, then $\mathbf{u}$ and $\mathbf{v}$ are orthogonal iff. $\langle \mathbf{u},\mathbf{v}\rangle = 0$

Lemma 10.4 (Orthogonality in $\mathbb{R}^2$)

$\mathbf{u},\ \mathbf{v} \in \mathbb{R}^2$, then $\mathbf{u} \perp \mathbf{v} \Leftrightarrow ||\mathbf{u}+\mathbf{v}||^2 = ||\mathbf{u}||^2 + ||\mathbf{v}||^2$

Lemma 10.5 (Orthogonality in $\mathbb{R}^n$)

$\mathbf{u},\ \mathbf{v} \in \mathbb{R}^n,\ \mathbf{v} \neq \mathbf{0},\ \lambda = \frac{\langle \mathbf{u}, \mathbf{v} \rangle}{\langle \mathbf{v}, \mathbf{v} \rangle}$, then for $\mathbf{w} = \mathbf{u} - \lambda \mathbf{v}$, $\mathbf{w} \perp \mathbf{v}$ and $\mathbf{u} = \mathbf{w} + \lambda \mathbf{v}$

Theorem 10.6 (Cauchy-Schwartz Inequality)

$\mathbf{u},\ \mathbf{v} \in \mathbb{R}^n$, then $|\langle \mathbf{u}, \mathbf{v} \rangle|| \leq ||\mathbf{u}|| \cdot ||\mathbf{v}||$

Fact

$\alpha \in \mathbb{R}$ then $||\alpha \mathbf{u}|| = |\alpha| \cdot ||\mathbf{u}||$

Theorem 10.7 (Triangle Inequality in $\mathbb{R}^n$)

$\mathbf{u},\ \mathbf{v} \in \mathbb{R}^n$, then $||\mathbf{u} + \mathbf{v}|| \leq ||\mathbf{u}|| + ||\mathbf{v}||$

Section 10.2: Convergence of Sequences in $\mathbb{R}^n$

Definition

A sequence in $\mathbb{R}^n$ is a function from $\mathbb{N}$ to $\mathbb{R}^n$. We denote the functional value for each k by $\mathbf{u}_k$. The set of all such functional values is denoted by $\{\mathbf{u}_k\}_{k=1}^{\infty}$

Definition

$\{\mathbf{u}_k\}_{k=1}^{\infty} \subset \mathbb{R}^n,\ \mathbf{u} \in \mathbb{R}^n$, then we say $\{\mathbf{u}_k\}_{k=1}^{\infty}$ converges to $\mathbf{u}$ (Namely, $\mathbf{u}_k \to \mathbf{u}$ and $\lim_{k\to\infty} \mathbf{u}_k = \mathbf{u}$) If given $\epsilon > 0\ \exists\ N \ni k \geq N \Rightarrow$ $||\mathbf{u}_k - \mathbf{u}|| < \epsilon$ (Namely, $\text{dist}(\mathbf{u}_k, \mathbf{u}) < \epsilon$)

Corollary 10.8

$\mathbf{u},\ \mathbf{v}\ , \mathbf{w} \in \mathbb{R}^n$, then $||\mathbf{u} - \mathbf{v}|| \leq ||\mathbf{u} - \mathbf{w}|| + ||\mathbf{w} - \mathbf{v}||$

Fact 1

$\mathbf{u}_k \to \mathbf{u}$ in $\mathbb{R}^n \Leftrightarrow ||\mathbf{u}_k - \mathbf{u}|| \to 0$

Fact 2

$\mathbf{u}_k \to \mathbf{u},\ \mathbf{u}_k \to \mathbf{u}' \Rightarrow \mathbf{u} = \mathbf{u}'$

Definition (ith Component Projection Function)

$P_i : \mathbb{R}^n \to \mathbb{R},\ i = 1, \cdots, n$ is $P_i(\mathbf{u}) = \mathbf{u}_i$

Note

(i) $\mathbf{u} = (u_1, \cdots, u_n) = (P_1(\mathbf{u}), \cdots, P_n(\mathbf{u}))$
(ii) $P_i(\alpha\mathbf{u} + \beta\mathbf{v}) = \alpha P_i(\mathbf{u}) + \beta P_i(\mathbf{v})$
(iii) $|P_i(\mathbf{u})| \leq ||\mathbf{u}||$

Definition

$\{\mathbf{u}_k\} \in \mathbb{R}^n$ converges to $\mathbf{u} \in \mathbb{R}^n$ componentwise if $\lim_{k\to\infty} P_i(\mathbf{u}_k) = P_i(\mathbf{u})$

Theorem 10.9 (Componentwise Convergence Criterion)

$\{\mathbf{u}_k\} \subset \mathbb{R}^n,\ \mathbf{u} \in \mathbb{R}^n$, then $\mathbf{u}_k \to \mathbf{u} \Leftrightarrow \mathbf{u}_k$ converges to $\mathbf{u}$ componentwise

Theorem 10.10

$\{\mathbf{u}_k\},\ \{\mathbf{v}_k\} \subset \mathbb{R}^n,\ \mathbf{u},\ \mathbf{v} \in \mathbb{R}^n,\ \mathbf{u}_k \to \mathbf{u},\ \mathbf{v}_k \to \mathbf{v}$, then for $\alpha,\ \beta \in \mathbb{R}$,
$\alpha\mathbf{u}_k + \beta\mathbf{v}_k \to \alpha\mathbf{u} + \beta\mathbf{v}$

Section 10.3: Open and Closed Sets in $\mathbb{R}^n$

Definition (Open Ball of Radius r)

$\mathbf{u} \in \mathbb{R}^n,\ r > 0$, $B_r(\mathbf{u}) = \{\mathbf{v} \in \mathbb{R}^n \mid ||\mathbf{v} - \mathbf{u}|| < r\}$

Definition (Interior Point)

$A \subset \mathbb{R}^n,\ \mathbf{u} \in A,\ \mathbf{u}$ is an interior point of A if $\exists\ r > 0 \ni B_r(\mathbf{u}) \subset A$

Definition

$\text{int}(A)$ is the set of all interior points of A and is called the interior of A

$\text{int}(A) = \{\mathbf{u} \in \mathbb{R}^n \mid \mathbf{u} \text{ is an interior point of } A\}$

Definition (Open)

$A \subset \mathbb{R}^n$, then A is open if $\text{int}(A) = A$

Proposition 10.13

$\mathbf{u} \in \mathbb{R}^n,\ r > 0,\ \text{then } B_r(\mathbf{u}) \text{ is open}$

Definition (Closed)

$A \subset \mathbb{R}^n$ is closed if whenever $\{\mathbf{u}_k\} \subset A,\ \mathbf{u}_k \to \mathbf{u},\ \text{then } \mathbf{u} \in A$

De Morgan's Laws

$\{A_s\}_{s\in S},\ \text{each } A_s \subset \mathbb{R}^n$
(a) $\mathbb{R}^n \setminus (\cap_{s\in S} A_s) = \cup_{s\in S}(\mathbb{R}^n \setminus A_s)$
(b) $\mathbb{R}^n \setminus (\cup_{s\in S} A_s) = \cap_{s\in S}(\mathbb{R}^n \setminus A_s)$

Theorem 10.16

$A \subset \mathbb{R}^n$, then A is open $\Leftrightarrow \mathbb{R}^n \setminus A$ is closed

Proposition 10.17

(i) $O_s \subset \mathbb{R}^n,\ O_s$ is open, $s \in S \Rightarrow \cup_{s\in S} O_s = O$ is open (Infinite union of open sets is open)
(ii) $C_s \subset \mathbb{R}^n,\ C_s$ is closed, $s \in S \Rightarrow \cap_{s\in S} C_s = C$ is closed (Infinite intersection of closed sets is closed)

Proposition 10.18

(i) $O_i \subset \mathbb{R}^n,\ O_i$ is open, $i = 1, \cdots, n \Rightarrow \cap_{i=1}^n O_i$ is open (Finite intersection of open sets is open)
(ii) $C_i \subset \mathbb{R}^n,\ C_i$ is closed, $i = 1, \cdots, n \Rightarrow \cup_{i=1}^n C_i$ is closed (Finite union of closed sets is closed)

Definition (Exterior Point)

$A \subset \mathbb{R}^n,\ \mathbf{u} \in \mathbb{R}^n$ is an exterior point of A if $\exists\ r > 0 \ni B_r(\mathbf{u}) \subset \mathbb{R}^n \setminus A$

Definition

$\mathrm{ext}(A)$ is the set of all exterior points of A and is called the exterior of A

$\mathrm{ext}(A) = \{\mathbf{u} \in \mathbb{R}^n \mid \mathbf{u}$ is an exterior point of $A\}$

Definition (Boundary Point)

$A \subset \mathbb{R}^n,\ \mathbf{u} \in \mathbb{R}^n$, is a boundary point of A if for each $r > 0,\ B_r(\mathbf{u}) \cap A \neq \emptyset$ and $B_r(\mathbf{u}) \cap (\mathbb{R}^n \setminus A) \neq \emptyset$

Definition

$\mathrm{bd}(A)$ is the set of all boundary points of A and is called the boundary of A

$\mathrm{bd}(A) = \{\mathbf{u} \in \mathbb{R}^n \mid \mathbf{u}$ is an boundary point of $A\}$

Note

$\mathbb{R}^n = \mathrm{int}(A) \cup \mathrm{ext}(A) \cup \mathrm{bd}(A)$ (Disjoint Union)

Fact

$\mathrm{int}(A) = \mathrm{ext}(\mathbb{R}^n \setminus A)$
$\mathrm{bd}(A) = \mathrm{bd}(\mathbb{R}^n \setminus A)$

Lemma A

$S \subset \mathbb{R}^n,\ \mathbf{u} \in \mathbb{R}^n$, then
$B_r(\mathbf{u}) \cap S \neq \emptyset$ for each $r > 0 \Leftrightarrow \exists\ \{\mathbf{u}_n\} \subset S \ni \mathbf{u}_n \to \mathbf{u}$

Proposition 10.19

$A \subset \mathbb{R}^n$, then
(i) A is open $\Leftrightarrow A \cap \mathrm{bd}(A) = \emptyset$
(ii) A is closed $\Leftrightarrow \mathrm{bd}(A) \subseteq A$

Definition

$A \subset \mathbb{R}^n$, then the closure of A or $\text{cl}(A)$ is defined by $\text{cl}(A) = \text{int}(A) \cup \text{bd}(A)$

Fact 2

(i) $A \subseteq \text{cl}(A)$
(ii) $\text{cl}(A)$ is closed
(iii) A is closed $\Leftrightarrow A = \text{cl}(A)$

Fact 3

$A \subset \mathbb{R}^n$, then
(i) $\text{int}(A)$ is open
(ii) $\text{ext}(A)$ is open
(iii) $\text{bd}(A)$ is closed

Section 11.1: Continuous Functions and Mappings

Note

$A \subseteq \mathbb{R}^n$ and $F : A \to \mathbb{R}^m$, then
(i) $m = 1 : F$ is a function
(ii) $m > 1 : F$ is a mapping

Definition

(i) $F : A \to \mathbb{R}^m$ is continuous at $\mathbf{u} \in A$ if whenever $\{\mathbf{u}_k\} \subset A,\ \mathbf{u}_k \to \mathbf{u}$, then $F(\mathbf{u}_k) \to F(\mathbf{u})$

(ii) $F : A \to \mathbb{R}^m$ is continuous if F is continous at each $\mathbf{u} \in A$

Proposition 11.1

$P_i : \mathbb{R}^n \to \mathbb{R}$ is continuous, $\ i = 1, \cdots, n$

Theorem 11.3

$\mathbf{u} \in A \subset \mathbb{R}^n,\ h : A \to \mathbb{R}$ and $g : A \to \mathbb{R}$ are continuous at $\mathbf{u}$, then
(i) $\alpha h + \beta g$ is continuous at $\mathbf{u},\ \alpha, \beta \in \mathbb{R}$
(ii) $h \cdot g$ is continuous at $\mathbf{u}$
(iii) if $g(\mathbf{v}) \neq 0\ \forall\ \mathbf{v} \in A,$ then $\frac{h}{g}$ is continuous at $\mathbf{u}$

Theorem 11.5

$A \subset \mathbb{R}^n,\ \mathbf{u} \in A,\ G : A \to \mathbb{R}^n$ is continuous at $\mathbf{u},\ B \subset \mathbb{R}^n \ni G(A) \subset B,\ H : B \to \mathbb{R}^k$ is continuous at $G(\mathbf{u})$, then $(H \circ G)(\mathbf{v}) = H(G(\mathbf{v}))$ is continuous at vu; namely, $H : A \to \mathbb{R}^k$ is continuous at $\mathbf{u}$

Example 11.7

$f : \mathbb{R}^n \to \mathbb{R}$ defined by $f(\mathbf{u}) = ||\mathbf{u}||$ is continuous

Fact A (Reverse Triangle Inequality)

$\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$, then $||\mathbf{u} - \mathbf{v}|| \geq ||\mathbf{u}|| - ||\mathbf{v}||$

Fact B

$f(\mathbf{u}) = ||\mathbf{u}||$ is continuous at each $\mathbf{u} \in \mathbb{R}^n$

Definition

$A \subset \mathbb{R}^n,\ F : A \to \mathbb{R}^m,\ \mathbf{u} \in A$, set $F_i(\mathbf{u}) = P_i(F(\mathbf{u})),\ i = 1, \cdots, m$
Then $F(\mathbf{u}) = (F_1(\mathbf{u}), \cdots, F_m(\mathbf{u}))$, and $F_i : A \to \mathbb{R}, i = 1, \cdots, m$ is called the ith component function of F

Theorem 11.9 (Componentwise Continuity Criterion)

$A \subset \mathbb{R}^n,\ \mathbf{u} \in A,\ F : A \to \mathbb{R}^n\ (F = (F_1, \cdots, F_n)$, then
F is continuous at $\mathbf{u} \Leftrightarrow F_i$ is continuous at $\mathbf{u}, i = 1, \cdots, n$

Corollary

$F : O \to \mathbb{R}^n$ given by $F(\mathbf{u}) = \frac{\mathbf{u}}{||\mathbf{u}||}$ is continuous (Note that this called the unit vector in the direction of $\mathbf{u}$)

Corollary 11.10

$A \subset \mathbb{R}^n,\ \mathbf{u} \in A,\ H : A \to \mathbb{R}^m$ and $G : A \to \mathbb{R}^m$ are continuous at $\mathbf{u}$, then
$\alpha H + \beta G : A \to \mathbb{R}^m$ is continuous at $\mathbf{u}$

Theorem 11.11 ($\epsilon - \delta$ Criterion)

$A \subset \mathbb{R}^n,\ \mathbf{u} \in A,\ F : A \to \mathbb{R}^m$, then the following are equivalent:
(i) F is continuous at $\mathbf{u}$
(ii) Given $\epsilon > 0\ \exists\ \delta > 0 \ni \mathbf{v} \in A,\ ||\mathbf{v} - \mathbf{u}|| < \delta \Rightarrow ||F(\mathbf{v}) - F(\mathbf{u})|| < \epsilon$

Theorem 11.12

$O \subset \mathbb{R}^n,\ O$ is open, $F : O \to \mathbb{R}^m$, then TFAE:
(i) F is continuous on O
(ii) $V \subset \mathbb{R}^m,\ V$ is open $\Rightarrow F^{-1}(V)$ is open

Corollary 11.13

$f : \mathbb{R}^n \to \mathbb{R}$ is continuous, $c \in \mathbb{R}$, then

$\{\mathbf{u} \in \mathbb{R}^n \mid f(\mathbf{u}) < c\} = O_{c^-}$
$\{\mathbf{u} \in \mathbb{R}^n \mid f(\mathbf{u}) > c\} = O_{c^+}$
Are open sets

$\{\mathbf{u} \in \mathbb{R}^n \mid f(\mathbf{u}) \leq c\} = C_{c^-}$
$\{\mathbf{u} \in \mathbb{R}^n \mid f(\mathbf{u}) \geq c\} = C_{c^+}$
Are closed sets

Section 11.2: Sequential Compactness / Extreme Values / Uniform Continuity

Definition

$\{\mathbf{x}_k\} \subset \mathbb{R}^n,\ \{k_i\} \subset \mathbb{N}$ is strictly increasing, then $\{\mathbf{x}_{k_i}\} \subset \mathbb{R}^n$ is a subsequence of $\{\mathbf{x}_k\}$

Fact

$\mathbf{x}_k \to \mathbf{x} \Rightarrow \mathbf{x}_{k_i} \to \mathbf{x}$

Definition

$A \subset \mathbb{R}^n$ is sequentially compact if $\{\mathbf{x}_k\} \subset A \Rightarrow \ \exists \ \{\mathbf{x}_{k_i}\}, \ \mathbf{x}_0 \in A \ni \mathbf{x}_{k_i} \to \mathbf{x}_0$

Definition

$A \subset \mathbb{R}^n$ is bounded if $\exists \ M \geq 0 \ni ||\mathbf{u}|| \leq M \ \forall \ \mathbf{u} \in A$

This is equivalent to saying $A \subset \overline{B_M(0)}$ "A is contained in the closed ball of radius M about 0"

Theorem 11.17

$\{\mathbf{x}_k\} \subset \mathbb{R}^n, \ \{\mathbf{x}\}$ is bounded $\Rightarrow \{\mathbf{x}_k\}$ has a convergent subsequence

Theorem 11.18 (Sequential Compactness Theorem)

$A \subset \mathbb{R}^n$ is sequentially compact $\Leftrightarrow A$ is closed and bounded

Fact (Closed Ball)

$\overline{B_r(\mathbf{u})} = \{\mathbf{v} \mid ||\mathbf{v} - \mathbf{u}|| \leq r$ is bounded and closed $\Rightarrow \overline{B_r(\mathbf{u})}$ is sequentially compact

Corollary 11.19 (Generalized Rectangle)

$I = [a_1, b_1] \times [a_2, b_2] \times \cdots \times [a_n, b_n]$ is sequentially compact

Theorem 11.20

$A \subset \mathbb{R}^n, \ A$ is sequentially compact, $F : A \to \mathbb{R}^m$ is continuous $\Rightarrow F(A)$ is sequentially compact in $\mathbb{R}^m$

Lemma 11.21

$A \subset \mathbb{R}$ is sequentially compact $\Rightarrow A$ has a max and min

Theorem 11.22 (Extreme Value Theorem)

$A \subset \mathbb{R}^n, A \neq \emptyset, A$ is sequentially compact, $f : A \to \mathbb{R}$ is continuous, then f attains its max and min on A

Definition (Extreme Value Property)

$A \subset \mathbb{R}^n$ has the extreme value property if every continuous function $f : A \to \mathbb{R}$ attains its max and min on A

Theorem 11.24

$A \subset \mathbb{R}^n$, then A has the extreme value property $\Leftrightarrow$ A is sequentially compact

Definition

$A \subset \mathbb{R}^n,\ F : A \to \mathbb{R}^m$ is uniformly continuous if $\{\mathbf{u}_k\},\ \{\mathbf{v}_k\} \subset A,$ then
$||\mathbf{u}_k - \mathbf{v}_k|| \to 0 \Rightarrow ||F(\mathbf{u}_k) - F(\mathbf{v}_k)|| \to 0$

Theorem 11.25

$A \subset \mathbb{R}^n, A$ is sequentially compact, $F : A \to \mathbb{R}^m$ is continuous, then
F is uniformly continous

Theorem 11.27

$A \subset \mathbb{R}^n,\ F : A \to \mathbb{R}^m$, then TFAE:
(i) F is uniformly continuous
(ii) Given $\epsilon > 0\ \exists\ \delta > 0 \ni \mathbf{u}, \mathbf{v} \in A$, then $||\mathbf{u} - \mathbf{v}|| < \delta \Rightarrow ||F(\mathbf{u}) - F(\mathbf{v})|| < \epsilon$

Section 13.1: Limits

Definition

$A \subset \mathbb{R}^n, \mathbf{x}_0 \in \mathbb{R}^n$, then $\mathbf{x}_0$ is a limit point of A if
there exists $\{\mathbf{x}_k\} \subset A \setminus \{\mathbf{x}_0\}$ such that $\mathbf{x}_k \to \mathbf{x}_0$

Definition

$A \subset \mathbb{R}^n, \mathbf{x}_0$ is a limit point of A, then for $f : A \to \mathbb{R}, l \in \mathbb{R}$, then we say

$$\lim_{\mathbf{x}\to\mathbf{x}_0} f(\mathbf{x}) = l \text{ if whenever}$$
$$\{\mathbf{x}_k\} \subset A \setminus \{\mathbf{x}_0\}, \mathbf{x}_k \to \mathbf{x}_0, \text{ then } f(\mathbf{x}_k) \to l$$

Example

$f : \mathbb{R}^n \to \mathbb{R}, f$ continuous at $\mathbf{x}_0 \in \mathbb{R}^n$, if $\mathbf{x}_k \to \mathbf{x}_0, \mathbf{x}_k \in \mathbb{R}^n \setminus \{\mathbf{x}_0\},$ then f continuous $\Rightarrow$ $f(\mathbf{x}_k) \to f(\mathbf{x}_0)$, so for $l = f(\mathbf{x}_0)$ we have that $\lim_{\mathbf{x}\to\mathbf{x}_0} f(\mathbf{x}) = f(\mathbf{x}_0)$

Example

P is a polynomial, $P : \mathbb{R}^n \to \mathbb{R}$, then

$$\lim_{\mathbf{x}\to\mathbf{x}_0} P(\mathbf{x}) = P(\mathbf{x}_0) \text{ for all } \mathbf{x}_0 \in \mathbb{R}^n$$

Example

$g(\mathbf{x}) = ||\mathbf{x}||$, then

$$\lim_{\mathbf{x}\to\mathbf{x}_0} g(\mathbf{x}) = g(\mathbf{x}_0) \text{ for all } \mathbf{x}_0 \in \mathbb{R}^n$$

Theorem 13.3

$A \subset \mathbb{R}^n, \mathbf{x}_0$ is a limit point of A, then for $f : A \to \mathbb{R}, g : A \to \mathbb{R}$ such that

$$\lim_{\mathbf{x}\to\mathbf{x}_0} f(\mathbf{x}) = l_1 \text{ and } \lim_{\mathbf{x}\to\mathbf{x}_0} g(\mathbf{x}) = l_2$$

then we have the following:

(i) $\lim_{\mathbf{x}\to\mathbf{x}_0}(f+g)(\mathbf{x}) = l_1 + l_2$
(ii) $\lim_{\mathbf{x}\to\mathbf{x}_0}(fg)(\mathbf{x}) = l_1 \cdot l_2$
(iii) $\lim_{\mathbf{x}\to\mathbf{x}_0}(\frac{f}{g})(\mathbf{x}) = \frac{l_1}{l_2}$ for $l_2 \neq 0$

Theorem 13.7

$A \subset \mathbb{R}^n, \mathbf{x}_0$ is a limit point of A, then for $f : A \to \mathbb{R}$ TFAE:

(i) $\lim_{\mathbf{x}\to\mathbf{x}_0} f(\mathbf{x}) = l$
(ii) Given $\epsilon > 0$, then there exists $\delta > 0$ such that $\mathbf{x} \in A \setminus \{\mathbf{x}_0\}$, then
$||\mathbf{x} - \mathbf{x}_0|| < \delta \Rightarrow |f(\mathbf{x}) - l| < \epsilon$

Definition

$\mathbf{x}_0 \in A$ is an isolated point of A if there exists some $r > 0$ such that
$B_r(\mathbf{x}_0) \cap A \setminus \{\mathbf{x}_0\} = \emptyset$

Fact 1

$\mathbf{x}_0$ is a limit point of $A \Leftrightarrow$ for every $r > 0$ there exists $\mathbf{x} \in A \setminus \{\mathbf{x}_0\}$ such that
$\mathbf{x} \in B_r(\mathbf{x}_0)$

Fact 2

$\mathbf{x} \in A$, then $\mathbf{x}$ is a limit point of A or $\mathbf{x}$ is an isolated point of A

Continuity and Limits Theorem

$f : A \subset \mathbb{R}^n \to \mathbb{R}, \mathbf{x}_0 \in A$, then
f continuous at $\mathbf{x}_0 \Leftrightarrow \lim \mathbf{x} \to \mathbf{x}_0 f(\mathbf{x}) = f(\mathbf{x}_0)$ whenever $\mathbf{x}_0 \in A$ and $\mathbf{x}_0$ is a limit point of A

Section 13.2: Partial Derivatives

Note

$\mathbf{u}, \mathbf{v} \in \mathbb{R}^n,\ f : \mathbb{R}^n \to \mathbb{R}$

Definition

$f : \mathbb{R}^n \to \mathbb{R}, \mathbf{x} \in \mathbb{R}^n, O$ open, $\mathbf{x} \in O$, then we say that f has a partial derivative with respect to the ith component at $\mathbf{x}, i \in \{1, 2, \cdots, n\}$ if

$$\frac{\partial f}{\partial x_i}(\mathbf{x}) = \lim_{t \to 0} \frac{f(\mathbf{x} + t\mathbf{e}_i) - f(\mathbf{x})}{t} \text{ exists}$$

Where $\mathbf{e}_i$ is the ith element of the standard basis for $\mathbb{R}^n$ (the ith component of this vector is 1 and everything else is 0)

Here we have that $\frac{\partial f}{\partial x_i}(\mathbf{x})$ is the ith partial derivative of f at $\mathbf{x}$

Generalization

$O \subset \mathbb{R}^n, \mathbf{x}_0 \in O, \mathbf{x}_0 = (x_1^0, \cdots, x_n^0)$, then $\dfrac{f(\mathbf{x}_0 + t\mathbf{e}_i) - f(\mathbf{x}_0)}{t} = \dfrac{f(x_1^0, \cdots, x_i^0 + t, \cdots, x_n^0) - f(x_1^0, \cdots, x_n^0)}{t}$

$$= \frac{\phi_i(x_i^0 + t) - \phi_i(x_i^0)}{t}$$

Where $\phi_i(x_i^0) = f(x_1^0, \cdots, x_i, \cdots, x_n^0)$, thus
the limit exists as $t \to 0$ if $\frac{d}{dx_i}\phi_i(x_i^0)$ exists, in which case $\frac{\partial f}{\partial x_i}(\mathbf{x}_0) = \frac{d}{dx_i}\phi_i(x_i^0)$

Definition

If the generalization holds for each $i = 1, \cdots, n$ and each $\mathbf{x}_0 \in O$, then f has first-order partial derivatives

Definition

$f : O \to \mathbb{R}$ is continuously differentiable if it has first order partial derivatives and each $\frac{\partial f}{\partial x_i} : O \to \mathbb{R}$ is continuous for $i = 1, \cdots, n$

Fact

$f : O \to \mathbb{R}$ is continuously differentiable $\Rightarrow f : O \to \mathbb{R}$ is continuous

Definition

$f : O \to \mathbb{R}$

(i) f has second-order partial derivatives if f has first-order partial derivatives and for each $i = 1, \cdots, n$ then $\frac{\partial f}{\partial x_i}$ has first-order partial derivatives

(ii) f has continuous second-order partial derivatives if (i) holds and for each $i = 1, \cdots, n$ and $j = 1, \cdots, n$ then $\frac{\partial^2 f}{\partial x_i \partial x_j} : O \to \mathbb{R}$ is continuous

Theorem 13.10

Suppose $f : O \to \mathbb{R}$ has continuous second-order partial derivatives. Then for any i, j with $1 \le i \le n,\ 1 \le j \le n$ and any $\mathbf{x} \in O$ we have that

$$\frac{\partial^2 f}{\partial x_i \partial x_j}(\mathbf{x}) = \frac{\partial^2 f}{\partial x_j \partial x_i}(\mathbf{x})$$

Section 13.3: Mean Value Theorem and Directional Derivatives

Lemma 13.14 (Mean Value Lemma)

$O \subset \mathbb{R}^n,\ O$ open and $f : O \to \mathbb{R}$ has $\frac{\partial f}{\partial x_i}$ for some i

Let $\mathbf{x} \in O$ and $a \in \mathbb{R}$ such that $\gamma(t) = \mathbf{x} + ta\mathbf{e}_i \in O$, $0 \leq t \leq 1$, then there exists some $\theta \in (0,1)$ such that

$$f(\mathbf{x} + a\mathbf{e}_i) - f(\mathbf{x}) = \frac{\partial f}{\partial x_i}(\mathbf{x} + \theta a \mathbf{e}_i)a$$

Proposition 13.15 (Mean Value Proposition)

$\mathbf{x} \in \mathbb{R}^n,\ r > 0$, and say that $f : B_r(\mathbf{x}) \to \mathbb{R}$ has first-order partial derivatives. Then for $\mathbf{h} \in \mathbb{R}^n$ such that $\mathbf{x} + \mathbf{h} \in B_r(\mathbf{x})$; namely, $||\mathbf{h}|| < r$, then there exists some $\mathbf{z}_1, \cdots, \mathbf{z}_n \in B_r(\mathbf{x})$ such that

$$f(\mathbf{x} + \mathbf{h}) - f(\mathbf{x}) = \sum_{i=1}^{n} h_i \frac{\partial f}{\partial x_i}(\mathbf{z}_i) \text{ and } ||\mathbf{x} - \mathbf{z}_i|| < ||\mathbf{h}||$$

Definition (Directional Derivative)

$f : O \to \mathbb{R},\ \mathbf{x} \in O, \mathbf{p} \in \mathbb{R}^n$, then the directional derivative of f in the direction of $\mathbf{p}$ at $\mathbf{x}$ is given by the following limit if it exists:

$$\lim_{t \to 0} \frac{f(\mathbf{x} + t\mathbf{p}) - f(\mathbf{x})}{t} = \frac{\partial f}{\partial \mathbf{p}}(\mathbf{x})$$

Definition (Gradient)

$\nabla f(\mathbf{x}) = (\frac{\partial f}{\partial x_1}(\mathbf{x}), \cdots, \frac{\partial f}{\partial x_n}(\mathbf{x}))$

Theorem 13.16 (Directional Derivative Theorem)

$f : O \to \mathbb{R},\ \mathbf{x} \in O, \mathbf{p} \in \mathbb{R}^n, f$ continuously differentiable, then for each $\mathbf{x} \in O$ and each $\mathbf{p} \in \mathbb{R}^n$ then $\frac{\partial f}{\partial \mathbf{p}}(\mathbf{x})$ exists and we have that

$$\frac{\partial f}{\partial \mathbf{p}}(\mathbf{x}) = \sum_{i=1}^{n} p_i \frac{\partial f}{\partial x_i}(\mathbf{x}) = \langle \nabla f(\mathbf{x}), \mathbf{p} \rangle$$

Theorem 13.17 (Mean Value Theorem)

$f : O \to \mathbb{R}$ is continuously differentiable, $\mathbf{x} \in O, \mathbf{h} \in \mathbb{R}^n$ such that $\mathbf{x} + t\mathbf{h} \in)$ for each $0 \leq t \leq 1$, then there exists some $\theta \in (0, 1)$ such that
$f(\mathbf{x} + \mathbf{h}) - f(\mathbf{x}) = \langle \nabla f(\mathbf{x} + \theta\mathbf{h}), \mathbf{h} \rangle$

Note that $\mathbf{x} + \theta\mathbf{h}$ is on the line segment joining $\mathbf{x}$ to $\mathbf{x} + \mathbf{h}$ and $\mathbf{h} = (\mathbf{x} + \mathbf{h}) - \mathbf{x}$

Corollary 13.18

$f : O \to \mathbb{R}$ is continuously differentiable, $\mathbf{x} \in O, \nabla f(\mathbf{x}) \neq 0$, then for $||\mathbf{p}|| = 1$, we have that $|\frac{\partial f}{\partial \mathbf{p}}(\mathbf{x})|$ is maximum when $\mathbf{p} = \mathbf{p}_0 = \frac{\nabla f(\mathbf{x})}{||\nabla f(\mathbf{x})||}$

Theorem 13.20

$f : O \to \mathbb{R}$ is continuously differentiable $\Rightarrow f$ is continuous at each $\mathbf{x} \in O$

Section 14.1: First-Order Approximation and Tangent Planes

Note

Recall for $n = 1,\ f : I \to \mathbb{R}$ is differentiable if for each $x \in I$ we have that the following limit exists

$$\lim_{h \to 0} \frac{f(x + h) - f(x_0)}{h} = f'(x)$$

Where $x = x_0,\ x + h = x$, and $h = x - x_0 = (x + h) - x$

Now rewrite as follows

$$\begin{aligned} 0 &= \lim_{h \to 0} \frac{f(x + h) - f(x_0)}{h} - f'(x) \\ &= \lim_{h \to 0} \frac{f(x + h) - [f(x) + f'(x) \cdot h]}{h} \end{aligned}$$

Definition

$f : O \to \mathbb{R},\ g : O \to \mathbb{R}, \mathbf{x} \in O \subset \mathbb{R}^n$, then f and g are kth-order approximations of each other at $\mathbf{x}$ if

$$\lim_{\mathbf{h} \to \mathbf{0}} \frac{f(\mathbf{x} + \mathbf{h}) - g(\mathbf{x} + \mathbf{h})}{||\mathbf{h}||^k} = 0$$

Theorem 14.2 (First-Order Approximation)

$f : O \to \mathbb{R}$ is continuously differentiable, $\mathbf{x} \in O$, then

$$\lim_{\mathbf{h}\to\mathbf{0}} \frac{f(\mathbf{x}+\mathbf{h}) - [f(\mathbf{x}) + \langle \nabla f(\mathbf{x}), \mathbf{h}\rangle]}{||\mathbf{h}||} = 0$$

Corollary 14.3

Suppose that O is an open subset of the plane $\mathbb{R}^2$ that contains the point (x_0, y_0) and that the function $f : O \to \mathbb{R}$ is continuously differentiable. Then there exists a tangent plane to the graph of the function $f : O \to \mathbb{R}$ at the point $(x_0, y_0, f(x_0, y_0))$.

Then the tangent plane is the graph of the function $\psi : \mathbb{R}^2 \to \mathbb{R}$ defined for $(x, y) \in \mathbb{R}^2$ by

$$\psi(x, y) = f(x_0, y_0) + \frac{\partial f}{\partial x}(x_0, y_0)(x - x_0) + \frac{\partial f}{\partial y}(x_0, y_0)(y - y_0)$$

Note (Tangent Hyperplanes)

$f(\mathbf{x}+\mathbf{h}) \cong f(\mathbf{x}) + (\nabla f(\mathbf{x}), \mathbf{h})$ for $||\mathbf{h}||$ close to zero; i.e., $\mathbf{x}+\mathbf{h}$ is nearby $\mathbf{x}$, then set $\mathbf{x} = \mathbf{x}_0, \mathbf{h} = \mathbf{x} - \mathbf{x}_0$, then $\mathbf{x} = \mathbf{x}_0 + \mathbf{h} \Rightarrow$
$f(\mathbf{x}) \cong f(\mathbf{x}_0) + (\nabla f(\mathbf{x}_0), \mathbf{x} - \mathbf{x}_0)$; namely,
$f(\mathbf{x}) \cong f(\mathbf{x}_0) + \sigma_{i=1}^{n} \frac{\partial f}{\partial x_i}(\mathbf{x}_0)(\mathbf{x} - \mathbf{x}_0)_i$

Note that this generates the tangent (hyper) plane

Section 15.1: Linear Algebra

Definition

$T : \mathbb{R}^n \to \mathbb{R}^m$ is linear if for each $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$ and $\alpha, \beta \in \mathbb{R}$, then
$T(\alpha\mathbf{u} + \beta\mathbf{v}) = \alpha \cdot T(\mathbf{u}) + \beta \cdot T(\mathbf{v})$

Example

$T : \mathbb{R}^n \to \mathbb{R}$ defined by $P_i(\mathbf{u})$, for $i = 1, \cdots, n$ is linear

Proposition 15.2

$T : \mathbb{R}^n \to \mathbb{R}$ is any linear mapping, then $T(\mathbf{x}) = \langle \mathbf{a}, \mathbf{x} \rangle$
where $P_i(\mathbf{a}) = T(\mathbf{e}_i$ for $i = 1, \cdots, n$

Theorem 15.6

$T : \mathbb{R}^n \to \mathbb{R}^m$ defined by $T(\mathbf{x}) = A\mathbf{x}$is linear

Matrix Products

Suppose $\mathbf{x} \in \mathbb{R}^n,$ and $A \in \mathbb{R}_{n\times n}$; namely,

$$A = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{bmatrix} \Rightarrow A\mathbf{x} = \begin{bmatrix} \mathbf{A}_1 \\ \vdots \\ \mathbf{A}_n \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} \langle \mathbf{A}_1, \mathbf{x} \rangle \\ \vdots \\ \langle \mathbf{A}_n, \mathbf{x} \rangle \end{bmatrix}$$

Corollary 15.20

$T : \mathbb{R}^n \to \mathbb{R}^n$ for the $n \times n$ matrix A, then TFAE:

(i) $\det A \neq 0$
(ii) A is invertible
(iii) T is invertible

Fact

For any $A \in \mathbb{R}_{n\times n}$ we have that there exists some $c_1 > 0$ such that
$||A\mathbf{x}|| \leq c_1 ||\mathbf{x}||$ for all $\mathbf{x} \in \mathbb{R}^n$

Corollary 15.21

Suppose $A \in \mathbb{R}_{n\times n}$ is invertible then it is equivalent to say that that there exists some $c_2 > 0$ such that $||A\mathbf{x}|| \geq c_2 ||\mathbf{x}||$ for all $\mathbf{x} \in \mathbb{R}^n$

Namely, by Fact 1, we have A invertible $\Rightarrow c_2||\mathbf{x}|| \leq ||A\mathbf{x}|| \leq c_1||\mathbf{x}||$

Section 15.2: Derivative Matrix

Note

$\mathbf{F} : O \to \mathbb{R}^m,\ O \subset \mathbb{R}^n,\ \mathbf{F} = (F_1, \cdots, F_m)$

Definition

(i) $\mathbf{F}$ has first-order partial derivatives at $\mathbf{x} \in O$ if each F_i has first-order partial derivatives at $\mathbf{x}$, for $i = 1, \cdots, m$

(ii) $\mathbf{F}$ has first-order partial derivatives if each F_i has first-order partial derivatives, for $i = 1, \cdots, m$

(iii) $\mathbf{F}$ is continuously differentiable if each F_i is continuously differentiable, for $i = 1, \cdots, m$

Proposition 15.25

$\mathbf{F} : O \to \mathbb{R}^m$ is continuously differentiable $\Rightarrow$ $\mathbf{F}$ is continuous

Definition

$\mathbf{F} : O \to \mathbb{R}^m$ has first-order partial derivatives at $\mathbf{x} \in O$, then the derivative matrix is defined by $\mathbf{DF}(\mathbf{x}) \in \mathbb{R}_{m\times n}$ as follows:

$$\mathbf{DF}(\mathbf{x}) = \begin{bmatrix} \frac{\partial F_1}{\partial x_1} & \cdots & \frac{\partial F_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial F_m}{\partial x_1} & \cdots & \frac{\partial F_m}{\partial x_n} \end{bmatrix} = \begin{bmatrix} \nabla F_1(\mathbf{x}) \\ \vdots \\ \nabla F_m(\mathbf{x}) \end{bmatrix}$$

Example

Suppose $\mathbf{F}(x, y) = (2xy, x^2 - y^2)$, then $\mathbf{DF}(x, y) = \begin{bmatrix} 2y & 2x \\ 2x & -2y \end{bmatrix}$

Theorem 15.29 (Mean Value Theorem)

$\mathbf{F} : O \to \mathbb{R}^m$ is continuously differnetiable, $\mathbf{x}$ and $\mathbf{x} + t\mathbf{h} \in O$, $0 \le t \le 1$,
then there exists some $\theta_1, \cdots, \theta_m \in (0, 1)$ such that
$F_i(\mathbf{x} + \mathbf{h}) - F_i(\mathbf{x}) = \langle \nabla F_i(\mathbf{x} + \theta_i \mathbf{h}), \mathbf{h} \rangle$ for all $i = 1, \cdots, m$

Namely, $\mathbf{F}(\mathbf{x} + \mathbf{h}) - \mathbf{F}(\mathbf{x}) = A\mathbf{h}$, where $A = \begin{bmatrix} \nabla F_1(\mathbf{x} + \theta_1 \mathbf{h}) \\ \vdots \\ \nabla F_m(\mathbf{x} + \theta_m \mathbf{h}) \end{bmatrix}$

Theorem 15.31

$\mathbf{F} : O \to \mathbb{R}^m$ is continuously differentiable and $\mathbf{x} \in O$, then

$$\lim_{\mathbf{h}\to\mathbf{0}} \frac{||\mathbf{F}(\mathbf{x}+\mathbf{h}) - [\mathbf{F}(\mathbf{x}) + \mathbf{DF}(\mathbf{x})\mathbf{h}]||}{||\mathbf{h}||} = 0$$

Theorem 15.32

$\mathbf{F} : O \to \mathbb{R}^m$ and $\mathbf{x} \in O$ and suppose $A \in \mathbb{R}_{m\times n}$ such that

$$\lim_{\mathbf{h}\to\mathbf{0}} \frac{||\mathbf{F}(\mathbf{x}+\mathbf{h}) - [\mathbf{F}(\mathbf{x}) + A\mathbf{h}]||}{||\mathbf{h}||} = 0$$

Then $\mathbf{F}$ has first-order partial derivatives at $\mathbf{x}$ and we have that $A = \mathbf{DF}(\mathbf{x})$

Example

Find the first-order approximation $\mathbf{F}(\mathbf{x}_0) + \mathbf{DF}(\mathbf{x}_0)(\mathbf{x} - \mathbf{x}_0)$ to $\mathbf{F}(\mathbf{x})$ for $\mathbf{F}(x, y) = (2xy, x^2 - y^2)$ and $\mathbf{x}_0 = (1, 2)$

$\mathbf{h} = (\mathbf{x} - \mathbf{x}_0) = (x - 1, y - 2)$, $\mathbf{DF}(\mathbf{x}) = \begin{bmatrix} 2y & 2x \\ 2x & -2y \end{bmatrix}$, $\mathbf{DF}(\mathbf{x}_0) = \begin{bmatrix} 4 & 2 \\ 2 & -4 \end{bmatrix}$,

$\mathbf{F}(1, 2) = (4, -3) \Rightarrow \mathbf{F}(x, y) \cong \begin{bmatrix} 4 \\ -3 \end{bmatrix} + \begin{bmatrix} 4 & 2 \\ 2 & -4 \end{bmatrix} \cdot \begin{bmatrix} x - 1 \\ y - 2 \end{bmatrix}$