# Real Analysis I

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## Section 1.1: The Completeness Axiom

### Definition

$S \subseteq \mathbb{R}$ is inductive if
(i) $1 \in S$
(ii) $x \in S \Rightarrow x + 1 \in S$

### Definition

$\mathbb{N}$ is the intersection of all inductive subsets of $\mathbb{R}$

### Principle of Mathematical Induction

For each $n \in N$ let $S(n)$ be some mathematical assertion. Suppose also that
(i) $S(1)$ is true
(ii) Whenever $S(n)$ is true, then $S(n+1)$ is true
Then $S(n)$ is true $\forall n \in N$

### Definition

$\mathbb{Z} = \mathbb{N} \cup \{0\} \cup \{-n | n \in \mathbb{N}\}$

### Fact

$m, n \in \mathbb{Z} \Rightarrow$
(i) $m + n \in \mathbb{Z}$
(ii) $m - n \in \mathbb{Z}$
(iii) $mn \in \mathbb{Z}$

## Definition

$\mathbb{Q} = \{\frac{m}{n} | m, n \in \mathbb{Z}, n \neq 0\}$

## Fact

(i) Each $x \in \mathbb{Q} \Rightarrow x = \frac{m}{n}, m$ or $n$ is odd
(ii) $n^2$ is even $\Rightarrow n$ is even

## Proposition 1.2

$\exists$ No $x \in \mathbb{Q} \ni x^2 = 2$

## Definition

$S \subset \mathbb{R}, S \neq \emptyset$ is Bounded Above if $\exists c \in \mathbb{R} \ni x \leq c \ \forall x \in S \Rightarrow c$ is an Upper Bound for $S$

## Completeness Axiom

If $S \subset \mathbb{R}, S \neq \emptyset$, and $S$ is Bounded Above, set $U_S = \{c \in \mathbb{R} | c$ is an upper bound for $S\}$
Then $\exists a \in U_S \ni a \leq c \ \forall c \in U_S$
$a = supS =$ supremum of S (least upper bound)

("Given a bounded, nonempty set S, and the set of all upper bounds of S, $U_S$, then there exists a least element in $U_S$ that is the least upper bound for S (its supremum)")

## Proposition 1.3

If $c > 0$, then $\exists! \ x > 0 \ni x^2 = c$

## Theorem 1.4

$S \subset \mathbb{R}, S \neq \emptyset$, and $S$ is Bounded Below, set $L_S = \{b \in \mathbb{R} | b$ is an lower bound for $S\}$
Then $\exists d \in L_S \ni d \geq b \ \forall b \in U_S$
$d = infS =$ infimum of S (greatest lower bound)

("Given a bounded, nonempty set S, and the set of all lower bounds of S, $L_S$, then there exists a greatest element in $L_S$ that is the greatest lower bound for S (its infimum)")

## Section 1.2: The Distribution of $\mathbb{Z}$ & $\mathbb{Q}$

### Theorem 1.5 (Archimedian Property)

(i) $c > 0 \Rightarrow \exists n \in \mathbb{N} \ni n > c$
(ii) $\epsilon > 0 \Rightarrow \exists n \in \mathbb{N} \ni \frac{1}{n} < \epsilon$

### Proposition 1.6

Let $n \in \mathbb{Z}$, then $\exists$ No $k \in \mathbb{Z} \ni k \in (n, n+1)$

### Proposition 1.7

Suppose $S \neq \emptyset, S \subset \mathbb{Z}$, and $S$ is Bounded Above, then $S$ has a Maximum $m \in S$
Note: $m \in S \Rightarrow m = supS$

### Theorem 1.8

For any $c \in \mathbb{R}$ $\exists!$ $k \in \mathbb{Z} \ni k \in [c, c+1)$

### Definition

$S \subset \mathbb{R}$ is Dense in $\mathbb{R}$ if for any $I = (a, b), a < b, S \cap I \neq \emptyset$

### Theorem 1.9

$\mathbb{Q}$ is Dense in $\mathbb{R}$

### Corollary 1.10

$\mathbb{R} \setminus \mathbb{Q}$ is Dense in $\mathbb{R}$

## Section 1.3: Inequalities and Identities

## Definition

$x \in \mathbb{R}$, $|x| = \begin{cases} x, & x \geq 0 \\ -x, & x < 0 \end{cases}$

## Fact 1

$d > 0$, $|c| \leq d \Leftrightarrow -d \leq c \leq d$

## Fact 2

$x \in \mathbb{R}, -|x| \leq x \leq |x|$

## Theorem 1.11 (Triangle Inequality)

If $a, b \in \mathbb{R}$, Then $|a + b| \leq |a| + |b|$

## Proposition 1.12

$a, r \in \mathbb{R}, r > 0$, TFAE:
(i) $|x - a| < r$
(ii) $a - r < x < a + r$
(iii) $x \in (a - r, a + r)$

## Difference of Powers Formula

$n \in \mathbb{N}$ and $a, b \in \mathbb{R}$,
$a^n - b^n = (a - b) \sum_{k=0}^{n-1} a^{n-1-k} b^k$

## Geometric Series Formula

$r \in \mathbb{R}, r \neq 1$, then $\frac{1-r^{n+1}}{1-r} = 1 + r + \cdots + r^n = \sum_{k=0}^{n} r^k$

## Definition

$n! = \begin{cases} 1, & n = 0, 1 \\ n * (n-1) * \cdots * 2 * 1, & n > 1 \end{cases}$

## Definition

$\binom{n}{k} = \frac{n!}{k!(n-k)!}$

## Binomial Formula

$n \in \mathbb{N}$ and $a, b \in \mathbb{R}$,
$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k$

# Section 2.1: Convergence of Sequences

## Definition

A sequence of real numbers is a function $f : \mathbb{N} \to \mathbb{R}$
set $a_n = f(n)$, then characterize $f$ by $\{a_1, a_2, \cdots\} = \{a_n\}_{n=1}^{\infty} = \{a_n\}$

## Definition

$\{a_n\}$ Converges to $a \in \mathbb{R}$ provided that
for each $\epsilon > 0$ $\exists N \in \mathbb{N} \ni |a_n - a| < \epsilon$ $\forall n \geq N$

## Proposition 2.6

$\{\frac{1}{n}\}$ converges to 0

## Fact

$\{1 + \frac{1}{2} + \cdots + \frac{1}{2^n}\}$ converges to 2

## Fact

$a_n \to a$, $a_n \to b \Rightarrow a = b$
(limits are unique)

## Fact

$\{(-1)^n\}$ does not converge

## Lemma 2.9 (Comparison Lemma)

Suppose we have $\{a_n\}, \{b_n\}$ with $a_n \to a$. Then $b_n \to b$ if
$\exists c \geq 0$ and $N_1 \in \mathbb{N} \ni |b_n - b| \leq c|a_n - a|$ $\forall n \geq N_1$

## Theorem 2.10 (Sum Property)

$a_n \to a,\ b_n \to b \Rightarrow a_n + b_n \to a + b$

## Lemma 2.11

$a_n \to a,\ \alpha \in \mathbb{R} \Rightarrow (\alpha)a_n \to (\alpha)a$

## Theorem 2.13 (Product property)

$a_n \to a,\ b_n \to b \Rightarrow a_n b_n \to ab$

## Fact 1

$a_n \to a \Rightarrow |a_n| \to |a|$

## Proposition 2.14

$b_n \to b \neq 0 \Rightarrow \frac{1}{b_n} \to \frac{1}{b}$

## Theorem 2.15 (Quotient property)

$a_n \to a,\ b_n \to b \neq 0 \Rightarrow \frac{a_n}{b_n} \to \frac{a}{b}$

## Proposition 2.16 (Linear property)

$a_n \to a,\ b_n \to b,\ \alpha, \beta \in \mathbb{R} \Rightarrow (\alpha)a_n + (\beta)b_n \to (\alpha)a + (\beta)b$

## Fact 2

$a_n = c\ \forall n \Rightarrow a_n \to c$

## Proposition 2.17

$P : \mathbb{R} \to \mathbb{R},\ a_n \to a \Rightarrow P(a_n) \to P(a)$

# Section 2.2: Sequences & Sets

## Theorem 2.18

$\{a_n\}$ converges $\Rightarrow \{a_n\}$ is bounded

## Proposition 2.19

$S$ is dense in $\mathbb{R} \Leftrightarrow$ each $x \in \mathbb{R}$ is a limit of a sequence in $S$

## Theorem 2.20 (Sequential Density of $\mathbb{Q}$)

Every $x \in \mathbb{R}$ is the limit of a sequence of rational numbers

## Lemma 2.21

$d_n \to d,\ d_n \geq 0 \Rightarrow d \geq 0$

## Theorem 2.22

$\{c_n\} \subset [a, b],\ c_n \to c \Rightarrow c \in [a, b]$

## Definition

$S \subset \mathbb{R}$ is closed if whenever$\{a_n\} \subset S$ and $a_n \to a$ then $a \in S$

## Fact

$[a, b]$ is closed

# Section 2.3: The Monotone Convergence Theorem

## Definition

$\{a_n\}$ is monotonically increasing if $a_{n+1} \geq a_n$ for each $n$

## Definition

$\{a_n\}$ is monotonically decreasing if $a_{n+1} \leq a_n$ for each $n$

## Definition

$\{a_n\}$ is monotone if it is either monotonically increasing or decreasing

## Theorem 2.25 (Monotone Convergence Theorem)

If $\{a_n\}$ is monotone, then
$\{a_n\}$ converges $\Leftrightarrow$ $\{a_n\}$ is bounded

Note: if $\{a_n\}$ is monotonically increasing, $a_n \to sup\{a_n\}$
Note: if $\{a_n\}$ is monotonically decreasing, $a_n \to inf\{a_n\}$

## Proposition 2.28

Let $c \in \mathbb{R},\ |c| < 1 \Rightarrow \lim_{n\to\infty} c^n = 0$

## Theorem 2.29 (Nested Interval Theorem)

Let $\{a_n\}$ and $\{b_n\}$ be such that $a_n < b_n$ and set $I_n = [a_n, b_n]$.
Assume that $I_{n+1} \subset I_n$ and that $\lim_{n\to\infty}[b_n - a_n] = 0$. Then $\exists!\ x \in \cap_{n=1}^{\infty} I_n$

# Section 2.4: The Sequential Compactness Theorem

## Definition

For a given $\{a_n\}$ let $\{n_k\}$ be a sequence of natural numbers that is strictly increasing. Then the sequence $\{b_k\}$ defined by $b_k = a_{n_k}$, with $k = 1, 2, \cdots$ is called a subsequence of $\{a_n\}$, denoted $\{b_k\} = \{a_{n_k}\}_{k=1}^{\infty} = \{a_{n_k}\}$

## Fact

Given a sequence $\{n_k\}_{k=1}^{\infty}$ of natural numbers that is strictly increasing, we have that $n_k \geq k$ for every $k \in \mathbb{N}$

## Proposition 2.30

Let $\{a_n\}$ converge to $a$, i.e., $a_n \to a$
Then $\lim_{n\to\infty} a_{n_k} = a$ for every subsequence $\{a_{n_k}\}$ of $\{a_n\}$

## Theorem 2.32

For every $\{a_n\}$ $\exists\{n_k\} \ni \{a_{n_k}\}$ is monotone

## Theorem 2.33

Every bounded sequence has a convergent subsequence

## Definition

$S \subseteq \mathbb{R}$ is sequentially compact if every sequence $\{a_n\} \subset S$ has a convergent subsequence whose limit is in $S$

## Theorem 2.36 (Sequential Compactness Theorem)

$a, b \in \mathbb{R}$ with $a < b \Rightarrow [a, b]$ is sequentially compact

# Section 3.1: Continuity

## Definition

For $D \subseteq \mathbb{R}, f : D \to \mathbb{R}$ is continuous at $x_0 \in D$ if whenever $\{x_n\} \subset D$
and $x_n \to x_0$ then $f(x_n) \to f(x_0)$

($f : D \to \mathbb{R}$ is continuous if it is continuous $\forall\ x_0 \in D$)

## Fact

$P : \mathbb{R} \to \mathbb{R} \Rightarrow x_0 \in \mathbb{R}, \{x_n\} \subset \mathbb{R}, x_n \to x_0 \Rightarrow P(x_n) \to P(x_0) \Rightarrow P$ is continuous

## Theorem 3.4

Suppose $f : D \to \mathbb{R}, g : D \to \mathbb{R}$ are continuous at $x_0 \in D,$ then
$f + g : D \to \mathbb{R}$ and $fg : D \to \mathbb{R}$ are continuous at $x_0 \in D$
and if $g(x) \neq 0\ \forall\ x \in D$ then $\frac{f}{g} : D \to \mathbb{R}$ is continuous at $x_0 \in D$

## Corollary 3.5

Let $P$ and $Q$ be polynomials, then $\frac{P}{Q} : D \to \mathbb{R}$ is continuous where $D = \{x \in \mathbb{R} \mid Q(x_0) \neq 0\}$

## Theorem 3.6

$f : D \to \mathbb{R}, g : U \to \mathbb{R}, f(D) \subseteq U$ and suppose that
$f$ is continuous at $x_0 \in D$ and $g$ is continuous at $f(x_0) \in U$
then $g \circ f$ is continuous at $x_0$; namely, $g \circ f : D \to \mathbb{R}$

## Fact

$f(x) = \sqrt{x}$ is continuous on $D = [0, +\infty)$

# Section 3.1 (Sup): Trigonometric Continuity

## Fact 1

if $\theta_n \to 0,$ then $\sin\theta_n \to 0$

## Fact 2

if $\theta_n \to 0,$ then $\cos\theta_n \to 1$

## Fact

$\sin\theta$ is continuous,
$\cos\theta$ is continuous,
$\tan\theta$ is continuous at $\cos\theta \neq 0$ $(\theta \neq (2n+1) * \frac{\pi}{2})$,
$\csc\theta$ is continuous at $\sin\theta \neq 0$ $(\theta \neq n\pi)$,
$\sec\theta$ is continuous at $\cos\theta \neq 0$ $(\theta \neq ((2n+1) * \frac{\pi}{2})$,
$\cot\theta$ is continuous at $\sin\theta \neq 0$ $(\theta \neq n\pi)$

# Section 3.2: Extreme Value Theorem

## Definition

For $f : D \to \mathbb{R}$ we define $f(D) = \{y \in \mathbb{R} \mid y = f(x) \text{ for some } x \in D\}$
$f(D)$ is the image of $f$

## Definition

$f : D \to \mathbb{R}$ attains a maximum (max value) if $\exists\ x_0 \in D \ni f(x) \leq f(x_0)\ \forall x \in D$
Such a point $x_0$ is a maximizer of $f$

$f : D \to \mathbb{R}$ attains a minimum (min value) if $\exists\ x_0' \in D \ni f(x_0') \leq f(x)\ \forall x \in D$
Such a point $x_0'$ is a minimizer of $f$

## Lemma 3.10

If $f : [a, b] \to \mathbb{R}$ is continuous, then $f([a, b])$ is bounded above ($\exists m \ni f(x) \leq m\ \forall x \in [a, b]$)

## Theorem 3.9 (Extreme Value Theorem)

If $f : [a, b] \to \mathbb{R}$ is continuous, then $f$ attains both a max and a min

## Fact

Let $S \subset [a, b]$, then $infS \in [a, b]$, and $supS \in [a, b]$

# Section 3.3: Intermediate Value Theorem

## Theorem 3.11 (Intermediate Value Theorem)

Suppose $f : [a, b] \to \mathbb{R}$ is continuous and let $c \in \mathbb{R}$ be any number strictly between $f(a)$ and $f(b)$; i.e., $f(a) < c < f(b)$ or $f(b) < c < f(a)$, then $\exists x_0 \in (a, b) \ni f(x_0) = c$

## Fact

Suppose $f : D \to \mathbb{R}$ is continuous. If $\exists [a, b] \subset D \ni f(a) < 0$
and $f(b) > 0$ (or vice-versa), then $\exists x_0 \in (a, b) \ni f(x_0) = 0$
"A real, continuous function that is positive on one side and negative on the other contains a root"

## Definition

$D \subseteq \mathbb{R}$ is convex if $u, v \in D,\ (u < v) \Rightarrow [u, v] \subset D$

## Fact

If $D \subset \mathbb{R}$ is convex then $D$ is an interval

## Theorem 3.14

If $I$ is an interval and $f : I \to \mathbb{R}$ is continuous then $f(I)$ is an interval

# Section 3.4: Uniform Continuity

## Definition

$f : D \to \mathbb{R}$ is uniformly continuous on $D$ if whenever $\{u_n\}, \{v_n\} \subset D \ni u_n - v_n \to 0$, then $f(u_n) - f(v_n) \to 0$

Note: if $v_n = x_0\ \forall n$, then $u_n - v_n \to 0 \Rightarrow u_n \to x_0$, so uniform continuity $\Rightarrow$ continuity at each $x_0 \in D$

## Fact

$f(x) = x$ is uniformly continuous but $f(x) = x^2$ is not

## Theorem 3.17

$f : [a, b] \to \mathbb{R}$ is continuous $\Rightarrow f$ is uniformly continuous on $[a, b]$

## Fact

If $f : D \to \mathbb{R}$ satisfies Lipschitz Continuity: $|f(u) - f(v)| \leq c|u - v|$, $\forall u, v \in D$ and for some $c \geq 0$, then $f$ is uniformly continuous.

# Section 3.5: The $\epsilon - \delta$ Criteria for Continuity

## Definition

"The $\epsilon-\delta$ Criterion At a Point" - $f : D \to \mathbb{R}$ satisfies the $\epsilon-\delta$ criterion at a point $x_0 \in D$, if for each $\epsilon > 0$ $\exists \delta > 0 \ni$ for $x \in D$, $|x - x_0| < \delta \Rightarrow |f(x) - f(x_0)| < \epsilon$

## Theorem 3.20

For $f : D \to \mathbb{R}$ and $x_0 \in D$, TFAE:

(i) $f$ is continuous at $x_0$
(ii) The $\epsilon - \delta$ criterion at $x_0$ holds

## Definition

"The $\epsilon - \delta$ Criterion On the Domain of a Function" - $f : D \to \mathbb{R}$ satisfies the $\epsilon - \delta$ criterion on $D$, if for each $\epsilon > 0$ $\exists \delta > 0 \ni$ $u, v \in D$, $|u - v| < \delta \Rightarrow |f(u) - f(v)| < \epsilon$

## Theorem 3.22

For $f : D \to \mathbb{R}$, TFAE:

(i) $f : D \to \mathbb{R}$ is uniformly continuous
(ii) $f : D \to \mathbb{R}$ satisfies the $\epsilon - \delta$ criterion on $D$