

Real Analysis

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Section 1.1: The Completeness Axiom

Definition

$S \subseteq \mathbb{R}$ is inductive if

- (i) $1 \in S$
- (ii) $x \in S \Rightarrow x + 1 \in S$

Definition

\mathbb{N} is the intersection of all inductive subsets of \mathbb{R}

Principle of Mathematical Induction

For each $n \in \mathbb{N}$ let $S(n)$ be some mathematical assertion. Suppose also that

- (i) $S(1)$ is true
- (ii) Whenever $S(n)$ is true, then $S(n + 1)$ is true

Then $S(n)$ is true $\forall n \in \mathbb{N}$

Definition

$$\mathbb{Z} = \mathbb{N} \cup \{0\} \cup \{-n | n \in \mathbb{N}\}$$

Fact

$m, n \in \mathbb{Z} \Rightarrow$

- (i) $m + n \in \mathbb{Z}$
- (ii) $m - n \in \mathbb{Z}$
- (iii) $mn \in \mathbb{Z}$

Definition

$$\mathbb{Q} = \{\frac{m}{n} | m, n \in \mathbb{Z}, n \neq 0\}$$

Fact

- (i) Each $x \in \mathbb{Q} \Rightarrow x = \frac{m}{n}, m$ or n is odd
- (ii) n^2 is even $\Rightarrow n$ is even

Proposition 1.2

$$\exists \text{ No } x \in \mathbb{Q} \ni x^2 = 2$$

Definition

$S \subset \mathbb{R}, S \neq \emptyset$ is Bounded Above if $\exists c \in \mathbb{R} \ni x \leq c \forall x \in S \Rightarrow c$ is an Upper Bound for S

Completeness Axiom

If $S \subset \mathbb{R}, S \neq \emptyset$, and S is Bounded Above, set $U_S = \{c \in \mathbb{R} | c \text{ is an upper bound for } S\}$

Then $\exists a \in U_S \ni a \leq c \forall c \in U_S$

$a = \sup S = \text{supremum of } S \text{ (least upper bound)}$

("Given a bounded, nonempty set S , and the set of all upper bounds of S , U_S , then there exists a least element in U_S that is the least upper bound for S (its supremum)")

Proposition 1.3

$$\text{If } c > 0, \text{ then } \exists! x > 0 \ni x^2 = c$$

Theorem 1.4

$S \subset \mathbb{R}, S \neq \emptyset$, and S is Bounded Below, set $L_S = \{b \in \mathbb{R} | b \text{ is a lower bound for } S\}$

Then $\exists d \in L_S \ni d \geq b \forall b \in L_S$

$d = \inf S = \text{infimum of } S \text{ (greatest lower bound)}$

("Given a bounded, nonempty set S , and the set of all lower bounds of S , L_S , then there exists a greatest element in L_S that is the greatest lower bound for S (its infimum)")

Section 1.2: The Distribution of \mathbb{Z} & \mathbb{Q}

Theorem 1.5 (Archimedian Property)

- (i) $c > 0 \Rightarrow \exists n \in \mathbb{N} \ni n > c$
- (ii) $\epsilon > 0 \Rightarrow \exists n \in \mathbb{N} \ni \frac{1}{n} < \epsilon$

Proposition 1.6

Let $n \in \mathbb{Z}$, then \exists No $k \in \mathbb{Z} \ni k \in (n, n+1)$

Proposition 1.7

Suppose $S \neq \emptyset, S \subset \mathbb{Z}$, and S is Bounded Above, then S has a Maximum $m \in S$
Note: $m \in S \Rightarrow m = \sup S$

Theorem 1.8

For any $c \in \mathbb{R} \exists! k \in \mathbb{Z} \ni k \in [c, c+1)$

Definition

$S \subset \mathbb{R}$ is Dense in \mathbb{R} if for any $I = (a, b), a < b, S \cap I \neq \emptyset$

Theorem 1.9

\mathbb{Q} is Dense in \mathbb{R}

Corollary 1.10

$\mathbb{R} \setminus \mathbb{Q}$ is Dense in \mathbb{R}

Section 1.3: Inequalities and Identities

Definition

$$x \in \mathbb{R}, |x| = \begin{cases} x, & x \geq 0 \\ -x, & x < 0 \end{cases}$$

Fact 1

$$d > 0, |c| \leq d \Leftrightarrow -d \leq c \leq d$$

Fact 2

$$x \in \mathbb{R}, -|x| \leq x \leq |x|$$

Theorem 1.11 (Triangle Inequality)

If $a, b \in \mathbb{R}$, Then $|a + b| \leq |a| + |b|$

Proposition 1.12

$a, r \in \mathbb{R}, r > 0$, TFAE:

- (i) $|x - a| < r$
- (ii) $a - r < x < a + r$
- (iii) $x \in (a - r, a + r)$

Difference of Powers Formula

$$n \in \mathbb{N} \text{ and } a, b \in \mathbb{R}, \\ a^n - b^n = (a - b) \sum_{k=0}^{n-1} a^{n-1-k} b^k$$

Geometric Series Formula

$$r \in \mathbb{R}, r \neq 1, \text{ then } \frac{1-r^{n+1}}{1-r} = 1 + r + \cdots + r^n = \sum_{k=0}^n r^k$$

Definition

$$n! = \begin{cases} 1, & n = 0, 1 \\ n * (n-1) * \cdots * 2 * 1, & n > 1 \end{cases}$$

Definition

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}$$

Binomial Formula

$$n \in \mathbb{N} \text{ and } a, b \in \mathbb{R}, \\ (a+b)^n = \sum_{k=0}^n \binom{n}{k} a^{n-k} b^k$$

Section 2.1: Convergence of Sequences

Definition

A sequence of real numbers is a function $f : \mathbb{N} \rightarrow \mathbb{R}$
set $a_n = f(n)$, then characterize f by $\{a_1, a_2, \dots\} = \{a_n\}_{n=1}^{\infty} = \{a_n\}$

Definition

$\{a_n\}$ Converges to $a \in \mathbb{R}$ provided that
for each $\epsilon > 0 \exists N \in \mathbb{N} \ni |a_n - a| < \epsilon \forall n \geq N$

Proposition 2.6

$\{\frac{1}{n}\}$ converges to 0

Fact

$\{1 + \frac{1}{2} + \dots + \frac{1}{2^n}\}$ converges to 2

Fact

$a_n \rightarrow a, a_n \rightarrow b \Rightarrow a = b$
(limits are unique)

Fact

$\{(-1)^n\}$ does not converge

Lemma 2.9 (Comparison Lemma)

Suppose we have $\{a_n\}, \{b_n\}$ with $a_n \rightarrow a$. Then $b_n \rightarrow b$ if
 $\exists c \geq 0$ and $N_1 \in \mathbb{N} \ni |b_n - b| \leq c|a_n - a| \forall n \geq N_1$

Theorem 2.10 (Sum Property)

$$a_n \rightarrow a, b_n \rightarrow b \Rightarrow a_n + b_n \rightarrow a + b$$

Lemma 2.11

$$a_n \rightarrow a, \alpha \in \mathbb{R} \Rightarrow (\alpha)a_n \rightarrow (\alpha)a$$

Theorem 2.13 (Product property)

$$a_n \rightarrow a, b_n \rightarrow b \Rightarrow a_n b_n \rightarrow ab$$

Fact 1

$$a_n \rightarrow a \Rightarrow |a_n| \rightarrow |a|$$

Proposition 2.14

$$b_n \rightarrow b \neq 0 \Rightarrow \frac{1}{b_n} \rightarrow \frac{1}{b}$$

Theorem 2.15 (Quotient property)

$$a_n \rightarrow a, b_n \rightarrow b \neq 0 \Rightarrow \frac{a_n}{b_n} \rightarrow \frac{a}{b}$$

Proposition 2.16 (Linear property)

$$a_n \rightarrow a, b_n \rightarrow b, \alpha, \beta \in \mathbb{R} \Rightarrow (\alpha)a_n + (\beta)b_n \rightarrow (\alpha)a + (\beta)b$$

Fact 2

$$a_n = c \forall n \Rightarrow a_n \rightarrow c$$

Proposition 2.17

$$P: \mathbb{R} \rightarrow \mathbb{R}, a_n \rightarrow a \Rightarrow P(a_n) \rightarrow P(a)$$

Section 2.2: Sequences & Sets

Theorem 2.18

$\{a_n\}$ converges $\Rightarrow \{a_n\}$ is bounded

Proposition 2.19

S is dense in $\mathbb{R} \Leftrightarrow$ each $x \in \mathbb{R}$ is a limit of a sequence in S

Theorem 2.20 (Sequential Density of \mathbb{Q})

Every $x \in \mathbb{R}$ is the limit of a sequence of rational numbers

Lemma 2.21

$d_n \rightarrow d, d_n \geq 0 \Rightarrow d \geq 0$

Theorem 2.22

$\{c_n\} \subset [a, b], c_n \rightarrow c \Rightarrow c \in [a, b]$

Definition

$S \subset \mathbb{R}$ is closed if whenever $\{a_n\} \subset S$ and $a_n \rightarrow a$ then $a \in S$

Fact

$[a, b]$ is closed

Section 2.3: The Monotone Convergence Theorem

Definition

$\{a_n\}$ is monotonically increasing if $a_{n+1} \geq a_n$ for each n

Definition

$\{a_n\}$ is monotonically decreasing if $a_{n+1} \leq a_n$ for each n

Definition

$\{a_n\}$ is monotone if it is either monotonically increasing or decreasing

Theorem 2.25 (Monotone Convergence Theorem)

If $\{a_n\}$ is monotone, then

$\{a_n\}$ converges $\Leftrightarrow \{a_n\}$ is bounded

Note: if $\{a_n\}$ is monotonically increasing, $a_n \rightarrow \sup\{a_n\}$

Note: if $\{a_n\}$ is monotonically decreasing, $a_n \rightarrow \inf\{a_n\}$

Proposition 2.28

Let $c \in \mathbb{R}$, $|c| < 1 \Rightarrow \lim_{n \rightarrow \infty} c^n = 0$

Theorem 2.29 (Nested Interval Theorem)

Let $\{a_n\}$ and $\{b_n\}$ be such that $a_n < b_n$ and set $I_n = [a_n, b_n]$.

Assume that $I_{n+1} \subset I_n$ and that $\lim_{n \rightarrow \infty} [b_n - a_n] = 0$. Then $\exists! x \in \bigcap_{n=1}^{\infty} I_n$

Section 2.4: The Sequential Compactness Theorem

Definition

For a given $\{a_n\}$ let $\{n_k\}$ be a sequence of natural numbers that is strictly increasing. Then the sequence $\{b_k\}$ defined by $b_k = a_{n_k}$, with $k = 1, 2, \dots$ is called a subsequence of $\{a_n\}$, denoted $\{b_k\} = \{a_{n_k}\}_{k=1}^{\infty} = \{a_{n_k}\}$

Fact

Given a sequence $\{n_k\}_{k=1}^{\infty}$ of natural numbers that is strictly increasing, we have that $n_k \geq k$ for every $k \in \mathbb{N}$

Proposition 2.30

Let $\{a_n\}$ converge to a , i.e., $a_n \rightarrow a$

Then $\lim_{n \rightarrow \infty} a_{n_k} = a$ for every subsequence $\{a_{n_k}\}$ of $\{a_n\}$

Theorem 2.32

For every $\{a_n\}$ $\exists \{n_k\} \ni \{a_{n_k}\}$ is monotone

Theorem 2.33

Every bounded sequence has a convergent subsequence

Definition

$S \subseteq \mathbb{R}$ is sequentially compact if every sequence $\{a_n\} \subset S$ has a convergent subsequence whose limit is in S

Theorem 2.36 (Sequential Compactness Theorem)

$a, b \in \mathbb{R}$ with $a < b \Rightarrow [a, b]$ is sequentially compact

Section 3.1: Continuity

Definition

For $D \subseteq \mathbb{R}$, $f : D \rightarrow \mathbb{R}$ is continuous at $x_0 \in D$ if whenever $\{x_n\} \subset D$ and $x_n \rightarrow x_0$ then $f(x_n) \rightarrow f(x_0)$

($f : D \rightarrow \mathbb{R}$ is continuous if it is continuous $\forall x_0 \in D$)

Fact

$P : \mathbb{R} \rightarrow \mathbb{R} \Rightarrow x_0 \in \mathbb{R}, \{x_n\} \subset \mathbb{R}, x_n \rightarrow x_0 \Rightarrow P(x_n) \rightarrow P(x_0) \Rightarrow P$ is continuous

Theorem 3.4

Suppose $f : D \rightarrow \mathbb{R}, g : D \rightarrow \mathbb{R}$ are continuous at $x_0 \in D$, then

$f + g : D \rightarrow \mathbb{R}$ and $fg : D \rightarrow \mathbb{R}$ are continuous at $x_0 \in D$

and if $g(x) \neq 0 \forall x \in D$ then $\frac{f}{g} : D \rightarrow \mathbb{R}$ is continuous at $x_0 \in D$

Corollary 3.5

Let P and Q be polynomials, then $\frac{P}{Q} : D \rightarrow \mathbb{R}$ is continuous where $D = \{x \in \mathbb{R} \mid Q(x) \neq 0\}$

Theorem 3.6

$f : D \rightarrow \mathbb{R}, g : U \rightarrow \mathbb{R}, f(D) \subseteq U$ and suppose that f is continuous at $x_0 \in D$ and g is continuous at $f(x_0) \in U$ then $g \circ f$ is continuous at x_0 ; namely, $g \circ f : D \rightarrow \mathbb{R}$

Fact

$f(x) = \sqrt{x}$ is continuous on $D = [0, +\infty)$

Section 3.1 (Sup): Trigonometric Continuity

Fact 1

if $\theta_n \rightarrow 0$, then $\sin \theta_n \rightarrow 0$

Fact 2

if $\theta_n \rightarrow 0$, then $\cos \theta_n \rightarrow 1$

Fact

$\sin \theta$ is continuous,
 $\cos \theta$ is continuous,
 $\tan \theta$ is continuous at $\cos \theta \neq 0$ ($\theta \neq (2n+1) * \frac{\pi}{2}$),
 $\csc \theta$ is continuous at $\sin \theta \neq 0$ ($\theta \neq n\pi$),
 $\sec \theta$ is continuous at $\cos \theta \neq 0$ ($\theta \neq (2n+1) * \frac{\pi}{2}$),
 $\cot \theta$ is continuous at $\sin \theta \neq 0$ ($\theta \neq n\pi$)

Section 3.2: Extreme Value Theorem

Definition

For $f : D \rightarrow \mathbb{R}$ we define $f(D) = \{y \in \mathbb{R} \mid y = f(x) \text{ for some } x \in D\}$
 $f(D)$ is the image of f

Definition

$f : D \rightarrow \mathbb{R}$ attains a maximum (max value) if $\exists x_0 \in D \ni f(x) \leq f(x_0) \forall x \in D$
Such a point x_0 is a maximizer of f

$f : D \rightarrow \mathbb{R}$ attains a minimum (min value) if $\exists x'_0 \in D \ni f(x'_0) \leq f(x) \forall x \in D$
Such a point x'_0 is a minimizer of f

Lemma 3.10

If $f : [a, b] \rightarrow \mathbb{R}$ is continuous, then $f([a, b])$ is bounded above ($\exists m \ni f(x) \leq m \forall x \in [a, b]$)

Theorem 3.9 (Extreme Value Theorem)

If $f : [a, b] \rightarrow \mathbb{R}$ is continuous, then f attains both a max and a min

$$\exists x_0, x'_0 \in [a, b] \ni f(x_0) \leq f(x) \leq f(x'_0) \forall x \in [a, b]$$

Fact

Let $S \subset [a, b]$, then $\inf S \in [a, b]$, and $\sup S \in [a, b]$

Section 3.3: Intermediate Value Theorem

Theorem 3.11 (Intermediate Value Theorem)

Suppose $f : [a, b] \rightarrow \mathbb{R}$ is continuous and let $c \in \mathbb{R}$ be any number strictly between $f(a)$ and $f(b)$; i.e., $f(a) < c < f(b)$ or $f(b) < c < f(a)$, then $\exists x_0 \in (a, b) \ni f(x_0) = c$

Fact

Suppose $f : D \rightarrow \mathbb{R}$ is continuous. If $\exists [a, b] \subset D \ni f(a) < 0$ and $f(b) > 0$ (or vice-versa), then $\exists x_0 \in (a, b) \ni f(x_0) = 0$

”A real, continuous function that is positive on one side and negative on the other contains a root”

Definition

$D \subseteq \mathbb{R}$ is convex if $u, v \in D, (u < v) \Rightarrow [u, v] \subset D$

Fact

If $D \subset \mathbb{R}$ is convex then D is an interval

Theorem 3.14

If I is an interval and $f : I \rightarrow \mathbb{R}$ is continuous then $f(I)$ is an interval

Section 3.4: Uniform Continuity

Definition

$f : D \rightarrow \mathbb{R}$ is uniformly continuous on D if whenever $\{u_n\}, \{v_n\} \subset D \ni u_n - v_n \rightarrow 0$, then $f(u_n) - f(v_n) \rightarrow 0$

Note: if $v_n = x_0 \forall n$, then $u_n - v_n \rightarrow 0 \Rightarrow u_n \rightarrow x_0$, so uniform continuity \Rightarrow continuity at each $x_0 \in D$

Fact

$f(x) = x$ is uniformly continuous but $f(x) = x^2$ is not

Theorem 3.17

$f : [a, b] \rightarrow \mathbb{R}$ is continuous $\Rightarrow f$ is uniformly continuous on $[a, b]$

Fact

If $f : D \rightarrow \mathbb{R}$ satisfies Lipschitz Continuity: $|f(u) - f(v)| \leq c|u - v|, \forall u, v \in D$ and for some $c \geq 0$, then f is uniformly continuous.

Fact

Let P be a polynomial. Then on each $[a, b]$, $P : [a, b] \rightarrow \mathbb{R}$ is lipschitz continuous

Section 3.5: The $\epsilon - \delta$ Criteria for Continuity

Definition

"The $\epsilon - \delta$ Criterion At a Point" - $f : D \rightarrow \mathbb{R}$ satisfies the $\epsilon - \delta$ criterion at a point $x_0 \in D$, if for each $\epsilon > 0 \exists \delta > 0 \ni$ for $x \in D$, $|x - x_0| < \delta \Rightarrow |f(x) - f(x_0)| < \epsilon$

Theorem 3.20

For $f : D \rightarrow \mathbb{R}$ and $x_0 \in D$, TFAE:

- (i) f is continuous at x_0
- (ii) The $\epsilon - \delta$ criterion at x_0 holds

Definition

"The $\epsilon - \delta$ Criterion On the Domain of a Function" - $f : D \rightarrow \mathbb{R}$ satisfies the $\epsilon - \delta$ criterion on D , if for each $\epsilon > 0 \exists \delta > 0 \ni u, v \in D$, $|u - v| < \delta \Rightarrow |f(u) - f(v)| < \epsilon$

Theorem 3.22

For $f : D \rightarrow \mathbb{R}$, TFAE:

- (i) $f : D \rightarrow \mathbb{R}$ is uniformly continuous
- (ii) $f : D \rightarrow \mathbb{R}$ satisfies the $\epsilon - \delta$ criterion on D

Fact

$I = (a, b)$, $f : I \rightarrow \mathbb{R}$ is continuous, then if $x_0 \in (a, b)$ with $f(x_0) > 0$, then $\exists I_1 = (a_1, b_1) \subset I \ni f(x) > 0 \forall x \in I_1$

Section 3.6: Images and Inverses; Monotone Functions

Definition

- (i) $f : D \rightarrow \mathbb{R}$ is monotonically increasing if $u, v \in D$ and $u < v \Rightarrow f(u) \leq f(v)$
- (ii) $f : D \rightarrow \mathbb{R}$ is monotonically decreasing if $u, v \in D$ and $u < v \Rightarrow f(u) \geq f(v)$
- (iii) $f : D \rightarrow \mathbb{R}$ is monotone if it is monotonically increasing or decreasing

Theorem 3.23

Suppose $f : D \rightarrow \mathbb{R}$ is monotone. If $f(D)$ is an interval, then f is continuous

Corollary 3.25

Suppose $f : I \rightarrow \mathbb{R}$ is monotone, then f is continuous $\Leftrightarrow f(I)$ is an interval

Definition

- (i) $f : D \rightarrow \mathbb{R}$ is strictly increasing if $u, v \in D$ and $u < v \Rightarrow f(u) < f(v)$
- (ii) $f : D \rightarrow \mathbb{R}$ is strictly decreasing if $u, v \in D$ and $u < v \Rightarrow f(u) > f(v)$

Definition

$f : D \rightarrow \mathbb{R}$ is one-to-one (injective) if for each $y \in f(D)$ $\exists! x \in D \ni f(x) = y$

"No element in the image may have more than one element in the domain mapping to it"

Fact

If f is strictly increasing or decreasing, then f is one-to-one

Fact

If $f : I \rightarrow \mathbb{R}$ is continuous and f is one-to-one, then f is strictly monotone

Definition

Suppose $f : D \rightarrow \mathbb{R}$ is one-to-one. If $y \in f(D)$, let $x \in D \ni f(x) = y$
Define $f^{-1} : f(D) \rightarrow D$ by $f^{-1}(y) = x$, so f^{-1} is well-defined since x is unique

Note:

- (i) $f^{-1}(f(x)) = x$, where $x \in D$
- (ii) $f(f^{-1}(y)) = y$, where $y \in f(D)$

Theorem 3.29

$f : I \rightarrow \mathbb{R}$ is continuous and strictly increasing or decreasing \Rightarrow
 $f^{-1} : f(I) \rightarrow \mathbb{R}$ is continuous

Section 3.7: Limits

Definition

$h(x) = \frac{f(x)-f(x_0)}{x-x_0}$ gives the slope of the line at point x_0
and $h(x)$ is continuous on $[a, b] \setminus \{x_0\}$

Definition

$D \subset \mathbb{R}$, $x_0 \in \mathbb{R}$ is a limit point of D if $\exists \{x_n\} \subset D \setminus \{x_0\} \ni x_n \rightarrow x_0$

Definition

If $f : D \rightarrow \mathbb{R}$ and x_0 is a limit point of D , then we denote $\lim_{x \rightarrow x_0} f(x) = l$
If whenever $\{x_n\} \subset D \setminus \{x_0\}$ and $x_n \rightarrow x_0$ we have that $\lim_{n \rightarrow \infty} f(x_n) = l$
(x_0 may or may not be in D)

Example

$D = \mathbb{R} \setminus \{x_0\}$, $f(x) = x^2 \Rightarrow h(x) = \frac{x^2-(x_0)^2}{x-x_0}$ and suppose
 $\{x_n\} \subset D$, $x_n \rightarrow x_0 \Rightarrow h(x_n) = \frac{(x_n)^2-(x_0)^2}{x_n-x_0} = \frac{(x_n+x_0)(x_n-x_0)}{x_n-x_0} = x_n + x_0$
So $\lim_{n \rightarrow \infty} h(x_n) = \lim_{n \rightarrow \infty} (x_n + x_0) = x_0 + x_0 = 2x_0$

Theorem 3.36

Suppose $f : D \rightarrow \mathbb{R}$, $g : D \rightarrow \mathbb{R}$, and x_0 is a limit point of D , so that

$$\lim_{x \rightarrow x_0} f(x) = A, \quad \lim_{x \rightarrow x_0} g(x) = B \Rightarrow$$

- (i) $\lim_{x \rightarrow x_0} [f(x) + g(x)] = A + B$
- (ii) $\lim_{x \rightarrow x_0} [f(x)g(x)] = AB$
- (ii)(a) $\alpha \in \mathbb{R}, \quad \lim_{x \rightarrow x_0} [\alpha f(x)] = \alpha A$
- (iii) $B \neq 0, \quad g(x) \neq 0 \quad \forall x \in D, \quad \lim_{x \rightarrow x_0} \left[\frac{f(x)}{g(x)} \right] = \frac{A}{B}$

Theorem 3.37

$f : D \rightarrow \mathbb{R}$, $g : U \rightarrow \mathbb{R}$ and
 x_0 is a limit point of $D \ni \lim_{x \rightarrow x_0} f(x) = y_0$,
 y_0 is a limit point of $U \ni \lim_{y \rightarrow y_0} g(y) = e$,
and suppose that $f(D \setminus \{x_0\}) \subset U \setminus \{y_0\}$, then
 $\lim_{x \rightarrow x_0} (g \circ f)(x) = e$

Definition

$x_0 \in D$ is an isolated point if $\exists r > 0 \ni (x_0 - r, x_0 + r) \cap D = \{x_0\}$

Fact

$x_0 \in D \Rightarrow x_0$ is either a limit point or an isolated point of D

Limits and Continuity Theorem

For $f : D \rightarrow \mathbb{R}$, $x_0 \in D$, then
 f is continuous at $x_0 \Leftrightarrow x_0$ is an isolated point of D or $\lim_{x \rightarrow x_0} f(x) = f(x_0)$

So f is continuous at $x_0 \Leftrightarrow \lim_{x \rightarrow x_0} f(x) = f(x_0)$

Fact in Review

If $h(x) = g(x)$ on $D \setminus \{x_0\}$ where $g : D \rightarrow \mathbb{R}$ is continuous on D , then
 $\lim_{x \rightarrow x_0} h(x) = \lim_{x \rightarrow x_0} g(x) = g(x_0)$

Section 4.1: The Algebra of Derivatives

Definition

$x_0 \in \mathbb{R}, I \subset \mathbb{R} \ni I = (a, b)$ and $x_0 \in I \Rightarrow I$ is a neighborhood of x_0

Definition

$x_0 \in \mathbb{R}$ and I is a neighborhood of $x_0 \Rightarrow f : I \rightarrow \mathbb{R}$ is differentiable at x_0 IF $\lim_{x \rightarrow x_0} \frac{f(x) - f(x_0)}{x - x_0}$ exists.
We say $f'(x_0) := \lim_{x \rightarrow x_0} \frac{f(x) - f(x_0)}{x - x_0}$ and is called the derivative of f at x_0

Definition

If $f : I \rightarrow \mathbb{R}$ is differentiable at each $x_0 \in I$ then f is differentiable and $f' : I \rightarrow \mathbb{R}$ is the derivative of f

Definition

The line determined by $y = f(x_0) + f'(x_0)(x - x_0)$ is the tangent line to the graph of f at $(x_0, f(x_0))$

For $y_0 = f(x_0)$, $y - y_0 = f'(x_0)(x - x_0)$

Proposition 4.4

$n \in \mathbb{N}, f(x) = x^n \forall x \in I = \mathbb{R} \Rightarrow f : \mathbb{R} \rightarrow \mathbb{R}$ is differentiable and $f'(x) = nx^{n-1}$

Proposition 4.5

$x_0 \in \mathbb{R}, I$ is a neighborhood of $x_0, I = (a, b)$ and $f : I \rightarrow \mathbb{R}$ is differentiable at $x_0 \Rightarrow f$ is continuous at x_0

Theorem 4.6

$x_0 \in \mathbb{R}, I$ is a neighborhood of $x_0, f : I \rightarrow \mathbb{R}$ and $g : I \rightarrow \mathbb{R}$ is differentiable at x_0 , then

(i) $f + g : I \rightarrow \mathbb{R}$ is differentiable at x_0 and
 $(f + g)'(x_0) = f'(x_0) + g'(x_0)$

(ii) $fg : I \rightarrow \mathbb{R}$ is differentiable at x_0 and
 $(fg)'(x_0) = f(x_0)g'(x_0) + f'(x_0)g(x_0)$

(iii) If $g(x) \neq 0 \forall x \in I$ then $\frac{1}{g} : I \rightarrow \mathbb{R}$ is differentiable at x_0 and

$$\left(\frac{1}{g}\right)'(x_0) = \frac{-g'(x_0)}{[g(x_0)]^2}$$

(iv) If $g(x) \neq 0 \forall x \in I$ then $\frac{f}{g} : I \rightarrow \mathbb{R}$ is differentiable at x_0 and

$$\left(\frac{f}{g}\right)'(x_0) = \frac{f'(x_0)g(x_0) - f(x_0)g'(x_0)}{[g(x_0)]^2}$$

Fact

For $x_0, \alpha \in \mathbb{R}$, $(\alpha f)'(x_0) = \alpha f'(x_0)$

Fact

$f(x) = c \Rightarrow f'(x) = 0 \forall x \in D$

Proposition 4.7

$n \in \mathbb{Z}$, $D = \mathbb{R}$ if $n \geq 0$ and $D = \mathbb{R} \setminus \{0\}$ if $n < 0$, then for $f : D \rightarrow \mathbb{R}$ defined by $f(x) = x^n$, f is differentiable and $f'(x) = nx^{n-1}$

Corollary 4.8

$p, q : \mathbb{R} \rightarrow \mathbb{R}$ are polynomials, $D = \mathbb{R} \setminus \{x \mid q(x) = 0\}$, then $\frac{p}{q} : D \rightarrow \mathbb{R}$ is differentiable

Section 4.2: Differentiating Inverses & Compositions

Theorem 4.11

Suppose $x_0 \in I$, and $f : I \rightarrow \mathbb{R}$ is strictly monotone, continuous, differentiable at x_0 , and $f'(x_0) \neq 0$. Let $J = f(I)$ then $f^{-1} : J \rightarrow \mathbb{R}$ is differentiable at $y_0 = f(x_0)$ and $(f^{-1})'(y_0) = \frac{1}{f'(x_0)} = \frac{1}{f'(f^{-1}(y_0))}$

Corollary 4.12

Suppose $f : I \rightarrow \mathbb{R}$ is strictly monotone, differentiable, and f' is nonzero on I . Let $J = f(I)$, then $(f^{-1}) : J \rightarrow \mathbb{R}$ is differentiable and $\forall x \in J$

$$(f^{-1})'(x) = \frac{1}{f'(f^{-1}(x))}$$

Proposition 4.13

Let $g(x) = x^{\frac{1}{n}}$ where $n \in \mathbb{N}$ and $x > 0$, then
 $g : (0, \infty) \rightarrow \mathbb{R}$ is differentiable and $g'(x) = \frac{1}{n}x^{\frac{1}{n}-1} \forall x > 0$

Theorem 4.14 (Chain Rule)

Suppose $x_0 \in I$ with $f : I \rightarrow \mathbb{R}$ is differentiable. Say $f(I) \subseteq J$ and suppose
 $g : J \rightarrow \mathbb{R}$ is differentiable at $f(x_0)$, then $g \circ f : I \rightarrow \mathbb{R}$ is differentiable at x_0
and $(g \circ f)'(x_0) = g'(f(x_0)) \cdot f'(x_0)$

Proposition 4.15

For $r = \frac{m}{n}$ where $n \neq 0, m \in \mathbb{Z}, n \in \mathbb{N}$, set $h(x) = x^r$, where $x > 0$, then
 h is differentiable and $h'(x) = rx^{r-1} \forall x > 0$

Section 4.3: The Mean Value Theorem

Lemma 4.16

Suppose I is a neighborhood of x_0 and $f : I \rightarrow \mathbb{R}$ is differentiable at x_0 .
If x_0 is a maximizer or a minimizer, then $f'(x_0) = 0$

Theorem 4.17 (Rolle's Theorem)

Suppose $f : [a, b] \rightarrow \mathbb{R}$ is continuous and $f : (a, b) \rightarrow \mathbb{R}$ is differentiable.
Assume that $f(a) = f(b)$, then $\exists x_0 \in (a, b) \ni f'(x_0) = 0$

Theorem 4.18 (Mean Value Theorem)

If $f : [a, b] \rightarrow \mathbb{R}$ is continuous and $f : (a, b) \rightarrow \mathbb{R}$ is differentiable, then
 $\exists x_0 \in (a, b) \ni f'(x_0) = \frac{f(b)-f(a)}{b-a}$

Lemma 4.19

Suppose $I = (a, b)$ and $f : I \rightarrow \mathbb{R}$ is differentiable. Then
 f is constant $\Leftrightarrow f'(x) = 0 \forall x \in I$

Proposition 4.20 (Identity Criterion)

Suppose $g, h : I \rightarrow \mathbb{R}$ are differentiable. Then
 $g = h + c \Leftrightarrow g'(x) = h'(x) \forall x \in I$

Corollary 4.21

- (i) $f : I \rightarrow \mathbb{R}$ is differentiable $\ni f'(x) > 0 \forall x \in I \Rightarrow f$ is strictly increasing
- (ii) $f : I \rightarrow \mathbb{R}$ is differentiable $\ni f'(x) < 0 \forall x \in I \Rightarrow f$ is strictly decreasing

Definition

Suppose $f : D \rightarrow \mathbb{R}$, then $x_0 \in D$ is a

- (i) local maximizer if $\exists \delta > 0 \ni x_0$ is a maximizer for f on $D \cap (x_0 - \delta, x_0 + \delta)$
- (ii) local minimizer if $\exists \delta > 0 \ni x_0$ is a minimizer for f on $D \cap (x_0 - \delta, x_0 + \delta)$

Definition

Suppose $f : I \rightarrow \mathbb{R}$ is differentiable on I . If $f' : I \rightarrow \mathbb{R}$ is differentiable on I , then define $f'' : I \rightarrow \mathbb{R}$ by $f''(x) = (f')'(x) = f^{(2)}(x)$ for each $x \in I$
Inductively define $f^{(k)} : I \rightarrow \mathbb{R}, k \in \mathbb{N}$

Theorem 4.22 (2nd Derivative Test)

Suppose $f, f' : I \rightarrow \mathbb{R}$ are differentiable and $x_0 \in I \ni f'(x_0) = 0$. Then

- (i) $f''(x_0) > 0 \Rightarrow x_0$ is a local minimizer for f (concave up)
- (ii) $f''(x_0) < 0 \Rightarrow x_0$ is a local maximizer for f (concave down)

Fact

If f is continuous on $[a, b]$ and f is differentiable on (a, b) , then f attains its max and min at either

- (i) The endpoints a or b
- (ii) $x_0 \in (a, b) \ni f'(x_0) = 0$

Section 4.4: Cauchy Mean Value Theorem

Theorem 4.23 (Cauchy Mean Value Theorem)

If $f : [a, b] \rightarrow \mathbb{R}$ and $g : [a, b] \rightarrow \mathbb{R}$ are continuous on $[a, b]$ and differentiable on (a, b) with $g'(x) \neq 0 \forall x \in (a, b)$, then $\exists x_0 \in (a, b) \ni \frac{f(b)-f(a)}{g(b)-g(a)} = \frac{f'(x_0)}{g'(x_0)}$

Lemma 1

If $h_1(x) = (x - x_0)^n$, then $h_1^{(k)}(x) = \begin{cases} \frac{n!}{(n-k)!} \cdot (x - x_0)^{n-k}, & 0 \leq k \leq n \\ 0, & k > n \end{cases}$

Theorem 4.24

Suppose $f : I \rightarrow \mathbb{R}$ has n derivatives on I and suppose at $x_0 \in I$ that $f^{(k)}(x_0) = 0$ for $0 \leq k \leq n-1$, then for each $x \in I$ with $x \neq x_0 \ni z$ strictly between x and $x_0 \ni$
 $f(x) = \frac{f^{(n)}(z)}{n!} \cdot (x - x_0)^n$

Application

Let $g : I \rightarrow \mathbb{R}$ have $n+1$ derivatives and set for $x_0 \in I$

$$h(x) = \sum_{j=0}^n \frac{g^{(j)}(x_0)}{j!} \cdot (x - x_0)^j$$

Then $g(x) = h(x) + \frac{g^{(n+1)}(z)}{(n+1)!} \cdot (x - x_0)^{n+1}$
(Taylor's Formula with Remainder)

Section 4.4 (sup): Trigonometric Differentiability

Fact 1

- (i) $\lim_{\theta \rightarrow 0} \frac{\sin \theta}{\theta} = 1$
- (ii) $\sin \theta \rightarrow 0$ as $\theta \rightarrow 0$
- (iii) $\cos \theta \rightarrow 1$ as $\theta \rightarrow 0$
- (iv) $\sin(x + y) = \sin x \cos y + \cos x \sin y$
- (v) $\cos(x + y) = \cos x \cos y - \sin x \sin y$
- (vi) $\frac{d}{dx} \sin x = \cos x$
- (vii) $\frac{d}{dx} \cos x = -\sin x$

Section 6.1: Darboux Sums; Upper and Lower Integrals

Definition

The approximation of an area under a curve using rectangles over subintervals with $P = \{x_0, x_1, \dots, x_n\}$ is a partition of $[a, b]$ with $a = x_0 < x_1 < \dots < x_n = b$. Then the total area of the rectangles is

$$\sum_{i=1}^n M_i(x_i - x_{i-1})$$

Where $x_i - x_{i-1}$ is the width of the i th partition and M_i is the height $M_i = \sup\{f(x) \mid x \in [x_{i-1}, x_i]\}$

If f is continuous then $M_i = f(x_i^*)$ where x_i^* is a maximizer for f on $[x_{i-1}, x_i]$

We can also consider sums of the form

$$\sum_{i=1}^n m_i(x_i - x_{i-1})$$

Where $m_i = \inf\{f(x) \mid x \in [x_{i-1}, x_i]\}$

Definition

$U(f, P) = \sum_{i=1}^n M_i(x_i - x_{i-1})$ is the Upper Darboux Sum

$L(f, P) = \sum_{i=1}^n m_i(x_i - x_{i-1})$ is the Lower Darboux Sum

Fact

$$L(f, P) \leq U(f, P)$$

Lemma 6.1

Suppose $f : [a, b] \rightarrow \mathbb{R}$ is bounded and $\exists m, M \in \mathbb{R} \ni m \leq f(x) \leq M \forall x \in [a, b]$

Then for any partition P of $[a, b]$ we have

$$m(b - a) \leq L(f, P) \leq U(f, P) \leq M(b - a)$$

Definition

Given a partition P of $[a, b]$, another partition P^* of $[a, b]$ is a refinement of P if $P \subset P^*$; i.e., if $P = \{x_0, x_1, \dots, x_n\}$ then each x_i is in P^* also

"A nontrivial refinement takes all the points in the given partition and adds at least one more"

Definition

Let $P_i = P^* \cap [x_{i-1}, x_i]$, then

$$U(f, P^*) = \sum_{i=1}^n U(f, P_i)$$

$$L(f, P^*) = \sum_{i=1}^n L(f, P_i)$$

Lemma 6.2 (Refinement Lemma)

Suppose $f : [a, b] \rightarrow \mathbb{R}$ is bounded and P is a partition of $[a, b]$ and P^* is a refinement of P , then

$$L(f, P) \leq L(f, P^*) \leq U(f, P^*) \leq U(f, P)$$

Definition

Let P_1, P_2 be two partitions of $[a, b]$ and set $P^* = P_1 \cup P_2$, then P^* refines both P_1 and P_2 and is called a common refinement of P_1 and P_2

Lemma 6.3

Suppose $f : [a, b] \rightarrow \mathbb{R}$ is bounded, P_1, P_2 are partitions of $[a, b]$, then

$$L(f, P_1) \leq U(f, P_2)$$

"Any lower sum is less than or equal to any upper sum"

Definition

Lower Integral

$$\int_a^b f \equiv \sup\{L(f, P) \mid P \text{ is a partition of } [a, b]\}$$

Upper Integral

$$\int_a^b f \equiv \inf\{U(f, P) \mid P \text{ is a partition of } [a, b]\}$$

Lemma 6.4

Suppose $f : [a, b] \rightarrow \mathbb{R}$ is bounded, then

$$\int_a^b f \leq \int_a^b f$$

Fact

A telescoping sum is a sum in which subsequent terms cancel each other and only leave the first and last terms

Example

$$m(b-a) = m \sum_{i=1}^n (x_i - x_{i-1}) = m[(x_1 - x_0) + (x_2 - x_1) + \cdots + (x_{n-1} - x_{n-2}) + (x_n - x_{n-1})] = m(x_n - x_0) = m(b-a)$$

Definition

Dirichlet's function: $f : [0, 1] \rightarrow \mathbb{R}$

$$\begin{cases} 0, & x \in [0, 1] \ni x \in \mathbb{Q} \\ 1, & x \in [0, 1] \ni x \in \mathbb{R} \setminus \mathbb{Q} \end{cases}$$

Section 6.2: The Archimedes-Riemman Theorem

Fact (Comparison Lemma for Positive Sequences)

Suppose $\{a_n\}, \{b_n\}$ satisfy $0 \leq a_n \leq b_n \forall n \in \mathbb{N}$ and $b_n \rightarrow 0$, then $a_n \rightarrow 0$

Fact (Order Preserving Property for Sequences)

Suppose $\{a_n\}, \{b_n\}$ with $a_n \rightarrow a$, $b_n \rightarrow b$ and $\forall n \in \mathbb{N} a_n \leq b_n$, then $a \leq b$

Definition

Suppose $f : [a, b] \rightarrow \mathbb{R}$ is bounded, then we say f is integrable (on $[a, b]$) if

$$\int_a^b f = \int_a^b f = I$$

If this is the case then we define

$$\int_a^b f = I = \text{the integral of } f \text{ on } [a, b]$$

Lemma 6.7

Suppose $f : [a, b] \rightarrow \mathbb{R}$ is bounded, and P is a partition of $[a, b]$, then

$$\begin{aligned} (*) \quad L(f, P) &\leq \int_a^b f \leq \int_a^b f \leq U(f, P) \Rightarrow \\ (i) \quad 0 &\leq \int_a^b f - \int_a^b f \leq U(f, P) - L(f, P) \\ (ii) \quad 0 &\leq U(f, P) - \int_a^b f \leq U(f, P) - L(f, P) \\ (ii) \quad 0 &\leq \int_a^b f - L(f, P) \leq U(f, P) - L(f, P) \end{aligned}$$

Theorem 6.8 (Archimedes-Riemann Theorem)

Suppose $f : [a, b] \rightarrow \mathbb{R}$ is bounded, then

$$f \text{ is integrable} \Leftrightarrow \exists \text{ a sequence of partitions } \{P_n\} \text{ of } [a, b] \ni \lim_{n \rightarrow \infty} [U(f, P_n) - L(f, P_n)] = 0$$

Definition

$f : [a, b] \rightarrow \mathbb{R}$ is bounded, $\{P_n\}$ is a sequence of partitions of $[a, b]$, then $\{P_n\}$ is an Archimedian Sequence if $U(f, P_n) - L(f, P_n) \rightarrow 0$

Definition

A partition of $P_n = \{x_0, x_1, \dots, x_n\}$ of $[a, b] \ni x_i = a + i(\frac{b-a}{n})$ is a regular partition of $[a, b]$

Definition

Let $P_n = \{x_0, x_1, \dots, x_n\}$ is a partition of $[a, b]$, then $\text{gap}(P) = \max_{1 \leq i \leq n} (x_i - x_{i-1})$

Fact 1

$f : [a, b] \rightarrow \mathbb{R}$ is bounded and f is monotonically increasing $\Rightarrow f$ is integrable

Definition

$f : [a, b] \rightarrow \mathbb{R}$ is a step function if $\exists P^* = \{z_0, z_1, \dots, z_k\}$ of $[a, b]$ and $c_1, \dots, c_k \in \mathbb{R} \ni f(x) = c_k, x \in (z_{k-1}, z_k)$

Note: $z_0 = a, z_k = b$

Fact 2

$f : [a, b] \rightarrow \mathbb{R}$ is a step function $\Rightarrow f$ is integrable

Fact 3

$f : [a, b] \rightarrow \mathbb{R}$ is Lipschitz continuous $\Rightarrow f$ is integrable

Leibniz Notation

$f : [a, b] \rightarrow \mathbb{R}$ is integrable

$$\int_a^b f = \int_a^b f(x)dx = \int_a^b f(*)d*$$

Section 6.3: Additivity, Monotonicity, Linearity

Theorem 6.12 (Additivity)

Suppose $f : [a, b] \rightarrow \mathbb{R}$ is integrable and $c \in (a, b)$, then f is integrable on $[a, c]$ and $[c, b]$ and

$$\int_a^b f = \int_a^c f + \int_c^b f$$

Theorem 6.13 (Monotonicity)

Suppose $f : [a, b] \rightarrow \mathbb{R}$ and $g : [a, b] \rightarrow \mathbb{R}$ are integrable and that $f(x) \leq g(x) \forall x \in [a, b]$, then

$$\int_a^b f \leq \int_a^b g$$

Lemma 6.14

$f : [a, b] \rightarrow \mathbb{R}$ and $g : [a, b] \rightarrow \mathbb{R}$ are bounded and P is a partition of $[a, b]$, then

$$L(f, P) + L(g, P) \leq L(f + g, P)$$

$$U(f + g, P) \leq U(f, P) + U(g, P)$$

And for any $\alpha \in \mathbb{R}$

$$\begin{cases} U(\alpha f, P) = \alpha U(f, P) \text{ and } L(\alpha f, P) = \alpha L(f, P), & \alpha \geq 0 \\ U(\alpha f, P) = \alpha L(f, P) \text{ and } L(\alpha f, P) = \alpha U(f, P), & \alpha < 0 \end{cases}$$

Theorem 6.15 (Linearity)

Let $f : [a, b] \rightarrow \mathbb{R}$ and $g : [a, b] \rightarrow \mathbb{R}$ be integrable. Then for $\alpha, \beta \in \mathbb{R}$ $\alpha f + \beta g$ is integrable and

$$\int_a^b (\alpha f + \beta g) = \alpha \int_a^b f + \beta \int_a^b g$$

Corollary 6.16

Let $f : [a, b] \rightarrow \mathbb{R}$ and $|f| : [a, b] \rightarrow \mathbb{R}$ be integrable. Then

$$\left| \int_a^b f(x) dx \right| \leq \int_a^b |f(x)| dx$$

Fact A

Suppose a set $S \subseteq \mathbb{R}$ and S is bounded, let $\alpha S = \{\alpha x \mid x \in S\}$, then

$$\begin{cases} \sup \alpha S = \alpha \sup S, & \alpha \geq 0 \\ \sup \alpha S = \alpha \inf S, & \alpha < 0 \end{cases}$$

Section 6.4: Continuity and Integrability

Theorem 6.18

$f : [a, b] \rightarrow \mathbb{R}$ is continuous $\Rightarrow f$ is integrable on $[a, b]$

Theorem 6.19

$f : [a, b] \rightarrow \mathbb{R}$ is bounded and continuous on $(a, b) \Rightarrow f$ is integrable on $[a, b]$
and $\int_a^b f$ does not depend on $f(a), f(b)$

Section 6.4 (sup): Continuity and Integrability

Theorem (6.3, 6)

$f : [a, b] \rightarrow \mathbb{R}$ is bounded and $a < c < b$. If f is integrable on $[a, c]$ and $[c, b]$, then f is integrable on $[a, b]$

Definition

$f : [a, b] \rightarrow \mathbb{R}$ is bounded and $a < c_1 < c_2 < \cdots < c_k < b$, then f is piecewise integrable if
 f is integrable on each of $[a, c_1], [c_1, c_2], \dots, [c_k, b]$
So f is piecewise integrable $\Rightarrow f$ is integrable

Corollary

Suppose f is bounded and $a < c_1 < c_2 < \cdots < c_k < b$, then

- (i) f is continuous on $(a, c_1), (c_1, c_2), \dots, (c_k, b) \Rightarrow f$ is integrable (f is piecewise continuous)
- (ii) f is monotone on $(a, c_1), (c_1, c_2), \dots, (c_k, b) \Rightarrow f$ is integrable (f is piecewise monotone)

Section 6.5: First Fundamental Theorem of Calculus

Theorem 6.22 (FTC 1)

$f : [a, b] \rightarrow \mathbb{R}$ is continuous and is differentiable on (a, b) and
 $f' : (a, b) \rightarrow \mathbb{R}$ is continuous and bounded, then

$$\int_a^b f'(x)dx = F(b) - F(a)$$

Section 6.6: Second Fundamental Theorem of Calculus

Theorem 6.26 (MVT for Integrals)

Let $f : [a, b] \rightarrow \mathbb{R}$ be continuous. Then $\exists c \in (a, b) \ni f(c)(b - a) = \int_a^b f$

$$f(c) = \frac{1}{b - a} \int_a^b f$$

Proposition 6.27

For $f : [a, b] \rightarrow \mathbb{R}$ is continuous, set $F(x) = \int_a^x f(t)dt$ for $x \in [a, b]$, then F is continuous on $[a, b]$

Theorem 6.29 (FTC 2)

$f : [a, b] \rightarrow \mathbb{R}$ is continuous, $F(x) = \int_a^x f(t)dt$, then F is differentiable on (a, b) and $F'(x) = f(x)$ for $x \in (a, b)$

Corollary 6.30

Suppose that the function $f : [a, b] \rightarrow \mathbb{R}$ is continuous. Then

$$\frac{d}{dx} \left[\int_x^b f \right] = -f(x) \text{ for all } x \in (a, b)$$

Definition

Let $f : [a, b] \rightarrow \mathbb{R}$ be integrable. Let $c, d \in [a, b] \ni c < d$

Set $\int_d^c f = -\int_c^d f, \int_c^c f = 0$

Fact

For any $x_1, x_2, x_3 \in [a, b]$

$$\int_{x_1}^{x_3} f = \int_{x_1}^{x_2} f + \int_{x_2}^{x_3} f$$

Corollary 6.31

$I = (a, b)$, f is continuous on I , $x_0 \in I \Rightarrow$

$$\frac{d}{dx} \int_{x_0}^x f = f(x) \quad \forall x \in I$$

Corollary 6.32

$I = (a, b)$, f is continuous on $J = (c, d)$, $\phi : J \rightarrow I$ is differentiable and $\phi(J) \subseteq I$, then for $x \in J$

$$\frac{d}{dx} \int_{x_0}^{\phi(x)} f = f(\phi(x)) \cdot \phi'(x) \quad \forall x \in J$$

Section 6.6 (sup): The Logarithm and Exponential Functions

Fact

Divergence of the Harmonic Series:

$S_n = \sum_{k=1}^n \frac{1}{k}$ is a harmonic series and diverges (does not converge and grows without bound)

Definitions

$$\ln x = \int_1^x \frac{1}{t} dt, \quad x > 0$$

$$\frac{d}{dx} \ln x = \frac{1}{x}$$

$$\ln 1 = \int_1^1 \frac{1}{t} dt = 0$$

Proposition 1

- (a) $\ln x$ is strictly increasing
- (b) $\ln x \rightarrow \infty$; $x \rightarrow \infty$
- (c) $\ln x \rightarrow -\infty$; $x \rightarrow 0$

Proposition 2

$a, b > 0, r \in \mathbb{Q}, r > 0$

- (i) $\ln ab = \ln a + \ln b$
- (ii) $\ln a^r = r \ln a$
- (iii) $\ln \frac{1}{b} = -\ln b$

Definition

$$f(x) = \ln x \Rightarrow f^{-1}(x) = \exp x = e^x$$

$\exp x$ is strictly increasing and

- (1) $\ln 1 = 0 \Rightarrow \exp 0 = 1$
- (2) $D(f) = (0, +\infty) \Rightarrow R(f^{-1}) = (0, +\infty)$
- (3) $R(f) = (-\infty, +\infty) \Rightarrow D(f^{-1}) = (-\infty, +\infty)$

Proposition 3

- (i) $\exp(a + b) = \exp a \cdot \exp b$
- (ii) $\exp ab = (\exp a)^b, b \in \mathbb{Q}, b > 0$
- (iii) $\frac{d}{dx} \exp x = \exp x$
- (iv) $\exp(-a) = \frac{1}{\exp a}$

Definition

$$e = \exp 1 = e^1 = \sum_{k=0}^{\infty} \frac{1}{k!}$$

Fact

$$x = b^y \Rightarrow y = \log_b(x)$$

Section 8.1: Taylor Polynomials

Definition

$I = (a, b), x_0 \in I. f : I \rightarrow \mathbb{R}, g : I \rightarrow \mathbb{R}$, have contact of order n at x_0 provided that f and g have derivatives of order n at x_0 and $f^{(k)}(x_0) = g^{(k)}(x_0)$,

$$k = 0, 1, \dots, n$$

Fact

$$\frac{d^k}{dx^k}(x - x_0)^l = \begin{cases} k!, & k = l \\ 0, & k \neq l \end{cases}$$

Proposition 8.2

$I = (a, b)$, $x_0 \in I$, $f : I \rightarrow \mathbb{R}$ has n derivatives (at x_0), set:

$$P_n(x) = \sum_{k=0}^n \frac{f^{(k)}(x_0)(x - x_0)^k}{k!}$$

Then P_n is the unique polynomial of degree $\leq n$ $\ni P_n$ has a contact of order n at x_0 with f (P_n is the Taylor Polynomial for f)

Fact

$$f(x) = e^x \Rightarrow P_n(x) = \sum_{k=0}^n \frac{x^k}{k!}$$

Fact

$$f(x) = \sin x \Rightarrow P_n(x) = \sum_{k=0}^n \frac{(-1)^k}{(2k+1)!} \cdot x^{2k+1}$$

Section 8.2: Lagrange Remainder Theorem

Lemma 8.7

$I = (a, b)$, $x_0 \in I$, $h : I \rightarrow \mathbb{R}$ has $n+1$ derivatives and $h^{(k)}(x_0) = 0$, $k = 0, 1, \dots, n$. Then for $x \in I$, $x \neq x_0 \ni z$ strictly between x and $x_0 \ni$

$$h(x) = \frac{h^{(n+1)}(z)(x - x_0)^{n+1}}{(n+1)!}$$

Theorem 8.8 (Lagrange Remainder Theorem)

$I = (a, b)$, $x_0 \in I$, $f : I \rightarrow \mathbb{R}$ has $n + 1$ derivatives. Then for $x \in I$
 $x \neq x_0 \exists c$ strictly between x and x_0 :

$$f(x) = P_n(x) + \frac{f^{(n+1)}(c)(x - x_0)^{n+1}}{(n + 1)!} = \sum_{k=0}^n \frac{f^{(k)}(x_0)(x - x_0)^k}{k!} + \frac{f^{(n+1)}(c)(x - x_0)^{n+1}}{(n + 1)!}$$

Section 8.3: Convergence of Taylor Polynomials

Lemma 8.20

Suppose $\{C_n\} \subset \mathbb{R}$: $\lim_{n \rightarrow \infty} \frac{|C_{n+1}|}{|C_n|} = l$, then

- (i) $l < 1 \Rightarrow C_n \rightarrow 0$
- (ii) $l > 1 \Rightarrow \{C_n\}$ is unbounded

Theorem 8.14

$I = (a, b)$, $f : I \rightarrow \mathbb{R}$ has derivatives up to all orders,
 $x_0 \in I$, $r > 0$: $[x_0 - r, x_0 + r] \subset I$. Suppose also that $\exists M$ for each $n \in \mathbb{N}$,
 $x \in [x_0 - r, x_0 + r]$, $|f^{(n)}(x)| \leq M^n$, then

$$f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(x_0)}{k!} (x - x_0)^k, \quad x \in [x_0 - r, x_0 + r]$$

Section 8.7: The Weierstrass Approximation Theorem

Lemma 8.24

For each $x \in \mathbb{R}$ and each $n \in \mathbb{N}$, $n \geq 2$

$$\sum_{k=0}^n (x - kn^{-1})^2 \binom{n}{k} x^k (1 - x)^{n-k} - \frac{x(1 - x)}{n}$$

Theorem 8.23 (Weirstrass Approximation)

Let $I = [a, b]$ and suppose $f : I \rightarrow \mathbb{R}$ is continuous. Then for each $\epsilon > 0$ there exists a polynomial $P : \mathbb{R} \rightarrow \mathbb{R}$ $\ni |f(x) - P(x)| < \epsilon \forall x \in I$

Where

$$P(x) = \sum_{k=0}^n f\left(\frac{k}{n}\right) \binom{n}{k} x^k (1-x)^{n-k}$$

Fact

Let f be continuous on $I = [a, b]$, then there exists a sequence of polynomials $\{P_n\} \ni \{P_n\}$ converges uniformly to f

Section 9.1: Sequences and Series

Definition

$\{a_n\}$ is a Cauchy Sequence if given $\epsilon > 0 \exists N \ni n, m \geq N \Rightarrow |a_n - a_m| < \epsilon$

Proposition 9.2

$\{a_n\}$ converges $\Rightarrow \{a_n\}$ is a Cauchy Sequence

Lemma 9.3

$\{a_n\}$ is a Cauchy Sequence $\Rightarrow \{a_n\}$ is bounded

Theorem 9.4

$\{a_n\}$ converges $\Leftrightarrow \{a_n\}$ is a Cauchy Sequence

Definition

For a given $\{a_n\}$ set $S_n = \sum_{k=1}^n a_k$

If $\{S_n\}$ converges we write

$$\sum_{k=1}^{\infty} a_k = \lim_{n \rightarrow \infty} S_n = \lim_{n \rightarrow \infty} \sum_{k=1}^n a_k$$

Fact

The Harmonic Series

$$\sum_{k=1}^{\infty} \frac{1}{k} \text{ diverges}$$

Proposition 9.5

$$\sum_{n=1}^{\infty} a_n \text{ converges} \Rightarrow \lim_{n \rightarrow \infty} a_n = 0$$

Proposition 9.6

$$|r| < 1 \Rightarrow \sum_{k=0}^{\infty} r^k = \frac{1}{1-r}$$

Theorem 9.7

Suppose $a_k \geq 1$, then

$$\sum_{k=0}^{\infty} a_k \text{ converges} \Leftrightarrow \exists M \ni \sum_{k=1}^n a_k \leq M$$

Corollary 9.8 (Comparison Test)

$$0 \leq a_k \leq b_k$$

$$(i) \sum_{k=1}^{\infty} b_k \text{ converges} \Rightarrow \sum_{k=1}^{\infty} a_k \text{ converges}$$

$$(ii) \sum_{k=1}^{\infty} a_k \text{ diverges} \Rightarrow \sum_{k=1}^{\infty} b_k \text{ diverges}$$

Corollary 9.11 (Integral Test)

Suppose $a_k \geq 0$ and suppose $\exists f : [0, +\infty) \rightarrow \mathbb{R}$, f is continuous, $f(k) = a_k$, and f is monotonically decreasing, then

$$\int_1^n f \leq M \text{ for some } M \forall n \in \mathbb{N} \Leftrightarrow \sum_{k=1}^{\infty} a_k \text{ converges}$$

Corollary 9.13 (p-Test)

$$\sum_{k=1}^{\infty} \frac{1}{k^p} \text{ converges} \Leftrightarrow p > 1$$

Theorem 9.15 (Alternating Series Test)

Suppose $a_n \geq 0$, $\{a_n\}$ is monotonically decreasing, and $a_n \rightarrow 0$, then

$$\sum_{k=1}^{\infty} (-1)^{k+1} a_k \text{ converges}$$

Theorem 9.17 (Cauchy Convergence Criterion for Series)

$$\sum_{k=1}^{\infty} a_k \text{ converges} \Leftrightarrow \text{Given } \epsilon > 0 \exists N \ni n \geq N \text{ implies for each } k \in \mathbb{N} |a_{n+1} + \cdots + a_{n+k}| < \epsilon$$

Definition

The series $\sum_{k=1}^{\infty} a_k$ is said to converge absolutely provided that the series $\sum_{k=1}^{\infty} |a_k|$ converges

Corollary 9.18 (Absolute Convergence Test)

$$\sum_{k=1}^{\infty} |a_k| \text{ converges} \Rightarrow \sum_{k=1}^{\infty} a_k \text{ converges (absolutely)}$$

Theorem 9.20

Suppose for $\sum_{k=1}^{\infty} a_k$ that $\exists N$ and $r \in [0, 1] \ni |a_{n+1}| \leq r|a_n| \forall n \geq N$, then

$$\sum_{k=1}^{\infty} a_k \text{ converges absolutely}$$

Corollary 9.21 (Ratio Test)

For $\sum_{k=1}^{\infty} a_k$, suppose that

$$\lim_{n \rightarrow \infty} \frac{|a_{n+1}|}{|a_n|} = l$$

(i) $l < 1 \Rightarrow \sum_{k=1}^{\infty} a_k$ converges (absolutely)

(ii) $l > 1 \Rightarrow \sum_{n=k}^{\infty} a_k$ diverges