Advanced Electromagnetics

Chapter 3 –Radiation

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Outline

3.1 Radiation in Free Space

3.1.1 Potentials

Consider no the magnetic sources, we have $\vec{\nabla} \cdot \vec{B} = 0$. Thus, from vector identity, the magnetic flux density can be expressed as

$$\vec{B} = \vec{\nabla} \times \vec{A} \tag{3.1.1}$$

where $\vec{A}$ is called the magnetic vector potential. Substitute (3.1.1) into (1.5.5) with $\vec{M} = 0$, we get

$$\vec{\nabla} \times \left(\vec{E} + i\omega\vec{A}\right) = 0 \tag{3.1.2}$$

from which we can express the electric field intensity as

$$\vec{E} = -\vec{\nabla}\mathcal{V} - i\omega\vec{A} \tag{3.1.3}$$

where $\mathcal{V}$ is called the electric scalar potential.

3.1.1 Potentials

Substitute (3.1.1) and (3.1.3) into (1.5.6), we get

$$\vec{\nabla} \times \vec{\nabla} \times \vec{A} = \mu \vec{J} + \omega^2 \mu \epsilon \vec{A} - i\omega\mu\epsilon\vec{\nabla}\mathcal{V} \tag{3.1.4}$$

Also, substitute (3.1.3) into (1.5.7), we get

$$\nabla^2 \mathcal{V} + i\omega\vec{\nabla} \cdot \vec{A} = -\rho/\epsilon \tag{3.1.5}$$

Using the vector identity $\vec{\nabla} \times \vec{\nabla} \times \vec{A} = \vec{\nabla}\left(\vec{\nabla} \cdot \vec{A}\right) - \nabla^2 \vec{A}$, and rearranging the terms, (3.1.4) and (3.1.5) can be re-expressed as

$$\nabla^2 \vec{A} + k^2 \vec{A} = -\mu \vec{J} + \vec{\nabla}\left(\vec{\nabla} \cdot \vec{A} + i\omega\mu\epsilon\mathcal{V}\right) \tag{3.1.6}$$

$$\nabla^2 \mathcal{V} + k^2 \mathcal{V} = -\rho/\epsilon - i\omega\left(\vec{\nabla} \cdot \vec{A} + i\omega\mu\epsilon\mathcal{V}\right) \tag{3.1.7}$$

3.1.1 Potentials

where $k^2 = \omega^2\mu\epsilon$. So far, we have specified the curl of $\vec{A}$, but not its divergence. To fully determine a vector field (up to a constant), both curl and divergence must be defined. We can use this freedom to simplify (3.1.6) and (3.1.7). Specifically, by choosing

$$\vec{\nabla} \cdot \vec{A} + i\omega\mu\epsilon\mathcal{V} = 0 \tag{3.1.8}$$

which is known as the Lorenz gauge condition, the equations become decoupled:

$$\nabla^2\vec{A} + k^2\vec{A} = -\mu\vec{J} \tag{3.1.9}$$

$$\nabla^2\mathcal{V} + k^2\mathcal{V} = -\frac{1}{\epsilon}\rho = \frac{1}{i\omega\epsilon}\vec{\nabla} \cdot \vec{J} \tag{3.1.10}$$

3.1.2 Green Function

Both (3.1.10) and the Catesian components of (3.1.9) satisfy the inhomonegnous scalar Helmholtz equation. Here, we are showing that the solution to the Helmholtz equation with a unit impulse

$$\nabla^2 G + k^2 G = -\delta(\vec{r} - \vec{r}') \tag{3.1.11}$$

is

$$G(\vec{r};\vec{r}') = e^{-ikR}/4\pi R \tag{3.1.12}$$

where $R = |\vec{r} - \vec{r}'|$, and

$$\begin{cases} \delta(\vec{r} - \vec{r}') = 0, & \vec{r} \neq \vec{r}' \\ \int_V \delta(\vec{r} - \vec{r}') dv = 1, & \vec{r}' \text{ in } V \end{cases} \tag{3.1.13}$$

G is known as the Green function solution.

3.1.2 Green Function

From (3.1.13), (3.1.11) can be re-expressed as

$$\begin{cases} \nabla^2 G + k^2 G = 0, & \vec{r} \neq \vec{r}\,' \\ \int_V (\nabla^2 G + k^2 G)\, dv = -1, & \vec{r}\,' \text{ in } V \end{cases} \tag{3.1.14}$$

We first consider $\vec{r} \neq \vec{r}\,'$, or $R \neq 0$. Then

$$\nabla^2 G = \frac{1}{R^2} \partial_R \left(R^2 \partial_R \frac{e^{-ikR}}{4\pi R} \right) = -k^2 \frac{e^{-ikR}}{4\pi R} \tag{3.1.15}$$

Thus, we have $\nabla^2 G + k^2 G = 0$ when $R \neq 0$, which is the first equation in (3.1.14).

3.1.2 Green Function

Now, for the second equation, let us consider an infinitesimal spherical volume V_0 with its center located at $\vec{r}'$ and its radius R_0, then we have

$$\int_{V_0} \nabla^2 G dv = \oint_{S_0} \vec{\nabla} G \cdot d\vec{s} = \oint_{S_0} \partial_R G|_{R_0} \hat{R} \cdot d\vec{s} - e^{-ikR_0}(1 + ikR_0) \tag{3.1.16}$$

and

$$\int_{V_0} k^2 G dv = e^{-ikR_0}(1 + ikR_0) - 1 \tag{3.1.17}$$

Adding (3.1.16) and (3.1.17), we get $\int_{V_0} (\nabla^2 G + k^2 G) dv = -1$, which is basically the second equation in (3.1.14).

3.1.3 Radiation Solution

Multiply (3.1.11) with $\mu\vec{J}(\vec{r}')$ and perform integration over a volume contraining all sources, we get

$$\int_{V'}\mu\vec{J}(\vec{r}')[(\nabla^2+k^2)G(\vec{r};\vec{r}')]dv' = (\nabla^2+k^2)\int_{V'}\mu\vec{J}(\vec{r}')G(\vec{r};\vec{r}')dv'$$

$$= -\int_{V'}\mu\vec{J}(\vec{r}')\delta(\vec{r}-\vec{r}')dv' = -\mu\vec{J}(\vec{r}) \qquad (3.1.18)$$

By comparing (3.1.18) and (3.1.9), we have

$$\vec{A}(\vec{r}) = \int_{V'}\mu\vec{J}(\vec{r}')G(\vec{r};\vec{r}')dv' \qquad (3.1.19)$$

Similarly, from (3.1.11) and (3.1.10), we have

$$\mathcal{V}(\vec{r}) = \frac{-1}{i\omega\epsilon}\int_{V'}\vec{\nabla}'\cdot\vec{J}(\vec{r}')G(\vec{r};\vec{r}')dv' \qquad (3.1.20)$$

3.1.3 Radiation Solution

Substituting (3.1.19-20) into (3.1.3) and use $\omega\mu = k\eta$, we get

$$\vec{E}(\vec{r}) = -ik\eta \int_{V'} \left[\vec{J}(\vec{r}') + \frac{1}{k^2}\vec{\nabla}\left(\vec{\nabla}' \cdot \vec{J}(\vec{r}')\right)\right] G(\vec{r};\vec{r}')dv'$$

(3.1.21)

Substituting (3.1.19) into (3.1.1), we get

$$\vec{H}(\vec{r}) = -\int_{V'} \vec{J}(\vec{r}') \times \vec{\nabla}G(\vec{r};\vec{r}')dv' \qquad (3.1.22)$$

Define the following operators

$$\mathfrak{L}(\vec{X}) \equiv -ik \int_{V'} \left[\vec{X} + \frac{1}{k^2}\vec{\nabla}(\vec{\nabla}' \cdot \vec{X})\right] G dv' \qquad (3.1.23)$$

$$\mathfrak{K}(\vec{X}) \equiv -\int_{V'} \vec{X} \times \vec{\nabla} G dv' \qquad (3.1.24)$$

3.1.3 Radiation Solution

Then, we can express the electric and magnetic field as

$$\vec{E} = \eta\mathfrak{L}(\vec{J}) \tag{3.1.25}$$

$$\vec{H} = \mathfrak{K}(\vec{J}) \tag{3.1.26}$$

Apply the duality transform (1.1.24), we get

$$\vec{E} = -\mathfrak{K}(\vec{M}) \tag{3.1.27}$$

$$\vec{H} = \mathfrak{L}(\vec{M})/\eta \tag{3.1.28}$$

By superposition:

$$\vec{E} = \eta\mathfrak{L}(\vec{J}) - \mathfrak{K}(\vec{M}) \tag{3.1.29}$$

$$\vec{H} = \mathfrak{K}(\vec{J}) + \mathfrak{L}(\vec{M})/\eta \tag{3.1.30}$$

3.1.3 Radiation Solution

It is noted that we can also derive the electric field representation by Substituting (3.1.19) and (3.1.8) into (3.1.3)

$$\vec{E}(\vec{r}) = -ik\eta \int_{V'} \left(1 + \frac{1}{k^2}\vec{\nabla}\,\vec{\nabla}\cdot\right)\left[\vec{J}(\vec{r}')G(\vec{r};\vec{r}')\right]dv' \qquad (3.1.31)$$

Note the distinction between (3.1.21) and (3.1.31). In (3.1.21), one $\vec{\nabla}$ acts on $\vec{r}$ (on G), while the other $\vec{\nabla}'$ acts on $\vec{r}'$ (on $\vec{J}(\vec{r}')$). The resulting singularity is weaker.

In (3.1.31), both $\vec{\nabla}$ operators act on $\vec{r}$, and thus on the Green function G, leading to a higher-order singularity in the integrand. This form is typically used for far-field calculations, where simplifications are possible.

3.1.4 Far-Field Approximation

For far-field approximation, we have $r \gg r'$, or $kR \gg 1$. Thus, the denominator of Green function solution (3.1.12) is$\cong 4\pi r$, and the nominator is approximated as

$$e^{-ikR} = e^{-ik\left[(\vec{r}-\vec{r}')\cdot(\vec{r}-\vec{r}')\right]^{1/2}} \cong e^{-ikr\left(1-\hat{r}\cdot\vec{r}'\right)} \qquad (3.1.32)$$

Thus, in far field, the Green function soliution is

$$G(\vec{r};\vec{r}') \cong \frac{e^{-ikr}}{4\pi r} e^{ik\hat{r}\cdot\vec{r}'} = G_r(r)G_a(\theta,\phi) \qquad (3.1.33)$$

where $G_r(r) = \dfrac{e^{-ikr}}{4\pi r}$ is the part containing only radial component, and$G_a(\theta,\phi) = e^{ik\hat{r}\cdot\vec{r}'}$containing only angular component.

3.1.4 Far-Field Approximation

In order to apply (3.1.31), let us first find $\vec{\nabla} G_r$ and $\vec{\nabla} G_a$:

$$\vec{\nabla} G_r = \hat{r}\partial_r\left(\frac{e^{-ikr}}{4\pi r}\right) = \hat{r}\left[-ikG_r + O\left(\frac{1}{r^2}\right)\right] \tag{3.1.34}$$

$$\vec{\nabla} G_a = \hat{\theta}\frac{1}{r}\partial_\theta\left(e^{ik\hat{r}\cdot\vec{r}'}\right) + \hat{\phi}\frac{1}{r\sin\theta}\partial_\phi\left(e^{ik\hat{r}\cdot\vec{r}'}\right) = O\left(\frac{1}{r}\right) \tag{3.1.35}$$

Thus

$$\vec{\nabla} G = G_a\vec{\nabla} G_r + G_r\vec{\nabla} G_a = -ikG\hat{r} + O\left(\frac{1}{r^2}\right) \cong -ikG\hat{r} \tag{3.1.36}$$

Then

$$\begin{aligned}\vec{\nabla}\,\vec{\nabla}\cdot\left[\vec{J}(\vec{r}')G\right] &= \vec{\nabla}\left[\vec{J}(\vec{r}')\vec{\nabla}\cdot G\right] \cong -ik\vec{\nabla}\left[\hat{r}\cdot\vec{J}(\vec{r}')G\right] \\ &= -ik\{\hat{r}\cdot\vec{J}(\vec{r}')\vec{\nabla} G + G\vec{\nabla}[\hat{r}\cdot\vec{J}(\vec{r}')]\}\end{aligned} \tag{3.1.37}$$

3.1.4 Far-Field Approximation

To proceed the derivation, let us first calculate the following

$$\vec{\nabla}\vec{r} = \left(\hat{x}\partial_x + \hat{y}\partial_y + \hat{z}\partial_z\right)(\hat{x}x + \hat{y}y + \hat{z}z) = \hat{x}\hat{x} + \hat{y}\hat{y} + \hat{z}\hat{z} = \bar{\bar{I}} \quad (3.1.38)$$

where $\bar{\bar{F}}$ is the dyadic notation with

$$\begin{aligned}\bar{\bar{F}} &= F_{xx}\hat{x}\hat{x} + F_{yx}\hat{y}\hat{x} + F_{zx}\hat{z}\hat{x} + F_{xy}\hat{x}\hat{y} + \\ &\quad F_{yy}\hat{y}\hat{y} + F_{zy}\hat{z}\hat{y} + F_{xz}\hat{x}\hat{z} + F_{yz}\hat{y}\hat{z} + F_{zz}\hat{z}\hat{z} \\ &= \begin{bmatrix} F_{xx} & F_{xy} & F_{xz} \\ F_{yx} & F_{yy} & F_{yz} \\ F_{zx} & F_{zy} & F_{zz} \end{bmatrix} \end{aligned} \quad (3.1.39)$$

and $\bar{\bar{I}}$ is the unit dyadic.

3.1.4 Far-Field Approximation

The juxtaposition of two vectors $\bar{\bar{F}} = \vec{a}\vec{b}$ is called a dyadic product with $F_{mn} = a_m b_n$. A component of the dyadic is called a dyad. We have the following rule for dyadic calculations:

$$\vec{c} \cdot \left(\vec{a}\vec{b}\right) = (\vec{c} \cdot \vec{a})\vec{b}$$

$$\left(\vec{a}\vec{b}\right) \cdot \vec{c} = \vec{a}\left(\vec{b} \cdot \vec{c}\right)$$

$$\vec{c} \times \left(\vec{a}\vec{b}\right) = (\vec{c} \times \vec{a})\vec{b}$$

$$\left(\vec{a}\vec{b}\right) \times \vec{c} = \vec{a}\left(\vec{b} \times \vec{c}\right)$$

Then, from $\vec{\nabla}\vec{r} = \vec{\nabla}(r\hat{r}) = \vec{\nabla}(r)\hat{r} + \vec{\nabla}(\hat{r})r = \hat{r}\hat{r} + \vec{\nabla}(\hat{r})r = \bar{\bar{I}}$

$$\vec{\nabla}\hat{r} = \left(\bar{\bar{I}} - \hat{r}\hat{r}\right)/r \qquad (3.1.40)$$

3.1.4 Far-Field Approximation

The term $\vec{\nabla}[\hat{r}\cdot\vec{J}(\vec{r}')]$ in (3.1.37) is, by the vector identity $\vec{\nabla}(\vec{a}\cdot\vec{b}) = \vec{a}\times\vec{\nabla}\times\vec{b} + \vec{b}\times\vec{\nabla}\times\vec{a} + (\vec{a}\cdot\vec{\nabla})\vec{b} + (\vec{b}\cdot\vec{\nabla})\vec{a}$:

$$\vec{\nabla}[\hat{r}\cdot\vec{J}(\vec{r}')] = [\vec{J}(\vec{r}')\cdot\vec{\nabla}]\hat{r} = \vec{J}(\vec{r}')\cdot\vec{\nabla}\hat{r} = \frac{\vec{J} - J_r\,\hat{r}}{r} = \frac{J_\theta\hat{\theta} + J_\phi\hat{\phi}}{r} = O\left(\frac{1}{r}\right) \tag{3.1.41}$$

Thus (3.1.37) continues

$$\begin{aligned} \ldots &= -ik\left\{\hat{r}\cdot\vec{J}(\vec{r}')\left[-ikG\hat{r} + O\left(\frac{1}{r^2}\right)\right] + G\times O\left(\frac{1}{r}\right)\right\} \\ &= -k^2[\vec{J}(\vec{r}')\cdot\hat{r}]\hat{r}G + O\left(\frac{1}{r^2}\right) \end{aligned} \tag{3.1.42}$$

3.1.4 Far-Field Approximation

Applying the calculations above, (3.1.31) becomes

$$\begin{aligned}\vec{E} &\cong -ik\eta \int_{V'} G\left[\vec{J}(\vec{r}') - J_r\,\hat{r}\right]dv' \\ &= -ik\eta \frac{e^{-ikr}}{4\pi r} \int_{V'} \left(J_\theta \hat{\theta} + J_\phi \hat{\phi}\right) e^{ik\hat{r}\cdot\vec{r}'} dv' \\ &= -ik\eta \frac{e^{-ikr}}{4\pi r} \int_{V'} \left(\hat{\theta}\hat{\theta} + \hat{\phi}\hat{\phi}\right) \cdot \vec{J}(\vec{r}') e^{ik\hat{r}\cdot\vec{r}'} dv' \\ &= ik\eta \frac{e^{-ikr}}{4\pi r} \int_{V'} \hat{r} \times \left[\hat{r} \times \vec{J}(\vec{r}')\right] e^{ik\hat{r}\cdot\vec{r}'} dv'\end{aligned}$$

(3.1.43)

3.1.4 Far-Field Approximation

From (3.1.22), we can get

$$\vec{H} \cong -ik\frac{e^{-ikr}}{4\pi r}\int_{V'}[\hat{r}\times\vec{J}(\vec{r}')]e^{ik\hat{r}\cdot\vec{r}'}dv' = \frac{1}{\eta}\hat{r}\times\vec{E} \quad (3.1.44)$$

For general cases, by applying the duality theorem, we have

$$\vec{E} \cong -ik\frac{e^{-ikr}}{4\pi r}\int_{V'}\left[\eta\left(\hat{\theta}\hat{\theta}+\hat{\phi}\hat{\phi}\right)\cdot\vec{J}(\vec{r}')+\hat{r}\times\vec{M}(\vec{r}')\right]e^{ik\hat{r}\cdot\vec{r}'}dv' \quad (3.1.45)$$

$$\vec{H} \cong -ik\frac{e^{-ikr}}{4\pi r}\int_{V'}\left[\frac{1}{\eta}\left(\hat{\theta}\hat{\theta}+\hat{\phi}\hat{\phi}\right)\cdot\vec{M}(\vec{r}')-\hat{r}\times\vec{J}(\vec{r}')\right]e^{ik\hat{r}\cdot\vec{r}'}dv' \quad (3.1.46)$$

It is noted that by expressing $k\hat{r}$ in Cartesian coordinates, the far field can be interpreted as the inverse Fourier transform (up to a constant factor) of the components of the source distribution.

3.1.5 Stratton-Chu Formulation

From surface equivalence principle (Section 1.6.4B), if all sources are included in a closed surface S_0, then by placing the surface currents

$$\begin{cases} \vec{J}_s = \hat{n} \times \vec{H} \\ \vec{M}_s = -\hat{n} \times \vec{E} \end{cases} \tag{3.1.47}$$

where $\hat{n}$ is the unit normal vector on S_0, we can set the field inside S_0 to be zero. Thus, using (3.1.29), the electric field outside S_0 is

$$\vec{E} = \eta\mathfrak{L}(\vec{J}_s) - \mathfrak{K}(\vec{M}_s)$$

$$= -ik\eta \oint_{S_0} \left[\vec{J}_s G - \frac{1}{k^2}(\vec{\nabla}' \cdot \vec{J}_s)\vec{\nabla} G\right] ds' + \oint_{S_0} (\vec{M}_s \times \vec{\nabla} G) ds' \tag{3.1.48}$$

3.1.5 Stratton-Chu Formulation

From continuity equation (1.1.3) and the matching condition (1.3.8)

$$\vec{\nabla}' \cdot \vec{J}_s = -i\omega\rho_s = -i\omega\epsilon\left(\hat{n} \cdot \vec{E}\right) \qquad (3.1.49)$$

Substitute (3.1.47) and (3.1.49) in (3.1.48) and apply the property $\vec{\nabla}' G = -\vec{\nabla} G$, we get

$$\vec{E} = \oint_{S_0}\left[-ik\eta\left(\hat{n} \times \vec{H}\right)G + \left(\hat{n} \cdot \vec{E}\right)\vec{\nabla}' G + \left(\hat{n} \times \vec{E}\right) \times \vec{\nabla}' G\right] ds' \qquad (3.1.50)$$

Applying duality transform (1.1.23), we get the magnetic field

$$\vec{H} = \oint_{S_0}\left[i\frac{k}{\eta}\left(\hat{n} \times \vec{E}\right)G + \left(\hat{n} \cdot \vec{H}\right)\vec{\nabla}' G + \left(\hat{n} \times \vec{H}\right) \times \vec{\nabla}' G\right] ds' \qquad (3.1.51)$$

This is the Stratton-Chu formulation.

3.2 Hertzian Dipole Radiation

3.2 Hertzian Dipole Radiation

Hertzian dipole is the simplest and the most fundamental radiator. Consider on an infinitesimal line dl, a charge q occilates with an agular frequency ω, then we have the current expressed as $\mathcal{I} = i\omega q$. Suppose the line is oriented along the z-axis at the origin, we have $\vec{J} dv' = \mathcal{I}\hat{z}dz'$. Thus, from (3.1.31), the electric field is

$$\vec{E}(\vec{r}) = -ik\eta\mathcal{I}dl\left(1 + \frac{1}{k^2}\vec{\nabla}\vec{\nabla}\cdot\right)\hat{z}G \qquad (3.2.1)$$

From (3.1.22), the magnetic field is

$$\vec{H}(\vec{r}) = -\mathcal{I}dl\,\hat{z} \times \vec{\nabla}G \qquad (3.2.2)$$

3.2 Hertzian Dipole Radiation

To express (3.2.1-2) in spherical coordinates, let us calculate the following first:

$$\vec{\nabla} G = \vec{\nabla}\left(\frac{e^{-ikr}}{4\pi r}\right) = -\left(ik + \frac{1}{r}\right) G\hat{r} \tag{3.2.3}$$

$$\vec{\nabla} \cdot (\hat{z}G) = \hat{z} \cdot \vec{\nabla} G = -\left(ik + \frac{1}{r}\right) G \cos\theta \tag{3.2.4}$$

$$\vec{\nabla}\left[\vec{\nabla} \cdot (\hat{z}G)\right] = G\left[\left(-k^2 + \frac{2ik}{r} + \frac{1}{r^2}\right)\cos\theta\,\hat{r} + \left(ik + \frac{1}{r}\right)\sin\theta\,\hat{\theta}\right] \tag{3.2.5}$$

$$\hat{z} \times \vec{\nabla} G = \left(\cos\theta\,\hat{r} - \sin\theta\,\hat{\theta}\right) \times \vec{\nabla} G = -\left(ik + \frac{1}{r}\right) G \sin\theta\,\hat{\phi} \tag{3.2.6}$$

3.2 Hertzian Dipole Radiation

Thus, the electric field of an Hertzian dipole can be expressed as

$$\vec{E} = \frac{\eta \mathcal{I} dl}{r}\left(1 + \frac{1}{ikr}\right) 2\cos\theta\, G\hat{r} + ik\eta\mathcal{I}dl\left(1 + \frac{1}{ikr} - \frac{1}{k^2r^2}\right)\sin\theta\, G\hat{\theta} \tag{3.2.7}$$

Accordingly, the magnetic field can be expressed as

$$\vec{H} = ik\mathcal{I}dl\left(1 + \frac{1}{ikr}\right)\sin\theta\, G\hat{\phi} \tag{3.2.8}$$

Notice that the fields can be devided into dependent parts on r^{-1},r^{-2}, and r^{-3} terms, and we characterize the region with $kr \ll 1$ as the near field and $kr \gg 1$ as the far field.

3.2 Hertzian Dipole Radiation

For the near-field region r^{-2} and r^{-3} terms dominate. Also using the approximation $e^{-ikr} \cong 1$, we get

$$\vec{E} \cong -i\frac{\eta \mathcal{J} dl}{4\pi k r^3}\left(2\cos\hat{r} + \sin\theta\,\hat{\theta}\right) \tag{3.2.9}$$

$$\vec{H} \cong \frac{\mathcal{J} dl}{4\pi r^2}\sin\theta\,\hat{\phi} \tag{3.2.10}$$

For the far-field region r^{-1} terms dominate and we get

$$\vec{E} \cong ik\eta\mathcal{J} dl\sin\theta\; G\hat{\theta} \tag{3.2.11}$$

$$\vec{H} \cong ik\mathcal{J} dl\sin\theta\; G\hat{\phi} \tag{3.2.12}$$

Problems

1. Complete the intermediate steps in (3.1.15-17).
2. Verify (3.1.27-28).
3. Verify (3.1.51).
4. Complete the intermediate steps in (3.2.5-8).
5. Complete the intermediate steps in (3.2.9-12).