Real Analysis I

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Section 1.1: The Completeness Axiom

Definition

$S \subseteq \mathbb{R}$ is inductive if
(i) $1 \in S$
(ii) $x \in S \Rightarrow x + 1 \in S$

Definition

$\mathbb{N}$ is the intersection of all inductive subsets of $\mathbb{R}$

Principle of Mathematical Induction

For each $n \in N$ let $S(n)$ be some mathematical assertion. Suppose also that
(i) $S(1)$ is true
(ii) Whenever $S(n)$ is true, then $S(n+1)$ is true
Then $S(n)$ is true $\forall n \in N$

Definition

$\mathbb{Z} = \mathbb{N} \cup \{0\} \cup \{-n | n \in \mathbb{N}\}$

Fact

$m, n \in \mathbb{Z} \Rightarrow$
(i) $m + n \in \mathbb{Z}$
(ii) $m - n \in \mathbb{Z}$
(iii) $mn \in \mathbb{Z}$

Definition

$\mathbb{Q} = \{\frac{m}{n} | m, n \in \mathbb{Z}, n \neq 0\}$

Fact

(i) Each $x \in \mathbb{Q} \Rightarrow x = \frac{m}{n}, m$ or n is odd
(ii) n^2 is even $\Rightarrow n$ is even

Proposition 1.2

$\exists$ No $x \in \mathbb{Q} \ni x^2 = 2$

Definition

$S \subset \mathbb{R}, S \neq \emptyset$ is Bounded Above if $\exists c \in \mathbb{R} \ni x \leq c \ \forall x \in S \Rightarrow c$ is an Upper Bound for S

Completeness Axiom

If $S \subset \mathbb{R}, S \neq \emptyset$, and S is Bounded Above, set $U_S = \{c \in \mathbb{R} | c$ is an upper bound for $S\}$
Then $\exists a \in U_S \ni a \leq c \ \forall c \in U_S$
$a = supS =$ supremum of S (least upper bound)

("Given a bounded, nonempty set S, and the set of all upper bounds of S, U_S, then there exists a least element in U_S that is the least upper bound for S (its supremum)")

Proposition 1.3

If $c > 0$, then $\exists! \ x > 0 \ni x^2 = c$

Theorem 1.4

$S \subset \mathbb{R}, S \neq \emptyset$, and S is Bounded Below, set $L_S = \{b \in \mathbb{R} | b$ is an lower bound for $S\}$
Then $\exists d \in L_S \ni d \geq b \ \forall b \in U_S$
$d = infS =$ infimum of S (greatest lower bound)

("Given a bounded, nonempty set S, and the set of all lower bounds of S, L_S, then there exists a greatest element in L_S that is the greatest lower bound for S (its infimum)")

Section 1.2: The Distribution of $\mathbb{Z}$ & $\mathbb{Q}$

Theorem 1.5 (Archimedean Property)

(i) $c > 0 \Rightarrow \exists n \in \mathbb{N} \ni n > c$
(ii) $\epsilon > 0 \Rightarrow \exists n \in \mathbb{N} \ni \frac{1}{n} < \epsilon$

Proposition 1.6

Let $n \in \mathbb{Z}$, then $\exists$ No $k \in \mathbb{Z} \ni k \in (n, n+1)$

Proposition 1.7

Suppose $S \neq \emptyset, S \subset \mathbb{Z}$, and S is Bounded Above, then S has a Maximum $m \in S$
Note: $m \in S \Rightarrow m = supS$

Theorem 1.8

For any $c \in \mathbb{R}$ $\exists!$ $k \in \mathbb{Z} \ni k \in [c, c+1)$

Definition

$S \subset \mathbb{R}$ is Dense in $\mathbb{R}$ if for any $I = (a, b), a < b, S \cap I \neq \emptyset$

Theorem 1.9

$\mathbb{Q}$ is Dense in $\mathbb{R}$

Corollary 1.10

$\mathbb{R} \setminus \mathbb{Q}$ is Dense in $\mathbb{R}$

Section 1.3: Inequalities and Identities

Definition

$x \in \mathbb{R}$, $|x| = \begin{cases} x, & x \geq 0 \\ -x, & x < 0 \end{cases}$

Fact 1

$d > 0$, $|c| \leq d \Leftrightarrow -d \leq c \leq d$

Fact 2

$x \in \mathbb{R}, -|x| \leq x \leq |x|$

Theorem 1.11 (Triangle Inequality)

If $a, b \in \mathbb{R}$, Then $|a + b| \leq |a| + |b|$

Proposition 1.12

$a, r \in \mathbb{R}, r > 0$, TFAE:
(i) $|x - a| < r$
(ii) $a - r < x < a + r$
(iii) $x \in (a - r, a + r)$

Difference of Powers Formula

$n \in \mathbb{N}$ and $a, b \in \mathbb{R}$,
$a^n - b^n = (a - b) \sum_{k=0}^{n-1} a^{n-1-k} b^k$

Geometric Series Formula

$r \in \mathbb{R}, r \neq 1$, then $\frac{1-r^{n+1}}{1-r} = 1 + r + \cdots + r^n = \sum_{k=0}^{n} r^k$

Definition

$n! = \begin{cases} 1, & n = 0, 1 \\ n * (n-1) * \cdots * 2 * 1, & n > 1 \end{cases}$

Definition

$\binom{n}{k} = \frac{n!}{k!(n-k)!}$

Binomial Formula

$n \in \mathbb{N}$ and $a, b \in \mathbb{R}$,
$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k$

Section 2.1: Convergence of Sequences

Definition

A sequence of real numbers is a function $f : \mathbb{N} \to \mathbb{R}$
set $a_n = f(n)$, then characterize f by $\{a_1, a_2, \cdots\} = \{a_n\}_{n=1}^{\infty} = \{a_n\}$

Definition

$\{a_n\}$ Converges to $a \in \mathbb{R}$ provided that
for each $\epsilon > 0$ $\exists N \in \mathbb{N} \ni |a_n - a| < \epsilon$ $\forall n \geq N$

Proposition 2.6

$\{\frac{1}{n}\}$ converges to 0

Fact

$\{1 + \frac{1}{2} + \cdots + \frac{1}{2^n}\}$ converges to 2

Fact

$a_n \to a$, $a_n \to b \Rightarrow a = b$
(limits are unique)

Fact

$\{(-1)^n\}$ does not converge

Lemma 2.9 (Comparison Lemma)

Suppose we have $\{a_n\}, \{b_n\}$ with $a_n \to a$. Then $b_n \to b$ if
$\exists c \geq 0$ and $N_1 \in \mathbb{N} \ni |b_n - b| \leq c|a_n - a|$ $\forall n \geq N_1$

Theorem 2.10 (Sum Property)

$a_n \to a,\ b_n \to b \Rightarrow a_n + b_n \to a + b$

Lemma 2.11

$a_n \to a,\ \alpha \in \mathbb{R} \Rightarrow (\alpha)a_n \to (\alpha)a$

Theorem 2.13 (Product property)

$a_n \to a,\ b_n \to b \Rightarrow a_n b_n \to ab$

Fact 1

$a_n \to a \Rightarrow |a_n| \to |a|$

Proposition 2.14

$b_n \to b \neq 0 \Rightarrow \frac{1}{b_n} \to \frac{1}{b}$

Theorem 2.15 (Quotient property)

$a_n \to a,\ b_n \to b \neq 0 \Rightarrow \frac{a_n}{b_n} \to \frac{a}{b}$

Proposition 2.16 (Linear property)

$a_n \to a,\ b_n \to b,\ \alpha, \beta \in \mathbb{R} \Rightarrow (\alpha)a_n + (\beta)b_n \to (\alpha)a + (\beta)b$

Fact 2

$a_n = c\ \forall n \Rightarrow a_n \to c$

Proposition 2.17

$P : \mathbb{R} \to \mathbb{R},\ a_n \to a \Rightarrow P(a_n) \to P(a)$

Section 2.2: Sequences & Sets

Theorem 2.18

$\{a_n\}$ converges $\Rightarrow \{a_n\}$ is bounded

Proposition 2.19

S is dense in $\mathbb{R} \Leftrightarrow$ each $x \in \mathbb{R}$ is a limit of a sequence in S

Theorem 2.20 (Sequential Density of $\mathbb{Q}$)

Every $x \in \mathbb{R}$ is the limit of a sequence of rational numbers

Lemma 2.21

$d_n \to d,\ d_n \geq 0 \Rightarrow d \geq 0$

Theorem 2.22

$\{c_n\} \subset [a, b],\ c_n \to c \Rightarrow c \in [a, b]$

Definition

$S \subset \mathbb{R}$ is closed if whenever$\{a_n\} \subset S$ and $a_n \to a$ then $a \in S$

Fact

$[a, b]$ is closed

Section 2.3: The Monotone Convergence Theorem

Definition

$\{a_n\}$ is monotonically increasing if $a_{n+1} \geq a_n$ for each n

Definition

$\{a_n\}$ is monotonically decreasing if $a_{n+1} \leq a_n$ for each n

Definition

$\{a_n\}$ is monotone if it is either monotonically increasing or decreasing

Theorem 2.25 (Monotone Convergence Theorem)

If $\{a_n\}$ is monotone, then
$\{a_n\}$ converges $\Leftrightarrow$ $\{a_n\}$ is bounded

Note: if $\{a_n\}$ is monotonically increasing, $a_n \to sup\{a_n\}$
Note: if $\{a_n\}$ is monotonically decreasing, $a_n \to inf\{a_n\}$

Proposition 2.28

Let $c \in \mathbb{R},\ |c| < 1 \Rightarrow \lim_{n\to\infty} c^n = 0$

Theorem 2.29 (Nested Interval Theorem)

Let $\{a_n\}$ and $\{b_n\}$ be such that $a_n < b_n$ and set $I_n = [a_n, b_n]$.
Assume that $I_{n+1} \subset I_n$ and that $\lim_{n\to\infty}[b_n - a_n] = 0$. Then $\exists!\ x \in \cap_{n=1}^{\infty} I_n$

Section 2.4: The Sequential Compactness Theorem

Definition

For a given $\{a_n\}$ let $\{n_k\}$ be a sequence of natural numbers that is strictly increasing. Then the sequence $\{b_k\}$ defined by $b_k = a_{n_k}$, with $k = 1, 2, \cdots$ is called a subsequence of $\{a_n\}$, denoted $\{b_k\} = \{a_{n_k}\}_{k=1}^{\infty} = \{a_{n_k}\}$

Fact

Given a sequence $\{n_k\}_{k=1}^{\infty}$ of natural numbers that is strictly increasing, we have that $n_k \geq k$ for every $k \in \mathbb{N}$

Proposition 2.30

Let $\{a_n\}$ converge to a, i.e., $a_n \to a$
Then $\lim_{n\to\infty} a_{n_k} = a$ for every subsequence $\{a_{n_k}\}$ of $\{a_n\}$

Theorem 2.32

For every $\{a_n\}$ $\exists\{n_k\} \ni \{a_{n_k}\}$ is monotone

Theorem 2.33

Every bounded sequence has a convergent subsequence

Definition

$S \subseteq \mathbb{R}$ is sequentially compact if every sequence $\{a_n\} \subset S$ has a convergent subsequence whose limit is in S

Theorem 2.36 (Sequential Compactness Theorem)

$a, b \in \mathbb{R}$ with $a < b \Rightarrow [a, b]$ is sequentially compact

Section 3.1: Continuity

Definition

For $D \subseteq \mathbb{R}, f : D \to \mathbb{R}$ is continuous at $x_0 \in D$ if whenever $\{x_n\} \subset D$
and $x_n \to x_0$ then $f(x_n) \to f(x_0)$

($f : D \to \mathbb{R}$ is continuous if it is continuous $\forall\ x_0 \in D$)

Fact

$P : \mathbb{R} \to \mathbb{R} \Rightarrow x_0 \in \mathbb{R}, \{x_n\} \subset \mathbb{R}, x_n \to x_0 \Rightarrow P(x_n) \to P(x_0) \Rightarrow P$ is continuous

Theorem 3.4

Suppose $f : D \to \mathbb{R}, g : D \to \mathbb{R}$ are continuous at $x_0 \in D,$ then
$f + g : D \to \mathbb{R}$ and $fg : D \to \mathbb{R}$ are continuous at $x_0 \in D$
and if $g(x) \neq 0\ \forall\ x \in D$ then $\frac{f}{g} : D \to \mathbb{R}$ is continuous at $x_0 \in D$

Corollary 3.5

Let P and Q be polynomials, then $\frac{P}{Q} : D \to \mathbb{R}$ is continuous where $D = \{x \in \mathbb{R} \mid Q(x_0) \neq 0\}$

Theorem 3.6

$f : D \to \mathbb{R}, g : U \to \mathbb{R}, f(D) \subseteq U$ and suppose that
f is continuous at $x_0 \in D$ and g is continuous at $f(x_0) \in U$
then $g \circ f$ is continuous at x_0; namely, $g \circ f : D \to \mathbb{R}$

Fact

$f(x) = \sqrt{x}$ is continuous on $D = [0, +\infty)$

Section 3.1 (Sup): Trigonometric Continuity

Fact 1

if $\theta_n \to 0,$ then $\sin \theta_n \to 0$

Fact 2

if $\theta_n \to 0,$ then $\cos \theta_n \to 1$

Fact

$\sin \theta$ is continuous,
$\cos \theta$ is continuous,
$\tan \theta$ is continuous at $\cos \theta \neq 0$ $(\theta \neq (2n+1) * \frac{\pi}{2})$,
$\csc \theta$ is continuous at $\sin \theta \neq 0$ $(\theta \neq n\pi)$,
$\sec \theta$ is continuous at $\cos \theta \neq 0$ $(\theta \neq ((2n+1) * \frac{\pi}{2})$,
$\cot \theta$ is continuous at $\sin \theta \neq 0$ $(\theta \neq n\pi)$

Section 3.2: Extreme Value Theorem

Definition

For $f : D \to \mathbb{R}$ we define $f(D) = \{y \in \mathbb{R} \mid y = f(x) \text{ for some } x \in D\}$
$f(D)$ is the image of f

Definition

$f : D \to \mathbb{R}$ attains a maximum (max value) if $\exists\ x_0 \in D \ni f(x) \leq f(x_0)\ \forall x \in D$
Such a point x_0 is a maximizer of f

$f : D \to \mathbb{R}$ attains a minimum (min value) if $\exists\ x_0' \in D \ni f(x_0') \leq f(x)\ \forall x \in D$
Such a point x_0' is a minimizer of f

Lemma 3.10

If $f : [a, b] \to \mathbb{R}$ is continuous, then $f([a, b])$ is bounded above ($\exists m \ni f(x) \leq m\ \forall x \in [a, b]$)

Theorem 3.9 (Extreme Value Theorem)

If $f : [a, b] \to \mathbb{R}$ is continuous, then f attains both a max and a min

$\exists x_0, x_0' \in [a, b] \ni f(x_0) \leq f(x) \leq f(x_0') \forall x \in [a, b]$

Fact

Let $S \subset [a, b]$, then $inf S \in [a, b]$, and $sup S \in [a, b]$

Section 3.3: Intermediate Value Theorem

Theorem 3.11 (Intermediate Value Theorem)

Suppose $f : [a, b] \to \mathbb{R}$ is continuous and let $c \in \mathbb{R}$ be any number strictly between $f(a)$ and $f(b)$; i.e., $f(a) < c < f(b)$ or $f(b) < c < f(a)$, then $\exists x_0 \in (a, b) \ni f(x_0) = c$

Fact

Suppose $f : D \to \mathbb{R}$ is continuous. If $\exists [a, b] \subset D \ni f(a) < 0$
and $f(b) > 0$ (or vice-versa), then $\exists x_0 \in (a, b) \ni f(x_0) = 0$

"A real, continuous function that is positive on one side and negative on the other contains a root"

Definition

$D \subseteq \mathbb{R}$ is convex if $u, v \in D,\ (u < v) \Rightarrow [u, v] \subset D$

Fact

If $D \subset \mathbb{R}$ is convex then D is an interval

Theorem 3.14

If I is an interval and $f : I \to \mathbb{R}$ is continuous then $f(I)$ is an interval

Section 3.4: Uniform Continuity

Definition

$f : D \to \mathbb{R}$ is uniformly continuous on D if whenever $\{u_n\}, \{v_n\} \subset D \ni u_n - v_n \to 0$, then $f(u_n) - f(v_n) \to 0$

Note: if $v_n = x_0\ \forall n$, then $u_n - v_n \to 0 \Rightarrow u_n \to x_0$, so uniform continuity $\Rightarrow$ continuity at each $x_0 \in D$

Fact

$f(x) = x$ is uniformly continuous but $f(x) = x^2$ is not

Theorem 3.17

$f : [a, b] \to \mathbb{R}$ is continuous $\ \Rightarrow f$ is uniformly continuous on $[a, b]$

Fact

If $f : D \to \mathbb{R}$ satisfies Lipschitz Continuity: $|f(u) - f(v)| \leq c|u - v|$, $\forall u, v \in D$ and for some $c \geq 0$, then f is uniformly continuous.

Fact

Let P be a polynomial. Then on each $[a, b]$, $P : [a, b] \to \mathbb{R}$ is lipschitz continous

Section 3.5: The $\epsilon - \delta$ Criteria for Continuity

Definition

"The $\epsilon - \delta$ Criterion At a Point" - $f : D \to \mathbb{R}$ satisfies the $\epsilon - \delta$ criterion at a point $x_0 \in D$, if for each $\epsilon > 0$ $\exists \delta > 0 \ni$ for $x \in D$, $|x - x_0| < \delta \Rightarrow |f(x) - f(x_0)| < \epsilon$

Theorem 3.20

For $f : D \to \mathbb{R}$ and $x_0 \in D$, TFAE:

(i) f is continuous at x_0
(ii) The $\epsilon - \delta$ criterion at x_0 holds

Definition

"The $\epsilon - \delta$ Criterion On the Domain of a Function" - $f : D \to \mathbb{R}$ satisfies the $\epsilon - \delta$ criterion on D, if for each $\epsilon > 0$ $\exists \delta > 0 \ni$ $u, v \in D$, $|u - v| < \delta \Rightarrow |f(u) - f(v)| < \epsilon$

Theorem 3.22

For $f : D \to \mathbb{R}$, TFAE:

(i) $f : D \to \mathbb{R}$ is uniformly continuous
(ii) $f : D \to \mathbb{R}$ satisfies the $\epsilon - \delta$ criterion on D

Fact

$I = (a, b)$, $f : I \to \mathbb{R}$ is continuous, then if $x_0 \in (a, b)$ with $f(x_0) > 0$, then
$\exists$ $I_1 = (a_1, b_1) \subset I \ni f(x) > 0$ $\forall$ $x \in I_1$

Section 3.6: Images and Inverses; Monotone Functions

Definition

(i) $f : D \to \mathbb{R}$ is monotonically increasing if $u, v \in D$ and $u < v \Rightarrow f(u) \leq f(v)$
(ii) $f : D \to \mathbb{R}$ is monotonically decreasing if $u, v \in D$ and $u < v \Rightarrow f(u) \geq f(v)$
(iii) $f : D \to \mathbb{R}$ is monotone if it is monotonically increasing or decreasing

Theorem 3.23

Suppose $f : D \to \mathbb{R}$ is monotone. If $f(D)$ is an interval, then f is continuous

Corollary 3.25

Suppose $f : I \to \mathbb{R}$ is monotone, then f is continuous $\Leftrightarrow f(I)$ is an interval

Definition

(i) $f : D \to \mathbb{R}$ is strictly increasing if $u, v \in D$ and $u < v \Rightarrow f(u) < f(v)$
(ii) $f : D \to \mathbb{R}$ is strictly decreasing if $u, v \in D$ and $u < v \Rightarrow f(u) > f(v)$

Definition

$f : D \to \mathbb{R}$ is one-to-one (injective) if for each $y \in f(D)$ $\exists! \ x \in D \ni f(x) = y$

"No element in the image may have more than one element in the domain mapping to it"

Fact

If f is strictly increasing or decreasing, then f is one-to-one

Fact

If $f : I \to \mathbb{R}$ is continuous and f is one-to-one, then f is strictly monotone

Definition

Suppose $f : D \to \mathbb{R}$ is one-to-one. If $y \in f(D)$, let $x \in D \ni f(x) = y$
Define $f^{-1} : f(D) \to D$ by $f^{-1}(y) = x$, so f^{-1} is well-defined since x is unique

Note:
(i) $f^{-1}(f(x)) = x$, where $x \in D$
(ii) $f(f^{-1}(y)) = y$, where $y \in f(D)$

Theorem 3.29

$f : I \to \mathbb{R}$ is continuous and strictly increasing or decreasing $\Rightarrow$
$f^{-1} : f(I) \to \mathbb{R}$ is continuous

Section 3.7: Limits

Definition

$h(x) = \frac{f(x)-f(x_0)}{x-x_0}$ gives the slope of the line at point x_0
and $h(x)$ is continuous on $[a,b] \setminus \{x_0\}$

Definition

$D \subset \mathbb{R}$, $x_0 \in \mathbb{R}$ is a limit point of D if $\exists \{x_n\} \subset D \setminus \{x_0\} \ni x_n \to x_0$

Definition

If $f : D \to \mathbb{R}$ and x_0 is a limit point of D, then we denote $\lim_{x \to x_0} f(x) = l$
If whenever $\{x_n\} \subset D \setminus \{x_0\}$ and $x_n \to x_0$ we have that $\lim_{n\to\infty} f(x_n) = l$
(x_0 may or may not be in D)

Example

$D = \mathbb{R} \setminus \{x_0\}$, $f(x) = x^2 \Rightarrow h(x) = \frac{x^2-(x_0)^2}{x-x_0}$ and suppose
$\{x_n\} \subset D, x_n \to x_0 \Rightarrow h(x_n) = \frac{(x_n)^2-(x_0)^2}{x_n-x_0} = \frac{(x_n+x_0)(x_n-x_0)}{x_n-x_0} = x_n + x_0$
So $\lim_{n\to\infty} h(x_n) = \lim_{n\to\infty}(x_n + x_0) = x_0 + x_0 = 2x_0$

Theorem 3.36

Suppose $f : D \to \mathbb{R},\ g : D \to \mathbb{R},$ and x_0 is a limit point of D, so that

$$\lim_{x \to x_0} f(x) = A, \quad \lim_{x \to x_0} g(x) = B \Rightarrow$$

(i) $\lim_{x\to x_0}[f(x) + g(x)] = A + B$
(ii) $\lim_{x\to x_0}[f(x)g(x)] = AB$
(ii)(a) $\alpha \in \mathbb{R},\ \lim_{x\to x_0}[\alpha f(x)] = \alpha A$
(iii) $B \neq 0,\ g(x) \neq 0\ \forall x \in D,\ \lim_{x\to x_0}[\frac{f(x)}{g(x)}] = \frac{A}{B}$

Theorem 3.37

$f : D \to \mathbb{R},\ g : U \to \mathbb{R}$ and
x_0 is a limit point of $D \ni \lim_{x\to x_0} f(x) = y_0$,
y_0 is a limit point of $U \ni \lim_{y\to y_0} g(y) = e$,
and suppose that $f(D \setminus \{x_0\}) \subset U \setminus \{y_0\}$, then
$\lim_{x\to x_0}(g \circ f)(x) = e$

Definition

$x_0 \in D$ is an isolated point if $\exists\ r > 0 \ni (x_0 - r, x_0 + r) \cap D = \{x_0\}$

Fact

$x_0 \in D \Rightarrow x_0$ is either a limit point or an isolated point of D

Limits and Continuity Theorem

For $f : D \to \mathbb{R},\ x_0 \in D$, then
f is continuous at $x_0 \Leftrightarrow x_0$ is an isolated point of D or $\lim_{x\to x_0} f(x) = f(x_0)$

So f is continuous at $x_0 \Leftrightarrow \lim_{x\to x_0} f(x) = f(x_0)$

Fact in Review

If $h(x) = g(x)$ on $D \setminus \{x_0\}$ where $g : D \to \mathbb{R}$ is continuous on D, then
$\lim_{x\to x_0} h(x) = \lim_{x\to x_0} g(x) = g(x_0)$

Section 4.1: The Algebra of Derivatives

Definition

$x_0 \in \mathbb{R}, I \subset \mathbb{R} \ni I = (a, b)$ and $x_0 \in I \Rightarrow I$ is a neighborhood of x_0

Definition

$x_0 \in \mathbb{R}$ and I is a neighborhood of $x_0 \Rightarrow f : I \to \mathbb{R}$ is differentiable at x_0 IF $\lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}$ exists. We say $f'(x_0) \coloneqq \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}$ and is called the derivative of f at x_0

Definition

If $f : I \to \mathbb{R}$ is differentiable at each $x_0 \in I$ then f is differentiable and $f' : I \to \mathbb{R}$ is the derivative of f

Definition

The line determined by $y = f(x_0) + f'(x_0)(x - x_0)$ is the tangent line to the graph of f at $(x_0, f(x_0))$

For $y_0 = f(x_0)$, $y - y_0 = f'(x_0)(x - x_0)$

Proposition 4.4

$n \in \mathbb{N}, f(x) = x^n \ \forall x \in I = \mathbb{R} \Rightarrow f : \mathbb{R} \to \mathbb{R}$ is differentiable and $f'(x) = nx^{n-1}$

Proposition 4.5

$x_0 \in \mathbb{R}, I$ is a neighborhood of $x_0, I = (a, b)$ and $f : I \to \mathbb{R}$ is differentiable at $x_0 \Rightarrow$ f is continuous at x_0

Theorem 4.6

$x_0 \in \mathbb{R}, I$ is a neighborhood of $x_0, f : I \to \mathbb{R}$ and $f : I \to \mathbb{R}$ is differentiable at x_0, then

(i) $f + g : I \to \mathbb{R}$ is differentiable at x_0 and
$(f + g)'(x_0) = f'(x_0) + g'(x_0)$

(ii) $fg : I \to \mathbb{R}$ is differentiable at x_0 and
$(fg)'(x_0) = f(x_0)g'(x_0) + f'(x_0)g(x_0)$

(iii) If $g(x) \neq 0 \ \forall x \in I$ then $\frac{1}{g} : I \to \mathbb{R}$ is differentiable at x_0 and

$(\frac{1}{g})'(x_0) = \frac{-g'(x_0)}{[g(x_0)]^2}$

(iv) If $g(x) \neq 0 \ \forall x \in I$ then $\frac{f}{g} : I \to \mathbb{R}$ is differentiable at x_0 and
$(\frac{f}{g})'(x_0) = \frac{f'(x_0)g(x_0) - f(x_0)g'(x_0)}{[g(x_0)]^2}$

Fact

For $x_0, \alpha \in \mathbb{R},\ (\alpha f)'(x_0) = \alpha f'(x_0)$

Fact

$f(x) = c \Rightarrow f'(x) = 0 \ \forall x \in D$

Proposition 4.7

$n \in \mathbb{Z}, D = \mathbb{R}$ if $n \geq 0$ and $D = \mathbb{R} \backslash \{0\}$ if $n < 0,$ then for $f : D \to \mathbb{R}$ defined by $f(x) = x^n,\ f$ is differentiable and $f'(x) = nx^{n-1}$

Corollary 4.8

$p, q : \mathbb{R} \to \mathbb{R}$ are polynomials, $D = \mathbb{R} \setminus \{x \mid q(x) = 0\}$, then
$\frac{p}{q} : D \to \mathbb{R}$ is differentiable

Section 4.2: Differentiating Inverses & Compositions

Theorem 4.11

Suppose $x_0 \in I,$ and $f : I \to \mathbb{R}$ is strictly monotone, continuous, differentiable at x_0, and $f'(x_0) \neq 0$. Let $J = f(I)$ then
$f^{-1} : J \to \mathbb{R}$ is differentiable at $y_0 = f(x_0)$ and $(f^{-1})'(y_0) = \frac{1}{f'(x_0)} = \frac{1}{f'(f^{-1}(y_0))}$

Corollary 4.12

Suppose $f : I \to \mathbb{R}$ is strictly monotone, differentiable, and f' is nonzero on I. Let $J = f(I)$, then $(f^{-1}) : J \to \mathbb{R}$ is differentiable and $\forall x \in J$
$(f^{-1})'(x) = \frac{1}{f'(f^{-1}(x))}$

Proposition 4.13

Let $g(g) = x^{\frac{1}{n}}$ where $n \in \mathbb{N}$ and $x > 0$, then
$g : (0, \infty) \to \mathbb{R}$ is differentiable and $g'(x) = \frac{1}{n} x^{\frac{1}{n}-1}$ $\forall x > 0$

Theorem 4.14 (Chain Rule)

Suppose $x_0 \in I$ with $f : I \to \mathbb{R}$ is differentiable. Say $f(I) \subseteq J$ and suppose $g : J \to \mathbb{R}$ is differentiable at $f(x_0)$, then $g \circ f : I \to \mathbb{R}$ is differentiable at x_0 and $(g \circ f)'(x_0) = g'(f(x_0)) \cdot f'(x_0)$

Proposition 4.15

For $r = \frac{m}{n}$ where $n \neq 0, m \in \mathbb{Z}, n \in \mathbb{N}$, set $h(x) = x^r$, where $x > 0$, then
h is differentiable and $h'(x) = rx^{r-1}$ $\forall x > 0$

Section 4.3: The Mean Value Theorem

Lemma 4.16

Suppose I is a neighborhood of x_0 and $f : I \to \mathbb{R}$ is differentiable at x_0.
If x_0 is a maximizer or a minimizer, then $f'(x_0) = 0$

Theorem 4.17 (Rolle's Theorem)

Suppose $f : [a, b] \to \mathbb{R}$ is continuous and $f : (a, b) \to \mathbb{R}$ is differentiable.
Assume that $f(a) = f(b)$, then $\exists\ x_0 \in (a, b) \ni f'(x_0) = 0$

Theorem 4.18 (Mean Value Theorem)

If $f : [a, b] \to \mathbb{R}$ is continuous and $f : (a, b) \to \mathbb{R}$ is differentiable, then
$\exists\ x_0 \in (a, b) \ni f'(x_0) = \frac{f(b)-f(a)}{b-a}$

Lemma 4.19

Suppose $I = (a, b)$ and $f : I \to \mathbb{R}$ is differentiable. Then
f is constant $\Leftrightarrow f'(x) = 0\ \forall\ x \in I$

Proposition 4.20

Suppose $g,\ h : I \to \mathbb{R}$ are differentiable. Then
$g = h + c \Leftrightarrow g'(x) = h'(x)\ \forall\ x \in I$

Corollary 4.21

(i) $f : I \to \mathbb{R}$ is differentiable $\ni f'(x) > 0\ \forall\ x \in I \Rightarrow f$ is strictly increasing

(ii) $f : I \to \mathbb{R}$ is differentiable $\ni f'(x) < 0\ \forall\ x \in I \Rightarrow f$ is strictly decreasing

Definition

Suppose $f : D \to \mathbb{R}$, then $x_0 \in D$ is a
(i) local maximizer if $\exists\ \delta > 0 \ni x_0$ is a maximizer for f on $D\ \cap (x_0 - \delta, x_0 + \delta)$
(ii) local minimizer if $\exists\ \delta > 0 \ni x_0$ is a minimizer for f on $D\ \cap (x_0 - \delta, x_0 + \delta)$

Definition

Suppose $f : I \to \mathbb{R}$ is differentiable on I. If $f' : I \to \mathbb{R}$ is differentiable on I, then define $f'' : I \to \mathbb{R}$ by $f''(x) = (f')'(x) = f^{(2)}(x)$ for each $x \in I$
Inductively define $f^{(k)} : I \to \mathbb{R}$, $k \in \mathbb{N}$

Theorem 4.22 (2nd Derivative Test)

Suppose $f, f' : I \to \mathbb{R}$ are differentiable and $x_0 \in I \ni f'(x_0) = 0$. Then
(i) $f''(x_0) > 0 \Rightarrow x_0$ is a local minimizer for f (concave up)
(ii) $f''(x_0) < 0 \Rightarrow x_0$ is a local maximizer for f (concave down)

Fact

If f is continuous on $[a, b]$ and f is differentiable on (a, b),
then f attains its max and min at either
(i) The endpoints a or b
(ii) $x_0 \in (a, b) \ni f'(x_0) = 0$

Section 4.4: Cauchy Mean Value Theorem

Theorem 4.23 (Cauchy Mean Value Theorem)

If $f : [a, b] \to \mathbb{R}$ and $g : [a, b] \to \mathbb{R}$ are continuous on $[a, b]$ and differentiable on (a, b) with $g'(x) \neq 0 \ \forall \ x \in (a, b)$, then $\exists \ x_0 \in (a, b) \ni \frac{f(b)-f(a)}{g(b)-g(a)} = \frac{f'(x_0)}{g'(x_0)}$

Lemma 1

If $h_1(x) = (x - x_0)^n$, then $h_1^{(k)}(x) = \begin{cases} \frac{n!}{(n-k)!} \cdot (x - x_0)^{n-k}, & 0 \leq k \leq n \\ 0, & k > n \end{cases}$

Theorem 4.24

Suppose $f : I \to \mathbb{R}$ has n derivatives on I and suppose at $x_0 \in I$ that
$f^{(k)}(x_0) = 0$ for $0 \leq k \leq n-1$, then for each $x \in I$ with $x \neq x_0 \ \exists \ z$ strictly between x and $x_0 \ni$
$f(x) = \frac{f^{(n)}(z)}{n!} \cdot (x - x_0)^n$

Application

Let $g : I \to \mathbb{R}$ have $n + 1$ derivatives and set for $x_0 \in I$
$h(x) = \sum_{j=0}^{n} \frac{g^{(j)}(x_0)}{j!} \cdot (x - x_0)^j$
Then $g(x) = h(x) + \frac{g^{(n+1)}(z)}{(n+1)!} \cdot (x - x_0)^{n+1}$
(Taylor's Formula with Remainder)

Section 4.4 (sup): Trigonometric Differentiability

Fact 1

(i) $\lim_{\theta \to 0} \frac{\sin \theta}{\theta} = 1$
(ii) $\sin \theta \to 0$ as $\theta \to 0$
(iii) $\cos \theta \to 1$ as $\theta \to 0$
(iv) $\sin(x + y) = \sin x \cos y + \cos x \sin y$
(v) $\cos(x + y) = \cos x \cos y - \sin x \sin y$
(vi) $\frac{d}{dx} \sin x = \cos x$
(vii) $\frac{d}{dx} \cos x = -\sin x$