Electromagnetic Theory

Chapter 3 –Radiation

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Outline

3.1 Radiation in Free Space

3.1.1 Potentials

Consider no the magnetic sources, we have $\vec{\nabla} \cdot \vec{B} = 0$. Thus, from vector identity, the magnetic flux density can be expressed as

$$\vec{B} = \vec{\nabla} \times \vec{A} \tag{3.1.1}$$

where $\vec{A}$ is called the magnetic vector potential. Substitute (3.1.1) into (1.4.5) with $\vec{M} = 0$, we get

$$\vec{\nabla} \times \left(\vec{E} + i\omega\vec{A}\right) = 0 \tag{3.1.2}$$

from which we can express the electric field intensity as

$$\vec{E} = -\vec{\nabla}\mathcal{V} - i\omega\vec{A} \tag{3.1.3}$$

where $\mathcal{V}$ is called the electric scalar potential.

3.1.1 Potentials

Substitute (3.1.1) and (3.1.3) into (1.5.6), we get

$$\vec{\nabla} \times \vec{\nabla} \times \vec{A} = \mu \vec{J} + \omega^2 \mu \epsilon \vec{A} - i\omega\mu\epsilon\vec{\nabla}\mathcal{V} \tag{3.1.4}$$

Also, substitute (3.1.3) into (1.5.7), we get

$$\nabla^2 \mathcal{V} + i\omega\vec{\nabla} \cdot \vec{A} = -\rho/\epsilon \tag{3.1.5}$$

Using the vector identity $\vec{\nabla} \times \vec{\nabla} \times \vec{A} = \vec{\nabla}\left(\vec{\nabla} \cdot \vec{A}\right) - \nabla^2 \vec{A}$, and rearranging the terms, (3.1.4) and (3.1.5) can be re-expressed as

$$\nabla^2 \vec{A} + k^2 \vec{A} = -\mu \vec{J} + \vec{\nabla}\left(\vec{\nabla} \cdot \vec{A} + i\omega\mu\epsilon\mathcal{V}\right) \tag{3.1.6}$$

$$\nabla^2 \mathcal{V} + k^2 \mathcal{V} = -\rho/\epsilon - i\omega\left(\vec{\nabla} \cdot \vec{A} + i\omega\mu\epsilon\mathcal{V}\right) \tag{3.1.7}$$

3.1.1 Potentials

where $k^2 = \omega^2\mu\epsilon$. So far, we have specified the curl of $\vec{A}$, but not its divergence. To fully determine a vector field (up to a constant), both curl and divergence must be defined. We can use this freedom to simplify (3.1.6) and (3.1.7). Specifically, by choosing

$$\vec{\nabla} \cdot \vec{A} + i\omega\mu\epsilon\mathcal{V} = 0 \tag{3.1.8}$$

which is known as the Lorenz gauge condition, the equations become decoupled:

$$\nabla^2\vec{A} + k^2\vec{A} = -\mu\vec{J} \tag{3.1.9}$$

$$\nabla^2\mathcal{V} + k^2\mathcal{V} = -\frac{1}{\epsilon}\rho = \frac{1}{i\omega\epsilon}\vec{\nabla} \cdot \vec{J} \tag{3.1.10}$$

3.1.2 Green Function

Both (3.1.10) and the Catesian components of (3.1.9) satisfy the inhomonegnous scalar Helmholtz equation. Here, we are showing that the solution to the Helmholtz equation with a unit impulse

$$\nabla^2 G + k^2 G = -\delta(\vec{r} - \vec{r}') \qquad (3.1.11)$$

is

$$G(\vec{r};\vec{r}') = e^{-ikR}/4\pi R \qquad (3.1.12)$$

where $R = |\vec{r} - \vec{r}'|$, and

$$\begin{cases} \delta(\vec{r} - \vec{r}') = 0, & \vec{r} \neq \vec{r}' \\ \int_V \delta(\vec{r} - \vec{r}')dv = 1, & \vec{r}' \text{ in } V \end{cases} \qquad (3.1.13)$$

G is known as the Green function solution.

3.1.2 Green Function

From (3.1.13), (3.1.11) can be re-expressed as

$$\begin{cases} \nabla^2 G + k^2 G = 0, & \vec{r} \neq \vec{r}' \\ \int_V (\nabla^2 G + k^2 G) dv = -1, & \vec{r}' \text{ in } V \end{cases} \tag{3.1.14}$$

We first consider $\vec{r} \neq \vec{r}'$, or $R \neq 0$. Then

$$\nabla^2 G = \frac{1}{R^2} \partial_R \left(R^2 \partial_R \frac{e^{-ikR}}{4\pi R} \right) = -k^2 \frac{e^{-ikR}}{4\pi R} \tag{3.1.15}$$

Thus, we have $\nabla^2 G + k^2 G = 0$ when $R \neq 0$, which is the first equation in (3.1.14).

3.1.2 Green Function

Now, for the second equation, let us consider an infinitesimal spherical volume V_0 with its center located at $\vec{r}'$ and its radius R_0, then we have

$$\int_{V_0} \nabla^2 G dv = \oint_{S_0} \vec{\nabla} G \cdot d\vec{s} = \oint_{S_0} \partial_R G|_{R_0} \hat{R} \cdot d\vec{s} = -e^{-ikR_0}(1 + ikR_0)$$

(3.1.16)

and

$$\int_{V_0} k^2 G dv = e^{-ikR_0}(1 + ikR_0) - 1 \qquad (3.1.17)$$

Adding (3.1.16) and (3.1.17), we get $\int_{V_0} (\nabla^2 G + k^2 G) dv = -1$, which is basically the second equation in (3.1.14).

3.1.3 Radiation Solution

Multiply (3.1.11) with $\mu\vec{J}(\vec{r}')$ and perform integration over a volume contraining all sources, we get

$$\int_{V'} \mu\vec{J}(\vec{r}')[(\nabla^2 + k^2)G(\vec{r};\vec{r}')]dv' = (\nabla^2 + k^2)\int_{V'} \mu\vec{J}(\vec{r}')G(\vec{r};\vec{r}')dv'$$

$$= -\int_{V'} \mu\vec{J}(\vec{r}')\delta(\vec{r} - \vec{r}')dv' = -\mu\vec{J}(\vec{r}) \qquad (3.1.18)$$

By comparing (3.1.18) and (3.1.9), we have

$$\vec{A}(\vec{r}) = \int_{V'} \mu\vec{J}(\vec{r}')G(\vec{r};\vec{r}')dv' \qquad (3.1.19)$$

Similarly, from (3.1.11) and (3.1.10), we have

$$\mathcal{V}(\vec{r}) = \frac{-1}{i\omega\epsilon}\int_{V'} \vec{\nabla}' \cdot \vec{J}(\vec{r}')G(\vec{r};\vec{r}')dv' \qquad (3.1.20)$$

3.1.3 Radiation Solution

Substituting (3.1.19-20) into (3.1.3) and use $\omega\mu = k\eta$, we get

$$\vec{E}(\vec{r}) = -ik\eta \int_{V'} \left[\vec{J}(\vec{r}') + \frac{1}{k^2}\vec{\nabla}\left(\vec{\nabla}' \cdot \vec{J}(\vec{r}')\right)\right] G(\vec{r};\vec{r}')dv' \tag{3.1.21}$$

Substituting (3.1.19) into (3.1.1), we get

$$\vec{H}(\vec{r}) = -\int_{V'} \vec{J}(\vec{r}') \times \vec{\nabla}G(\vec{r};\vec{r}')dv' \tag{3.1.22}$$

Define the following operators

$$\mathfrak{L}(\vec{X}) \equiv -ik \int_{V'} \left[\vec{X} + \frac{1}{k^2}\vec{\nabla}(\vec{\nabla}' \cdot \vec{X})\right] G dv' \tag{3.1.23}$$

$$\mathfrak{K}(\vec{X}) \equiv -\int_{V'} \vec{X} \times \vec{\nabla} G dv' \tag{3.1.24}$$

3.1.3 Radiation Solution

Then, we can express the electric and magnetic field as

$$\vec{E} = \eta\mathfrak{L}(\vec{J}) \tag{3.1.25}$$

$$\vec{H} = \mathfrak{K}(\vec{J}) \tag{3.1.26}$$

Apply the duality transform (1.1.24), we get

$$\vec{E} = -\mathfrak{K}(\vec{M}) \tag{3.1.27}$$

$$\vec{H} = \mathfrak{L}(\vec{M})/\eta \tag{3.1.28}$$

By superposition:

$$\vec{E} = \eta\mathfrak{L}(\vec{J}) - \mathfrak{K}(\vec{M}) \tag{3.1.29}$$

$$\vec{H} = \mathfrak{K}(\vec{J}) + \mathfrak{L}(\vec{M})/\eta \tag{3.1.30}$$

3.1.3 Radiation Solution

It is noted that we can also derive the electric field representation by Substituting (3.1.19) and (3.1.8) into (3.1.3)

$$\vec{E}(\vec{r}) = -ik\eta \int_{V'} \left(1 + \frac{1}{k^2}\vec{\nabla}\,\vec{\nabla}\cdot\right)\left[\vec{J}(\vec{r}')G(\vec{r};\vec{r}')\right]dv' \tag{3.1.31}$$

Note the distinction between (3.1.21) and (3.1.31). In (3.1.21), one $\vec{\nabla}$ acts on $\vec{r}$ (on G), while the other $\vec{\nabla}'$ acts on $\vec{r}'$ (on $\vec{J}(\vec{r}')$). The resulting singularity is weaker.

In (3.1.31), both $\vec{\nabla}$ operators act on $\vec{r}$, and thus on the Green function G, leading to a higher-order singularity in the integrand. This form is typically used for far-field calculations, where simplifications are possible.

3.1.4 Stratton-Chu Formulation

From surface equivalence principle (Section 1.5.4B), if all sources are included in a closed surface S_0, then by placing the surface currents

$$\begin{cases} \vec{J}_s = \hat{n} \times \vec{H} \\ \vec{M}_s = -\hat{n} \times \vec{E} \end{cases} \tag{3.1.32}$$

where $\hat{n}$ is the unit normal vector on S_0, we can set the field inside S_0 to be zero. Thus, using (3.1.29), the electric field outside S_0 is

$$\begin{aligned} \vec{E} &= \eta \mathfrak{L}(\vec{J}_s) - \mathfrak{K}(\vec{M}_s) \\ &= -ik\eta \oint_{S_0} \left[\vec{J}_s G - \frac{1}{k^2} (\vec{\nabla}' \cdot \vec{J}_s) \vec{\nabla} G \right] ds' + \oint_{S_0} (\vec{M}_s \times \vec{\nabla} G) ds' \end{aligned} \tag{3.1.33}$$

3.1.5 Stratton-Chu Formulation

From continuity equation and the matching condition (1.2.8)

$$\vec{\nabla}' \cdot \vec{J}_s = -i\omega\rho_s = -i\omega\epsilon\left(\hat{n} \cdot \vec{E}\right) \qquad (3.1.34)$$

Substitute (3.1.32) and (3.1.434) in (3.1.33) and apply the property $\vec{\nabla}'G = -\vec{\nabla}G$, we get

$$\vec{E} = \oint_{S_0}\left[-ik\eta\left(\hat{n} \times \vec{H}\right)G + \left(\hat{n} \cdot \vec{E}\right)\vec{\nabla}'G + \left(\hat{n} \times \vec{E}\right) \times \vec{\nabla}'G\right]ds' \qquad (3.1.35)$$

Applying duality transform (1.1.36), we get the magnetic field

$$\vec{H} = \oint_{S_0}\left[i\frac{k}{\eta}\left(\hat{n} \times \vec{E}\right)G + \left(\hat{n} \cdot \vec{H}\right)\vec{\nabla}'G + \left(\hat{n} \times \vec{H}\right) \times \vec{\nabla}'G\right]ds' \qquad (3.1.36)$$

This is the Stratton-Chu formulation.

3.2 Far-Field Approximation

3.2 Far-Field Approximation

For far-field approximation, we have $r \gg r'$, or $kR \gg 1$. Thus, the denominator of Green function solution (3.1.12) is$\cong 4\pi r$, and the nominator is approximated as

$$e^{-ikR} = e^{-ik\left[(\vec{r}-\vec{r}')\cdot(\vec{r}-\vec{r}')\right]^{1/2}} \cong e^{-ikr(1-\hat{r}\cdot\vec{r}')} \qquad (3.2.1)$$

Thus, in far field, the Green function soliution is

$$G(\vec{r};\vec{r}') \cong \frac{e^{-ikr}}{4\pi r} e^{ik\hat{r}\cdot\vec{r}'} = G_r(r)G_a(\theta,\phi) \qquad (3.2.2)$$

where $G_r(r) = \frac{e^{-ikr}}{4\pi r}$ is the part containing only radial component, and$G_a(\theta,\phi) = e^{ik\hat{r}\cdot\vec{r}'}$containing only angular component.

3.2 Far-Field Approximation

In order to apply (3.1.31), let us first find $\vec{\nabla} G_r$ and $\vec{\nabla} G_a$:

$$\vec{\nabla} G_r = \hat{r}\partial_r\left(\frac{e^{-ikr}}{4\pi r}\right) = \hat{r}\left[-ikG_r + O\left(\frac{1}{r^2}\right)\right] \tag{3.2.3}$$

$$\vec{\nabla} G_a = \hat{\theta}\frac{1}{r}\partial_\theta\left(e^{ik\hat{r}\cdot\vec{r}\,'}\right) + \hat{\phi}\frac{1}{r\sin\theta}\partial_\phi\left(e^{ik\hat{r}\cdot\vec{r}\,'}\right) = O\left(\frac{1}{r}\right) \tag{3.2.4}$$

Thus

$$\vec{\nabla} G = G_a\vec{\nabla} G_r + G_r\vec{\nabla} G_a = -ikG\hat{r} + O\left(\frac{1}{r^2}\right) \cong -ikG\hat{r} \tag{3.2.5}$$

Then

$$\begin{aligned}\vec{\nabla}\,\vec{\nabla}\cdot\left[\vec{J}(\vec{r}\,')G\right] &= \vec{\nabla}\left[\vec{J}(\vec{r}\,')\vec{\nabla}\cdot G\right] \cong -ik\vec{\nabla}\left[\hat{r}\cdot\vec{J}(\vec{r}\,')G\right] \\ &= -ik\left\{\hat{r}\cdot\vec{J}(\vec{r}\,')\vec{\nabla} G + G\vec{\nabla}\left[\hat{r}\cdot\vec{J}(\vec{r}\,')\right]\right\}\end{aligned} \tag{3.2.6}$$

3.2 Far-Field Approximation

To proceed the derivation, let us first calculate the following

$$\vec{\nabla}\vec{r} = \left(\hat{x}\partial_x + \hat{y}\partial_y + \hat{z}\partial_z\right)(\hat{x}x + \hat{y}y + \hat{z}z) = \hat{x}\hat{x} + \hat{y}\hat{y} + \hat{z}\hat{z} = \bar{\bar{I}} \quad (3.2.7)$$

where $\bar{\bar{F}}$ is the dyadic notation with

$$\begin{aligned}\bar{\bar{F}} &= F_{xx}\hat{x}\hat{x} + F_{yx}\hat{y}\hat{x} + F_{zx}\hat{z}\hat{x} + F_{xy}\hat{x}\hat{y} \\ &\quad + F_{yy}\hat{y}\hat{y} + F_{zy}\hat{z}\hat{y} + F_{xz}\hat{x}\hat{z} + F_{yz}\hat{y}\hat{z} + F_{zz}\hat{z}\hat{z} \\ &= \begin{bmatrix} F_{xx} & F_{xy} & F_{xz} \\ F_{yx} & F_{yy} & F_{yz} \\ F_{zx} & F_{zy} & F_{zz} \end{bmatrix}\end{aligned} \quad (3.2.8)$$

and $\bar{\bar{I}}$ is the unit dyadic.

3.2 Far-Field Approximation

The juxtaposition of two vectors $\bar{\bar{F}} = \vec{a}\vec{b}$ is called a dyadic product with $F_{mn} = a_m b_n$. A component of the dyadic is called a dyad. We have the following rule for dyadic calculations:

$$\vec{c} \cdot \left(\vec{a}\vec{b}\right) = (\vec{c} \cdot \vec{a})\vec{b}$$

$$\left(\vec{a}\vec{b}\right) \cdot \vec{c} = \vec{a}\left(\vec{b} \cdot \vec{c}\right)$$

$$\vec{c} \times \left(\vec{a}\vec{b}\right) = (\vec{c} \times \vec{a})\vec{b}$$

$$\left(\vec{a}\vec{b}\right) \times \vec{c} = \vec{a}\left(\vec{b} \times \vec{c}\right)$$

Then, from $\vec{\nabla}\vec{r} = \vec{\nabla}(r\hat{r}) = \vec{\nabla}(r)\hat{r} + \vec{\nabla}(\hat{r})r = \hat{r}\hat{r} + \vec{\nabla}(\hat{r})r = \bar{\bar{I}}$

$$\vec{\nabla}\hat{r} = \left(\bar{\bar{I}} - \hat{r}\hat{r}\right)/r \qquad (3.2.9)$$

3.2 Far-Field Approximation

The term $\vec{\nabla}\left[\hat{r}\cdot\vec{J}(\vec{r}')\right]$ in (3.2.6) is, by the vector identity $\vec{\nabla}\left(\vec{a}\cdot\vec{b}\right) = \vec{a}\times\vec{\nabla}\times\vec{b}+\vec{b}\times\vec{\nabla}\times\vec{a}+\left(\vec{a}\cdot\vec{\nabla}\right)\vec{b}+\left(\vec{b}\cdot\vec{\nabla}\right)\vec{a}$:

$$\vec{\nabla}\left[\hat{r}\cdot\vec{J}(\vec{r}')\right] = \left[\vec{J}(\vec{r}')\cdot\vec{\nabla}\right]\hat{r} = \vec{J}(\vec{r}')\cdot\vec{\nabla}\hat{r} = \frac{\vec{J}-J_r\,\hat{r}}{r} = \frac{J_\theta\hat{\theta}+J_\phi\hat{\phi}}{r} = O\left(\frac{1}{r}\right)$$

(3.2.10)

Thus (3.2.6) continues

$$\ldots = -ik\left\{\hat{r}\cdot\vec{J}(\vec{r}')\left[-ikG\hat{r}+O\left(\frac{1}{r^2}\right)\right]+G\times O\left(\frac{1}{r}\right)\right\}$$

$$= -k^2\left[\vec{J}(\vec{r}')\cdot\hat{r}\right]\hat{r}G+O\left(\frac{1}{r^2}\right) \qquad (3.2.11)$$

3.2 Far-Field Approximation

Applying the calculations above, (3.1.31) becomes

$$\vec{E} \cong -ik\eta \int_{V'} G\left[\vec{J}(\vec{r}') - J_r \hat{r}\right] dv'$$

$$= -ik\eta \frac{e^{-ikr}}{4\pi r} \int_{V'} \left(J_\theta \hat{\theta} + J_\phi \hat{\phi}\right) e^{ik\hat{r}\cdot\vec{r}'} dv'$$

$$= -ik\eta \frac{e^{-ikr}}{4\pi r} \int_{V'} \left(\hat{\theta}\hat{\theta} + \hat{\phi}\hat{\phi}\right) \cdot \vec{J}(\vec{r}') e^{ik\hat{r}\cdot\vec{r}'} dv'$$

$$= ik\eta \frac{e^{-ikr}}{4\pi r} \int_{V'} \hat{r} \times \left[\hat{r} \times \vec{J}(\vec{r}')\right] e^{ik\hat{r}\cdot\vec{r}'} dv'$$

(3.2.12)

3.2 Far-Field Approximation

From (3.1.22), we can get

$$\vec{H} \cong -ik\frac{e^{-ikr}}{4\pi r}\int_{V'}\left[\hat{r}\times\vec{J}(\vec{r}')\right]e^{ik\hat{r}\cdot\vec{r}'}dv' = \frac{1}{\eta}\hat{r}\times\vec{E} \quad (3.2.13)$$

For general cases, by applying the duality theorem, we have

$$\vec{E} \cong -ik\frac{e^{-ikr}}{4\pi r}\int_{V'}\left[\eta\left(\hat{\theta}\hat{\theta}+\hat{\phi}\hat{\phi}\right)\cdot\vec{J}(\vec{r}') - \hat{r}\times\vec{M}(\vec{r}')\right]e^{ik\hat{r}\cdot\vec{r}'}dv' \quad (3.2.14)$$

$$\vec{H} \cong -ik\frac{e^{-ikr}}{4\pi r}\int_{V'}\left[\frac{1}{\eta}\left(\hat{\theta}\hat{\theta}+\hat{\phi}\hat{\phi}\right)\cdot\vec{M}(\vec{r}') + \hat{r}\times\vec{J}(\vec{r}')\right]e^{ik\hat{r}\cdot\vec{r}'}dv' \quad (3.2.15)$$

It is noted that by expressing $k\hat{r}$ in Cartesian coordinates, the far field can be interpreted as the inverse Fourier transform (up to a constant factor) of the components of the source distribution.

3.3 Hertzian Dipole Radiation

3.3 Hertzian Dipole Radiation

Hertzian dipole is the simplest and the most fundamental radiator. Consider on an infinitesimal line dl, a charge q occilates with an agular frequency ω, then we have the current expressed as $\mathcal{I} = i\omega q$. Suppose the line is oriented along the z-axis at the origin, we have $\vec{J}dv' = \mathcal{I}\hat{z}dz'$. Thus, from (3.1.31), the electric field is

$$\vec{E}(\vec{r}) = -ik\eta\mathcal{I}dl\left(1 + \frac{1}{k^2}\vec{\nabla}\vec{\nabla}\cdot\right)\hat{z}G \qquad (3.3.1)$$

From (3.1.22), the magnetic field is

$$\vec{H}(\vec{r}) = -\mathcal{I}dl\,\hat{z}\times\vec{\nabla}G \qquad (3.3.2)$$

3.3 Hertzian Dipole Radiation

To express (3.3.1-2) in spherical coordinates, let us calculate the following first:

$$\vec{\nabla}G = \vec{\nabla}\left(\frac{e^{-ikr}}{4\pi r}\right) = -\left(ik + \frac{1}{r}\right)G\hat{r} \tag{3.3.3}$$

$$\vec{\nabla}\cdot(\hat{z}G) = \hat{z}\cdot\vec{\nabla}G = -\left(ik + \frac{1}{r}\right)G\cos\theta \tag{3.3.4}$$

$$\vec{\nabla}[\vec{\nabla}\cdot(\hat{z}G)] = G\left[\left(-k^2 + \frac{2ik}{r} + \frac{1}{r^2}\right)\cos\theta\,\hat{r} + \left(ik + \frac{1}{r}\right)\sin\theta\,\hat{\theta}\right] \tag{3.3.5}$$

$$\hat{z}\times\vec{\nabla}G = \left(\cos\theta\,\hat{r} - \sin\theta\,\hat{\theta}\right)\times\vec{\nabla}G = -\left(ik + \frac{1}{r}\right)G\sin\theta\,\hat{\phi} \tag{3.3.6}$$

3.3 Hertzian Dipole Radiation

Thus, the electric field of an Hertzian dipole can be expressed as

$$\vec{E} = \frac{\eta \mathcal{J} dl}{r}\left(1 + \frac{1}{ikr}\right) 2\cos\theta\, G\hat{r} + ik\eta\mathcal{J}dl\left(1 + \frac{1}{ikr} - \frac{1}{k^2r^2}\right)\sin\theta\, G\hat{\theta} \tag{3.3.7}$$

Accordingly, the magnetic field can be expressed as

$$\vec{H} = ik\mathcal{J}dl\left(1 + \frac{1}{ikr}\right)\sin\theta\, G\hat{\phi} \tag{3.3.8}$$

Notice that the fields can be devided into dependent parts on r^{-1},r^{-2}, and r^{-3} terms, and we characterize the region with $kr \ll 1$ as the near field and $kr \gg 1$ as the far field.

3.3 Hertzian Dipole Radiation

For the near-field region r^{-2} and r^{-3} terms dominate. Also using the approximation $e^{-ikr} \cong 1$, we get

$$\vec{E} \cong -i \frac{\eta \mathcal{J} dl}{4\pi k r^3} \left(2\cos\hat{r} + \sin\theta\, \hat{\theta}\right) \qquad (3.3.9)$$

$$\vec{H} \cong \frac{\mathcal{J} dl}{4\pi r^2} \sin\theta\, \hat{\phi} \qquad (3.3.10)$$

For the far-field region r^{-1} terms dominate and we get

$$\vec{E} \cong ik\eta\mathcal{J}dl \sin\theta\, G\hat{\theta} \qquad (3.3.11)$$

$$\vec{H} \cong ik\mathcal{J}dl \sin\theta\, G\hat{\phi} \qquad (3.3.12)$$

3.3 Hertzian Dipole Radiation

Finally, the directivity is the ratio of the radiation intensity in a given direction to the average intensity

$$D = \frac{U}{P_{\mathrm{rad}}/4\pi} = \frac{3}{2}\sin^2\theta \tag{3.3.16}$$

Applying the duality principle (1.1.36), we can also obtain the far field of a magnetic dipole source

$$\vec{E} \cong -ik\mathcal{J}_m dl \sin\theta\, G\hat{\phi} \tag{3.3.17}$$

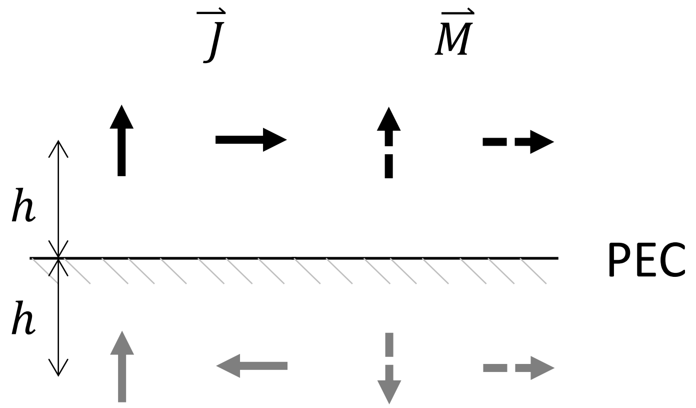
$$\vec{H} \cong ik\frac{\mathcal{J}_m}{\eta} dl \sin\theta\, G\hat{\theta} \tag{3.3.18}$$

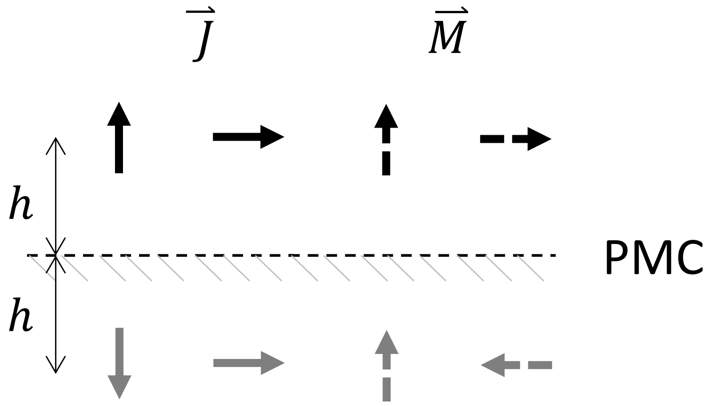
where $\mathcal{J}_m$ is the magnitude of the magnetic current.

3.4 Image Theory

3.4 Image Theory

Image theory simplifies the analysis of current sources near a flat conducting surface by replacing the source–plane system with the source and its image in free space. The image is oriented and phased to satisfy boundary conditions: for a PEC, the tangential electric field vanishes; for a PMC, the tangential magnetic field vanishes.





3.5 Radiation From an Aperture

3.5 Radiation From an Aperture

When an electromagnetic wave passes through an aperture in a conducting screen, it radiates into the space beyond. By the equivalence principle, the field can be represented by equivalent surface currents on the aperture on the $z = 0$ plane:

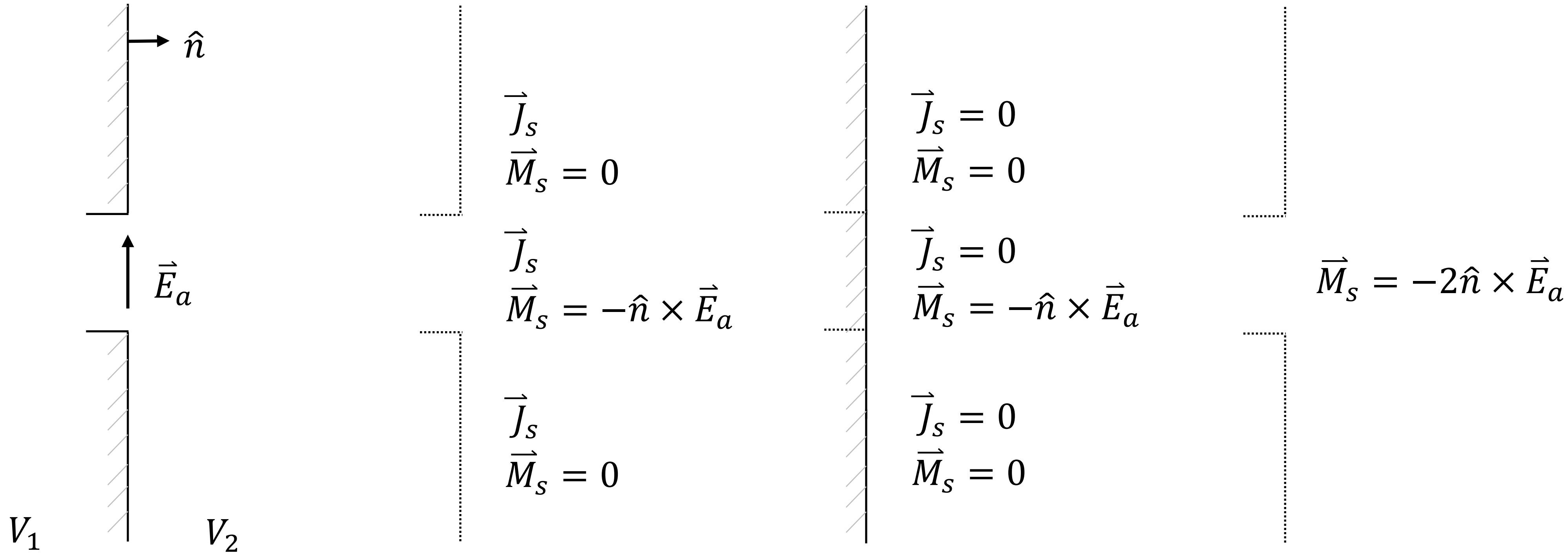
$$\vec{J}_s = \hat{z} \times \vec{H}_a, \quad \vec{M}_s = -\hat{z} \times \vec{E}_a \tag{3.5.1}$$

where $\vec{E}_a$ and $\vec{H}_a$ are the fields on the aperture.

First, the surface equivalence principle is applied by replacing the PEC with surface currents. Since the tangential electric field on a PEC is zero, the equivalent magnetic current vanishes. Applying the equivalence principle again by a PEC sets the electric surface current in the PEC region to zero, leaving only the magnetic current on the aperture. Finally, image theory removes the PEC and doubles the magnetic current, giving

$$\vec{M}_s = -2\hat{z} \times \vec{E}_a \tag{3.5.2}$$

3.5 Radiation From an Aperture



3.5 Radiation From an Aperture

The far-field electric field follows from the radiation integral with the equivalent magnetic current:

$$\vec{E} \approx -ik \frac{e^{-ikr}}{2\pi r} \int_{S_A} \hat{r} \times \left(\hat{z} \times \vec{E}_a \right) e^{ik\hat{r}\cdot\vec{r}\,'} ds' \qquad (3.5.3)$$

where S_A is the aperture area.

For a rectangular aperture of size $a{\times}b$, illuminated by a normally incident plane wave with $\vec{E}_a = E_0\hat{y}$, the above becomes

$$\vec{E} \approx ikE_0 \frac{e^{-ikr}}{2\pi r} \int_{-a/2}^{a/2} \int_{-b/2}^{b/2} (\sin\phi\ \hat{\theta} + \cos\theta \cos\phi\ \hat{\phi})\, e^{ik(x'u+y'v)} dx'dy' \qquad (3.5.4)$$

3.5 Radiation From an Aperture

with $u = \sin\theta\cos\phi$ and $v = \sin\theta\cos\phi$. Evaluating the integral yields

$$\vec{E} \approx ikE_0ab\frac{e^{-ikr}}{4\pi\, r}\left(sin\,\phi\ \hat{\theta} + cos\,\theta\, cos\,\phi\ \hat{\phi}\right)\mathrm{sinc}\left(\frac{ka}{2}u\right)\mathrm{sinc}\left(\frac{kb}{2}v\right) \tag{3.5.5}$$

The radiation pattern is the product of two sinc functions, characteristic of rectangular apertures.

Problems

1. Complete the intermediate steps in (3.1.15-17).
2. Verify (3.1.27-28) by applying the duality transform.
3. Show that (3.1.31) is equal to

$$-ik\eta \int_{V'} \left[\left(\bar{\bar{I}} + \frac{1}{k^2} \vec{\nabla}\,\vec{\nabla} \right) G(\vec{r};\vec{r}\,') \right] \cdot \vec{J}(\vec{r}\,') dv'$$

$$= ik\eta \int_{V'} \bar{\bar{G}}(\vec{r};\vec{r}\,') \cdot \vec{J}(\vec{r}\,') dv'$$

   where

$$\bar{\bar{G}}(\vec{r};\vec{r}\,') = -\left[\left(\bar{\bar{I}} + \frac{1}{k^2} \vec{\nabla}\,\vec{\nabla} \right) G(\vec{r};\vec{r}\,') \right]$$

4. Complete the intermediate steps in (3.2.3-8).