

# Real Analysis I

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## Section 1.1: The Completeness Axiom

### Definition

$S \subseteq \mathbb{R}$  is inductive if

- (i)  $1 \in S$
- (ii)  $x \in S \Rightarrow x + 1 \in S$

### Definition

$\mathbb{N}$  is the intersection of all inductive subsets of  $\mathbb{R}$

### Principle of Mathematical Induction

For each  $n \in \mathbb{N}$  let  $S(n)$  be some mathematical assertion. Suppose also that

- (i)  $S(1)$  is true
- (ii) Whenever  $S(n)$  is true, then  $S(n + 1)$  is true

Then  $S(n)$  is true  $\forall n \in \mathbb{N}$

### Definition

$$\mathbb{Z} = \mathbb{N} \cup \{0\} \cup \{-n | n \in \mathbb{N}\}$$

### Fact

$m, n \in \mathbb{Z} \Rightarrow$

- (i)  $m + n \in \mathbb{Z}$
- (ii)  $m - n \in \mathbb{Z}$
- (iii)  $mn \in \mathbb{Z}$

## Definition

$$\mathbb{Q} = \left\{ \frac{m}{n} \mid m, n \in \mathbb{Z}, n \neq 0 \right\}$$

## Fact

- (i) Each  $x \in \mathbb{Q} \Rightarrow x = \frac{m}{n}, m$  or  $n$  is odd
- (ii)  $n^2$  is even  $\Rightarrow n$  is even

## Proposition 1.2

$$\exists \text{ No } x \in \mathbb{Q} \ni x^2 = 2$$

## Definition

$S \subset \mathbb{R}, S \neq \emptyset$  is Bounded Above if  $\exists c \in \mathbb{R} \ni x \leq c \forall x \in S \Rightarrow c$  is an Upper Bound for  $S$

## Completeness Axiom

If  $S \subset \mathbb{R}, S \neq \emptyset$ , and  $S$  is Bounded Above, set  $U_S = \{c \in \mathbb{R} \mid c \text{ is an upper bound for } S\}$

Then  $\exists a \in U_S \ni a \leq c \forall c \in U_S$

$a = \sup S =$  supremum of  $S$  (least upper bound)

("Given a bounded, nonempty set  $S$ , and the set of all upper bounds of  $S$ ,  $U_S$ , then there exists a least element in  $U_S$  that is the least upper bound for  $S$  (its supremum)")

## Proposition 1.3

$$\text{If } c > 0, \text{ then } \exists! x > 0 \ni x^2 = c$$

## Theorem 1.4

$S \subset \mathbb{R}, S \neq \emptyset$ , and  $S$  is Bounded Below, set  $L_S = \{b \in \mathbb{R} \mid b \text{ is a lower bound for } S\}$

Then  $\exists d \in L_S \ni d \geq b \forall b \in L_S$

$d = \inf S =$  infimum of  $S$  (greatest lower bound)

("Given a bounded, nonempty set  $S$ , and the set of all lower bounds of  $S$ ,  $L_S$ , then there exists a greatest element in  $L_S$  that is the greatest lower bound for  $S$  (its infimum)")

## Section 1.2: The Distribution of $\mathbb{Z}$ & $\mathbb{Q}$

### Theorem 1.5 (Archimedian Property)

- (i)  $c > 0 \Rightarrow \exists n \in \mathbb{N} \ni n > c$
- (ii)  $\epsilon > 0 \Rightarrow \exists n \in \mathbb{N} \ni \frac{1}{n} < \epsilon$

### Proposition 1.6

Let  $n \in \mathbb{Z}$ , then  $\exists$  No  $k \in \mathbb{Z} \ni k \in (n, n+1)$

### Proposition 1.7

Suppose  $S \neq \emptyset$ ,  $S \subset \mathbb{Z}$ , and  $S$  is Bounded Above, then  $S$  has a Maximum  $m \in S$   
Note:  $m \in S \Rightarrow m = \sup S$

### Theorem 1.8

For any  $c \in \mathbb{R} \exists! k \in \mathbb{Z} \ni k \in [c, c+1)$

### Definition

$S \subset \mathbb{R}$  is Dense in  $\mathbb{R}$  if for any  $I = (a, b)$ ,  $a < b$ ,  $S \cap I \neq \emptyset$

### Theorem 1.9

$\mathbb{Q}$  is Dense in  $\mathbb{R}$

### Corollary 1.10

$\mathbb{R} \setminus \mathbb{Q}$  is Dense in  $\mathbb{R}$

## Section 1.3: Inequalities and Identities

## Definition

$$x \in \mathbb{R}, |x| = \begin{cases} x, & x \geq 0 \\ -x, & x < 0 \end{cases}$$

## Fact 1

$$d > 0, |c| \leq d \Leftrightarrow -d \leq c \leq d$$

## Fact 2

$$x \in \mathbb{R}, -|x| \leq x \leq |x|$$

## Theorem 1.11 (Triangle Inequality)

If  $a, b \in \mathbb{R}$ , Then  $|a + b| \leq |a| + |b|$

## Proposition 1.12

$a, r \in \mathbb{R}, r > 0$ , TFAE:

- (i)  $|x - a| < r$
- (ii)  $a - r < x < a + r$
- (iii)  $x \in (a - r, a + r)$

## Difference of Powers Formula

$$n \in \mathbb{N} \text{ and } a, b \in \mathbb{R}, \\ a^n - b^n = (a - b) \sum_{k=0}^{n-1} a^{n-1-k} b^k$$

## Geometric Series Formula

$$r \in \mathbb{R}, r \neq 1, \text{ then } \frac{1-r^{n+1}}{1-r} = 1 + r + \cdots + r^n = \sum_{k=0}^n r^k$$

## Definition

$$n! = \begin{cases} 1, & n = 0, 1 \\ n * (n-1) * \cdots * 2 * 1, & n > 1 \end{cases}$$

## Definition

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}$$

## Binomial Formula

$$n \in \mathbb{N} \text{ and } a, b \in \mathbb{R},$$
$$(a + b)^n = \sum_{k=0}^n \binom{n}{k} a^{n-k} b^k$$

## Section 2.1: Convergence of Sequences

### Definition

A sequence of real numbers is a function  $f : \mathbb{N} \rightarrow \mathbb{R}$   
set  $a_n = f(n)$ , then characterize  $f$  by  $\{a_1, a_2, \dots\} = \{a_n\}_{n=1}^{\infty} = \{a_n\}$

### Definition

$\{a_n\}$  Converges to  $a \in \mathbb{R}$  provided that  
for each  $\epsilon > 0 \exists N \in \mathbb{N} \ni |a_n - a| < \epsilon \forall n \geq N$

### Proposition 2.6

$\{\frac{1}{n}\}$  converges to 0

### Fact

$\{1 + \frac{1}{2} + \dots + \frac{1}{2^n}\}$  converges to 2

### Fact

$a_n \rightarrow a, a_n \rightarrow b \Rightarrow a = b$   
(limits are unique)

### Fact

$\{(-1)^n\}$  does not converge

### Lemma 2.9 (Comparison Lemma)

Suppose we have  $\{a_n\}, \{b_n\}$  with  $a_n \rightarrow a$ . Then  $b_n \rightarrow b$  if  
 $\exists c \geq 0$  and  $N_1 \in \mathbb{N} \ni |b_n - b| \leq c|a_n - a| \forall n \geq N_1$

### **Theorem 2.10 (Sum Property)**

$$a_n \rightarrow a, b_n \rightarrow b \Rightarrow a_n + b_n \rightarrow a + b$$

### **Lemma 2.11**

$$a_n \rightarrow a, \alpha \in \mathbb{R} \Rightarrow (\alpha)a_n \rightarrow (\alpha)a$$

### **Theorem 2.13 (Product property)**

$$a_n \rightarrow a, b_n \rightarrow b \Rightarrow a_n b_n \rightarrow ab$$

### **Fact 1**

$$a_n \rightarrow a \Rightarrow |a_n| \rightarrow |a|$$

### **Proposition 2.14**

$$b_n \rightarrow b \neq 0 \Rightarrow \frac{1}{b_n} \rightarrow \frac{1}{b}$$

### **Theorem 2.15 (Quotient property)**

$$a_n \rightarrow a, b_n \rightarrow b \neq 0 \Rightarrow \frac{a_n}{b_n} \rightarrow \frac{a}{b}$$

### **Proposition 2.16 (Linear property)**

$$a_n \rightarrow a, b_n \rightarrow b, \alpha, \beta \in \mathbb{R} \Rightarrow (\alpha)a_n + (\beta)b_n \rightarrow (\alpha)a + (\beta)b$$

### **Fact 2**

$$a_n = c \ \forall n \Rightarrow a_n \rightarrow c$$

### **Proposition 2.17**

$$P : \mathbb{R} \rightarrow \mathbb{R}, a_n \rightarrow a \Rightarrow P(a_n) \rightarrow P(a)$$

## Section 2.2: Sequences & Sets

### Theorem 2.18

$\{a_n\}$  converges  $\Rightarrow \{a_n\}$  is bounded

### Proposition 2.19

$S$  is dense in  $\mathbb{R} \Leftrightarrow$  each  $x \in \mathbb{R}$  is a limit of a sequence in  $S$

### Theorem 2.20 (Sequential Density of $\mathbb{Q}$ )

Every  $x \in \mathbb{R}$  is the limit of a sequence of rational numbers

### Lemma 2.21

$d_n \rightarrow d, d_n \geq 0 \Rightarrow d \geq 0$

### Theorem 2.22

$\{c_n\} \subset [a, b], c_n \rightarrow c \Rightarrow c \in [a, b]$

### Definition

$S \subset \mathbb{R}$  is closed if whenever  $\{a_n\} \subset S$  and  $a_n \rightarrow a$  then  $a \in S$

### Fact

$[a, b]$  is closed

## Section 2.3: The Monotone Convergence Theorem

### Definition

$\{a_n\}$  is monotonically increasing if  $a_{n+1} \geq a_n$  for each  $n$

## Definition

$\{a_n\}$  is monotonically decreasing if  $a_{n+1} \leq a_n$  for each  $n$

## Definition

$\{a_n\}$  is monotone if it is either monotonically increasing or decreasing

## Theorem 2.25 (Monotone Convergence Theorem)

If  $\{a_n\}$  is monotone, then  
 $\{a_n\}$  converges  $\Leftrightarrow \{a_n\}$  is bounded

Note: if  $\{a_n\}$  is monotonically increasing,  $a_n \rightarrow \sup\{a_n\}$

Note: if  $\{a_n\}$  is monotonically decreasing,  $a_n \rightarrow \inf\{a_n\}$

## Proposition 2.28

Let  $c \in \mathbb{R}$ ,  $|c| < 1 \Rightarrow \lim_{n \rightarrow \infty} c^n = 0$

## Theorem 2.29 (Nested Interval Theorem)

Let  $\{a_n\}$  and  $\{b_n\}$  be such that  $a_n < b_n$  and set  $I_n = [a_n, b_n]$ .  
Assume that  $I_{n+1} \subset I_n$  and that  $\lim_{n \rightarrow \infty} [b_n - a_n] = 0$ . Then  $\exists! x \in \cap_{n=1}^{\infty} I_n$

## Section 2.4: The Sequential Compactness Theorem

## Definition

For a given  $\{a_n\}$  let  $\{n_k\}$  be a sequence of natural numbers that is strictly increasing. Then the sequence  $\{b_k\}$  defined by  $b_k = a_{n_k}$ , with  $k = 1, 2, \dots$  is called a subsequence of  $\{a_n\}$ , denoted  $\{b_k\} = \{a_{n_k}\}_{k=1}^{\infty} = \{a_{n_k}\}$

## Fact

Given a sequence  $\{n_k\}_{k=1}^{\infty}$  of natural numbers that is strictly increasing, we have that  $n_k \geq k$  for every  $k \in \mathbb{N}$



### Proposition 2.30

Let  $\{a_n\}$  converge to  $a$ , i.e.,  $a_n \rightarrow a$   
Then  $\lim_{n \rightarrow \infty} a_{n_k} = a$  for every subsequence  $\{a_{n_k}\}$  of  $\{a_n\}$

### Theorem 2.32

For every  $\{a_n\}$   $\exists \{n_k\} \ni \{a_{n_k}\}$  is monotone

### Theorem 2.33

Every bounded sequence has a convergent subsequence

### Definition

$S \subseteq \mathbb{R}$  is sequentially compact if every sequence  $\{a_n\} \subset S$  has a convergent subsequence whose limit is in  $S$

### Theorem 2.36 (Sequential Compactness Theorem)

$a, b \in \mathbb{R}$  with  $a < b \Rightarrow [a, b]$  is sequentially compact

## Section 3.1: Continuity

### Definition

For  $D \subseteq \mathbb{R}$ ,  $f : D \rightarrow \mathbb{R}$  is continuous at  $x_0 \in D$  if whenever  $\{x_n\} \subset D$   
and  $x_n \rightarrow x_0$  then  $f(x_n) \rightarrow f(x_0)$

( $f : D \rightarrow \mathbb{R}$  is continuous if it is continuous  $\forall x_0 \in D$ )

### Fact

$P : \mathbb{R} \rightarrow \mathbb{R} \Rightarrow x_0 \in \mathbb{R}, \{x_n\} \subset \mathbb{R}, x_n \rightarrow x_0 \Rightarrow P(x_n) \rightarrow P(x_0) \Rightarrow P$  is continuous

### Theorem 3.4

Suppose  $f : D \rightarrow \mathbb{R}, g : D \rightarrow \mathbb{R}$  are continuous at  $x_0 \in D$ , then  
 $f + g : D \rightarrow \mathbb{R}$  and  $fg : D \rightarrow \mathbb{R}$  are continuous at  $x_0 \in D$   
and if  $g(x) \neq 0 \forall x \in D$  then  $\frac{f}{g} : D \rightarrow \mathbb{R}$  is continuous at  $x_0 \in D$

### Corollary 3.5

Let  $P$  and  $Q$  be polynomials, then  $\frac{P}{Q} : D \rightarrow \mathbb{R}$  is continuous where  $D = \{x \in \mathbb{R} \mid Q(x) \neq 0\}$

### Theorem 3.6

$f : D \rightarrow \mathbb{R}, g : U \rightarrow \mathbb{R}, f(D) \subseteq U$  and suppose that  $f$  is continuous at  $x_0 \in D$  and  $g$  is continuous at  $f(x_0) \in U$  then  $g \circ f$  is continuous at  $x_0$ ; namely,  $g \circ f : D \rightarrow \mathbb{R}$

### Fact

$f(x) = \sqrt{x}$  is continuous on  $D = [0, +\infty)$

## Section 3.1 (Sup): Trigonometric Continuity

### Fact 1

if  $\theta_n \rightarrow 0$ , then  $\sin \theta_n \rightarrow 0$

### Fact 2

if  $\theta_n \rightarrow 0$ , then  $\cos \theta_n \rightarrow 1$

### Fact

$\sin \theta$  is continuous,

$\cos \theta$  is continuous,

$\tan \theta$  is continuous at  $\cos \theta \neq 0$  ( $\theta \neq (2n+1) * \frac{\pi}{2}$ ),

$\csc \theta$  is continuous at  $\sin \theta \neq 0$  ( $\theta \neq n\pi$ ),

$\sec \theta$  is continuous at  $\cos \theta \neq 0$  ( $\theta \neq (2n+1) * \frac{\pi}{2}$ ),

$\cot \theta$  is continuous at  $\sin \theta \neq 0$  ( $\theta \neq n\pi$ )

## Section 3.2: Extreme Value Theorem

## Definition

For  $f : D \rightarrow \mathbb{R}$  we define  $f(D) = \{y \in \mathbb{R} \mid y = f(x) \text{ for some } x \in D\}$   
 $f(D)$  is the image of  $f$

## Definition

$f : D \rightarrow \mathbb{R}$  attains a maximum (max value) if  $\exists x_0 \in D \ni f(x) \leq f(x_0) \forall x \in D$   
Such a point  $x_0$  is a maximizer of  $f$

$f : D \rightarrow \mathbb{R}$  attains a minimum (min value) if  $\exists x'_0 \in D \ni f(x'_0) \leq f(x) \forall x \in D$   
Such a point  $x'_0$  is a minimizer of  $f$

## Lemma 3.10

If  $f : [a, b] \rightarrow \mathbb{R}$  is continuous, then  $f([a, b])$  is bounded above ( $\exists m \ni f(x) \leq m \forall x \in [a, b]$ )

## Theorem 3.9 (Extreme Value Theorem)

If  $f : [a, b] \rightarrow \mathbb{R}$  is continuous, then  $f$  attains both a max and a min

## Fact

Let  $S \subset [a, b]$ , then  $\inf S \in [a, b]$ , and  $\sup S \in [a, b]$

## Section 3.3: Intermediate Value Theorem

### Theorem 3.11 (Intermediate Value Theorem)

Suppose  $f : [a, b] \rightarrow \mathbb{R}$  is continuous and let  $c \in \mathbb{R}$  be any number strictly between  $f(a)$  and  $f(b)$ ; i.e.,  $f(a) < c < f(b)$  or  $f(b) < c < f(a)$ , then  $\exists x_0 \in (a, b) \ni f(x_0) = c$

## Definition

$D \subseteq \mathbb{R}$  is convex if  $u, v \in D$ ,  $(u < v) \Rightarrow [u, v] \subset D$

**Fact**

If  $D \subset \mathbb{R}$  is convex then  $D$  is an interval

**Theorem 3.14**

If  $I$  is an interval and  $f : I \rightarrow \mathbb{R}$  is continuous then  $f(I)$  is an interval