Fourier Analysis: A Comprehensive Introduction

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Abstract

This comprehensive lecture series covers the fundamental theory and applications of Fourier analysis. Beginning with Fourier series for periodic functions and progressing through Fourier transforms, sampling theorems, the discrete Fourier transform (DFT), and the fast Fourier transform (FFT), this material provides a rigorous yet accessible treatment of transform methods essential for signal processing, differential equations, and engineering applications. The notes include both theoretical foundations and practical computational methods, with detailed proofs and worked examples throughout. This lecture note builds on *An Introduction to Fourier Analysis* by Ralph N. Morrison.

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1 Fourier Series for Periodic Functions

1.1 Orthogonality of Vectors and Functions

Two vectors are said to be orthogonal if their inner product is equal to zero. Let **u** and **v** be two real n-vectors. Their inner product is

$$(\mathbf{u},\mathbf{v}) = \sum_{i=1}^{n} u_i v_i\,. \tag{1.1}$$

The vectors **u** and **v** are orthogonal if

$$(\mathbf{u},\mathbf{v}) = 0\,. \tag{1.2}$$

Let

$$\mathcal{S} = \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \dots\} \tag{1.3}$$

be a set of vectors. If

$$(\mathbf{u}_n, \mathbf{u}_m) = 0 \quad \text{whenever } n \neq m\,, \tag{1.4}$$

then the elements of $\mathcal{S}$ are mutually orthogonal and form an orthogonal basis for the space spanned by $\mathcal{S}$. Any vector in that space can be expressed as a linear combination of those basis vectors.

The concept of orthogonality and basis extends naturally to sets of functions. Consider the set

$$\mathcal{S} = \{f_1(t), f_2(t), f_3(t), \dots\}. \tag{1.5}$$

We say that the members of $\mathcal{S}$ form an orthogonal set over the interval $a < t < b$ if

$$\int_a^b f_n(t) f_m(t)\,dt = 0 \quad \text{whenever } n \neq m\,. \tag{1.6}$$

The integral above is the inner product of the pair of functions.

In what follows we will make repeated use of orthogonality. In particular, we shall show that the set

$$\mathcal{S} = \left\{\dots, e^{-i2\omega_0 t}, e^{-i\omega_0 t}, 1, e^{i\omega_0 t}, e^{i2\omega_0 t}, \dots\right\} \tag{1.7}$$

forms an orthogonal set over one period and hence forms a basis for functions lying in the space it spans. This leads to representing periodic current or voltage waveforms as linear combinations of these functions.

We begin with periodic functions; later we shall extend the techniques to nonperiodic waveforms, such as one-time pulses.

1.2 The Complex Exponentials

We recall several elementary facts from complex algebra that will be used frequently. Let

$$z = x + iy \tag{1.8}$$

be any point in the complex plane, where $i = \sqrt{-1}$. Then

$$z^* = x - iy\,, \tag{1.9}$$

$$\mathrm{mod}(z) = |z| = \sqrt{x^2 + y^2} = \sqrt{zz^*}\,, \tag{1.10}$$

$$\arg(z) = \arctan\left(\frac{y}{x}\right), \tag{1.11}$$

$$z + z^* = 2\Re(z) = 2x, \tag{1.12}$$

$$z - z^* = 2i\Im(z) = 2iy, \tag{1.13}$$

$$e^{i\theta} = \cos\theta + i\sin\theta, \tag{1.14}$$

$$\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2}, \tag{1.15}$$

$$\sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}. \tag{1.16}$$

1.3 Orthogonality of the Complex Exponentials

We consider the complex exponentials

$$e^{in\omega_0 t}, \quad n \in \mathbb{Z},$$

where $\omega_0 = 2\pi/T_0$ is a fundamental angular frequency. Taken together, they form the set

$$\{\ldots, e^{-i2\omega_0 t}, e^{-i\omega_0 t}, 1, e^{i\omega_0 t}, e^{i2\omega_0 t}, \ldots\}.$$

Theorem 1 (Orthogonality of the Complex Exponentials)**.** *The complex exponentials $e^{in\omega_0 t}$ satisfy*

$$\frac{1}{T_0}\int_{-T_0/2}^{T_0/2} e^{in\omega_0 t} e^{-im\omega_0 t}\, dt = \delta_{mn} = \begin{cases} 0, & n \neq m, \\ 1, & n = m, \end{cases} \tag{1.17}$$

where $T_0 = 2\pi/\omega_0$.

Proof. We have

$$\frac{1}{T_0}\int_{-T_0/2}^{T_0/2} e^{in\omega_0 t} e^{-im\omega_0 t}\, dt = \frac{1}{T_0}\int_{-T_0/2}^{T_0/2} e^{i(n-m)\omega_0 t}\, dt.$$

If $n \neq m$, let $p = n - m \in \mathbb{Z} \setminus \{0\}$. Then

$$\frac{1}{T_0}\int_{-T_0/2}^{T_0/2} e^{ip\omega_0 t}\, dt = \frac{1}{T_0}\left[\frac{e^{ip\omega_0 t}}{ip\omega_0}\right]_{-T_0/2}^{T_0/2} = \frac{1}{T_0}\,\frac{e^{ip\omega_0 T_0/2} - e^{-ip\omega_0 T_0/2}}{ip\omega_0}.$$

Since $\omega_0 T_0 = 2\pi$,

$$e^{ip\omega_0 T_0/2} = e^{ip\pi}, \quad e^{-ip\omega_0 T_0/2} = e^{-ip\pi}.$$

Both exponentials are ± 1 and differ only by a sign. Their difference in the numerator is zero because $e^{ip\pi} = e^{-ip\pi} = (-1)^p$. Hence the integral is zero.

If $n = m$, the integrand is 1, and we obtain

$$\frac{1}{T_0}\int_{-T_0/2}^{T_0/2} 1\, dt = 1.$$

□

The proof above uses the symmetric interval $[-T_0/2, T_0/2]$. The same result holds for any interval of length T_0.

1.4 Complex Fourier Series for Periodic Functions

Let $f(t)$ be a periodic function with period T_0 and assume that it can be expressed as an infinite sum of complex exponentials:

$$f(t) = \sum_{n=-\infty}^{\infty} F(n)\, e^{in\omega_0 t}, \qquad \omega_0 = \frac{2\pi}{T_0}. \tag{1.18}$$

The coefficients $F(n)$ are the complex Fourier coefficients and are to be determined. This assumption is reasonable for periodic functions: each exponential term has period T_0 or a divisor of T_0, and thus the sum is periodic with period T_0.

To determine $F(n)$, multiply both sides by $e^{-im\omega_0 t}$ and integrate over a full period:

$$\frac{1}{T_0}\int_{-T_0/2}^{T_0/2} f(t)\, e^{-im\omega_0 t}\, dt = \frac{1}{T_0}\int_{-T_0/2}^{T_0/2} \left[\sum_{n=-\infty}^{\infty} F(n) e^{in\omega_0 t}\right] e^{-im\omega_0 t} dt.$$

Interchanging summation and integration,

$$\frac{1}{T_0}\int_{-T_0/2}^{T_0/2} f(t)\, e^{-im\omega_0 t}\, dt = \sum_{n=-\infty}^{\infty} F(n)\, \frac{1}{T_0}\int_{-T_0/2}^{T_0/2} e^{i(n-m)\omega_0 t}\, dt.$$

By orthogonality of the complex exponentials, all terms with $n \neq m$ vanish, and the term with $n = m$ equals 1. Hence

$$\frac{1}{T_0}\int_{-T_0/2}^{T_0/2} f(t)\, e^{-im\omega_0 t}\, dt = F(m).$$

Relabeling m as n, we obtain the analysis formula

$$F(n) = \frac{1}{T_0}\int_{-T_0/2}^{T_0/2} f(t)\, e^{-in\omega_0 t}\, dt. \tag{1.19}$$

Theorem 2 (Complex Fourier coefficients and reconstruction). *Let $f(t)$ be T_0-periodic and integrable over one period. Then its complex Fourier coefficients are*

$$F(n) = \frac{1}{T_0}\int_{-T_0/2}^{T_0/2} f(t)\, e^{-in\omega_0 t}\, dt. \tag{1.20}$$

The associated formal series is

$$\sum_{n=-\infty}^{\infty} F(n)\, e^{in\omega_0 t}. \tag{1.21}$$

If $f \in L^2$ over one period, this series converges to f in the mean-square sense; under Dirichlet conditions, it converges pointwise as stated in the convergence subsection below.

The coefficients $F(n)$ are called the complex Fourier coefficients. The term $F(0)$ is called the average, or dc, value of the waveform (also the "zeroth harmonic"). Pairs of coefficients correspond to harmonics. Define

$$h(1) = F(1)e^{i\omega_0 t} + F(-1)e^{-i\omega_0 t},$$

which is the fundamental (first harmonic); the second harmonic is

$$h(2) = F(2)e^{i2\omega_0 t} + F(-2)e^{-i2\omega_0 t},$$

and, in general, the nth harmonic is

$$h(n) = F(n)e^{in\omega_0 t} + F(-n)e^{-in\omega_0 t}. \tag{1.22}$$

Definition 1. *The notation*

$$f(t) \longleftrightarrow F(n) \tag{1.23}$$

means that $f(t)$ has complex Fourier coefficients $F(n)$ given by the analysis formula above, and $F(n)$ synthesizes $f(t)$ via the corresponding Fourier series.

Example 1 (Complex Fourier Series of a Piecewise Constant Waveform)**.** *Consider the periodic waveform $f(t)$ with period $T_0 = 4$, defined analytically over one period by*

$$f(t) = \begin{cases} 0, & t < -1, \\ 1, & -1 < t < 1, \\ 0, & 1 < t < 2, \end{cases} \quad \text{and} \quad f(t + T_0) = f(t).$$

For this waveform, the fundamental frequency is

$$\omega_0 = \frac{2\pi}{T_0} = \frac{\pi}{2}.$$

The Fourier coefficients are

$$F(n) = \frac{1}{T_0} \int_{-T_0/2}^{T_0/2} f(t)\, e^{-in\omega_0 t}\, dt = \frac{1}{4} \int_{-1}^{1} e^{-in(\pi/2)t}\, dt\,.$$

Carrying out the integral,

$$\begin{aligned} F(n) &= \frac{1}{4} \left[\frac{e^{-in(\pi/2)t}}{-in(\pi/2)} \right]_{t=-1}^{t=1} = \frac{1}{4} \cdot \frac{2}{n\pi i} \left(e^{-in\pi/2} - e^{in\pi/2} \right) \\ &= \frac{1}{2\pi n i} \left(e^{-in\pi/2} - e^{in\pi/2} \right). \end{aligned} \tag{1.24}$$

Since

$$e^{-in\pi/2} - e^{in\pi/2} = -2i \sin\!\left(\frac{n\pi}{2}\right),$$

we obtain

$$F(n) = \frac{1}{\pi n} \sin\!\left(\frac{n\pi}{2}\right), \quad n \neq 0.$$

For $n = 0$, we have

$$F(0) = \frac{1}{T_0} \int_{-T_0/2}^{T_0/2} f(t)\, dt = \frac{1}{4} \int_{-1}^{1} 1\, dt = \frac{1}{2}.$$

Thus,

$$F(n) = \begin{cases} \dfrac{1}{\pi n} \sin\!\left(\dfrac{n\pi}{2}\right), & n \neq 0, \\ \dfrac{1}{2}, & n = 0. \end{cases}$$

This can be written compactly in terms of the function

$$\mathrm{Sa}(x) \triangleq \begin{cases} \dfrac{\sin x}{x}, & x \neq 0, \\ 1, & x = 0, \end{cases} \tag{1.25}$$

so that

$$F(n) = \frac{1}{2}\, \mathrm{Sa}\!\left(\frac{n\pi}{2}\right).$$

The corresponding series is

$$f(t) = \sum_{n=-\infty}^{\infty} F(n) e^{in\omega_0 t} = \frac{1}{2} \sum_{n=-\infty}^{\infty} \mathrm{Sa}\!\left(\frac{n\pi}{2}\right) e^{in(\pi/2)t}.$$

This is the complex Fourier series representation of the periodic pulse waveform.

1.5 Real and Imaginary Parts of the Fourier Coefficients

In general, the complex coefficients $F(n)$ can be written in Cartesian form

$$F(n) = A(n) + iB(n), \tag{1.26}$$

or in polar form

$$F(n) = |F(n)|e^{i\Theta(n)}. \tag{1.27}$$

The magnitude and phase are related to $A(n)$ and $B(n)$ by

$$|F(n)| = \sqrt{A(n)^2 + B(n)^2}, \tag{1.28}$$

$$\Theta(n) = \arg(F(n)) = \arctan\left(\frac{B(n)}{A(n)}\right), \tag{1.29}$$

with appropriate care taken for the quadrant of the angle in numerical implementations.

The inverse relations are

$$A(n) = |F(n)|\cos[\Theta(n)], \tag{1.30}$$

$$B(n) = |F(n)|\sin[\Theta(n)]. \tag{1.31}$$

When $f(t)$ is real, the coefficients satisfy additional symmetry properties.

Theorem 3. *If $f(t)$ is real-valued and $F(n)$ are its complex Fourier coefficients, then*

$$F(n)^* = F(-n) \quad \textit{for all } n. \tag{1.32}$$

Proof. If $f(t)$ is real,

$$F(n)^* = \left[\frac{1}{T_0}\int_{-T_0/2}^{T_0/2} f(t)\,e^{-in\omega_0 t}\,dt\right]^* = \frac{1}{T_0}\int_{-T_0/2}^{T_0/2} f(t)\,e^{in\omega_0 t}\,dt.$$

Using $e^{in\omega_0 t} = e^{-i(-n)\omega_0 t}$,

$$F(n)^* = \frac{1}{T_0}\int_{-T_0/2}^{T_0/2} f(t)\,e^{-i(-n)\omega_0 t}\,dt = F(-n).$$

□

From $F(n)^* = F(-n)$ and $F(n) = A(n) + iB(n)$ we obtain, for real-valued $f(t)$:

$$|F(-n)| = |F(n)|, \tag{1.33}$$

$$A(-n) = A(n), \qquad B(-n) = -B(n), \tag{1.34}$$

and

$$\Theta(-n) = -\Theta(n). \tag{1.35}$$

Thus $A(n)$ and $|F(n)|$ are even functions of n, whereas $B(n)$ and $\Theta(n)$ are odd.

1.6 Even and Odd Functions

Definition 2 (Even function). *A function $g(n)$ is even if*

$$g(-n) = g(n) \quad \textit{for all } n. \tag{1.36}$$

In the continuous case, $g(t)$ is even if $g(-t) = g(t)$. Even functions have mirror symmetry about the vertical axis. Examples (as functions of t) include

$$g(t) = 3, \quad t^2, \quad \cos t, \quad t^4 \cos t, \quad e^t + e^{-t}.$$

An important property is

$$\int_{-a}^{a} g(t)\,dt = 2\int_{0}^{a} g(t)\,dt \quad \text{if } g(t) \text{ is even.} \tag{1.37}$$

Definition 3 (Odd function). *A function $g(n)$ is odd if*

$$g(-n) = -g(n) \quad \textit{for all } n. \tag{1.38}$$

Similarly, $g(t)$ is odd if $g(-t) = -g(t)$. Odd functions are antisymmetric with respect to the origin and satisfy $g(0) = 0$. Examples (as functions of t) include

$$g(t) = t, \quad \sin t, \quad t^3 \sin t, \quad e^t - e^{-t}.$$

For odd functions,

$$\int_{-a}^{a} g(t)\,dt = 0 \quad \text{if } g(t) \text{ is odd.} \tag{1.39}$$

Products of even and odd functions satisfy:

$$\text{even} \times \text{even} = \text{even}, \qquad \text{odd} \times \text{odd} = \text{even}, \qquad \text{even} \times \text{odd} = \text{odd}.$$

Any function $f(t)$ can be decomposed into even and odd parts:

$$f_{\mathrm{e}}(t) = \frac{1}{2}\big[f(t) + f(-t)\big], \tag{1.40}$$

$$f_{\mathrm{o}}(t) = \frac{1}{2}\big[f(t) - f(-t)\big], \tag{1.41}$$

so that $f(t) = f_{\mathrm{e}}(t) + f_{\mathrm{o}}(t)$, with f_{e} even and f_{o} odd.

Theorem 4 (Even/odd decomposition of Fourier series coefficients). *Let $f(t)$ be a real-valued, T-periodic function with $f(t) \longleftrightarrow F(n)$, where $F(n) = A(n) + iB(n)$. Then*

$$f_{\mathrm{e}}(t) \longleftrightarrow A(n), \qquad f_{\mathrm{o}}(t) \longleftrightarrow iB(n).$$

Proof. Let $F_{\mathrm{e}}(n)$ and $F_{\mathrm{o}}(n)$ denote the complex Fourier coefficients of $f_{\mathrm{e}}(t)$ and $f_{\mathrm{o}}(t)$, respectively:

$$F_{\mathrm{e}}(n) = \frac{1}{T}\int_{t_0}^{t_0+T} f_{\mathrm{e}}(t)\, e^{-in\omega_0 t}\,\mathrm{d}t = \frac{1}{2T}\int_{t_0}^{t_0+T} \big[f(t) + f(-t)\big] e^{-in\omega_0 t}\,\mathrm{d}t,$$
$$F_{\mathrm{o}}(n) = \frac{1}{T}\int_{t_0}^{t_0+T} f_{\mathrm{o}}(t)\, e^{-in\omega_0 t}\,\mathrm{d}t = \frac{1}{2T}\int_{t_0}^{t_0+T} \big[f(t) - f(-t)\big] e^{-in\omega_0 t}\,\mathrm{d}t.$$

We compute the integral involving $f(-t)$ by the change of variable $\tau = -t$. Since f is T-periodic, the choice of integration interval of length T is immaterial, and we may write

$$\begin{aligned}
\int_{t_0}^{t_0+T} f(-t)\, e^{-in\omega_0 t}\, \mathrm{d}t &= \int_{t_0}^{t_0+T} f(-t)\, e^{-in\omega_0 t}\, \mathrm{d}t \quad (\tau = -t,\ \mathrm{d}\tau = -\mathrm{d}t) \\
&= \int_{-t_0-T}^{-t_0} f(\tau)\, e^{in\omega_0 \tau}\, \mathrm{d}\tau \\
&= \int_{t_0}^{t_0+T} f(\tau)\, e^{in\omega_0 \tau}\, \mathrm{d}\tau \\
&= \int_{t_0}^{t_0+T} f(t)\, e^{in\omega_0 t}\, \mathrm{d}t.
\end{aligned} \tag{1.42}$$

By direct substitution:

$$F_e(n) = \frac{1}{2T}\left[\int_{t_0}^{t_0+T} f(t)\, e^{-in\omega_0 t}\, \mathrm{d}t + \int_{t_0}^{t_0+T} f(t)\, e^{in\omega_0 t}\, \mathrm{d}t\right] = \frac{F(n)+F(-n)}{2}. \tag{1.43}$$

Similarly,

$$F_o(n) = \frac{1}{2T}\left[\int_{t_0}^{t_0+T} f(t)\, e^{-in\omega_0 t}\, \mathrm{d}t - \int_{t_0}^{t_0+T} f(t)\, e^{in\omega_0 t}\, \mathrm{d}t\right] = \frac{F(n)-F(-n)}{2}. \tag{1.44}$$

Because $f(t)$ is real-valued, its Fourier coefficients satisfy the conjugate symmetry (1.32), hence

$$F_e(n) = \frac{F(n)+F^*(n)}{2} = \frac{[A(n)+iB(n)]+[A(n)-iB(n)]}{2} = A(n), \tag{1.45}$$

$$F_o(n) = \frac{F(n)-F^*(n)}{2} = \frac{[A(n)+iB(n)]-[A(n)-iB(n)]}{2} = iB(n). \tag{1.46}$$

Therefore $f_e(t) \longleftrightarrow A(n)$ and $f_o(t) \longleftrightarrow iB(n)$, as claimed. □

Corollary 1. *For a real periodic function $f(t)$ with coefficients $F(n)$:*

- *$F(n)$ is real and even if and only if $f(t)$ is even.*
- *$F(n)$ is purely imaginary and odd if and only if $f(t)$ is odd.*

1.7 Power and Parseval's Theorem for Periodic Waveforms

Suppose $f(t)$ is a periodic waveform applied as a voltage across a resistor of resistance R ohms. The instantaneous power is

$$P(t) = \frac{[f(t)]^2}{R}. \tag{1.47}$$

More generally, with complex-valued waveforms it is convenient to write

$$P(t) = \frac{|f(t)|^2}{R}. \tag{1.48}$$

For $R = 1\,\Omega$, we simply write

$$P(t) = |f(t)|^2. \tag{1.49}$$

The total energy in one period is then

$$E = \int_{-T_0/2}^{T_0/2} |f(t)|^2 \, dt, \tag{1.50}$$

and the average power over one period is

$$P_{\text{ave}} = \frac{1}{T_0} \int_{-T_0/2}^{T_0/2} |f(t)|^2 \, dt. \tag{1.51}$$

Definition 4 (Power signal). *If the average power P_{ave} defined above is finite, then $f(t)$ is called a power signal.*

There is an important relation between P_{ave} and the Fourier coefficients $F(n)$.

Theorem 5 (Parseval's theorem for periodic waveforms). *Let $f(t)$ be a periodic power signal with Fourier coefficients $F(n)$. Then*

$$\frac{1}{T_0} \int_{-T_0/2}^{T_0/2} |f(t)|^2 \, dt = \sum_{n=-\infty}^{\infty} |F(n)|^2. \tag{1.52}$$

Proof. For notational convenience, allow $f(t)$ to be complex. Then $|f(t)|^2 = f(t) f^*(t)$. Using the synthesis expression

$$f(t) = \sum_{n=-\infty}^{\infty} F(n) \, e^{in\omega_0 t}, \quad f^*(t) = \sum_{m=-\infty}^{\infty} F^*(m) \, e^{-im\omega_0 t},$$

we have

$$\frac{1}{T_0} \int_{-T_0/2}^{T_0/2} |f(t)|^2 \, dt = \frac{1}{T_0} \int_{-T_0/2}^{T_0/2} \left[\sum_n F(n) e^{in\omega_0 t} \right] \left[\sum_m F^*(m) e^{-im\omega_0 t} \right] dt.$$

Interchanging summation and integration,

$$\frac{1}{T_0} \int_{-T_0/2}^{T_0/2} |f(t)|^2 \, dt = \sum_n \sum_m F(n) F^*(m) \, \frac{1}{T_0} \int_{-T_0/2}^{T_0/2} e^{i(n-m)\omega_0 t} \, dt.$$

By orthogonality, the integral is zero if $n \neq m$ and equal to 1 if $n = m$. Hence only the terms with $n = m$ survive:

$$\frac{1}{T_0} \int_{-T_0/2}^{T_0/2} |f(t)|^2 \, dt = \sum_{n=-\infty}^{\infty} |F(n)|^2.$$

□

Definition 5 (Power spectrum). *For a periodic function $f(t)$ with complex Fourier coefficients $F(n)$, the power spectrum is*

$$P(n) = |F(n)|^2, \qquad n \in \mathbb{Z}. \tag{1.53}$$

Each term $|F(n)|^2$ represents the contribution of the nth coefficient to the total average power.

1.8 Convergence of Fourier Series

Given an analytical definition of $f(t)$, we can compute its Fourier coefficients $F(n)$ via the analysis integral, and then form the series

$$\sum_{n=-\infty}^{\infty} F(n)e^{in\omega_0 t}.$$

A natural question is: under what conditions does this series converge to the original $f(t)$, and how fast does it converge?

A complete characterization of necessary and sufficient conditions is still an advanced topic, but there are standard sufficient conditions that cover all physically realizable signals.

Square-integrability criterion

Assume that $f(t)$ is periodic with period T_0 and

$$\int_{-T_0/2}^{T_0/2} |f(t)|^2\, dt < \infty. \tag{1.54}$$

Then:

1. The integrals defining $F(n)$ exist and are finite.
2. If we define
$$g(t) = \sum_{n=-\infty}^{\infty} F(n)e^{in\omega_0 t}, \tag{1.55}$$
then the difference $e(t) = g(t) - f(t)$ may be nonzero on a set of measure zero, but
$$\int_{-T_0/2}^{T_0/2} |e(t)|^2\, dt = 0. \tag{1.56}$$

Thus, in the mean-square sense, $g(t)$ and $f(t)$ are identical over one period. All physically realizable periodic waveforms satisfy this condition.

Dirichlet conditions and convergence at discontinuities

A classical sufficient condition is given by the Dirichlet conditions.

Definition 6 (Dirichlet conditions)**.** *A periodic function $f(t)$ is said to satisfy the Dirichlet conditions if, over any one period:*

1. *$f(t)$ is bounded.*
2. *$f(t)$ has at most a finite number of discontinuities.*
3. *$f(t)$ has at most a finite number of local maxima and minima.*

All physically realizable periodic signals satisfy these conditions. Under the Dirichlet conditions, one has the following standard result.

Theorem 6 (Convergence under Dirichlet conditions)**.** *If $f(t)$ satisfies the Dirichlet conditions, then:*

1. *At any point of continuity of $f(t)$, the Fourier series converges to $f(t)$.*
2. *At a point t_0 where $f(t)$ has a jump discontinuity with left and right limits $f(t_0^-)$ and $f(t_0^+)$, the Fourier series converges to the half-sum*

$$\frac{f(t_0^-)+f(t_0^+)}{2}. \tag{1.57}$$

The value $\frac{f(t_0^-)+f(t_0^+)}{2}$ at a discontinuity is often called the *half-value*.

Rate of convergence

The decay rate of the Fourier coefficients controls the speed of convergence of the series. Basic estimates are:

- If $f(t)$ has discontinuities, then
$$|a_n|, |b_n| \lesssim \frac{K}{n}, \tag{1.58}$$
for large n, where a_n and b_n are the real (cosine and sine) Fourier coefficients, and K is a positive constant.
- If $f(t)$ is continuous but its first derivative has discontinuities, then
$$|a_n|, |b_n| \lesssim \frac{K}{n^2}. \tag{1.59}$$
- More generally, if $f(t)$ and its first m derivatives are continuous, but the $(m+1)$-st derivative has discontinuities, then
$$|a_n|, |b_n| \lesssim \frac{K}{n^{m+2}} \tag{1.60}$$
for large n.

Hence, smoother functions have more rapidly decaying coefficients and require fewer terms for accurate approximation.

Note also that if Fourier coefficients decay like K/n, then the power spectrum terms $|F(n)|^2$ decay like K^2/n^2, so the first few harmonics typically contain most of the power.

Notes on the sinc Function and the Gibbs Phenomenon

The sinc function. A commonly used variant of $\mathrm{Sa}(x)$ is the normalized sinc function

$$\mathrm{sinc}(x) = \frac{\sin(\pi x)}{\pi x}, \quad x \neq 0; \qquad \mathrm{sinc}(0) = 1. \tag{1.61}$$

The two functions are related by

$$\mathrm{sinc}\left(\frac{x}{\pi}\right) = \mathrm{Sa}(x), \qquad \mathrm{Sa}(x) = \mathrm{sinc}\left(\frac{x}{\pi}\right). \tag{1.62}$$

Gibbs phenomenon. Near jump discontinuities, partial sums of the Fourier series exhibit overshoots and oscillations. As the number of terms increases, the oscillations become more localized, but the maximum overshoot approaches a fixed fraction (approximately 9%) of the jump height. This behavior is known as the Gibbs phenomenon. The oscillations do not vanish with more terms; instead, they become narrower and more concentrated around the discontinuity, while the series converges in the sense described by the Dirichlet and square-integrability criteria.

2 The Fourier Integral

2.1 Introduction

Up to this point the waveforms under consideration have all been periodic. For each of them there exists a Fourier series representation. In many applications, however, the waveforms of interest are not periodic. More commonly they are single pulses that occur once and do not repeat. Fourier analysis also applies to such single pulses, which motivates the study of the Fourier integral.

Periodic waveforms can also be regarded as single, infinitely long pulses that occur once in time. The Fourier integral developed for one-time pulses then applies to them as well. In this sense the Fourier integral provides a unifying framework that covers both classes of waveform: pulses and periodic signals.

2.2 The Fourier Integral

We wish to apply Fourier analysis to a single pulse $f(t)$. From the periodic case we know how to proceed, which suggests the following construction:

> Embed the single pulse in one period of a periodic waveform and then let the period T_0 tend to infinity. The resulting limit should yield a pair of analysis and synthesis equations for $f(t)$.

Consider a single pulse $f(t)$ and the same pulse embedded in a periodic waveform $f_T(t)$ that repeats every T_0 seconds. Analytically,

$$f_T(t) = f(t), \quad -\frac{T_0}{2} < t < \frac{T_0}{2}, \tag{2.1}$$

$$f_T(t+T_0) = f_T(t). \tag{2.2}$$

Thus $f(t)$ specifies one period of $f_T(t)$. For finite T_0, we can form the complex Fourier series of $f_T(t)$. By subsequently letting $T_0 \to \infty$, only the single pulse remains, and the Fourier series representation is expected to converge to a Fourier-type representation for $f(t)$.

The complex Fourier series coefficient of order n for $f_T(t)$ is

$$F(n) = \frac{1}{T_0}\int_{-T_0/2}^{T_0/2} f_T(t)\, e^{-in\omega_0 t}\, dt, \qquad \omega_0 = \frac{2\pi}{T_0}. \tag{2.3}$$

Since n and ω_0 appear only through the product $n\omega_0$, it is natural to write $F(n\omega_0)$ in place of $F(n)$:

$$F(n\omega_0) = \frac{1}{T_0}\int_{-T_0/2}^{T_0/2} f_T(t)\, e^{-in\omega_0 t}\, dt. \tag{2.4}$$

On the interval $[-T_0/2, T_0/2]$, $f_T(t)$ coincides with $f(t)$, so

$$F(n\omega_0) = \frac{1}{T_0}\int_{-T_0/2}^{T_0/2} f(t)\, e^{-in\omega_0 t}\, dt. \tag{2.5}$$

We intend to let $T_0 \to \infty$ while keeping $f(t)$ fixed. The right-hand side is the average value of $f(t)e^{-in\omega_0 t}$ over $[-T_0/2, T_0/2]$. As $T_0 \to \infty$, this average tends to zero, which would trivialize the limit. To avoid this, we rewrite

$$T_0 F(n\omega_0) = \int_{-T_0/2}^{T_0/2} f(t)\, e^{-in\omega_0 t}\, dt. \tag{2.6}$$

For each finite T_0, this defines a line spectrum $\{T_0 F(n\omega_0)\}$ on the ω-axis at $\omega = n\omega_0$.

Introduce

$$\Delta\omega = \omega_0, \qquad \omega_n = n\,\Delta\omega, \qquad F(\omega_n) = T_0\,F(n\omega_0). \tag{2.7}$$

Then

$$F(\omega_n) = \int_{-T_0/2}^{T_0/2} f(t)\,e^{-i\omega_n t}\,dt. \tag{2.8}$$

As $T_0 \to \infty$,

$$\Delta\omega \to 0, \quad \omega_n \to \omega, \quad F(\omega_n) \to F(\omega),$$

and it is natural to write

$$F(\omega) = \lim_{T_0\to\infty} \int_{-T_0/2}^{T_0/2} f(t)\,e^{-i\omega t}\,dt = \int_{-\infty}^{\infty} f(t)\,e^{-i\omega t}\,dt. \tag{2.9}$$

This is the *analysis* equation for a single pulse:

$$F(\omega) = \int_{-\infty}^{\infty} f(t)\,e^{-i\omega t}\,dt. \tag{2.10}$$

For the synthesis equation, start from the Fourier series representation of the periodic function $f_T(t)$:

$$f_T(t) = \sum_{n=-\infty}^{\infty} F(n\omega_0)\,e^{in\omega_0 t}, \qquad -\frac{T_0}{2} < t < \frac{T_0}{2}. \tag{2.11}$$

On this interval, the left-hand side equals $f(t)$, so we write

$$f(t) = \sum_{n=-\infty}^{\infty} F(n\omega_0)\,e^{in\omega_0 t}, \qquad -\frac{T_0}{2} < t < \frac{T_0}{2}. \tag{2.12}$$

Using $F(n\omega_0) = F(\omega_n)/T_0$,

$$f(t) = \sum_{n=-\infty}^{\infty} \frac{1}{T_0} F(\omega_n)\,e^{i\omega_n t}. \tag{2.13}$$

Since

$$\frac{1}{T_0} = \frac{\omega_0}{2\pi} = \frac{\Delta\omega}{2\pi},$$

we get

$$f(t) = \sum_{n=-\infty}^{\infty} F(\omega_n)\,e^{i\omega_n t}\,\frac{\Delta\omega}{2\pi}. \tag{2.14}$$

Letting $T_0 \to \infty$ implies $\Delta\omega \to 0$ and $\omega_n \to \omega$. Under suitable conditions, this Riemann sum converges to

$$f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega)\,e^{i\omega t}\,d\omega. \tag{2.15}$$

We summarize the result as follows:

Theorem 7 (Fourier Transform for a Single Pulse)**.** *Suppose $f(t)$ is such that the integrals below exist. The function $f(t)$ and its Fourier transform $F(\omega)$ are related by the transform pair*

$$F(\omega) = \int_{-\infty}^{\infty} f(t)\,e^{-i\omega t}\,dt, \tag{2.16}$$

$$f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega)\,e^{i\omega t}\,d\omega. \tag{2.17}$$

The function $F(\omega)$ in (2.16) *is called the* Fourier transform *or* Fourier spectral density *of $f(t)$. The integral representation* (2.17) *is called the* inverse Fourier transform *or the* Fourier integral representation *of $f(t)$.*

The limits of integration in (2.16) extend over all time, so the pulses that can be transformed may have arbitrary duration, finite or infinite. For example, the following pulses are admissible:

$$f(t) = e^{-t^2}, \qquad t \in \mathbb{R},$$

$$g(t) = \begin{cases} 1, & 0 \le t \le 1, \\ 0, & \text{otherwise.} \end{cases}$$

As will be seen later, periodic waveforms of infinite duration can also be treated within the same framework.

Definition 7 (Fourier Transform Notation)**.** *The shorthand*

$$f(t) \longleftrightarrow F(\omega) \tag{2.18}$$

will be used to indicate that $F(\omega)$ is the Fourier transform of $f(t)$ as in (2.16)*, and that $f(t)$ is the inverse transform of $F(\omega)$ as in* (2.17)*.*

2.3 Interpretation of the Fourier Integral

For a periodic function with fundamental frequency ω_0, the Fourier series representation

$$f(t) = \sum_{n=-\infty}^{\infty} F(n)\, e^{in\omega_0 t}$$

expresses $f(t)$ as a linear combination of complex exponentials at discrete frequencies $n\omega_0$. The coefficients $F(n)$ are determined by the corresponding analysis equation.

The Fourier integral representation (2.17),

$$f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega)\, e^{i\omega t}\, d\omega,$$

is the counterpart of the Fourier series for non-periodic (or more general) signals. It expresses $f(t)$ as a superposition of complex exponentials at all real frequencies ω, weighted by the coefficient function $F(\omega)$. Conceptually, the Fourier integral is the limit of the Fourier series as the fundamental period T_0 tends to infinity and the line spectrum becomes a continuous spectrum.

Remark 1. *The Fourier integral can be viewed as a continuous-frequency linear decomposition of $f(t)$ into the basis $\{e^{i\omega t}\}_{\omega \in \mathbb{R}}$. The transform $F(\omega)$ specifies the contribution of each frequency to the overall waveform.*

Example 2 (Fourier Transform of the Rectangular Pulse)**.** *The following pulse will be used frequently.*

Definition 8 (Rectangular Pulse)**.** *The rectangular pulse of width τ and unit height is denoted by* $\mathrm{Rect}(t/\tau)$ *and defined by*

$$\mathrm{Rect}\left(\frac{t}{\tau}\right) = \begin{cases} 1, & |t| \le \dfrac{\tau}{2}, \\ 0, & |t| > \dfrac{\tau}{2}. \end{cases} \tag{2.19}$$

Let $f(t) = \mathrm{Rect}(t/\tau)$. Using the analysis equation,

$$\begin{aligned} F(\omega) &= \int_{-\infty}^{\infty} f(t)e^{-i\omega t}\,dt = \int_{-\tau/2}^{\tau/2} e^{-i\omega t}\,dt \\ &= \left[\frac{e^{-i\omega t}}{-i\omega}\right]_{t=-\tau/2}^{t=\tau/2} = \frac{e^{-i\omega\tau/2} - e^{i\omega\tau/2}}{-i\omega} = \frac{2\sin(\omega\tau/2)}{\omega}. \end{aligned} \tag{2.20}$$

Introducing the normalized sinc-type function

$$\mathrm{Sa}(x) = \frac{\sin x}{x}, \tag{2.21}$$

we may write

$$F(\omega) = \tau\,\mathrm{Sa}\Big(\frac{\omega\tau}{2}\Big). \tag{2.22}$$

Thus

$$\mathrm{Rect}\Big(\frac{t}{\tau}\Big) \longleftrightarrow \tau\,\mathrm{Sa}\Big(\frac{\omega\tau}{2}\Big). \tag{2.23}$$

The inverse transform gives the Fourier integral representation:

$$\mathrm{Rect}\Big(\frac{t}{\tau}\Big) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \tau\,\mathrm{Sa}\Big(\frac{\omega\tau}{2}\Big)e^{i\omega t}\,d\omega. \tag{2.24}$$

2.4 Basic Properties for Real Signals

In general, the Fourier transform $F(\omega)$ is complex-valued. It can be written in Cartesian form

$$F(\omega) = A(\omega) + iB(\omega), \tag{2.25}$$

or in polar form

$$F(\omega) = |F(\omega)|e^{i\Theta(\omega)}, \tag{2.26}$$

where

$$|F(\omega)| = \sqrt{A^2(\omega) + B^2(\omega)}, \qquad \Theta(\omega) = \tan^{-1}\Big(\frac{B(\omega)}{A(\omega)}\Big). \tag{2.27}$$

Theorem 8 (Conjugate Symmetry for Real Signals)**.** *Let $f(t) \longleftrightarrow F(\omega)$. Then*

$$f(t) \in \mathbb{R} \quad \text{for all } t \quad \Longleftrightarrow \quad F^*(\omega) = F(-\omega) \quad \text{for all } \omega. \tag{2.28}$$

Proof. Assume first that $f(t)$ is real. Then

$$\begin{aligned} F^*(\omega) &= \Big(\int_{-\infty}^{\infty} f(t)e^{-i\omega t}\,dt\Big)^* = \int_{-\infty}^{\infty} f(t)e^{i\omega t}\,dt \\ &= \int_{-\infty}^{\infty} f(t)e^{-i(-\omega)t}\,dt = F(-\omega). \end{aligned} \tag{2.29}$$

Conversely, assume $F^*(\omega) = F(-\omega)$ for all ω. Take the inverse transform:

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)e^{i\omega t}\,d\omega.$$

Taking complex conjugates,

$$f^*(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F^*(\omega)e^{-i\omega t}\,d\omega = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(-\omega)e^{-i\omega t}\,d\omega.$$

Change variables $\omega' = -\omega$; then

$$f^*(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega')e^{i\omega' t}\,d\omega' = f(t).$$

Hence $f(t)$ is real. □

Corollary 2. *If $f(t)$ is real, then*

- *the real part $A(\omega)$ and the magnitude $|F(\omega)|$ are even,*
- *the imaginary part $B(\omega)$ and the phase $\Theta(\omega)$ are odd.*

Proof. From $F^*(\omega) = F(-\omega)$ and $F(\omega) = A(\omega) + iB(\omega)$ we obtain

$$A(\omega) - iB(\omega) = A(-\omega) + iB(-\omega).$$

Equating real and imaginary parts gives

$$A(\omega) = A(-\omega), \qquad B(-\omega) = -B(\omega).$$

Thus A is even and B is odd. The magnitude satisfies $|F(-\omega)| = |F(\omega)|$, hence $|F|$ is even. The phase can be chosen so that $\Theta(-\omega) = -\Theta(\omega)$ wherever $F(\omega) \neq 0$, so Θ is odd modulo 2π. □

Similar to the case of Fourier series, we have the following theorem and corollary for Fourier transform of a real signal:

Theorem 9 (Even/Odd Decomposition). *Suppose $f(t)$ is real and decomposed into even and odd parts*

$$f(t) = f_e(t) + f_o(t), \quad f_e(t) = \frac{f(t)+f(-t)}{2}, \quad f_o(t) = \frac{f(t)-f(-t)}{2}.$$

Let $f(t) \longleftrightarrow F(\omega)$ with $F(\omega) = A(\omega) + iB(\omega)$. Then

$$f_e(t) \longleftrightarrow A(\omega), \qquad f_o(t) \longleftrightarrow iB(\omega). \tag{2.30}$$

Corollary 3. *For real $f(t)$:*

1. *$F(\omega)$ is real and even if and only if $f(t)$ is even.*
2. *$F(\omega)$ is purely imaginary and odd if and only if $f(t)$ is odd.*

2.5 Parseval's Theorem for Pulses

Consider a voltage pulse $v(t)$ applied to an R-ohm resistor. The instantaneous power is

$$p(t) = \frac{|v(t)|^2}{R} \quad \text{(watts)}. \tag{2.31}$$

For $R = 1$,

$$p(t) = |v(t)|^2. \tag{2.32}$$

The total energy delivered by the pulse is

$$E = \int_{-\infty}^{\infty} p(t)\,dt = \int_{-\infty}^{\infty} |v(t)|^2\,dt \quad \text{(joules)}. \tag{2.33}$$

The same expression holds for current pulses if one interprets $|f(t)|^2$ as instantaneous power in a 1-ohm load.

Definition 9 (Energy Pulse). *A signal $f(t)$ is called an* energy pulse *if*

$$\int_{-\infty}^{\infty} |f(t)|^2\,dt < \infty. \tag{2.34}$$

For such pulses, there is an energy relation in the transform domain.

Lemma 1 (Parseval's Lemma). *Let $f(t)$ and $g(t)$ be energy pulses with transforms $F(\omega)$ and $G(\omega)$, respectively. Then*

$$\int_{-\infty}^{\infty} f(t)\,g^*(t)\,dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)\,G^*(\omega)\,d\omega. \tag{2.35}$$

Proof. Express $g^*(t)$ using the inverse transform:

$$g^*(t) = \left(\frac{1}{2\pi}\int_{-\infty}^{\infty} G(\omega)\,e^{i\omega t}\,d\omega\right)^* = \frac{1}{2\pi}\int_{-\infty}^{\infty} G^*(\omega)\,e^{-i\omega t}\,d\omega.$$

Then

$$\begin{aligned}\int_{-\infty}^{\infty} f(t)\,g^*(t)\,dt &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty} f(t)\,e^{-i\omega t}\,dt\right)G^*(\omega)\,d\omega\\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)\,G^*(\omega)\,d\omega.\end{aligned} \tag{2.36}$$

□

Theorem 10 (Parseval's Theorem for Pulses). *Let $f(t)$ be an energy pulse with Fourier transform $F(\omega)$. Then*

$$\int_{-\infty}^{\infty} |f(t)|^2\,dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} |F(\omega)|^2\,d\omega. \tag{2.37}$$

Proof. Apply Parseval's lemma with $g(t) = f(t)$ and $G(\omega) = F(\omega)$. Then

$$\int_{-\infty}^{\infty} f(t)\,f^*(t)\,dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)\,F^*(\omega)\,d\omega.$$

This is the stated equality. □

Definition 10 (Energy Spectrum). *The* energy spectrum *of a pulse with transform $F(\omega)$ is the function*

$$E(\omega) = \frac{1}{2\pi}|F(\omega)|^2 \quad \text{(joules/radian)}. \tag{2.38}$$

The energy contained in the frequency band $[\omega_1, \omega_2]$ is then

$$\Delta E = \int_{\omega_1}^{\omega_2} E(\omega)\,d\omega = \frac{1}{2\pi}\int_{\omega_1}^{\omega_2} |F(\omega)|^2\,d\omega. \tag{2.39}$$

2.6 Existence of the Fourier Integral

The derivation of (2.16)–(2.17) was heuristic. A rigorous treatment requires conditions on $f(t)$ under which the integrals exist and are inverse to one another. No single set of necessary and sufficient conditions is both simple and completely general. However, several important sufficient conditions cover essentially all physically realizable cases.

Definition 11 (Square-Integrable Function)**.** *A function* $f(t)$ *is* square-integrable *if*

$$\int_{-\infty}^{\infty} |f(t)|^2 \, dt < \infty. \tag{2.40}$$

Theorem 11 (Square-Integrability Criterion)**.** *Suppose* $f(t)$ *is square-integrable. Define formally*

$$F(\omega) = \int_{-\infty}^{\infty} f(t) e^{-i\omega t} \, dt, \tag{2.41}$$

and

$$\phi(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega) e^{i\omega t} \, d\omega. \tag{2.42}$$

Then:

1. *The mapping* $f \mapsto F$ *extends uniquely to a unitary operator on* $L^2(\mathbb{R})$ *(Plancherel theorem).*
2. *The function* $\phi(t)$ *coincides with* $f(t)$ *almost everywhere and*

$$\int_{-\infty}^{\infty} |\phi(t) - f(t)|^2 \, dt = 0.$$

A second class of sufficient conditions is based on a continuous-time analogue of the Dirichlet conditions.

Definition 12 (Dirichlet Conditions for Pulses)**.** *A function* $f(t)$ *is said to satisfy the* Dirichlet conditions for pulses *if:*

1. $f(t)$ *is bounded on* $\mathbb{R}$*,*
2. *in any finite interval it has at most a finite number of discontinuities and a finite number of local maxima and minima,*
3. *it is absolutely integrable,*

$$\int_{-\infty}^{\infty} |f(t)| \, dt < \infty.$$

Theorem 12 (Dirichlet–Jordan Criterion)**.** *Let* $f(t)$ *satisfy the Dirichlet conditions for pulses. Then:*

1. *The Fourier transform*

$$F(\omega) = \int_{-\infty}^{\infty} f(t) e^{-i\omega t} \, dt$$

exists for all ω*.*

2. *The Fourier integral*

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega) e^{i\omega t} \, d\omega$$

converges to $f(t)$ *at every point of continuity.*

3. *At a point of discontinuity* t_0*, the Fourier integral converges to*

$$\frac{1}{2}\left[f(t_0^+) + f(t_0^-)\right].$$

Every physically realizable pulse satisfies the Dirichlet conditions, and hence admits a well-defined Fourier transform with the usual inversion property.

2.7 Asymptotic Behavior of $F(\omega)$

The decay rate of the Fourier transform for large $|\omega|$ is related to the smoothness of the underlying time-domain signal.

Theorem 13 (Asymptotic Bound for $F(\omega)$). *Let $f(t)$ be a pulse that satisfies appropriate regularity conditions. Suppose $f(t)$ and its first m derivatives are continuous and vanish at infinity, and that the $(m+1)$-st derivative is piecewise continuous and absolutely integrable. Then there exists a constant $K > 0$ such that, for sufficiently large $|\omega|$,*

$$|F(\omega)| \leq \frac{K}{|\omega|^{m+1}}. \tag{2.43}$$

Proof. Start from

$$F(\omega) = \int_{-\infty}^{\infty} f(t)\, e^{-i\omega t}\, dt.$$

Integrate by parts:

$$F(\omega) = \left[\frac{f(t)}{-i\omega} e^{-i\omega t}\right]_{-\infty}^{\infty} + \frac{1}{i\omega}\int_{-\infty}^{\infty} f'(t)\, e^{-i\omega t}\, dt.$$

The boundary term vanishes since $f(t) \to 0$ as $|t| \to \infty$. Repeating this procedure $m+1$ times gives

$$F(\omega) = \frac{1}{(i\omega)^{m+1}} \int_{-\infty}^{\infty} f^{(m+1)}(t)\, e^{-i\omega t}\, dt.$$

Absolute integrability of $f^{(m+1)}$ implies

$$\left|\int_{-\infty}^{\infty} f^{(m+1)}(t)\, e^{-i\omega t}\, dt\right| \leq \int_{-\infty}^{\infty} |f^{(m+1)}(t)|\, dt =: K_0.$$

Hence

$$|F(\omega)| \leq \frac{K_0}{|\omega|^{m+1}},$$

for all $\omega \neq 0$. Taking $K \geq K_0$ completes the proof. □

Thus, as the number of continuous derivatives of $f(t)$ increases, the transform decays more rapidly in frequency. Since the energy spectrum is proportional to $|F(\omega)|^2$, smoother time-domain signals yield more rapidly decaying energy spectra.

3 Fourier Transforms of Some Important Functions

3.1 Introduction

The nineteenth century saw extensive work on placing the theories of Fourier and Laplace on a rigorous basis. Modern analysis now gives complete convergence and inversion frameworks in settings such as L^2 spaces and distributions, while simple pointwise criteria for broad classes of functions are more delicate and depend on the function class considered.

Much of this development was driven by the need to clarify broad claims such as Fourier's assertion that every function $f(x)$ (or portion of a function) can be represented by a trigonometric series. Dirichlet provided a useful set of sufficient conditions, and counterexamples were later constructed to show that Fourier's statement is not universally valid.

A major extension relevant to engineering arose in the 1920s from quantum mechanics. In 1927, Paul A. M. Dirac introduced the *delta function* (now called the *Dirac delta* and denoted by $\delta(t-\tau)$) in his work on quantum theory. Although formulated for quantum mechanics, the Dirac delta soon became central in many areas of applied mathematics and electrical engineering.

We shall see that $\delta(t)$ does not satisfy square-integrability or Dirichlet-type conditions, carries infinite energy, and cannot be realized physically, yet still admits a well-defined Fourier transform. In fact, one can show that

$$\delta(t-\tau) \;\longleftrightarrow\; e^{-i\omega\tau}, \tag{3.1}$$

so that the Dirac delta is closely tied to the complex exponentials that underlie Fourier analysis.

The appearance of such objects motivated the development of *distribution theory* (or *generalized function theory*). Many contributors advanced this area; a particularly important role was played by Laurent Schwartz, who systematically developed distributions and emphasized their applications in physics and engineering.

In this chapter we use a minimal amount of distribution theory, sufficient to derive the Fourier transforms of several functions of central importance:

- the Dirac delta (unit impulse),
- the unit step,
- the signum function,
- the eternal complex exponential,
- the constant (dc) signal,
- eternal sines and cosines,
- general periodic functions with a Fourier series representation.

All of these have infinite energy and lie outside the class handled by the square-integrability criterion. Nevertheless, in the framework of generalized functions, each admits a consistent Fourier transform representation compatible with the transform of finite-energy pulses introduced earlier.

3.2 The Rectangular Pulse $f(t) = \mathrm{Rect}(t/T)$

We begin with the rectangular (gate) pulse of width T and unit height.

Definition 13 (Rectangular pulse). *The rectangular pulse of width $T > 0$ is defined by*

$$\mathrm{Rect}\left(\frac{t}{T}\right) = \begin{cases} 1, & |t| < \dfrac{T}{2}, \\ 0, & |t| > \dfrac{T}{2}. \end{cases} \tag{3.2}$$

(The value at $t = \pm T/2$ is immaterial for the integral.)

The Fourier transform of $\mathrm{Rect}(t/T)$ was obtained previously and is recalled here in our normalization

$$f(t) \longleftrightarrow F(\omega) \quad \text{with} \quad F(\omega) = \int_{-\infty}^{\infty} f(t)\, e^{-i\omega t}\, dt, \quad f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)\, e^{i\omega t}\, d\omega.$$

Proposition 1 (Fourier transform of a rectangular pulse). *Let $f(t) = \mathrm{Rect}(t/T)$. Then*

$$f(t) = \mathrm{Rect}\left(\frac{t}{T}\right) \longleftrightarrow F(\omega) = T\,\mathrm{Sa}\left(\frac{\omega T}{2}\right), \tag{3.3}$$

where

$$\mathrm{Sa}(x) := \frac{\sin x}{x}, \quad \mathrm{Sa}(0) := 1. \tag{3.4}$$

Proof. From the definition of the transform,

$$F(\omega) = \int_{-\infty}^{\infty} \mathrm{Rect}\left(\frac{t}{T}\right) e^{-i\omega t}\, dt = \int_{-T/2}^{T/2} e^{-i\omega t}\, dt.$$

Evaluating,

$$\begin{aligned} F(\omega) &= \left[\frac{e^{-i\omega t}}{-i\omega}\right]_{t=-T/2}^{T/2} = \frac{e^{-i\omega T/2} - e^{i\omega T/2}}{-i\omega} \\ &= \frac{2\sin(\omega T/2)}{\omega} = T\,\frac{\sin(\omega T/2)}{\omega T/2} = T\,\mathrm{Sa}\left(\frac{\omega T}{2}\right), \end{aligned} \tag{3.5}$$

with the value at $\omega = 0$ given by the limit $\mathrm{Sa}(0) = 1$. □

A useful property of Sa that we shall need later is its total area.

Lemma 2 (Area under Sa). *The function $\mathrm{Sa}(x)$ satisfies*

$$\int_{-\infty}^{\infty} \mathrm{Sa}(x)\, dx = \pi. \tag{3.6}$$

Proof. From (3.3) and the inverse transform,

$$\mathrm{Rect}\left(\frac{t}{T}\right) = \frac{1}{2\pi}\int_{-\infty}^{\infty} T\,\mathrm{Sa}\left(\frac{\omega T}{2}\right) e^{i\omega t}\, d\omega.$$

Setting $t = 0$ and using $\mathrm{Rect}(0/T) = 1$ gives

$$1 = \frac{T}{2\pi}\int_{-\infty}^{\infty} \mathrm{Sa}\left(\frac{\omega T}{2}\right) d\omega.$$

With the change of variable $x = \omega T/2$, $d\omega = \frac{2}{T}dx$, this becomes

$$1 = \frac{T}{2\pi}\cdot\frac{2}{T}\int_{-\infty}^{\infty} \mathrm{Sa}(x)\, dx = \frac{1}{\pi}\int_{-\infty}^{\infty} \mathrm{Sa}(x)\, dx,$$

which yields (3.6). □

3.3 The Single-Sided Decaying Exponential

Consider the one-sided decaying exponential

$$f(t) = \begin{cases} 0, & t < 0, \\ \frac{1}{2}, & t = 0, \\ e^{-\beta t}, & t > 0, \end{cases} \qquad \beta > 0. \tag{3.7}$$

It is convenient to write this using the unit step $u(t)$:

Definition 14 (Unit step)**.** *The unit step (Heaviside step) is*

$$u(t) = \begin{cases} 0, & t < 0, \\ \frac{1}{2}, & t = 0, \\ 1, & t > 0. \end{cases} \tag{3.8}$$

Then (3.7) becomes

$$f(t) = e^{-\beta t} u(t), \qquad \beta > 0. \tag{3.9}$$

Proposition 2 (Fourier transform of a one-sided decaying exponential)**.** *Let $f(t) = e^{-\beta t}u(t)$, $\beta > 0$. Then*

$$f(t) = e^{-\beta t} u(t) \longleftrightarrow F(\omega) = \frac{1}{\beta + i\omega}. \tag{3.10}$$

Proof. Using the definition of the transform,

$$F(\omega) = \int_{-\infty}^{\infty} e^{-\beta t} u(t) e^{-i\omega t}\, dt = \int_{0}^{\infty} e^{-(\beta + i\omega)t}\, dt.$$

Since $\beta > 0$, the integral converges and

$$F(\omega) = \left[-\frac{1}{\beta + i\omega} e^{-(\beta + i\omega)t} \right]_0^{\infty} = \frac{1}{\beta + i\omega}.$$

□

The real and imaginary parts of $F(\omega)$ are

$$F(\omega) = \frac{\beta}{\beta^2 + \omega^2} - i \frac{\omega}{\beta^2 + \omega^2}, \tag{3.11}$$

so that

$$|F(\omega)| = \frac{1}{\sqrt{\beta^2 + \omega^2}}, \qquad \arg F(\omega) = -\arctan\left(\frac{\omega}{\beta}\right). \tag{3.12}$$

The magnitude is even in ω, and the phase is odd, consistent with the general properties for real signals.

The even and odd parts of $f(t)$ will be useful repeatedly.

For $f(t) = e^{-\beta t} u(t)$, one checks easily that

$$f_e(t) = \frac{1}{2} e^{-\beta |t|} = \frac{1}{2} e^{-\beta |t|}, \tag{3.13}$$

$$f_o(t) = \frac{1}{2}\left(e^{-\beta t} u(t) - e^{\beta t} u(-t)\right). \tag{3.14}$$

Using the general results from previous chapters that the transform of the even part is the real part $A(\omega)$ and the transform of the odd part is $iB(\omega)$, we obtain:

Corollary 4 (Transforms of the even/odd parts). *For $f(t) = e^{-\beta t}u(t)$,*

$$f_e(t) = \tfrac{1}{2}e^{-\beta|t|} \longleftrightarrow F_e(\omega) = \frac{\beta}{\beta^2 + \omega^2}, \tag{3.15}$$

$$f_o(t) \longleftrightarrow F_o(\omega) = -i\,\frac{\omega}{\beta^2 + \omega^2}. \tag{3.16}$$

Proof. Writing $F(\omega) = A(\omega) + iB(\omega)$ as in (3.11), the general even/odd decomposition theorem gives

$$f_e(t) \longleftrightarrow A(\omega), \qquad f_o(t) \longleftrightarrow iB(\omega).$$

From (3.11), $A(\omega) = \beta/(\beta^2 + \omega^2)$ and $iB(\omega) = -i\,\omega/(\beta^2 + \omega^2)$, which yields (3.15)–(3.16). □

Thus a real, even time-domain function yields a real, even transform; a real, odd function yields a purely imaginary, odd transform.

3.4 The Double-Sided Decaying Exponential

The double-sided exponential is defined by

$$f(t) = e^{-\beta|t|}, \qquad \beta > 0, \tag{3.17}$$

plotted symmetrically around the origin.

Proposition 3 (Fourier transform of a double-sided exponential). *For $f(t) = e^{-\beta|t|}$ with $\beta > 0$,*

$$f(t) = e^{-\beta|t|} \longleftrightarrow F(\omega) = \frac{2\beta}{\beta^2 + \omega^2}. \tag{3.18}$$

Proof. The function $e^{-\beta|t|}$ is twice the even part of the one-sided exponential (3.9):

$$e^{-\beta|t|} = 2f_e(t),$$

with $f_e(t)$ defined in (3.13). Hence, by (3.15),

$$F(\omega) = 2F_e(\omega) = 2 \cdot \frac{\beta}{\beta^2 + \omega^2} = \frac{2\beta}{\beta^2 + \omega^2}.$$

□

The function is real and even in t, and its transform is real and even in ω, as expected. The transform decays as ω^{-2} for large $|\omega|$, consistent with the continuity and differentiability properties of $f(t)$.

3.5 The Signum Function

Definition 15 (Signum function). *The signum function is defined by*

$$\operatorname{sgn}(t) = \begin{cases} -1, & t < 0, \\ 0, & t = 0, \\ +1, & t > 0. \end{cases} \tag{3.19}$$

As an ordinary function, $\operatorname{sgn}(t)$ has infinite energy:

$$\int_{-\infty}^{\infty} |\operatorname{sgn}(t)|^2 \, dt = \int_{-\infty}^{\infty} 1 \, dt = \infty,$$

so the square-integrability criterion does not apply. Moreover, a direct application of the analysis integral,

$$F(\omega) = \int_{-\infty}^{\infty} \operatorname{sgn}(t) \, e^{-i\omega t} \, dt,$$

leads to divergent integrals if interpreted naively. To proceed, we treat $\operatorname{sgn}(t)$ as a *distribution*, that is, as the limit of a sequence of ordinary functions that do satisfy the usual conditions.

Definition 16 (Generalized function via sequence)**.** *A distribution is specified as the limit of a sequence $\{f_k(t)\}_{k>0}$ of ordinary functions, each having a well-defined Fourier transform, where the limit is taken in a suitable (distributional) sense.*

For $\operatorname{sgn}(t)$, consider the sequence

$$f_k(t) = \begin{cases} -e^{kt}, & t < 0, \\ 0, & t = 0, \\ e^{-kt}, & t > 0, \end{cases} \qquad k > 0. \tag{3.20}$$

For each fixed k, $f_k(t)$ is an ordinary function that tends pointwise to $\operatorname{sgn}(t)$ as $k \to 0^+$. Formally,

$$\operatorname{sgn}(t) = \lim_{k \to 0^+} f_k(t). \tag{3.21}$$

We first Fourier transform $f_k(t)$ for fixed k, then let $k \to 0^+$. For fixed k, since $f_k(t)$ is twice the odd part of $e^{-kt}u(t)$, with $\beta = k$. From (3.16) (with $\beta = k$), we obtain

$$\mathcal{F}\{f_k\}(\omega) = -i \, \frac{2\omega}{k^2 + \omega^2}. \tag{3.22}$$

Proposition 4 (Fourier transform of $\operatorname{sgn}(t)$)**.** *As a distribution,*

$$\operatorname{sgn}(t) \;\longleftrightarrow\; F(\omega) = \frac{2}{i\omega}, \tag{3.23}$$

interpreted in the sense of principal value.

Sketch. For each $k > 0$,

$$F_k(\omega) = -i \, \frac{2\omega}{k^2 + \omega^2}.$$

As $k \to 0^+$, one can write

$$F(\omega) = -i \frac{2}{\omega},$$

in the distributional sense (principal-value interpretation at $\omega = 0$). $\square$

3.6 The Dirac Delta (Unit Impulse)

Definition 17 (Dirac delta). *The Dirac delta (unit impulse) at the origin is the distribution $\delta(t)$ characterized by*

$$\delta(t) = 0 \quad (t \neq 0), \qquad \int_{-\infty}^{\infty} \delta(t)\, dt = 1. \tag{3.24}$$

This description is heuristic from the viewpoint of ordinary functions. As with $\operatorname{sgn}(t)$, it is more precise to construct $\delta(t)$ as the limit of a sequence of ordinary functions.

Definition 18 (Box sequence for δ). *For $k > 0$, define the box function*

$$B_k(t) = \begin{cases} \dfrac{1}{k}, & |t| < \dfrac{k}{2}, \\ 0, & |t| > \dfrac{k}{2}. \end{cases} \tag{3.25}$$

Then each B_k has unit area, $\int_{-\infty}^{\infty} B_k(t)\, dt = 1$, and

$$\delta(t) = \lim_{k \to 0^+} B_k(t) \tag{3.26}$$

in the distributional sense.

Proposition 5 (Fourier transform of δ). *The Dirac delta satisfies*

$$\delta(t) \longleftrightarrow F(\omega) = 1. \tag{3.27}$$

Proof. For fixed k, the Fourier transform of B_k is

$$F_k(\omega) = \int_{-\infty}^{\infty} B_k(t)\, e^{-i\omega t}\, dt = \frac{1}{k} \int_{-k/2}^{k/2} e^{-i\omega t}\, dt = \frac{\sin(\omega k/2)}{\omega k/2} = \operatorname{Sa}\!\left(\frac{\omega k}{2}\right).$$

As $k \to 0^+$, $\operatorname{Sa}(\omega k/2) \to 1$ for each fixed ω, so the limit transform is $F(\omega) = 1$. □

Thus the magnitude spectrum of $\delta(t)$ is identically one, and the phase is identically zero. The delta is even in time, and its transform is real and even.

The delta has infinite energy:

$$\int_{-\infty}^{\infty} |\delta(t)|^2\, dt = \infty,$$

as also seen from Parseval's theorem applied formally to $F(\omega) \equiv 1$.

Definition 19 (Weight and shifted delta). *For any $\tau \in \mathbb{R}$, the shifted impulse $\delta(t - \tau)$ is the delta located at $t = \tau$. For any complex constant a, the impulse $a\,\delta(t - \tau)$ is said to have* weight *a. Its integral is*

$$\int_{-\infty}^{\infty} a\,\delta(t - \tau)\, dt = a. \tag{3.28}$$

The delta has the fundamental *sampling (sifting) property*.

Theorem 14 (Sampling property of the delta). *Let $\varphi(t)$ be continuous at $t = \tau$. Then*

$$\varphi(t)\,\delta(t - \tau) = \varphi(\tau)\,\delta(t - \tau) \tag{3.29}$$

as distributions, and

$$\int_{-\infty}^{\infty} \varphi(t)\,\delta(t - \tau)\, dt = \varphi(\tau). \tag{3.30}$$

The shifted delta has the transform

$$\delta(t-\tau) \longleftrightarrow e^{-i\omega\tau}, \tag{3.31}$$

obtained by direct integration:

$$\int_{-\infty}^{\infty} \delta(t-\tau)e^{-i\omega t}\,dt = e^{-i\omega\tau}.$$

As an important integral identity, one has

Theorem 15 (Integral representation of δ). *In the sense of distributions,*

$$\lim_{K\to\infty} \int_{-K}^{K} e^{i\omega t}\,d\omega = 2\pi\,\delta(t). \tag{3.32}$$

Proof. From Proposition 5, $\delta(t) \longleftrightarrow 1$. The inverse transform formula gives

$$\delta(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} 1\cdot e^{i\omega t}\,d\omega.$$

Interpreting the improper integral as the distributional limit

$$\int_{-\infty}^{\infty} e^{i\omega t}\,d\omega = \lim_{K\to\infty}\int_{-K}^{K} e^{i\omega t}\,d\omega,$$

we obtain

$$\delta(t) = \frac{1}{2\pi}\lim_{K\to\infty}\int_{-K}^{K} e^{i\omega t}\,d\omega,$$

which is equivalent to (3.32). □

This can be viewed as a continuous superposition of complex exponentials yielding a sharply localized impulse in time.

3.7 The Eternal Constant

Consider the constant signal

$$f(t) = V_0, \qquad t \in \mathbb{R}, \tag{3.33}$$

representing an ideal dc voltage or current. This is not an energy signal and must be treated as a distribution.

One may regard it as the limit of a rectangular pulse of height V_0 and width k as $k \to \infty$:

$$f(t) = \lim_{k\to\infty} V_0\,\mathrm{Rect}\left(\frac{t}{k}\right). \tag{3.34}$$

For fixed k,

$$F_k(\omega) = V_0\int_{-\infty}^{\infty} \mathrm{Rect}\left(\frac{t}{k}\right)e^{-i\omega t}\,dt = V_0\,k\,\mathrm{Sa}\left(\frac{\omega k}{2}\right).$$

Using Theorem 15 (with appropriate scaling), one finds that in the limit this converges to a single delta at the origin.

Proposition 6 (Fourier transform of a constant). *For the eternal constant $f(t) = V_0$,*

$$V_0 \longleftrightarrow F(\omega) = 2\pi V_0\,\delta(\omega). \tag{3.35}$$

Proof. From (3.33), linearity of the transform and Proposition 5 applied in the frequency domain yield a direct argument. Consider

$$\delta(\omega) \longleftrightarrow \frac{1}{2\pi},$$

which follows from (3.31) with $\tau = 0$ and the inversion formula. By linearity, for any constant V_0,

$$2\pi V_0\,\delta(\omega) \longleftrightarrow V_0,$$

which is (3.35). Equivalently, the limit of $V_0 k\,\mathrm{Sa}((\omega k)/2)$ as $k \to \infty$ is $2\pi V_0\delta(\omega)$, by Theorem 15. □

Thus all the energy of an ideal dc signal is concentrated at $\omega = 0$ in the frequency domain.

3.8 The Unit Step as a Distribution

The unit step $u(t)$ can be expressed in terms of $\mathrm{sgn}(t)$ and a constant:

$$u(t) = \frac{1}{2}\,\mathrm{sgn}(t) + \frac{1}{2}. \tag{3.36}$$

Using (3.23) and (3.35), we obtain directly:

Proposition 7 (Fourier transform of the unit step)**.** *As a distribution,*

$$u(t) \longleftrightarrow F(\omega) = \frac{1}{i\omega} + \pi\,\delta(\omega), \tag{3.37}$$

with the term $1/(i\omega)$ interpreted in the principal-value sense.

Proof. From (3.36),

$$u(t) = \frac{1}{2}\,\mathrm{sgn}(t) + \frac{1}{2}.$$

By linearity and (3.23)–(3.35),

$$\mathcal{F}\{u(t)\}(\omega) = \frac{1}{2}\cdot\frac{2}{i\omega} + \frac{1}{2}\cdot 2\pi\delta(\omega) = \frac{1}{i\omega} + \pi\delta(\omega),$$

with $1/(i\omega)$ interpreted as a principal value. This yields (3.37). □

The odd part of $u(t)$ is $\frac{1}{2}\,\mathrm{sgn}(t)$, which transforms to the purely imaginary odd component $\frac{1}{i\omega}$, and the even part is the constant $\frac{1}{2}$, which transforms to $\pi\delta(\omega)$.

3.9 The Eternal Complex Exponential

Consider the complex exponential of single angular frequency ω_0:

$$f(t) = e^{i\omega_0 t}, \qquad \omega_0 \in \mathbb{R}. \tag{3.38}$$

As an energy signal it diverges, so we again treat it as a distribution, for example as the limit

$$e^{i\omega_0 t} = \lim_{k\to\infty} e^{i\omega_0 t}\,\mathrm{Rect}\!\left(\frac{t}{k}\right). \tag{3.39}$$

For fixed k,

$$F_k(\omega) = \int_{-\infty}^{\infty} e^{i\omega_0 t}\mathrm{Rect}\!\left(\frac{t}{k}\right)e^{-i\omega t}\,dt = \int_{-k/2}^{k/2} e^{-i(\omega-\omega_0)t}\,dt = k\,\mathrm{Sa}\!\left(\frac{(\omega-\omega_0)k}{2}\right).$$

As $k \to \infty$, this converges to a delta at $\omega = \omega_0$.

Proposition 8 (Fourier transform of an eternal complex exponential). *For $f(t) = e^{i\omega_0 t}$,*

$$e^{i\omega_0 t} \longleftrightarrow F(\omega) = 2\pi\,\delta(\omega - \omega_0). \tag{3.40}$$

Proof. Using the expression for $F_k(\omega)$, the limiting distribution is

$$\lim_{k\to\infty} k\,\mathrm{Sa}\left(\frac{(\omega-\omega_0)k}{2}\right) = 2\pi\,\delta(\omega-\omega_0),$$

which is the scaled and shifted version of Theorem 15. □

Thus an ideal monochromatic complex exponential has its entire spectral content concentrated at a single frequency.

3.10 Eternal Cosine and Sine

The cosine and sine of angular frequency ω_0 can be expressed in terms of complex exponentials:

$$\cos(\omega_0 t) = \frac{1}{2}\left(e^{i\omega_0 t} + e^{-i\omega_0 t}\right), \tag{3.41}$$

$$\sin(\omega_0 t) = \frac{1}{2i}\left(e^{i\omega_0 t} - e^{-i\omega_0 t}\right). \tag{3.42}$$

Using (3.40), we obtain:

Proposition 9 (Fourier transforms of cosine and sine). *For $\omega_0 \in \mathbb{R}$,*

$$\cos(\omega_0 t) \longleftrightarrow F_{\cos}(\omega) = \pi\big[\delta(\omega-\omega_0) + \delta(\omega+\omega_0)\big], \tag{3.43}$$

$$\sin(\omega_0 t) \longleftrightarrow F_{\sin}(\omega) = \frac{\pi}{i}\big[\delta(\omega-\omega_0) - \delta(\omega+\omega_0)\big]. \tag{3.44}$$

Proof. From (3.41),

$$\cos(\omega_0 t) = \frac{1}{2}e^{i\omega_0 t} + \frac{1}{2}e^{-i\omega_0 t}.$$

Using linearity and (3.40),

$$\mathcal{F}\{\cos(\omega_0 t)\}(\omega) = \frac{1}{2}\cdot 2\pi\delta(\omega-\omega_0) + \frac{1}{2}\cdot 2\pi\delta(\omega+\omega_0) = \pi\big[\delta(\omega-\omega_0)+\delta(\omega+\omega_0)\big].$$

Similarly, from (3.42),

$$\sin(\omega_0 t) = \frac{1}{2i}e^{i\omega_0 t} - \frac{1}{2i}e^{-i\omega_0 t},$$

and hence

$$\mathcal{F}\{\sin(\omega_0 t)\}(\omega) = \frac{1}{2i}\cdot 2\pi\delta(\omega-\omega_0) - \frac{1}{2i}\cdot 2\pi\delta(\omega+\omega_0) = \frac{\pi}{i}\big[\delta(\omega-\omega_0)-\delta(\omega+\omega_0)\big].$$

□

The cosine is real and even; its spectrum is real and even, consisting of two impulses of equal weight π at $\omega = \pm\omega_0$. The sine is real and odd; its spectrum is purely imaginary and odd, with impulses of weight π/i at $\omega = \omega_0$ and $-\pi/i$ at $\omega = -\omega_0$.

3.11 Periodic Functions

Let $f_{\mathrm{p}}(t)$ be a real-valued function periodic with period $T_{\mathrm{p}} > 0$,

$$f_{\mathrm{p}}(t+T_{\mathrm{p}}) = f_{\mathrm{p}}(t), \tag{3.45}$$

and assume that f_{p} has finite energy in each period. Then f_{p} admits a complex Fourier series

$$f_{\mathrm{p}}(t) = \sum_{n=-\infty}^{\infty} F_{\mathrm{FS}}(n)\, e^{in\omega_0 t}, \qquad \omega_0 := \frac{2\pi}{T_{\mathrm{p}}}, \tag{3.46}$$

where the Fourier series coefficients are

$$F_{\mathrm{FS}}(n) = \frac{1}{T_{\mathrm{p}}} \int_{t_0}^{t_0+T_{\mathrm{p}}} f_{\mathrm{p}}(t)\, e^{-in\omega_0 t}\, dt, \tag{3.47}$$

for any $t_0 \in \mathbb{R}$.

Using (3.40) term-by-term, we can write the Fourier transform of f_{p} as a sum of impulses in frequency:

Theorem 16 (Fourier transform of a periodic function). *Let $f_{\mathrm{p}}(t)$ be periodic with period T_{p} and Fourier series coefficients $F_{\mathrm{FS}}(n)$. Then in the sense of distributions,*

$$f_{\mathrm{p}}(t) \longleftrightarrow F(\omega) = 2\pi \sum_{n=-\infty}^{\infty} F_{\mathrm{FS}}(n)\, \delta(\omega - n\omega_0), \qquad \omega_0 = \frac{2\pi}{T_{\mathrm{p}}}. \tag{3.48}$$

Proof. Starting from the Fourier series (3.46),

$$f_{\mathrm{p}}(t) = \sum_{n=-\infty}^{\infty} F_{\mathrm{FS}}(n)\, e^{in\omega_0 t},$$

and using linearity of the transform together with (3.40) for each harmonic,

$$\mathcal{F}\{e^{in\omega_0 t}\}(\omega) = 2\pi\, \delta(\omega - n\omega_0),$$

we obtain

$$F(\omega) = \mathcal{F}\{f_{\mathrm{p}}(t)\}(\omega) = \sum_{n=-\infty}^{\infty} F_{\mathrm{FS}}(n)\, \mathcal{F}\{e^{in\omega_0 t}\}(\omega) = 2\pi \sum_{n=-\infty}^{\infty} F_{\mathrm{FS}}(n)\, \delta(\omega - n\omega_0),$$

with convergence understood in the distribution sense. This is (3.48). □

Thus the Fourier transform of a periodic function consists of a discrete set of spectral lines (Dirac deltas) at integer multiples of the fundamental frequency ω_0, with weights proportional to the Fourier series coefficients.

3.12 The Periodic Impulse Train (Dirac Comb)

An especially important periodic generalized function is the impulse train (Dirac comb) with period T_{p}:

Definition 20 (Periodic impulse train). *For $T_{\mathrm{p}} > 0$, the periodic impulse train in time is*

$$\mathrm{Sha}_{T_{\mathrm{p}}}(t) := \sum_{k=-\infty}^{\infty} \delta(t - kT_{\mathrm{p}}). \tag{3.49}$$

This signal is periodic with period T_p. We now compute its Fourier series coefficients and transform.

Over one period $(-T_\mathrm{p}/2, T_\mathrm{p}/2)$, only the impulse at the origin contributes. Using the Fourier series analysis formula,

$$F_{\mathrm{FS}}(n) = \frac{1}{T_\mathrm{p}} \int_{-T_\mathrm{p}/2}^{T_\mathrm{p}/2} \mathrm{Sha}_{T_\mathrm{p}}(t)\, e^{-in\omega_0 t}\, dt = \frac{1}{T_\mathrm{p}} \int_{-T_\mathrm{p}/2}^{T_\mathrm{p}/2} \delta(t)\, e^{-in\omega_0 t}\, dt = \frac{1}{T_\mathrm{p}},$$

for all integers n. Thus

$$\mathrm{Sha}_{T_\mathrm{p}}(t) = \sum_{n=-\infty}^{\infty} \frac{1}{T_\mathrm{p}} e^{in\omega_0 t}, \qquad \omega_0 = \frac{2\pi}{T_\mathrm{p}}, \tag{3.50}$$

in the distributional sense.

Applying Theorem 16, we obtain:

Theorem 17 (Fourier transform of the Dirac comb). *The Fourier transform of the periodic impulse train* $\mathrm{Sha}_{T_\mathrm{p}}(t)$ *is*

$$\mathrm{Sha}_{T_\mathrm{p}}(t) \longleftrightarrow F(\omega) = 2\pi \sum_{n=-\infty}^{\infty} \frac{1}{T_\mathrm{p}} \delta(\omega - n\omega_0), \qquad \omega_0 = \frac{2\pi}{T_\mathrm{p}}. \tag{3.51}$$

Equivalently, defining the frequency-domain comb

$$\mathrm{Sha}_{\omega_0}(\omega) := \sum_{n=-\infty}^{\infty} \delta(\omega - n\omega_0), \tag{3.52}$$

we can write

$$\mathrm{Sha}_{T_\mathrm{p}}(t) \longleftrightarrow F(\omega) = \frac{2\pi}{T_\mathrm{p}} \mathrm{Sha}_{\omega_0}(\omega), \qquad \omega_0 = \frac{2\pi}{T_\mathrm{p}}. \tag{3.53}$$

Proof. From the preceding computation, the Fourier series coefficients of $\mathrm{Sha}_{T_\mathrm{p}}(t)$ are $F_{\mathrm{FS}}(n) = 1/T_\mathrm{p}$ for all integers n. Applying Theorem 16,

$$\mathcal{F}\{\mathrm{Sha}_{T_\mathrm{p}}(t)\}(\omega) = 2\pi \sum_{n=-\infty}^{\infty} F_{\mathrm{FS}}(n)\, \delta(\omega - n\omega_0) = 2\pi \sum_{n=-\infty}^{\infty} \frac{1}{T_\mathrm{p}} \delta(\omega - n\omega_0),$$

which is (3.51). Factoring out $2\pi/T_\mathrm{p}$ and using the definition of Sha_{ω_0} gives (3.53). □

Thus the Dirac comb is self-replicating under the Fourier transform: a comb in time transforms into a comb in frequency, with reciprocal spacing and appropriate scaling. This structure is central in the analysis of sampling and discrete-time representations, to be revisited later.

4 The Method of Successive Differentiation

4.1 The Differentiation Property

The Fourier transform is particularly useful because differentiation in time corresponds to multiplication in frequency. We first introduce the differential operator.

Definition 21 (Differential operator). *The operator D denotes differentiation with respect to time:*

$$Dx(t) = \frac{d}{dt}x(t), \qquad D^2x(t) = \frac{d^2}{dt^2}x(t),$$

and, in general,

$$D^n x(t) = \frac{d^n}{dt^n}x(t).$$

We now state the basic relation between time-domain differentiation and the Fourier transform.

Theorem 18 (Time-domain differentiation). *Let $f(t) \longleftrightarrow F(\omega)$. Then*

$$Df(t) \longleftrightarrow i\omega F(\omega). \tag{4.1}$$

Proof. Using the synthesis formula,

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)e^{i\omega t}\,d\omega. \tag{4.2}$$

Differentiating with respect to t,

$$\begin{aligned} Df(t) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)\frac{d}{dt}\left(e^{i\omega t}\right)d\omega \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)(i\omega)e^{i\omega t}\,d\omega, \end{aligned} \tag{4.3}$$

which is the synthesis formula for the Fourier transform $i\omega F(\omega)$. Hence $Df(t) \longleftrightarrow i\omega F(\omega)$. □

Repeated differentiation follows immediately.

Corollary 5. *If $f(t) \longleftrightarrow F(\omega)$, then for any integer $n \geq 1$,*

$$D^n f(t) \longleftrightarrow (i\omega)^n F(\omega). \tag{4.4}$$

Polynomial differential operators act in an analogous way.

Corollary 6. *Let*

$$P(D) = a_nD^n + a_{n-1}D^{n-1} + \cdots + a_1D + a_0, \tag{4.5}$$

and let $f(t) \longleftrightarrow F(\omega)$. Then

$$P(D)f(t) \longleftrightarrow P(i\omega)F(\omega). \tag{4.6}$$

4.2 Differentiating Functions with Discontinuities

Differentiation of functions with jump discontinuities requires generalized functions. The key building block is the derivative of the unit step.

Recall that the Dirac delta $\delta(t)$ satisfies

$$\int_{-\infty}^{t} \delta(\tau)\, d\tau = U(t), \tag{4.7}$$

where $U(t)$ is the unit step.

Proposition 10 (Derivative of the unit step)**.**

$$\frac{d}{dt} U(t) = \delta(t). \tag{4.8}$$

By time shift, the same relation holds for shifted steps:

$$\frac{d}{dt} U(t-\tau) = \delta(t-\tau), \tag{4.9}$$

for any real τ.

Thus, for a function $x(t)$ with jump discontinuities, differentiation produces impulses at each jump.

Proposition 11 (Derivative of a piecewise-smooth function)**.** *Let $x(t)$ be piecewise continuously differentiable and suppose that at $t = t_k$ it has a jump of height*

$$h_k = \lim_{t \to t_k^+} x(t) - \lim_{t \to t_k^-} x(t).$$

Then, in the sense of generalized functions,

$$Dx(t) = x'_{\mathrm{cont}}(t) + \sum_k h_k\, \delta(t - t_k), \tag{4.10}$$

where $x'_{\mathrm{cont}}(t)$ denotes the ordinary derivative in the intervals where x is continuous.

Combining Proposition 11 with Theorem 18 gives a systematic way to handle discontinuities in the time domain.

4.3 The Method of Successive Differentiation

For many signals of interest, especially piecewise-polynomial signals with finite support, repeated differentiation eventually yields a finite sum of impulses (and possibly derivatives of impulses). Once this derivative is known, the Fourier transform can be obtained by inspection in the frequency domain and then divided by an appropriate power of $i\omega$ to recover the original transform.

Definition 22 (Finite span)**.** *A signal $x(t)$ is said to have finite span if there exists a finite interval $[a, b]$ such that $x(t) = 0$ for all $t \notin [a, b]$.*

For signals with finite span and piecewise-polynomial structure, the method proceeds as follows:

1. Differentiate $x(t)$ repeatedly until the result is a finite combination of impulses (and possibly impulse derivatives).

2. Use known Fourier pairs to obtain the transform of the differentiated signal.
3. Divide by the corresponding power of $i\omega$ according to Corollary 5 to recover $X(\omega)$.

Example 3 (First derivative suffices). *Let $x(t)$ be a piecewise-constant signal whose derivative is*

$$Dx(t) = \delta(t) + \delta(t-1) - 2\delta(t-2). \tag{4.11}$$

Then, by Theorem 18 and the time-shift property,

$$i\omega X(\omega) = 1 + e^{-i\omega} - 2e^{-2i\omega}, \tag{4.12}$$

so

$$X(\omega) = \frac{1 + e^{-i\omega} - 2e^{-2i\omega}}{i\omega}. \tag{4.13}$$

This expression is obtained without explicit use of the analysis integral.

Example 4 (Second derivative required). *Consider a triangular pulse symmetric about the origin with support $[-1, 1]$. Its second derivative can be represented as*

$$D^2 x(t) = \delta(t+1) - 2\delta(t) + \delta(t-1). \tag{4.14}$$

Applying Theorem 18,

$$(i\omega)^2 X(\omega) = e^{i\omega} - 2 + e^{-i\omega}, \tag{4.15}$$

so

$$X(\omega) = \frac{e^{i\omega} - 2 + e^{-i\omega}}{(i\omega)^2}. \tag{4.16}$$

This reproduces the known transform of a triangular pulse.

Impulses themselves can be differentiated.

Definition 23 (Doublet). *The derivative of the Dirac delta,*

$$\delta'(t) = D\delta(t), \tag{4.17}$$

is called a doublet.

From Theorem 18,

$$\delta'(t) \longleftrightarrow i\omega. \tag{4.18}$$

4.4 A Complication: Signals Without Finite Span

When the signal does not have finite span, repeated differentiation may remove constant components, which correspond to impulses at $\omega = 0$ in the frequency domain. These must be restored at the end of the procedure.

We distinguish two classes.

Definition 24 (Class I and Class II signals). Class I*: Signals that are piecewise polynomial and have finite span.* Class II*: Signals that are piecewise polynomial but do not have finite span.*

For Class I signals, the method of successive differentiation applies directly, and no additional correction at $\omega = 0$ is required.

For Class II signals, the average value over the entire real line may be nonzero. Since a nonzero average corresponds to a $\delta(\omega)$ term in the frequency domain, differentiation, which annihilates constants, removes such a term. After using successive differentiation, one must add back the contribution corresponding to the average value.

Proposition 12 (Restoring the dc component). *Let $f(t)$ be a Class II signal with average value*

$$\bar{f} = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} f(t)\,dt.$$

Then the Fourier transform includes the term

$$2\pi\bar{f}\,\delta(\omega). \tag{4.19}$$

If successive differentiation has been used to determine the rest of the spectrum, the term $2\pi\bar{f}\,\delta(\omega)$ must be added to obtain the complete $F(\omega)$.

Example 5 (Rectangular pulse plus constant). *Let*

$$f(t) = \mathrm{Rect}(t) + C, \tag{4.20}$$

where $\mathrm{Rect}(t)$ has support on $[-1/2, 1/2]$. Then $f(t)$ is piecewise constant but does not have finite span because of the constant C. Differentiation gives

$$Df(t) = D\,\mathrm{Rect}(t), \tag{4.21}$$

so the constant C is removed. Using the method of successive differentiation yields the transform of $\mathrm{Rect}(t)$,

$$F_{\mathrm{Rect}}(\omega) = \mathrm{Sa}\left(\frac{\omega}{2}\right). \tag{4.22}$$

The average of $f(t)$ over $\mathbb{R}$ is $\bar{f} = C$. Hence the complete transform is

$$F(\omega) = \mathrm{Sa}\left(\frac{\omega}{2}\right) + 2\pi C\,\delta(\omega). \tag{4.23}$$

Example 6 ($\mathrm{sgn}(t)$). *The signum function*

$$\mathrm{sgn}(t) = \begin{cases} -1, & t < 0, \\ 0, & t = 0, \\ +1, & t > 0, \end{cases} \tag{4.24}$$

is a Class II signal. Its derivative is

$$D\,\mathrm{sgn}(t) = 2\delta(t), \tag{4.25}$$

so

$$i\omega F(\omega) = 2, \tag{4.26}$$

and

$$F(\omega) = \frac{2}{i\omega}. \tag{4.27}$$

The average of $\mathrm{sgn}(t)$ over $\mathbb{R}$ is zero, so no $\delta(\omega)$ term is needed.

Example 7 (Unit step $U(t)$). *The unit step*

$$U(t) = \begin{cases} 0, & t < 0, \\ \frac{1}{2}, & t = 0, \\ 1, & t > 0, \end{cases} \tag{4.28}$$

is also a Class II signal. Its derivative is

$$DU(t) = \delta(t), \tag{4.29}$$

so

$$i\omega F(\omega) = 1, \tag{4.30}$$

and

$$F(\omega) = \frac{1}{i\omega}. \tag{4.31}$$

The average of $U(t)$ over $\mathbb{R}$ is $1/2$, hence the complete transform is

$$U(t) \longleftrightarrow \frac{1}{i\omega} + \pi\,\delta(\omega). \tag{4.32}$$

4.5 Nonpolynomial Sections

The method of successive differentiation extends to many signals that are not piecewise polynomial, provided that differentiation produces expressions whose Fourier transforms are known or easily identified.

For example, the one-sided decaying exponential

$$f(t) = e^{-\alpha t}U(t), \qquad \alpha > 0, \tag{4.33}$$

has the well-known transform

$$f(t) \longleftrightarrow \frac{1}{\alpha + i\omega}, \tag{4.34}$$

which is consistent with the differentiation property and the unit-step transform. Similar reasoning applies to gated sinusoids such as $\cos(\omega_0 t)\,\mathrm{Rect}(t/T)$ and related signals, where differentiation combined with time shifts and modulation can simplify the derivation of their Fourier transforms.

5 Frequency-Domain Analysis

5.1 Introduction

Perhaps the most important contribution that Fourier analysis makes to engineering, and especially to electrical engineering, is that it enables us to decompose input signals into their frequency components. Once this has been done, it becomes possible to understand how filters and other linear networks act on such signals, frequency by frequency, and to design them so that the resulting outputs have desired attributes.

In this chapter we consider three applications of Fourier analysis in the frequency domain:

1. The response of an electrical network to a pulse.
2. The response of a network to an ac input such as $\sin(\omega_0 t)$ or $\cos(\omega_0 t)$.
3. The response of a network to an arbitrary periodic signal.

The second application, usually referred to as ac linear circuit analysis, is one of the most frequently used analytical tools in electrical engineering (see, e.g., Edminster, 1972). Often it is taught in an algorithmic way at an early stage of study: students are shown that if they replace inductors and capacitors in a network by the quantities $i\omega L$ and $1/(i\omega C)$, respectively (sometimes termed “complex resistors”), and if they take as the input to the network the quantity $e^{i\omega_0 t}$ (a *phasor*), then the response may be computed directly using Kirchhoff’s voltage and current laws. Although this procedure may initially appear somewhat mysterious, its effectiveness leads to quick acceptance and the underlying justification is often not emphasized.

Historically, the method was first used by Oliver Heaviside in 1878 (Nahin, 1987). It can be derived via his operational calculus, but we shall avoid that route here. Instead, we show how it arises directly from Fourier analysis. In many introductory courses this connection is not made explicit because students typically lack the required background in Fourier transforms. Since we now have that background, one of the main objectives of this chapter is to show clearly how the techniques of ac circuit analysis are obtained from Fourier analysis and why they are valid. This provides a natural setting in which to apply the machinery developed in previous chapters.

Although the examples in this chapter are drawn from electrical networks, the methods are applicable to a wide range of systems composed of other types of components—mechanical, acoustical, hydraulic, and many others. More generally, the forthcoming results apply to any system whose behavior can be modeled by a constant-coefficient linear differential equation.

5.2 Response of a Linear, Time-Invariant System to a Pulse Function

We focus on systems composed of electrical components and, more specifically, on linear systems whose element values do not change with time. Such systems are called *linear, time-invariant* (LTI) systems.

Definition 25 (LTI system and CCL DE). *An LTI system is one that can be characterized by a* constant-coefficient linear differential equation *(CCL DE). The LTI system and its associated CCL DE are mathematically equivalent, and we shall use these descriptions interchangeably.*

The effort required to derive the differential equation of an electrical network of nontrivial complexity can be substantial. One of the main advantages of the frequency-domain approach that we develop in this chapter is that, in practice, we will often not need to write down the differential equation explicitly. Instead, we will derive an operational method that allows us to obtain the response without first forming the differential equation. If desired, the differential equation can then be reconstructed directly from the frequency-domain characterization.

To begin, we examine how the Fourier transform may be used to solve CCL DEs and, in particular, to determine the response of an LTI network to various pulse inputs.

5.2.1 General form of a constant-coefficient linear differential equation

The general form of a CCL DE describing an LTI system is

$$P_1(D)\,y(t) = P_2(D)\,x(t), \tag{5.1}$$

where

- $P_1(D)$ and $P_2(D)$ are polynomials in the operator $D = \dfrac{\mathrm{d}}{\mathrm{d}t}$ with constant coefficients,
- $x(t)$ is the input or forcing function,
- $y(t)$ is the output (response, or solution).

An example is

$$(5D^2 + 6D + 1)\,y(t) = (3D + 2)\,x(t). \tag{5.2}$$

The coefficients of the two polynomials are constants and the equation is linear, so (5.2) is indeed a CCL DE.

Two points are important:

1. Equation (5.2) is assumed to hold for all t; that is, the solution $y(t)$ must satisfy the equation for all t in $(-\infty, \infty)$.
2. No initial conditions are specified; that is, no constraints such as $y(0) = 1$ or $y'(5) = 2$ are imposed.

These two facts always apply, either explicitly or implicitly, when using the Fourier transform to solve CCL DEs.

From a system viewpoint, an LTI system/CCL DE can be represented by the block diagram

$$x(t) \longrightarrow \boxed{\text{LTI system/CCL DE}} \longrightarrow y(t)$$

To *solve* the CCL DE means: given $x(t)$, find $y(t)$ so that (5.1) is satisfied for all t. Thus, solving the CCL DE associated with an LTI system is equivalent to finding the response of the system to the given input.

5.2.2 Fourier transform of a CCL DE

We now solve (5.2) using the differentiation property of the Fourier transform. Taking the Fourier transform of both sides of (5.2) yields

$$\int_{-\infty}^{\infty} (5D^2 + 6D + 1)\,y(t)\,e^{-i\omega t}\,\mathrm{d}t = \int_{-\infty}^{\infty} (3D + 2)\,x(t)\,e^{-i\omega t}\,\mathrm{d}t. \tag{5.3}$$

The integration over $(-\infty, \infty)$ is consistent with the assumption that the differential equation holds for all t.

Using the differentiation property of the Fourier transform, i.e. $D^k x(t) \longleftrightarrow (i\omega)^k X(\omega)$, and denoting

$$x(t) \longleftrightarrow X(\omega), \qquad y(t) \longleftrightarrow Y(\omega),$$

we obtain

$$\big[5(i\omega)^2 + 6(i\omega) + 1\big] Y(\omega) = \big[3(i\omega) + 2\big] X(\omega). \tag{5.4}$$

This is a purely algebraic equation in the frequency domain. More generally, we have the following result.

Theorem 19. *Fourier transformation of the time-domain CCL DE*

$$P_1(D)\, y(t) = P_2(D)\, x(t) \tag{5.5}$$

converts it to the frequency-domain algebraic equation

$$P_1(i\omega) Y(\omega) = P_2(i\omega) X(\omega), \tag{5.6}$$

where $x(t) \longleftrightarrow X(\omega)$ and $y(t) \longleftrightarrow Y(\omega)$.

Solving (5.4) for $Y(\omega)$ gives

$$Y(\omega) = \frac{3\, i\omega + 2}{5(i\omega)^2 + 6\, i\omega + 1} X(\omega). \tag{5.7}$$

Two observations are useful:

1. The right-hand side of (5.7) is the product of two factors: one depending only on the differential equation, and one depending only on the input $x(t)$.
2. The expression is valid for any $x(t)$ possessing a Fourier transform; once $X(\omega)$ is known, (5.7) yields $Y(\omega)$, and hence $y(t)$ by inversion.

5.2.3 Frequency response (transfer function)

Equation (5.7) suggests representing the LTI system in the frequency domain, where the system is characterized by a multiplicative function of frequency.

We define the frequency response (or frequency transfer function) as follows.

Definition 26 (Frequency response / frequency transfer function)**.** *Let the CCL DE for an LTI system be*

$$P_1(D)\, y(t) = P_2(D)\, x(t). \tag{5.8}$$

The frequency response *or* frequency transfer function *of the system is*

$$H(i\omega) = \frac{P_2(i\omega)}{P_1(i\omega)}. \tag{5.9}$$

For the example (5.2), we have

$$H(i\omega) = \left.\frac{3D + 2}{5D^2 + 6D + 1}\right|_{D \mapsto i\omega} = \frac{3\, i\omega + 2}{5(i\omega)^2 + 6\, i\omega + 1}. \tag{5.10}$$

The function $H(i\omega)$ encapsulates all information about the LTI system: the component values and their interconnection are reflected in its numerator and denominator. It is a frequency-domain quantity. In the next chapter, we will introduce a corresponding time-domain quantity, the *impulse response* $h(t)$, which also completely characterizes the system. As is well known, $H(i\omega)$ and $h(t)$ form a Fourier-transform pair.

With Definition 26, equation (5.7) may be written in the compact form

$$Y(\omega) = H(i\omega)X(\omega), \tag{5.11}$$

and, by the synthesis formula,

$$y(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} H(i\omega)X(\omega)\,e^{i\omega t}\,\mathrm{d}\omega. \tag{5.12}$$

Equations (5.11) and (5.12) constitute a central result in the analysis of LTI systems.

Theorem 20. *Let the pulse $x(t) \longleftrightarrow X(\omega)$ be the input to an LTI system whose frequency transfer function is $H(i\omega)$. Then:*

1. *In the frequency domain, the output transform is*

$$Y(\omega) = H(i\omega)X(\omega). \tag{5.13}$$

2. *In the time domain, the output is given by*

$$y(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} H(i\omega)X(\omega)\,e^{i\omega t}\,\mathrm{d}\omega. \tag{5.14}$$

It is instructive to compare (5.14) with the Fourier integral representation of the input,

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)\,e^{i\omega t}\,\mathrm{d}\omega. \tag{5.15}$$

The output representation (5.14) differs from (5.15) only by the factor $H(i\omega)$ under the integral: the system acts in the frequency domain by pointwise multiplication of the input spectrum.

5.2.4 Magnitude and phase of the output spectrum

We now illustrate these ideas with two examples.

Example 8. *Assume that the frequency response of a network is*

$$H(i\omega) = \frac{1}{2+i\omega}. \tag{5.16}$$

Find the magnitude and phase spectra of the output when the input is

$$x(t) = e^{-t}u(t), \tag{5.17}$$

where $u(t)$ is the unit step.

Solution. *From* (5.16),

$$H(i\omega) = \frac{1}{2+i\omega} = \frac{1}{\sqrt{4+\omega^2}} \exp\!\Big(-i \tan^{-1}\frac{\omega}{2}\Big), \tag{5.18}$$

so the magnitude and phase spectra of the system are

$$\big|H(i\omega)\big| = \frac{1}{\sqrt{4+\omega^2}}, \qquad \phi_H(\omega) = -\tan^{-1}\frac{\omega}{2}. \tag{5.19}$$

From the standard transform pair

$$e^{-t}u(t) \longleftrightarrow \frac{1}{1+i\omega},$$

we obtain

$$X(\omega) = \frac{1}{1+i\omega} = \frac{1}{\sqrt{1+\omega^2}} \exp\!\big(-i \tan^{-1}\omega\big), \tag{5.20}$$

so

$$\big|X(\omega)\big| = \frac{1}{\sqrt{1+\omega^2}}, \qquad \phi_X(\omega) = -\tan^{-1}\omega. \tag{5.21}$$

Using Theorem 20, the output transform is

$$Y(\omega) = H(i\omega)X(\omega) = \big|H(i\omega)\big|\big|X(\omega)\big| \exp\!\big(i[\phi_H(\omega)+\phi_X(\omega)]\big). \tag{5.22}$$

Therefore,

$$\big|Y(\omega)\big| = \big|H(i\omega)\big|\big|X(\omega)\big| = \frac{1}{\sqrt{4+\omega^2}}\,\frac{1}{\sqrt{1+\omega^2}}, \tag{5.23}$$

and

$$\phi_Y(\omega) = \phi_H(\omega) + \phi_X(\omega) = -\tan^{-1}\frac{\omega}{2} - \tan^{-1}\omega. \tag{5.24}$$

Observe that in Example 8:

- The magnitude spectra multiply,
- The phase spectra add.

This is a general property of LTI systems.

Theorem 21. *Let the signal $x(t) \longleftrightarrow X(\omega)$ be the input to an LTI network whose frequency response is $H(i\omega)$, and let $y(t) \longleftrightarrow Y(\omega)$ be the output. Then*

$$\big|Y(\omega)\big| = \big|H(i\omega)\big|\,\big|X(\omega)\big|, \tag{5.25}$$

and

$$\phi_Y(\omega) = \phi_H(\omega) + \phi_X(\omega), \tag{5.26}$$

where ϕ_H, ϕ_X, and ϕ_Y denote the corresponding phase spectra.

In other words, an LTI network acts as a frequency-selective filter: it modifies the magnitude and phase of each frequency component of the input according to $H(i\omega)$.

Example 9. *Consider the electrical network characterized by the CCL DE*

$$12\,y''(t) + 7\,y'(t) + y(t) = 2\,x'(t) + x(t). \tag{5.27}$$

(1) Find the frequency response $H(i\omega)$.

(2) Find the magnitude and phase spectra of $y(t)$ when $x(t) = e^{-t}u(t)$.

(3) Find the explicit time-domain expression for $y(t)$.

(4) Verify that your result in (3) satisfies (5.27).

Solution. (1) Frequency response. *Using the operator* $D = \dfrac{\mathrm{d}}{\mathrm{d}t}$, (5.27) *can be written as*

$$(12D^2 + 7D + 1)\, y(t) = (2D + 1)\, x(t). \tag{5.28}$$

Hence, by Definition 26,

$$H(i\omega) = \frac{2D + 1}{12D^2 + 7D + 1}\bigg|_{D \mapsto i\omega} = \frac{2\, i\omega + 1}{12(i\omega)^2 + 7\, i\omega + 1}. \tag{5.29}$$

(2) Magnitude and phase spectra. *We first rewrite the denominator of* (5.29)*:*

$$12(i\omega)^2 + 7\, i\omega + 1 = -12\omega^2 + 7\, i\omega + 1.$$

Thus

$$H(i\omega) = \frac{2\, i\omega + 1}{1 - 12\omega^2 + 7\, i\omega}. \tag{5.30}$$

The magnitude and phase are obtained as the ratio and difference of the corresponding quantities for numerator and denominator:

$$\big|H(i\omega)\big| = \frac{\sqrt{1 + 4\omega^2}}{\sqrt{(1 - 12\omega^2)^2 + 49\omega^2}}, \tag{5.31}$$

$$\phi_H(\omega) = \tan^{-1}(2\omega) - \tan^{-1}\Big(\frac{7\omega}{1 - 12\omega^2}\Big). \tag{5.32}$$

For the input $x(t) = e^{-t}u(t)$*, we have*

$$x(t) = e^{-t}u(t) \longleftrightarrow X(\omega) = \frac{1}{1 + i\omega},$$

so

$$\big|X(\omega)\big| = \frac{1}{\sqrt{1 + \omega^2}}, \qquad \phi_X(\omega) = -\tan^{-1}\omega. \tag{5.33}$$

By Theorem 21, the output magnitude and phase spectra are

$$\big|Y(\omega)\big| = \big|H(i\omega)\big|\big|X(\omega)\big| = \frac{\sqrt{1 + 4\omega^2}}{\sqrt{(1 - 12\omega^2)^2 + 49\omega^2}}\, \frac{1}{\sqrt{1 + \omega^2}}, \tag{5.34}$$

$$\phi_Y(\omega) = \phi_H(\omega) + \phi_X(\omega) = \tan^{-1}(2\omega) - \tan^{-1}\Big(\frac{7\omega}{1 - 12\omega^2}\Big) - \tan^{-1}\omega. \tag{5.35}$$

(3) Time-domain expression for $y(t)$. *The output transform is*

$$Y(\omega) = H(i\omega)X(\omega) = \frac{2\, i\omega + 1}{12(i\omega)^2 + 7\, i\omega + 1}\, \frac{1}{1 + i\omega}. \tag{5.36}$$

We perform a partial fraction expansion of (5.36) *in the form*

$$Y(\omega) = \frac{A}{i\omega + \alpha} + \frac{B}{i\omega + \beta} + \frac{C}{i\omega + \gamma}, \tag{5.37}$$

where α, β, γ *are the poles of the denominator and* A, B, C *are constants determined algebraically. Inversion term by term then yields*

$$y(t) = \big[(2/3)\, e^{-t/4} - (1/2)\, e^{-t/3} - (1/6)\, e^{-t}\big]u(t). \tag{5.38}$$

(4) Verification. *Differentiating* (5.38) *and substituting into* (5.27) *shows that*

$$12\, y''(t) + 7\, y'(t) + y(t) = 2\, x'(t) + x(t)$$

for all t*, with* $x(t) = e^{-t}u(t)$*, confirming that* (5.38) *is indeed a solution of the CCL DE.*

6 Time-Domain Analysis

6.1 Introduction

Fourier analysis originated in the study of heat conduction and the solution of partial differential equations (PDEs). In its modern usage for electrical engineering, several shifts in emphasis have taken place:

- In Fourier's original work, a Fourier series was not required to represent a periodic function; it was viewed as an expansion of an initial temperature profile on a finite rod. In electrical engineering, the inherent periodicity of Fourier series makes them natural tools for representing periodic waveforms.
- The functions appearing in Fourier integrals were originally viewed as initial temperature distributions on an infinite rod, i.e. functions of the spatial variable x. In signal analysis, they are interpreted as one-time pulses $x(t)$, not necessarily of infinite duration.
- The inversion formula
$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)\, e^{i\omega t}\, d\omega \tag{6.1}$$
did not play a central role in Fourier's own applications. In the analysis of linear, time-invariant (LTI) systems, however, such inversion formulas are fundamental: the application of Fourier analysis to LTI networks depended crucially on this relationship.

Historically, Fourier's methods were adapted to electrical engineering problems through the work of several authors, including Kelvin and Heaviside. Kelvin applied Fourier techniques to the telegraph equation, focusing on transmission lines rather than lumped networks. Heaviside, in particular, made systematic use of operational methods for electrical circuits and is credited with the development of the classical ac circuit analysis framework, in which inductors and capacitors are represented by $i\omega L$ and $1/(i\omega C)$, respectively. The function now commonly called the unit step is still referred to as the Heaviside function in some contexts.

A key observation is that Fourier series and Fourier integrals resolve signals into sums or superpositions of complex exponentials. For LTI systems, complex exponentials are eigenfunctions, and this makes Fourier methods natural tools for studying network responses.

6.2 The Impulse Response

In Chapter 5 the analysis of LTI systems was carried out primarily in the frequency domain. For an LTI system with input $x(t)$, output $y(t)$, and frequency response (or transfer function) $H(i\omega)$, the central result was
$$Y(\omega) = H(i\omega)X(\omega), \tag{6.2}$$
where
$$x(t) \longleftrightarrow X(\omega), \qquad y(t) \longleftrightarrow Y(\omega).$$

The goal here is to interpret (6.2) purely in the time domain. The transform pairs for $x(t)$ and $y(t)$ are clear; it remains to identify the time-domain quantity corresponding to $H(i\omega)$.

Problem 1. *What is the time-domain counterpart of $H(i\omega)$ in (6.2)?*

Let the inverse Fourier transform of $H(i\omega)$ be denoted by $h(t)$. That is,

$$H(i\omega) \longleftrightarrow h(t), \tag{6.3}$$

in the sense that

$$h(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} H(i\omega)\, e^{i\omega t}\, d\omega. \tag{6.4}$$

Consider now an LTI system with frequency response $H(i\omega)$, and apply as input the Dirac delta

$$x(t) = \delta(t). \tag{6.5}$$

Since $\delta(t) \longleftrightarrow 1$, we have $X(\omega) = 1$, and (6.2) reduces to

$$Y(\omega) = H(i\omega). \tag{6.6}$$

By (6.3), the time-domain counterpart of $Y(\omega)$ in this case is precisely $h(t)$:

$$y(t) = h(t). \tag{6.7}$$

Thus, when the input to the LTI system is a unit impulse, the output is the inverse Fourier transform of $H(i\omega)$.

Theorem 22 (Impulse response). *Let $H(i\omega)$ be the frequency response of an LTI system. Its inverse Fourier transform*

$$h(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} H(i\omega)\, e^{i\omega t}\, d\omega \tag{6.8}$$

is the output of the system when the input is the unit impulse $\delta(t)$.

Definition 27 (Impulse response). *The function $h(t)$ defined by* (6.8) *is called the* impulse response *of the LTI system.*

Example 10. *Consider two first-order LTI systems with frequency responses*

$$H_1(i\omega) = \frac{B}{B + i\omega}, \qquad B > 0, \tag{6.9}$$

$$H_2(i\omega) = 1 - \frac{B}{B + i\omega} = \frac{i\omega}{B + i\omega}, \qquad B > 0. \tag{6.10}$$

Find their impulse responses.

- *For $H_1(i\omega)$, we recognize the standard transform pair*

  $$Be^{-Bt}u(t) \longleftrightarrow \frac{B}{B + i\omega}.$$

  Hence

  $$h_1(t) = Be^{-Bt}u(t). \tag{6.11}$$

- *For $H_2(i\omega)$,*

  $$H_2(i\omega) = 1 - \frac{B}{B + i\omega}$$

  and using linearity together with $\delta(t) \longleftrightarrow 1$, we obtain

  $$h_2(t) = \delta(t) - Be^{-Bt}u(t). \tag{6.12}$$

In both cases the time-domain impulse response follows directly by inverting $H(i\omega)$.

6.3 Convolution

Consider an LTI system with input $x(t)$, output $y(t)$, and impulse response $h(t)$, as indicated conceptually by

$$x(t) \longrightarrow \boxed{\text{LTI system}} \longrightarrow y(t),$$

with $h(t)$ defined as in Theorem 22. The goal is to obtain a time-domain formula expressing $y(t)$ in terms of $x(t)$ and $h(t)$.

Problem 2. *Given the input $x(t)$ and the impulse response $h(t)$ of an LTI system, what is the time-domain expression for the output $y(t)$?*

From frequency-domain analysis we already know that

$$Y(\omega) = H(i\omega)X(\omega), \tag{6.13}$$

with

$$H(i\omega) \longleftrightarrow h(t). \tag{6.14}$$

Thus $y(t)$ is the inverse Fourier transform of $H(i\omega)X(\omega)$:

$$y(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} H(i\omega)X(\omega)\,e^{i\omega t}\,d\omega. \tag{6.15}$$

The convolution theorem identifies a purely time-domain expression for this inverse transform.

Theorem 23 (Time-domain convolution theorem)**.** *Let $x(t) \longleftrightarrow X(\omega)$ and $h(t) \longleftrightarrow H(i\omega)$. Then*

$$\int_{-\infty}^{\infty} x(\tau)h(t-\tau)\,d\tau \longleftrightarrow X(\omega)H(i\omega). \tag{6.16}$$

Equivalently,

$$\mathcal{F}\{x * h\}(\omega) = X(\omega)H(i\omega), \tag{6.17}$$

where

$$(x * h)(t) := \int_{-\infty}^{\infty} x(\tau)h(t-\tau)\,d\tau. \tag{6.18}$$

Definition 28 (Convolution)**.** *Given two functions $x(t)$ and $h(t)$, their* convolution *is the function*

$$(x * h)(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)\,d\tau. \tag{6.19}$$

The same operation is sometimes written $x(t) * h(t)$ for emphasis.

Combining Theorem 23 with (6.13), we obtain the time-domain input–output relation for LTI systems.

Theorem 24 (Impulse response representation of an LTI system)**.** *Let an LTI system have impulse response $h(t)$, and let $x(t)$ be the input and $y(t)$ the output. Then*

$$Y(\omega) = H(i\omega)X(\omega), \tag{6.20}$$

$$y(t) = (x * h)(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)\,d\tau. \tag{6.21}$$

Thus, multiplication in the frequency domain corresponds to convolution in the time domain:

$$x * h \longleftrightarrow X(\omega)H(i\omega). \tag{6.22}$$

Convolution therefore provides a complete time-domain alternative to the frequency-domain techniques for computing the response of an LTI system to a pulse input.

6.4 Interpretation of the Convolution Product

The formula

$$y(t) = \int_{-\infty}^{\infty} x(\tau) h(t-\tau)\, d\tau \tag{6.23}$$

encapsulates the response of an LTI system to an arbitrary input $x(t)$ in terms of its impulse response $h(t)$. It is useful to interpret this expression in terms of superposition of scaled and shifted impulse responses.

Formally, any (sufficiently regular) pulse $x(t)$ can be represented as a superposition of weighted and shifted Dirac deltas:

$$x(t) = \int_{-\infty}^{\infty} x(\tau) \delta(t-\tau)\, d\tau. \tag{6.24}$$

This identity follows from the sampling property of the delta function:

$$\int_{-\infty}^{\infty} x(\tau) \delta(t-\tau)\, d\tau = x(t). \tag{6.25}$$

Theorem 25 (Representation by impulses)**.** *Any suitable pulse $x(t)$ can be written as a "continuous linear combination" of shifted Dirac deltas:*

$$x(t) = \int_{-\infty}^{\infty} x(\tau) \delta(t-\tau)\, d\tau, \tag{6.26}$$

where $x(\tau)$ acts as the coefficient function.

Now consider an LTI system with impulse response $h(t)$. By linearity and time invariance:

- The response to $\delta(t)$ is $h(t)$.
- The response to $\delta(t-\tau)$ is $h(t-\tau)$ (time shift).
- The response to $x(\tau)\delta(t-\tau)$ is $x(\tau)h(t-\tau)$ (scaling by $x(\tau)$).

Using the representation (6.24), the input can be viewed as a superposition of such weighted impulses. By superposition of responses,

$$y(t) = \int_{-\infty}^{\infty} x(\tau) h(t-\tau)\, d\tau, \tag{6.27}$$

which is precisely the convolution expression (6.23).

Thus, convolution can be interpreted as a superposition of all shifted impulse responses, each weighted by the value of the input at the corresponding shift. Equivalently, The input $x(t)$ is a "train" of weighted impulses; the output $y(t)$ is the corresponding train of weighted, shifted impulse responses, summed over all shifts.

This viewpoint is consistent with the eigenfunction interpretation. There, the input was expanded in complex exponentials and the output was obtained as a superposition of eigen-responses. Here, the expansion is in terms of Dirac deltas rather than exponentials, but the linear superposition principle is the same.

6.5 Convolution Symmetry

The convolution product is symmetric in its arguments.

Theorem 26 (Convolution symmetry). *For suitable functions $x(t)$ and $h(t)$,*

$$\int_{-\infty}^{\infty} x(\tau)\,h(t-\tau)\,d\tau = \int_{-\infty}^{\infty} x(t-\tau)\,h(\tau)\,d\tau. \tag{6.28}$$

Proof. Let $z = t-\tau$, so that $\tau = t-z$ and $d\tau = -dz$. Then

$$\begin{aligned}\int_{-\infty}^{\infty} x(\tau)\,h(t-\tau)\,d\tau &= \int_{-\infty}^{\infty} x(t-z)\,h(z)\,(-dz)\\ &= \int_{-\infty}^{\infty} x(t-z)\,h(z)\,dz\\ &= \int_{-\infty}^{\infty} x(t-\tau)\,h(\tau)\,d\tau.\end{aligned}$$

□

Because of this symmetry, one may write

$$(x*h)(t) = \int_{-\infty}^{\infty} x(\tau)\,h(t-\tau)\,d\tau = \int_{-\infty}^{\infty} x(t-\tau)\,h(\tau)\,d\tau, \tag{6.29}$$

and use whichever representation is more convenient in a given context, including graphical convolution constructions.

6.6 Convolution in the Frequency Domain

The convolution theorem has a dual counterpart: whereas convolution in time corresponds to multiplication in frequency, multiplication in time corresponds to convolution in frequency.

Theorem 27 (Frequency-domain convolution theorem). *Let $x(t) \longleftrightarrow X(\omega)$ and $g(t) \longleftrightarrow G(\omega)$. Then*

$$x(t)\,g(t) \;\longleftrightarrow\; \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\Omega)\,G(\omega-\Omega)\,d\Omega. \tag{6.30}$$

Equivalently,

$$\mathcal{F}\{xg\}(\omega) = \frac{1}{2\pi}(X*G)(\omega), \tag{6.31}$$

where the convolution in frequency is defined by

$$(X*G)(\omega) = \int_{-\infty}^{\infty} X(\Omega)\,G(\omega-\Omega)\,d\Omega. \tag{6.32}$$

Definition 29 (Frequency-domain convolution). *Given frequency-domain functions $X(\omega)$ and $G(\omega)$, their convolution is*

$$(X*G)(\omega) = \int_{-\infty}^{\infty} X(\Omega)\,G(\omega-\Omega)\,d\Omega = \int_{-\infty}^{\infty} G(\Omega)\,X(\omega-\Omega)\,d\Omega. \tag{6.33}$$

Proof. Start from the right-hand side of (6.30) and take its inverse Fourier transform:

$$\begin{aligned}\frac{1}{2\pi}\int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} X(\Omega)\,G(\omega-\Omega)\,d\Omega\right]e^{i\omega t}\,d\omega \\ = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\Omega)\left[\int_{-\infty}^{\infty} G(\omega-\Omega)e^{i\omega t}\,d\omega\right]d\Omega.\end{aligned} \tag{6.34}$$

Let $\sigma = \omega-\Omega$, so $\omega = \sigma+\Omega$ and $d\omega = d\sigma$. Then

$$\begin{aligned}\int_{-\infty}^{\infty} G(\omega-\Omega)\,e^{i\omega t}\,d\omega &= \int_{-\infty}^{\infty} G(\sigma)\,e^{i(\sigma+\Omega)t}\,d\sigma \\ &= e^{i\Omega t}\int_{-\infty}^{\infty} G(\sigma)\,e^{i\sigma t}\,d\sigma \\ &= 2\pi\,g(t)\,e^{i\Omega t},\end{aligned} \tag{6.35}$$

by inversion of $G(\omega)$. Substituting back, we obtain

$$\begin{aligned}\frac{1}{2\pi}\int_{-\infty}^{\infty} X(\Omega)[2\pi g(t)e^{i\Omega t}]\,d\Omega = g(t)\int_{-\infty}^{\infty} X(\Omega)e^{i\Omega t}\,d\Omega \\ = g(t)\cdot 2\pi\,x(t)\cdot\frac{1}{2\pi} \\ = x(t)g(t),\end{aligned} \tag{6.36}$$

as claimed. □

A basic and important application is modulation. If

$$y(t) = x(t)\cos(\omega_0 t), \tag{6.37}$$

then, using the transform pair

$$\cos(\omega_0 t) \longleftrightarrow \pi\big[\delta(\omega-\omega_0)+\delta(\omega+\omega_0)\big], \tag{6.38}$$

the convolution in frequency yields

$$Y(\omega) = \frac{1}{2}X(\omega-\omega_0)+\frac{1}{2}X(\omega+\omega_0), \tag{6.39}$$

i.e. the baseband spectrum $X(\omega)$ is shifted to $\pm\omega_0$ and scaled. This underlies standard amplitude modulation schemes.

In summary:

- Convolution in time corresponds to multiplication in frequency:

$$x * h \longleftrightarrow X(\omega)H(i\omega).$$

- Multiplication in time corresponds to convolution in frequency:

$$xg \longleftrightarrow \frac{1}{2\pi}(X*G)(\omega).$$

These dual relationships are central to the use of Fourier analysis in time-domain and frequency-domain system analysis.

7 The Properties of the Fourier Transform

In this chapter we collect the standard properties of the continuous-time Fourier transform and provide proofs for each. Throughout, we use the following convention:

$$F(\omega) = \int_{-\infty}^{\infty} f(t)\, e^{-i\omega t}\, dt, \tag{7.1}$$

$$f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega)\, e^{i\omega t}\, d\omega. \tag{7.2}$$

We write

$$f(t) \longleftrightarrow F(\omega)$$

to denote such a Fourier-transform pair.

7.1 Linearity

Theorem 28 (Linearity). *Let $f_1(t) \leftrightarrow F_1(\omega)$ and $f_2(t) \leftrightarrow F_2(\omega)$. For constants $c_1, c_2 \in \mathbb{C}$, define*

$$f(t) = c_1 f_1(t) + c_2 f_2(t).$$

Then

$$f(t) \longleftrightarrow F(\omega) = c_1 F_1(\omega) + c_2 F_2(\omega). \tag{7.3}$$

Proof. By definition,

$$\begin{aligned} F(\omega) &= \int_{-\infty}^{\infty} f(t)\, e^{-i\omega t}\, dt = \int_{-\infty}^{\infty} \big(c_1 f_1(t) + c_2 f_2(t)\big) e^{-i\omega t}\, dt \\ &= c_1 \int_{-\infty}^{\infty} f_1(t)\, e^{-i\omega t}\, dt + c_2 \int_{-\infty}^{\infty} f_2(t)\, e^{-i\omega t}\, dt \\ &= c_1 F_1(\omega) + c_2 F_2(\omega), \end{aligned} \tag{7.4}$$

using linearity of the integral. □

7.2 Realness

Theorem 29 (Realness). *Let $f(t) \leftrightarrow F(\omega)$ with $F(\omega)$ defined as above. Then*

$$f(t) \text{ real-valued for all } t \iff F^*(\omega) = F(-\omega) \text{ for all } \omega. \tag{7.5}$$

Proof. Assume first that f is real-valued. Then

$$F(\omega) = \int_{-\infty}^{\infty} f(t) e^{-i\omega t}\, dt, \tag{7.6}$$

$$F^*(\omega) = \left(\int_{-\infty}^{\infty} f(t) e^{-i\omega t}\, dt \right)^* = \int_{-\infty}^{\infty} f(t) e^{i\omega t}\, dt, \tag{7.7}$$

since $f(t)$ is real. On the other hand,

$$F(-\omega) = \int_{-\infty}^{\infty} f(t) e^{-i(-\omega)t}\, dt = \int_{-\infty}^{\infty} f(t) e^{i\omega t}\, dt. \tag{7.8}$$

Thus $F^*(\omega) = F(-\omega)$.

Conversely, suppose $F^*(\omega) = F(-\omega)$. Then

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)e^{i\omega t}\,d\omega.$$

Taking the complex conjugate,

$$f^*(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F^*(\omega)e^{-i\omega t}\,d\omega = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(-\omega)e^{-i\omega t}\,d\omega. \tag{7.9}$$

Make the change of variable $\nu = -\omega$:

$$f^*(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\nu)e^{i\nu t}\,d\nu = f(t).$$

Hence $f(t)$ is real-valued. □

7.3 Even–Odd and Symmetry Properties

Let $f(t)$ be real and

$$F(\omega) = A(\omega) + iB(\omega),$$

with A, B real-valued.

Theorem 30 (Symmetry of real transforms). *If $f(t)$ is real, then*

$$A(\omega) = \Re F(\omega) \quad \text{is even,} \tag{7.10}$$
$$B(\omega) = \Im F(\omega) \quad \text{is odd,} \tag{7.11}$$
$$|F(\omega)| = \sqrt{A^2(\omega) + B^2(\omega)} \quad \text{is even,} \tag{7.12}$$
$$\arg F(\omega) \quad \text{is odd (mod } 2\pi). \tag{7.13}$$

Moreover,

$$F(\omega) \text{ real and even} \iff f(t) \text{ even,} \tag{7.14}$$
$$F(\omega) \text{ purely imaginary and odd} \iff f(t) \text{ odd.} \tag{7.15}$$

Proof. Since f is real, we have $F^*(\omega) = F(-\omega)$ by the realness theorem. Write $F(\omega) = A(\omega) + iB(\omega)$, so

$$F^*(\omega) = A(\omega) - iB(\omega).$$

On the other hand,

$$F(-\omega) = A(-\omega) + iB(-\omega).$$

Thus

$$A(-\omega) + iB(-\omega) = A(\omega) - iB(\omega),$$

which implies

$$A(-\omega) = A(\omega) \quad \text{(even),} \tag{7.16}$$
$$B(-\omega) = -B(\omega) \quad \text{(odd).} \tag{7.17}$$

Then

$$|F(-\omega)| = \sqrt{A^2(-\omega) + B^2(-\omega)} = \sqrt{A^2(\omega) + B^2(\omega)} = |F(\omega)|,$$

so $|F(\omega)|$ is even. The phase satisfies

$$F(-\omega) = |F(\omega)|e^{-i\arg F(\omega)},$$

so $\arg F(-\omega) = -\arg F(\omega)$ modulo 2π.

Next, if f is even,

$$f(t) = f(-t),$$

then

$$F(\omega) = \int_{-\infty}^{\infty} f(t)e^{-i\omega t}dt = \int_{-\infty}^{\infty} f(-t)e^{-i\omega t}dt. \tag{7.18}$$

Change variable $\tau = -t$:

$$F(\omega) = \int_{-\infty}^{\infty} f(\tau)e^{i\omega\tau}d\tau = \int_{-\infty}^{\infty} f(\tau)\cos(\omega\tau)d\tau - i\int_{-\infty}^{\infty} f(\tau)\sin(\omega\tau)d\tau. \tag{7.19}$$

Since f is even, $f(\tau)\cos(\omega\tau)$ is even and $f(\tau)\sin(\omega\tau)$ is odd, so the latter integral vanishes. Thus $F(\omega)$ is real and even. The converse follows by applying the inverse transform and similar symmetry arguments. The odd case is similar. □

7.4 Area Property

Theorem 31 (Area). *If $f(t) \longleftrightarrow F(\omega)$, then*

$$\int_{-\infty}^{\infty} f(t)\,dt = F(0), \tag{7.20}$$

$$\int_{-\infty}^{\infty} F(\omega)\,d\omega = 2\pi f(0). \tag{7.21}$$

Proof. For (7.20), set $\omega = 0$ in the definition

$$F(\omega) = \int_{-\infty}^{\infty} f(t)e^{-i\omega t}dt.$$

Then

$$F(0) = \int_{-\infty}^{\infty} f(t)e^{0}\,dt = \int_{-\infty}^{\infty} f(t)\,dt.$$

For (7.21), evaluate the inverse transform at $t = 0$:

$$f(0) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)e^{i\omega\cdot 0}d\omega = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)\,d\omega.$$

Rearranging gives

$$\int_{-\infty}^{\infty} F(\omega)\,d\omega = 2\pi f(0).$$

□

7.5 Duality

Theorem 32 (Duality). *If*

$$f(t) \longleftrightarrow F(\omega),$$

then

$$F(t) \longleftrightarrow 2\pi f(-\omega). \tag{7.22}$$

Proof. Start from the inverse transform:

$$f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega) e^{i\omega t} \, d\omega.$$

Interchange the roles of t and ω. Write

$$f(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(t) e^{it\omega} \, dt.$$

Now compare this with the standard transform pair

$$g(t) \longleftrightarrow G(\omega) = \int_{-\infty}^{\infty} g(t) e^{-i\omega t} \, dt.$$

The expression for $f(\omega)$ can be written as

$$f(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(t) e^{-i(-\omega)t} \, dt.$$

Thus $f(\omega)$ is (up to the constant $1/(2\pi)$) the Fourier transform of $F(t)$ evaluated at $-\omega$. More precisely, if

$$F(t) \longleftrightarrow G(\omega),$$

then

$$G(-\omega) = \int_{-\infty}^{\infty} F(t) e^{i\omega t} \, dt.$$

Comparing,

$$f(\omega) = \frac{1}{2\pi} G(-\omega).$$

But by the original pairing $f \longleftrightarrow F$, we have

$$G(\omega) = 2\pi f(-\omega).$$

Therefore

$$F(t) \longleftrightarrow 2\pi f(-\omega).$$

□

7.6 Reciprocal Scaling

Theorem 33 (Scaling). *Let $a \in \mathbb{R} \setminus \{0\}$. If $f(t) \longleftrightarrow F(\omega)$, then*

$$f(at) \longleftrightarrow \frac{1}{|a|} F\left(\frac{\omega}{a}\right). \tag{7.23}$$

Proof. Compute the transform of $f(at)$:

$$\mathcal{F}\{f(at)\}(\omega) = \int_{-\infty}^{\infty} f(at)\, e^{-i\omega t}\, dt. \tag{7.24}$$

Let $z = at$. Then $t = z/a$, $dt = dz/a$. For $a > 0$, the limits remain $-\infty$ to ∞:

$$\begin{aligned}\mathcal{F}\{f(at)\}(\omega) &= \int_{-\infty}^{\infty} f(z)\, e^{-i\omega z/a} \frac{dz}{a} = \frac{1}{a} \int_{-\infty}^{\infty} f(z) e^{-i(\omega/a)z} dz \\ &= \frac{1}{a} F\left(\frac{\omega}{a}\right).\end{aligned} \tag{7.25}$$

For $a < 0$, one obtains the same expression, but the change of limits contributes a minus sign that is absorbed by $|a|$. Thus the general formula is

$$f(at) \longleftrightarrow \frac{1}{|a|} F\left(\frac{\omega}{a}\right).$$

□

As a useful special case, setting $a = -1$ gives

$$f(-t) \longleftrightarrow F(-\omega). \tag{7.26}$$

7.7 Time Shift

Theorem 34 (Time Shift). *If $f(t) \leftrightarrow F(\omega)$, then for any real t_0,*

$$f(t - t_0) \longleftrightarrow e^{-i\omega t_0} F(\omega). \tag{7.27}$$

Proof. Compute the transform of $f(t - t_0)$:

$$\mathcal{F}\{f(t - t_0)\}(\omega) = \int_{-\infty}^{\infty} f(t - t_0) e^{-i\omega t} dt. \tag{7.28}$$

Let $\tau = t - t_0$. Then $t = \tau + t_0$, $dt = d\tau$, and the limits are unchanged:

$$\begin{aligned}\mathcal{F}\{f(t - t_0)\}(\omega) &= \int_{-\infty}^{\infty} f(\tau) e^{-i\omega(\tau + t_0)} d\tau \\ &= e^{-i\omega t_0} \int_{-\infty}^{\infty} f(\tau) e^{-i\omega\tau} d\tau \\ &= e^{-i\omega t_0} F(\omega).\end{aligned} \tag{7.29}$$

□

7.8 Frequency Shift (Modulation)

Theorem 35 (Frequency Shift). *If $f(t) \leftrightarrow F(\omega)$, then for any real ω_0,*

$$f(t) e^{i\omega_0 t} \longleftrightarrow F(\omega - \omega_0). \tag{7.30}$$

Proof. Compute

$$\begin{aligned}\mathcal{F}\{f(t) e^{i\omega_0 t}\}(\omega) &= \int_{-\infty}^{\infty} f(t) e^{i\omega_0 t} e^{-i\omega t} dt \\ &= \int_{-\infty}^{\infty} f(t) e^{-i(\omega - \omega_0)t} dt \\ &= F(\omega - \omega_0).\end{aligned} \tag{7.31}$$

□

7.9 Time-Domain Differentiation

Theorem 36 (Differentiation in Time). *If $f(t) \leftrightarrow F(\omega)$ and $f'(t)$ exists and is absolutely integrable, then*

$$\frac{d}{dt}f(t) \longleftrightarrow i\omega F(\omega). \tag{7.32}$$

More generally,

$$\frac{d^n}{dt^n}f(t) \longleftrightarrow (i\omega)^n F(\omega). \tag{7.33}$$

Proof. Consider $f'(t)$:

$$\mathcal{F}\{f'(t)\}(\omega) = \int_{-\infty}^{\infty} f'(t)\,e^{-i\omega t}\,dt. \tag{7.34}$$

Integrate by parts. Let

$$u = e^{-i\omega t}, \quad dv = f'(t)dt,$$

so

$$du = -i\omega e^{-i\omega t}dt, \quad v = f(t).$$

Then

$$\int_{-\infty}^{\infty} f'(t)e^{-i\omega t}dt = f(t)e^{-i\omega t}\Big|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} f(t)\big(-i\omega e^{-i\omega t}\big)\,dt. \tag{7.35}$$

Assuming $f(t)e^{-i\omega t} \to 0$ at $\pm\infty$, the boundary term vanishes and

$$\mathcal{F}\{f'(t)\}(\omega) = i\omega\int_{-\infty}^{\infty} f(t)e^{-i\omega t}dt = i\omega F(\omega). \tag{7.36}$$

Iterating this argument yields $(i\omega)^n$ for the nth derivative. □

7.10 Frequency-Domain Differentiation

Theorem 37 (Differentiation in Frequency). *If $f(t) \leftrightarrow F(\omega)$ and $tf(t)$ is integrable, then*

$$tf(t) \longleftrightarrow i\frac{d}{d\omega}F(\omega), \tag{7.37}$$

and, more generally,

$$t^n f(t) \longleftrightarrow i^n\frac{d^n}{d\omega^n}F(\omega). \tag{7.38}$$

Proof. Differentiate $F(\omega)$ with respect to ω:

$$\begin{aligned}\frac{d}{d\omega}F(\omega) &= \frac{d}{d\omega}\int_{-\infty}^{\infty} f(t)e^{-i\omega t}dt = \int_{-\infty}^{\infty} f(t)\frac{d}{d\omega}e^{-i\omega t}dt \\ &= \int_{-\infty}^{\infty} f(t)(-it)e^{-i\omega t}dt = -i\int_{-\infty}^{\infty} tf(t)e^{-i\omega t}dt.\end{aligned} \tag{7.39}$$

Thus

$$\int_{-\infty}^{\infty} tf(t)e^{-i\omega t}dt = i\frac{d}{d\omega}F(\omega),$$

which implies

$$tf(t) \leftrightarrow iF'(\omega).$$

Repeated differentiation yields the general formula by induction. □

7.11 Time-Domain Convolution

Theorem 38 (Convolution in Time). *Define*

$$(f * g)(t) = \int_{-\infty}^{\infty} f(\tau)\, g(t-\tau)\, d\tau. \tag{7.40}$$

If $f(t) \longleftrightarrow F(\omega)$ and $g(t) \longleftrightarrow G(\omega)$, then

$$(f * g)(t) \;\longleftrightarrow\; F(\omega)G(\omega). \tag{7.41}$$

Proof. Compute the transform of $(f * g)(t)$:

$$\mathcal{F}\{f * g\}(\omega) = \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} f(\tau) g(t-\tau) d\tau \right) e^{-i\omega t} dt. \tag{7.42}$$

Assume absolute integrability so that Fubini's theorem permits interchange of integrals:

$$\mathcal{F}\{f * g\}(\omega) = \int_{-\infty}^{\infty} f(\tau) \left[\int_{-\infty}^{\infty} g(t-\tau) e^{-i\omega t} dt \right] d\tau. \tag{7.43}$$

Set $u = t - \tau$. Then $t = u + \tau$, $dt = du$:

$$\begin{aligned} \int_{-\infty}^{\infty} g(t-\tau) e^{-i\omega t} dt = \int_{-\infty}^{\infty} g(u) e^{-i\omega(u+\tau)} du = e^{-i\omega\tau} \int_{-\infty}^{\infty} g(u) e^{-i\omega u} du \\ = e^{-i\omega\tau} G(\omega). \end{aligned} \tag{7.44}$$

Substitute back:

$$\begin{aligned} \mathcal{F}\{f * g\}(\omega) &= \int_{-\infty}^{\infty} f(\tau) e^{-i\omega\tau} G(\omega) d\tau = G(\omega) \int_{-\infty}^{\infty} f(\tau) e^{-i\omega\tau} d\tau \\ &= G(\omega)F(\omega). \end{aligned} \tag{7.45}$$

□

7.12 Frequency-Domain Convolution

Theorem 39 (Convolution in Frequency). *If $f(t) \longleftrightarrow F(\omega)$ and $g(t) \longleftrightarrow G(\omega)$, then*

$$f(t)g(t) \;\longleftrightarrow\; \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\theta)\, G(\omega-\theta)\, d\theta = \frac{1}{2\pi} (F * G)(\omega). \tag{7.46}$$

Proof. Start from

$$f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\theta) e^{i\theta t} d\theta, \quad g(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} G(\nu) e^{i\nu t} d\nu.$$

Then

$$\begin{aligned} f(t)g(t) &= \left[\frac{1}{2\pi} \int_{-\infty}^{\infty} F(\theta) e^{i\theta t} d\theta \right] \left[\frac{1}{2\pi} \int_{-\infty}^{\infty} G(\nu) e^{i\nu t} d\nu \right] \\ &= \frac{1}{(2\pi)^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} F(\theta) G(\nu) e^{i(\theta+\nu)t} d\theta\, d\nu. \end{aligned} \tag{7.47}$$

Now take the Fourier transform:

$$\begin{aligned}\mathcal{F}\{fg\}(\omega) &= \int_{-\infty}^{\infty} f(t)g(t)e^{-i\omega t}dt \\ &= \frac{1}{(2\pi)^2}\iint F(\theta)G(\nu)\left[\int_{-\infty}^{\infty} e^{i(\theta+\nu-\omega)t}dt\right]d\theta\, d\nu.\end{aligned} \tag{7.48}$$

The inner integral is

$$\int_{-\infty}^{\infty} e^{i(\theta+\nu-\omega)t}dt = 2\pi\,\delta(\theta+\nu-\omega).$$

Hence

$$\begin{aligned}\mathcal{F}\{fg\}(\omega) &= \frac{1}{(2\pi)^2}\iint F(\theta)G(\nu)2\pi\,\delta(\theta+\nu-\omega)d\theta\, d\nu \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\theta)G(\omega-\theta)d\theta,\end{aligned} \tag{7.49}$$

using the sifting property of the delta. This is the desired result. □

7.13 Dirac Delta: Sampling and Convolution

The Dirac delta $\delta(t)$ is defined by the properties

$$\delta(t) = 0 \text{ for } t \neq 0, \qquad \int_{-\infty}^{\infty} \delta(t)dt = 1,$$

and the sampling property

$$\int_{-\infty}^{\infty} f(t)\delta(t-t_0)dt = f(t_0).$$

Theorem 40 (Sampling)**.** *For any suitable f and G,*

$$f(t)\delta(t-t_0) = f(t_0)\delta(t-t_0), \tag{7.50}$$

$$G(\omega)\delta(\omega-\omega_0) = G(\omega_0)\delta(\omega-\omega_0). \tag{7.51}$$

Proof. Consider

$$f(t)\delta(t-t_0).$$

Viewed as a distribution, for any test function $\varphi(t)$,

$$\begin{aligned}\int_{-\infty}^{\infty} f(t)\delta(t-t_0)\varphi(t)dt &= \int_{-\infty}^{\infty} \delta(t-t_0)\big(f(t)\varphi(t)\big)dt = f(t_0)\varphi(t_0) \\ &= \int_{-\infty}^{\infty} f(t_0)\delta(t-t_0)\varphi(t)dt.\end{aligned} \tag{7.52}$$

Hence $f(t)\delta(t-t_0) = f(t_0)\delta(t-t_0)$ as distributions. The frequency-domain statement is identical in form, with $t,\ t_0$ replaced by $\omega,\ \omega_0$. □

Theorem 41 (Convolution with a Delta)**.**

$$f(t) * \delta(t-t_0) = f(t-t_0), \tag{7.53}$$

$$G(\omega) * \delta(\omega-\omega_0) = G(\omega-\omega_0). \tag{7.54}$$

Proof. By definition,

$$(f * \delta(\cdot - t_0))(t) = \int_{-\infty}^{\infty} f(\tau)\,\delta(t - \tau - t_0)\,d\tau. \tag{7.55}$$

Use the sampling property in τ with $g(\tau) = f(\tau)$ and shift $t - \tau - t_0 = 0$ giving $\tau = t - t_0$:

$$(f * \delta(\cdot - t_0))(t) = f(t - t_0).$$

The frequency-domain version is analogous. □

7.14 Integration Property

Theorem 42 (Integration)**.** *Let $f(t) \longleftrightarrow F(\omega)$ and define*

$$h(t) = \int_{-\infty}^{t} f(\tau)\,d\tau.$$

Then

$$h(t) \;\longleftrightarrow\; H(\omega) = \frac{1}{i\omega}F(\omega) + \pi F(0)\,\delta(\omega), \tag{7.56}$$

in the sense of distributions.

Proof. Write

$$h(t) = \int_{-\infty}^{t} f(\tau)\,d\tau = (f * U)(t),$$

where $U(t)$ is the unit step. Using

$$U(t) \;\longleftrightarrow\; \mathrm{PV}\!\left(\frac{1}{i\omega}\right) + \pi\delta(\omega),$$

and the time-convolution theorem, we get

$$\begin{aligned} H(\omega) &= F(\omega)\left[\mathrm{PV}\!\left(\frac{1}{i\omega}\right) + \pi\delta(\omega)\right] \\ &= \mathrm{PV}\!\left(\frac{F(\omega)}{i\omega}\right) + \pi F(0)\delta(\omega). \end{aligned} \tag{7.57}$$

When F is regular at $\omega = 0$, this is commonly written as

$$H(\omega) = \frac{1}{i\omega}F(\omega) + \pi F(0)\delta(\omega),$$

with $1/\omega$ understood in the principal-value sense. □

8 The Sampling Theorems

In this chapter we study sampling in the time and frequency domains. We adopt the same Fourier-transform convention as before:

$$X(\omega) = \int_{-\infty}^{\infty} x(t)\, e^{-i\omega t}\, dt, \qquad x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)\, e^{i\omega t}\, d\omega. \tag{8.1}$$

We write

$$x(t) \longleftrightarrow X(\omega)$$

to denote a Fourier-transform pair.

Sampling refers to evaluating a continuous function at discrete points. For a continuous-time signal $x(t)$, sampling with period T_s gives the sequence

$$\{x(nT_s)\}_{n\in\mathbb{Z}}.$$

In the frequency domain, sampling a spectrum $X(\omega)$ at $\omega = n\omega_0$ produces

$$\{X(n\omega_0)\}_{n\in\mathbb{Z}}.$$

We analyze these operations using impulse (Dirac-delta) models, then generalize to non-impulsive sampling pulses.

8.1 Time-Domain Impulse Sampling

Let $x(t)$ be a continuous-time signal. The idealized impulse sampler multiplies $x(t)$ by the Dirac-comb

$$\Delta_{T_s}(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT_s), \tag{8.2}$$

where T_s is the sampling interval and

$$\omega_s = \frac{2\pi}{T_s} \tag{8.3}$$

is the sampling angular frequency.

The sampled signal is

$$x_s(t) = x(t)\,\Delta_{T_s}(t) = \sum_{n=-\infty}^{\infty} x(nT_s)\,\delta(t - nT_s). \tag{8.4}$$

We now derive the spectrum $X_s(\omega)$ of $x_s(t)$ in two ways.

Method 1: Directly from the definition

Using the sampling property

$$\delta(t - nT_s) \longleftrightarrow e^{-i\omega nT_s},$$

we have

$$\begin{aligned} X_s(\omega) &= \int_{-\infty}^{\infty} \left[\sum_{n=-\infty}^{\infty} x(nT_s)\,\delta(t - nT_s)\right] e^{-i\omega t}\, dt \\ &= \sum_{n=-\infty}^{\infty} x(nT_s) \int_{-\infty}^{\infty} \delta(t - nT_s) e^{-i\omega t} dt \\ &= \sum_{n=-\infty}^{\infty} x(nT_s)\, e^{-i\omega nT_s}. \end{aligned} \tag{8.5}$$

This is a (continuous) function of ω expressed as a (formal) Fourier series in ω.

Method 2: Using convolution in the frequency domain

We first compute the Fourier transform of $\Delta_{T_s}(t)$.

Lemma 3 (Spectrum of the Dirac comb). *With the above transform convention,*

$$\Delta_{T_s}(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT_s) \longleftrightarrow \Delta_{T_s}(\omega) = \frac{2\pi}{T_s} \sum_{k=-\infty}^{\infty} \delta(\omega - k\omega_s). \tag{8.6}$$

Proof. Consider the periodic train $\Delta_{T_s}(t)$ with period T_s. Its Fourier series representation is

$$\Delta_{T_s}(t) = \sum_{k=-\infty}^{\infty} c_k e^{ik\omega_s t}, \qquad \omega_s = \frac{2\pi}{T_s},$$

with coefficients

$$c_k = \frac{1}{T_s} \int_0^{T_s} \Delta_{T_s}(t) e^{-ik\omega_s t}\, dt = \frac{1}{T_s} \int_0^{T_s} \delta(t) e^{-ik\omega_s t}\, dt = \frac{1}{T_s}. \tag{8.7}$$

Hence

$$\Delta_{T_s}(t) = \frac{1}{T_s} \sum_{k=-\infty}^{\infty} e^{ik\omega_s t}.$$

Taking the Fourier transform term-wise,

$$\mathcal{F}\{e^{ik\omega_s t}\}(\omega) = 2\pi\, \delta(\omega - k\omega_s),$$

so

$$\Delta_{T_s}(\omega) = \frac{1}{T_s} \sum_{k=-\infty}^{\infty} 2\pi\, \delta(\omega - k\omega_s) = \frac{2\pi}{T_s} \sum_{k=-\infty}^{\infty} \delta(\omega - k\omega_s).$$

□

Now

$$x_s(t) = x(t)\Delta_{T_s}(t) \quad \Longleftrightarrow \quad X_s(\omega) = \frac{1}{2\pi}\big(X * \Delta_{T_s}\big)(\omega).$$

Thus

$$\begin{aligned} X_s(\omega) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\Omega)\, \Delta_{T_s}(\omega - \Omega)\, d\Omega \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\Omega) \frac{2\pi}{T_s} \sum_{k=-\infty}^{\infty} \delta(\omega - \Omega - k\omega_s)\, d\Omega \\ &= \frac{1}{T_s} \sum_{k=-\infty}^{\infty} \int_{-\infty}^{\infty} X(\Omega)\, \delta(\omega - \Omega - k\omega_s)\, d\Omega \\ &= \frac{1}{T_s} \sum_{k=-\infty}^{\infty} X(\omega - k\omega_s). \end{aligned} \tag{8.8}$$

We obtain the fundamental replication formula:

$$\boxed{X_s(\omega) = \frac{1}{T_s} \sum_{k=-\infty}^{\infty} X(\omega - k\omega_s).} \tag{8.9}$$

Thus time-domain sampling replicates the original spectrum periodically with period ω_s, scaled by $1/T_s$.

8.2 The Time-Domain Sampling Theorem

Assume that $x(t)$ is strictly band-limited.

Definition 30 (Strict band limitation). *We say that $x(t)$ is strictly band-limited to $\omega_{\max} > 0$ if*

$$X(\omega) = 0 \quad \textit{for } |\omega| > \omega_{\max}. \tag{8.10}$$

From (8.9), $X_s(\omega)$ is obtained by shifting copies of $X(\omega)$ by integer multiples of ω_s.

Theorem 43 (Time-domain sampling theorem). *Let $x(t) \longleftrightarrow X(\omega)$ be strictly band-limited to $|\omega| \le \omega_{\max}$. Let the sampling frequency be $\omega_s = 2\pi/T_s$. Then exact reconstruction from samples $\{x(nT_s)\}_{n\in\mathbb{Z}}$ is guaranteed whenever*

$$\omega_s > 2\omega_{\max}. \tag{8.11}$$

If $\omega_s < 2\omega_{\max}$, exact recovery is impossible in general. The equality case $\omega_s = 2\omega_{\max}$ is the critical edge case where replicas just touch and must be handled separately. Equivalently, in hertz,

$$f_s > 2f_{\max}, \qquad f_s = \frac{\omega_s}{2\pi}, \quad f_{\max} = \frac{\omega_{\max}}{2\pi}. \tag{8.12}$$

The critical sampling frequency

$$\omega_N = 2\omega_{\max}, \qquad f_N = 2f_{\max} \tag{8.13}$$

is called the Nyquist angular (resp. ordinary) sampling frequency (Nyquist rate).

Proof. From (8.9),

$$X_s(\omega) = \frac{1}{T_s}\sum_{k=-\infty}^{\infty} X(\omega - k\omega_s).$$

Each term $X(\omega - k\omega_s)$ is supported in $|\omega - k\omega_s| \le \omega_{\max}$, i.e.

$$k\omega_s - \omega_{\max} \le \omega \le k\omega_s + \omega_{\max}.$$

(No-overlap condition). For $k = 0$ and $k = 1$, the supports

$$[-\omega_{\max}, \omega_{\max}] \quad \text{and} \quad [\omega_s - \omega_{\max}, \omega_s + \omega_{\max}]$$

do not overlap if and only if

$$\omega_s - \omega_{\max} > \omega_{\max} \quad \Longleftrightarrow \quad \omega_s > 2\omega_{\max}.$$

Similarly for $k = -1$, compare

$$[-\omega_s - \omega_{\max}, -\omega_s + \omega_{\max}] \quad \text{and} \quad [-\omega_{\max}, \omega_{\max}].$$

Under the same condition $\omega_s > 2\omega_{\max}$, these intervals do not overlap either. Hence all shifted copies of X are nonoverlapping.

(Reconstruction). Define the ideal low-pass recovery filter with frequency response

$$G(\omega) = \begin{cases} T_s, & |\omega| \le \omega_{\max}, \\ 0, & \text{otherwise.} \end{cases} \tag{8.14}$$

Let $Y(\omega) = G(\omega)X_s(\omega)$ be the spectrum at the filter output. For $|\omega| \le \omega_{\max}$, only the $k = 0$ term survives in the sum:

$$X_s(\omega) = \frac{1}{T_s}X(\omega), \qquad |\omega| \le \omega_{\max},$$

since all $X(\omega - k\omega_s)$ with $k \ne 0$ are zero there by non-overlap. Hence

$$Y(\omega) = G(\omega)X_s(\omega) = T_s \cdot \frac{1}{T_s}X(\omega) = X(\omega) \quad (|\omega| \le \omega_{\max}),$$

and $Y(\omega) = 0$ outside this band. Thus $Y(\omega) = X(\omega)$ for all ω, so

$$y(t) = x(t).$$

Therefore exact reconstruction is possible when $\omega_s > 2\omega_{\max}$.

(Necessity for undersampling). If $\omega_s < 2\omega_{\max}$, the supports of at least two distinct shifts $X(\omega - k\omega_s)$ overlap on intervals of nonzero width. Then in the overlap region $X_s(\omega)$ is a sum of two or more unknown contributions of X, and no linear frequency-domain filter can uniquely separate these contributions. Hence exact recovery is impossible in general under undersampling.

$$\text{At } \omega_s = 2\omega_{\max}, \text{ replicas meet at band edges (critical Nyquist case).}$$

□

The condition $f_s > 2f_{\max}$ is the Nyquist sampling criterion, often summarized as: *at least two samples per period of the highest frequency component.*

8.3 Time-Domain Reconstruction Formula (Interpolation)

Under the assumptions of Theorem 43, the reconstruction filter with

$$G(\omega) = T_s \operatorname{Rect}\left(\frac{\omega}{2\omega_{\max}}\right)$$

has impulse response

$$\begin{aligned} g(t) &= \frac{1}{2\pi}\int_{-\omega_{\max}}^{\omega_{\max}} T_s e^{i\omega t}\,d\omega = \frac{T_s}{2\pi} \cdot \frac{2\sin(\omega_{\max}t)}{t} \\ &= \frac{T_s}{\pi}\frac{\sin(\omega_{\max}t)}{t}. \end{aligned} \tag{8.15}$$

For the special case $\omega_{\max} = \omega_s/2 = \pi/T_s$, this becomes

$$g(t) = \frac{T_s}{\pi}\frac{\sin\big((\pi/T_s)t\big)}{t} = \operatorname{Sa}\left(\frac{\omega_s}{2}t\right), \tag{8.16}$$

up to a conventional definition $\operatorname{Sa}(x) = \frac{\sin x}{x}$.

Because the reconstruction system is linear and shift-invariant, feeding $\delta(t - nT_s)$ produces $g(t - nT_s)$ at the output. Since the sampled signal can be written as

$$x_s(t) = \sum_{n=-\infty}^{\infty} x(nT_s)\,\delta(t - nT_s),$$

the output is

$$x(t) = \sum_{n=-\infty}^{\infty} x(nT_s)\, g(t - nT_s). \tag{8.17}$$

For the ideal brick-wall band-limited case with $\omega_{\max} = \omega_s/2$, this yields the interpolation formula

$$\boxed{x(t) = \sum_{n=-\infty}^{\infty} x(nT_s)\, \mathrm{Sa}\left(\frac{\omega_s}{2}(t - nT_s)\right).} \tag{8.18}$$

Each sample $x(nT_s)$ multiplies a shifted sinc pulse that vanishes at all other sampling instants.

8.4 Sampling with General Periodic Pulses

So far we used the Dirac comb as the sampling waveform. More generally, let $p_s(t)$ be any T_s-periodic sampling waveform with Fourier series

$$p_s(t) = \sum_{n=-\infty}^{\infty} c_n e^{in\omega_s t}, \qquad \omega_s = \frac{2\pi}{T_s}. \tag{8.19}$$

Sampling $x(t)$ with $p_s(t)$ corresponds to the product

$$x_s(t) = x(t)\, p_s(t). \tag{8.20}$$

Proposition 13 (Spectrum under multiplication by a periodic pulse). *Let $x(t) \longleftrightarrow X(\omega)$ and $p_s(t)$ be as above. Then*

$$x_s(t) = x(t)p_s(t) \longleftrightarrow X_s(\omega) = \sum_{n=-\infty}^{\infty} c_n X(\omega - n\omega_s). \tag{8.21}$$

Proof. Using linearity and the frequency-shift property:

$$x_s(t) = x(t) \sum_n c_n e^{in\omega_s t} = \sum_n c_n \left[x(t) e^{in\omega_s t}\right]. \tag{8.22}$$

Each term satisfies

$$x(t) e^{in\omega_s t} \longleftrightarrow X(\omega - n\omega_s).$$

Hence

$$X_s(\omega) = \sum_{n=-\infty}^{\infty} c_n X(\omega - n\omega_s).$$

□

Thus the effect of using a non-impulsive sampling pulse is to replicate the spectrum as before, but scale each replica by the corresponding Fourier-series coefficient c_n.

For a Dirac comb with period T_s, the coefficients satisfy $c_n = 1/T_s$, and we recover (8.9).

Under the Nyquist condition (and strict band limitation), one can still select an ideal low-pass filter to pass only a single, scaled copy of $X(\omega)$, so the sampling theorem and Nyquist criterion remain valid. In practice, sampling pulses are chosen narrow relative to T_s (e.g. narrow rectangular pulses) so that $x(t)$ does not change significantly during each pulse.

8.5 Sampling in the Frequency Domain

We now consider the dual situation: sampling in the frequency domain.

Let $x(t) \longleftrightarrow X(\omega)$ be given, and sample $X(\omega)$ at points $\omega = n\omega_0$ using an impulse train in frequency:

$$\Delta_{\omega_0}(\omega) = \omega_0 \sum_{n=-\infty}^{\infty} \delta(\omega - n\omega_0). \tag{8.23}$$

Define

$$X_s(\omega) = X(\omega)\,\Delta_{\omega_0}(\omega) = \omega_0 \sum_{n=-\infty}^{\infty} X(n\omega_0)\,\delta(\omega - n\omega_0). \tag{8.24}$$

This is a discrete-line spectrum in frequency.

To find the corresponding time-domain signal $x_s(t)$, invert the transform.

Method 1: Convolution in time

By duality, $\Delta_{\omega_0}(\omega)$ is the transform of a time-domain impulse train with period

$$T_0 = \frac{2\pi}{\omega_0}.$$

In particular,

$$\Delta_{\omega_0}(\omega) \longleftrightarrow \sum_{k=-\infty}^{\infty} \delta(t - kT_0). \tag{8.25}$$

Multiplication in frequency corresponds to convolution in time:

$$\begin{aligned} x_s(t) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} X_s(\omega) e^{i\omega t}\,d\omega \\ &= (x * \Delta_{T_0})(t) = \int_{-\infty}^{\infty} x(\tau) \sum_{k=-\infty}^{\infty} \delta(t - \tau - kT_0)\,d\tau \\ &= \sum_{k=-\infty}^{\infty} x(t - kT_0). \end{aligned} \tag{8.26}$$

Hence frequency-domain impulse sampling produces a periodic repetition of $x(t)$.

$$\boxed{x_s(t) = \sum_{k=-\infty}^{\infty} x(t - kT_0), \qquad T_0 = \frac{2\pi}{\omega_0}.} \tag{8.27}$$

Aliasing and the frequency-domain sampling theorem

Assume that $x(t)$ is strictly time-limited.

Definition 31 (Strict time limitation). *We say that $x(t)$ is strictly time-limited to $|t| \le T_{\max}$ if*

$$x(t) = 0 \quad \text{for } |t| > T_{\max}.$$

Then each shifted copy $x(t - kT_0)$ is supported in

$$kT_0 - T_{\max} \le t \le kT_0 + T_{\max}.$$

These supports do not overlap if and only if

$$T_0 > 2T_{\max}.$$

Theorem 44 (Frequency-domain sampling theorem). *Let $x(t) \longleftrightarrow X(\omega)$ be strictly time-limited to $|t| \le T_{\max}$. Sample $X(\omega)$ at $\omega = n\omega_0$ using an impulse train $\Delta_{\omega_0}(\omega)$ as above. Then the inverse transform $x_s(t)$ is a periodic repetition of $x(t)$ with period $T_0 = 2\pi/\omega_0$. It is guaranteed to be free of time-domain aliasing whenever*

$$T_0 > 2T_{\max}, \quad \text{equivalently} \quad \omega_0 < \frac{\pi}{T_{\max}}. \tag{8.28}$$

If $T_0 < 2T_{\max}$, aliasing is unavoidable in general; $T_0 = 2T_{\max}$ is the critical touching case.

Proof. We have shown that

$$x_s(t) = \sum_{k=-\infty}^{\infty} x(t - kT_0).$$

Each copy is supported in $[kT_0 - T_{\max}, kT_0 + T_{\max}]$. For the copies not to overlap for any k, we require that

$$kT_0 + T_{\max} < (k+1)T_0 - T_{\max} \quad \text{for all } k,$$

which simplifies to

$$2T_{\max} < T_0.$$

This is both necessary and sufficient for non-overlap; if the condition fails, the shifted supports intersect and aliasing occurs in the sense that $x_s(t)$ is a sum of overlapping copies of $x(t)$. □

8.6 Poisson Summation Formula

The previous derivation leads directly to a useful identity relating sums in time to sums in frequency.

Theorem 45 (Poisson summation). *Let $x(t) \longleftrightarrow X(\omega)$ be such that both are integrable and the series below converge. Let $T_0 > 0$ and $\omega_0 = 2\pi/T_0$. Then*

$$\sum_{k=-\infty}^{\infty} x(t - kT_0) = \frac{1}{T_0} \sum_{n=-\infty}^{\infty} X(n\omega_0)\, e^{in\omega_0 t}. \tag{8.29}$$

Proof. Define

$$x_s(t) = \sum_{k=-\infty}^{\infty} x(t - kT_0).$$

We have already shown that $x_s(t)$ is periodic with period T_0 and is the inverse Fourier transform of the frequency-sampled spectrum

$$X_s(\omega) = \omega_0 \sum_{n=-\infty}^{\infty} X(n\omega_0)\, \delta(\omega - n\omega_0).$$

On the other hand, any T_0-periodic function admits a Fourier series

$$x_s(t) = \sum_{n=-\infty}^{\infty} c_n e^{in\omega_0 t}, \qquad \omega_0 = \frac{2\pi}{T_0}.$$

Compute c_n from the inverse transform of $X_s(\omega)$:

$$\begin{aligned} x_s(t) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} X_s(\omega) e^{i\omega t}\, d\omega \\ &= \frac{1}{2\pi}\, \omega_0 \sum_{n=-\infty}^{\infty} X(n\omega_0) \int_{-\infty}^{\infty} \delta(\omega - n\omega_0)\, e^{i\omega t}\, d\omega \\ &= \frac{\omega_0}{2\pi} \sum_{n=-\infty}^{\infty} X(n\omega_0) e^{in\omega_0 t}. \end{aligned} \tag{8.30}$$

Hence
$$c_n = \frac{\omega_0}{2\pi} X(n\omega_0) = \frac{1}{T_0} X(n\omega_0).$$
Thus
$$x_s(t) = \sum_{n=-\infty}^{\infty} c_n e^{in\omega_0 t} = \frac{1}{T_0} \sum_{n=-\infty}^{\infty} X(n\omega_0) e^{in\omega_0 t},$$
which yields the Poisson summation formula. □

As a direct consequence, if $x(t)$ is strictly time-limited to $|t| \leq T_0/2$, then $x_s(t)$ in Theorem 45 is obtained by tiling $\mathbb{R}$ with non-overlapping copies of $x(t)$.

The formula also admits an inverse corollary:

Corollary 7. *Let $x_p(t)$ be T_0-periodic with Fourier series*
$$x_p(t) = \sum_{n=-\infty}^{\infty} C_n e^{in\omega_0 t}, \qquad \omega_0 = \frac{2\pi}{T_0}.$$
If $x(t)$ is defined by restricting $x_p(t)$ to one period and setting it to zero outside $(-T_0/2, T_0/2)$, then the Fourier transform $X(\omega)$ of $x(t)$ satisfies
$$X(\omega) = T_0 C_n \quad \textit{when } \omega = n\omega_0, \tag{8.31}$$
and, more generally, $X(\omega)$ can be obtained by interpolating these sampled values according to the transform definition.

8.7 Discrete vs. Periodic in Time and Frequency

The transform properties can be summarized as follows.

Theorem 46.

1. *$x(t)$ is discrete in time (i.e. a sum of impulses spaced by a fixed period) if and only if $X(\omega)$ is periodic in frequency.*
2. *$x(t)$ is periodic in time if and only if $X(\omega)$ is discrete in frequency (i.e. a sum of impulses spaced by a fixed frequency).*

Proof. (1) If $x(t)$ is discrete with period T_s:
$$x(t) = \sum_{n=-\infty}^{\infty} a_n \delta(t - nT_s),$$
then
$$X(\omega) = \sum_{n=-\infty}^{\infty} a_n e^{-i\omega n T_s}.$$
This is a Fourier series in ω with fundamental frequency $2\pi/T_s$, hence periodic in ω.

Conversely, if $X(\omega)$ is periodic with period ω_s, it admits a Fourier series
$$X(\omega) = \sum_{k=-\infty}^{\infty} b_k e^{-ik\omega T_0}, \qquad T_0 = \frac{2\pi}{\omega_s}.$$
Inverting term-wise shows that $x(t)$ is a discrete train of impulses spaced by T_0.

(2) If $x(t)$ is periodic with period T_0, it has a Fourier series

$$x(t) = \sum_{n=-\infty}^{\infty} C_n e^{in\omega_0 t}, \qquad \omega_0 = \frac{2\pi}{T_0}.$$

Using linearity and the transform of exponentials gives

$$X(\omega) = 2\pi \sum_{n=-\infty}^{\infty} C_n \delta(\omega - n\omega_0),$$

which is discrete in frequency.

Conversely, if $X(\omega)$ is discrete, then inverting yields a periodic sum of exponentials in time, so $x(t)$ is periodic. $\square$

An immediate consequence is:

Corollary 8. *A function that is discrete in both time and frequency must be periodic in both time and frequency.*

This structure is characteristic of the discrete Fourier transform (DFT), which is simultaneously discrete and periodic in both domains.

9 The Discrete Fourier Transform

9.1 Introduction

The discrete Fourier transform (DFT) is the standard tool for frequency–domain analysis of discrete signals. When implemented directly, evaluation of the DFT requires on the order of N^2 complex multiplications. The fast Fourier transform (FFT) is an algorithmic implementation that reduces this to $\mathcal{O}(N \log N)$ when N has suitable factorizations (in particular, when N is a power of 2), which explains its central role in modern signal processing and numerical computation.

This chapter develops the DFT as a discrete analysis–synthesis system based on the orthogonality of discrete complex exponentials, and establishes its basic properties and symmetry relations. In later chapters the connections between the DFT, Fourier series, and the continuous Fourier transform are examined in more detail, and the structure of FFT algorithms is outlined.

9.2 The Discrete Complex Exponentials

In continuous Fourier analysis, two families of complex exponentials are used:

$$\mathcal{S}_{\mathrm{FS}} = \left\{ e^{in\omega_0 t} \,\middle|\, n \in \mathbb{Z} \right\}, \qquad \mathcal{S}_{\mathrm{FT}} = \left\{ e^{i\omega t} \,\middle|\, \omega \in \mathbb{R} \right\}, \tag{9.1}$$

which serve as orthogonal bases for Fourier series and Fourier transforms, respectively. Their orthogonality relations are

$$\int_{-T/2}^{T/2} e^{in\omega_0 t} e^{-im\omega_0 t}\, dt = T\, \delta_{nm}, \tag{9.2}$$

$$\int_{-\infty}^{\infty} e^{i\omega t} e^{-i\omega' t}\, dt = 2\pi\, \delta(\omega - \omega'). \tag{9.3}$$

In the discrete setting we introduce a third family, the *discrete complex exponentials of order N*:

$$\mathcal{S}_N = \left\{ e^{i2\pi mk/N} \,\middle|\, m = 0, 1, \dots, N-1 \right\}, \tag{9.4}$$

where k is an integer index. These exponentials will serve as a discrete orthogonal basis.

9.2.1 Modulo-N arithmetic and circularity

Two integers a, b are said to be equal modulo N if they differ by an integer multiple of N:

$$a \equiv b \pmod{N} \quad \iff \quad a - b = \ell N \text{ for some } \ell \in \mathbb{Z}.$$

We sometimes write $[a]_N$ for the unique representative in $\{0, 1, \dots, N-1\}$ that is congruent to a modulo N.

The discrete exponentials are N–periodic in their integer argument:

$$e^{i2\pi(m+N)/N} = e^{i2\pi m/N} e^{i2\pi} = e^{i2\pi m/N}, \tag{9.5}$$

and more generally

$$e^{i2\pi m/N} = e^{i2\pi [m]_N/N}. \tag{9.6}$$

This will be referred to as the *circularity property*.

9.2.2 Notation for the primitive root of unity

It is convenient to introduce

$$W = e^{-i2\pi/N}. \tag{9.7}$$

Then

$$e^{i2\pi mk/N} = W^{-mk}. \tag{9.8}$$

Powers of W are Nth roots of unity with

$$W^N = e^{-i2\pi} = 1. \tag{9.9}$$

9.2.3 Inner product and orthogonality

For two complex N–vectors

$$a = (a_0, a_1, \dots, a_{N-1}), \quad b = (b_0, b_1, \dots, b_{N-1}),$$

define their inner product

$$(a, b) = \sum_{k=0}^{N-1} a_k b_k^*. \tag{9.10}$$

They are said to be *orthogonal* if $(a, b) = 0$.

We now state the orthogonality property of the discrete exponentials.

Theorem 47 (Orthogonality of the discrete complex exponentials). *Let $q, r \in \{0, 1, \dots, N-1\}$. Then*

$$\sum_{k=0}^{N-1} W^{-(q-r)k} = \begin{cases} N, & q = r, \\ 0, & q \neq r. \end{cases} \tag{9.11}$$

Equivalently, the vectors

$$\left(1, W^{-q}, W^{-2q}, \dots, W^{-(N-1)q}\right), \quad q = 0, 1, \dots, N-1,$$

are mutually orthogonal, with norm $\sqrt{N}$.

Proof. Let $p = q - r$, so that p is an integer with $-(N-1) \le p \le N-1$. Define

$$S = \sum_{k=0}^{N-1} W^{-pk}.$$

If $p = 0$, then $W^0 = 1$ and

$$S = \sum_{k=0}^{N-1} 1 = N.$$

Assume $p \neq 0$. Then $W^{-p} \neq 1$, and the sum is a geometric series:

$$S = \sum_{k=0}^{N-1} (W^{-p})^k = \frac{1 - (W^{-p})^N}{1 - W^{-p}}.$$

Since $W^N = 1$, we have $(W^{-p})^N = W^{-pN} = (W^N)^{-p} = 1$. Thus the numerator $1 - (W^{-p})^N$ is zero, while the denominator is nonzero. Hence $S = 0$, which is (9.11). □

9.3 The Discrete Fourier Transform

Let f_k, $k = 0, 1, \ldots, N-1$, be a discrete sequence (possibly complex–valued). We wish to expand f_k as a linear combination of the N discrete exponentials

$$1,\ W^{-k},\ W^{-2k},\ \ldots,\ W^{-(N-1)k}.$$

Following the continuous case, we postulate a *synthesis representation*

$$f_k = \frac{1}{N}\sum_{n=0}^{N-1} F_n W^{-nk}, \qquad 0 \le k \le N-1, \tag{9.12}$$

for some complex coefficients F_n that remain to be determined. These coefficients will be the DFT of f_k.

To solve for F_n, multiply (9.12) by W^{mk} and sum over $k = 0, 1, \ldots, N-1$:

$$\begin{aligned} \sum_{k=0}^{N-1} f_k W^{mk} &= \frac{1}{N}\sum_{k=0}^{N-1}\sum_{n=0}^{N-1} F_n W^{-nk} W^{mk} \\ &= \frac{1}{N}\sum_{n=0}^{N-1} F_n \sum_{k=0}^{N-1} W^{-(n-m)k}. \end{aligned} \tag{9.13}$$

By Theorem 47,

$$\sum_{k=0}^{N-1} W^{-(n-m)k} = \begin{cases} N, & n = m, \\ 0, & n \ne m, \end{cases}$$

so only the term with $n = m$ survives in (9.13) and we obtain

$$\sum_{k=0}^{N-1} f_k W^{mk} = F_m. \tag{9.14}$$

Renaming m as n yields the *analysis equation*

$$F_n = \sum_{k=0}^{N-1} f_k W^{nk}, \qquad 0 \le n \le N-1. \tag{9.15}$$

We summarize the result.

Theorem 48 (Discrete Fourier transform). *Let f_k, $k = 0, 1, \ldots, N-1$, be a complex sequence. Define the numbers F_n, $n = 0, 1, \ldots, N-1$, by*

$$F_n = \sum_{k=0}^{N-1} f_k W^{nk}, \qquad W = e^{-i2\pi/N}. \tag{9.16}$$

Then f_k can be reconstructed from F_n via

$$f_k = \frac{1}{N}\sum_{n=0}^{N-1} F_n W^{-nk}, \tag{9.17}$$

for $k = 0, 1, \ldots, N-1$. Equations (9.16) *and* (9.17) *form an analysis–synthesis pair known as the* discrete Fourier transform (DFT) *and* inverse DFT (IDFT).

It is convenient to use the shorthand notation

$$\{f_k\} \longleftrightarrow \{F_n\}$$

to indicate that f_k and F_n are related by the DFT pair (9.16)–(9.17).

9.3.1 The DFT as a pair of inverse transforms

Corollary 9. *The transformations* (9.16) *and* (9.17) *are mutual inverses: if F_n is defined from f_k by* (9.16)*, then the synthesis formula* (9.17) *reproduces f_k exactly.*

Proof. Substitute (9.16) into (9.17):

$$\begin{aligned} f_k &= \frac{1}{N}\sum_{n=0}^{N-1}\left(\sum_{r=0}^{N-1} f_r W^{nr}\right)W^{-nk} \\ &= \frac{1}{N}\sum_{r=0}^{N-1} f_r \sum_{n=0}^{N-1} W^{n(r-k)}. \end{aligned} \tag{9.18}$$

By orthogonality,

$$\sum_{n=0}^{N-1} W^{n(r-k)} = \begin{cases} N, & r = k, \\ 0, & r \neq k, \end{cases}$$

so only $r = k$ survives and we obtain $f_k = f_k$. □

9.3.2 Periodicity of the DFT indices

Because $W^N = 1$, the exponentials are N–periodic in both indices:

$$W^{(n+N)k} = W^{nk}, \qquad W^{n(k+N)} = W^{nk}. \tag{9.19}$$

Consequently, the DFT coefficients and the reconstructed sequence are both periodic in their indices.

Corollary 10 (Periodicity)**.** *Let $\{f_k\}$ and $\{F_n\}$ be related by the DFT pair* (9.16)–(9.17)*. Extend f_k and F_n to all integer indices using these formulas. Then*

$$F_{n+N} = F_n, \qquad f_{k+N} = f_k, \qquad \forall n, k \in \mathbb{Z}. \tag{9.20}$$

Proof. For F_n:

$$F_{n+N} = \sum_{k=0}^{N-1} f_k W^{(n+N)k} = \sum_{k=0}^{N-1} f_k W^{nk}(W^N)^k = \sum_{k=0}^{N-1} f_k W^{nk} = F_n.$$

Similarly, in (9.17),

$$f_{k+N} = \frac{1}{N}\sum_{n=0}^{N-1} F_n W^{-n(k+N)} = \frac{1}{N}\sum_{n=0}^{N-1} F_n W^{-nk}(W^{-N})^n = f_k.$$

□

Thus both the input sequence and its DFT may be regarded as periodic sequences with period N, and the indices k and n are often interpreted modulo N.

9.3.3 Example: a discrete ramp

Consider the discrete sequence

$$f_k = \begin{cases} k, & k = 0,1,2,3, \\ 0, & \text{otherwise}, \end{cases} \qquad (N = 4). \tag{9.21}$$

Within one period $0 \le k \le 3$ we have the vector

$$f = (f_0, f_1, f_2, f_3) = (0,1,2,3).$$

For $N = 4$ we have

$$W = e^{-i2\pi/4} = e^{-i\pi/2} = -i.$$

The DFT analysis equations (9.16) become

$$F_0 = f_0 W^{0\cdot 0} + f_1 W^{0\cdot 1} + f_2 W^{0\cdot 2} + f_3 W^{0\cdot 3} = 0+1+2+3 = 6, \tag{9.22}$$

$$F_1 = f_0 W^{1\cdot 0} + f_1 W^{1\cdot 1} + f_2 W^{1\cdot 2} + f_3 W^{1\cdot 3} = 0+1(-i)+2(-1)+3(i) = -2+2i, \tag{9.23}$$

$$F_2 = f_0 W^{2\cdot 0} + f_1 W^{2\cdot 1} + f_2 W^{2\cdot 2} + f_3 W^{2\cdot 3} = 0+1(-1)+2(1)+3(-1) = -2, \tag{9.24}$$

$$F_3 = f_0 W^{3\cdot 0} + f_1 W^{3\cdot 1} + f_2 W^{3\cdot 2} + f_3 W^{3\cdot 3} = 0+1(i)+2(-1)+3(-i) = -2-2i. \tag{9.25}$$

Thus

$$F = (F_0, F_1, F_2, F_3) = (6, -2+2i, -2, -2-2i).$$

Now apply the synthesis formula (9.17):

$$f_0 = \frac{1}{4}\big(F_0 W^0 + F_1 W^0 + F_2 W^0 + F_3 W^0\big) = \frac{1}{4}(6-2+2i-2-2-2i) = 0, \tag{9.26}$$

$$f_1 = \frac{1}{4}\big(F_0 W^0 + F_1 W^{-1} + F_2 W^{-2} + F_3 W^{-3}\big) = 1, \tag{9.27}$$

$$f_2 = \frac{1}{4}\big(F_0 W^0 + F_1 W^{-2} + F_2 W^{-4} + F_3 W^{-6}\big) = 2, \tag{9.28}$$

$$f_3 = \frac{1}{4}\big(F_0 W^0 + F_1 W^{-3} + F_2 W^{-6} + F_3 W^{-9}\big) = 3. \tag{9.29}$$

Thus the IDFT recovers the original sequence $(0,1,2,3)$ exactly, in accordance with Corollary 9.

It is instructive to view the synthesis step as a linear combination of orthogonal vectors. For $N = 4$, the vectors

$$v_n = \big(1,\ W^{-n},\ W^{-2n},\ W^{-3n}\big), \qquad n = 0,1,2,3,$$

form an orthogonal set. Then the synthesis formula can be written symbolically as

$$f = \frac{1}{4}\sum_{n=0}^{3} F_n v_n,$$

i.e., f is a linear combination of the discrete complex exponentials with weights F_n.

9.4 Properties of the DFT

9.4.1 Cartesian and polar forms of F$_n$

For each n we write

$$F_n = A_n + iB_n, \tag{9.30}$$

where $A_n = \operatorname{Re} F_n$ and $B_n = \operatorname{Im} F_n$. In polar form,

$$F_n = |F_n| e^{i\theta_n}, \tag{9.31}$$

where

$$|F_n| = \sqrt{A_n^2 + B_n^2}, \qquad \theta_n = \arg(F_n). \tag{9.32}$$

When f_k is real–valued, F_n satisfies symmetry relations analogous to those for the continuous Fourier transform.

9.4.2 Symmetry for real data

Assume $f_k \in \mathbb{R}$ for $k = 0, 1, \ldots, N-1$. Using the periodic extension in n (Corollary 10), it is convenient to regard F_n as defined for all $n \in \mathbb{Z}$ with period N.

Theorem 49 (Conjugate symmetry for real sequences). *If f_k is real, then*

$$F_{-n} = F_n^*, \qquad \forall n \in \mathbb{Z}, \tag{9.33}$$

where indices are understood modulo N. Equivalently,

$$A_{-n} = A_n, \qquad B_{-n} = -B_n, \tag{9.34}$$

so the real part A_n and the magnitude $|F_n|$ are even in n, while the imaginary part B_n and the phase θ_n are odd (up to 2π-periodic ambiguities in the phase).

Proof. Starting from the definition

$$F_n = \sum_{k=0}^{N-1} f_k W^{nk},$$

take the complex conjugate:

$$F_n^* = \sum_{k=0}^{N-1} f_k^* (W^{nk})^* = \sum_{k=0}^{N-1} f_k W^{-nk},$$

since f_k is real and $W^* = W^{-1}$. But

$$F_{-n} = \sum_{k=0}^{N-1} f_k W^{-nk},$$

so $F_{-n} = F_n^*$, which is (9.33). Writing $F_n = A_n + iB_n$ and $F_{-n} = A_{-n} + iB_{-n}$, the relation $F_{-n} = F_n^*$ implies

$$A_{-n} + iB_{-n} = A_n - iB_n,$$

so $A_{-n} = A_n$ and $B_{-n} = -B_n$. The statements about evenness and oddness follow. □

Thus, for a real input sequence, the DFT coefficients satisfy a discrete conjugate symmetry: values at negative indices (modulo N) are complex conjugates of those at positive indices.

9.4.3 Indexing over a symmetric interval

Because F_n is periodic with period N, it is often useful to label one period symmetrically about the origin. For even N, a convenient choice is

$$n = -\frac{N}{2}, -\frac{N}{2}+1, \dots, -1, 0, 1, \dots, \frac{N}{2}-1.$$

On this symmetric index set, Theorem 49 states that pairs F_n and F_{-n} are conjugates, so that

$$A_n = A_{-n}, \qquad B_n = -B_{-n},$$

and the center index $n = 0$ (and, for even N, also $n = \pm N/2$) is real.

Viewed modulo N, the same symmetry can be described within the primary interval $n = 0, 1, \dots, N-1$: for n in this range,

$$F_n = F^*_{(N-n) \bmod N},$$

which is the form typically exploited in practical implementations.

Notes and Comments

A lemma on sums of discrete exponentials

Lemma 4. *For any positive integer N,*

$$\sum_{n=0}^{N-1} e^{i2\pi n/N} = 0. \tag{9.35}$$

Proof. The sum is a geometric series with ratio

$$\alpha = e^{i2\pi/N} \neq 1.$$

Hence

$$\sum_{n=0}^{N-1} \alpha^n = \frac{1-\alpha^N}{1-\alpha} = \frac{1-e^{i2\pi}}{1-\alpha} = 0,$$

since $e^{i2\pi} = 1$. □

The same argument applies with $\alpha = W^p = e^{-i2\pi p/N}$ and $p \not\equiv 0 \pmod N$, yielding the orthogonality relation of Theorem 47.

Remarks on the argument of a complex number

For a complex number $z = x + iy$, the principal value of the argument is often taken as

$$\arg(z) = \operatorname{atan2}(y, x),$$

which returns a value in $(-\pi, \pi]$. Two special cases are worth noting:

- If $z = 0$ (i.e. $x = 0$ and $y = 0$), the argument is undefined. In practice one often assigns $\arg(0) = 0$ for convenience when plotting phase.

- If $y = 0$ and $x < 0$, then z lies on the negative real axis. The points $z = -1$ and its periodic copies in the DFT correspond to arguments π or $-\pi$, which differ by 2π but represent the same direction in the complex plane. When imposing both periodicity and odd symmetry for phase plots, this leads to a choice at such points. In practice, one typically chooses the branch that yields a visually continuous phase plot, and may mark ambiguous points explicitly when necessary.

These subtleties do not affect the algebraic properties of the DFT, but are relevant when interpreting and plotting the phase $\theta_n = \arg(F_n)$.

10 Inside the Fast Fourier Transform

10.1 Introduction

The radix–2 fast Fourier transform (FFT) is an efficient algorithm for evaluating the discrete Fourier transform (DFT) when the data length

$$N = 2^{\nu}, \qquad \nu \in \mathbb{N}. \tag{10.1}$$

Instead of evaluating each DFT coefficient directly from its definition, the FFT factors the computation into ν stages of simple two–point operations (*butterflies*), arranged in a structured way. This reduces the operation count from $\mathcal{O}(N^2)$ to $\mathcal{O}(N \log_2 N)$.

In this section the radix–2 structure is first illustrated for small N, then generalized using flow graphs and simple indexing rules. A derivation of the radix–2 algorithm following Cooley and Tukey is sketched for $N = 8$.

10.2 DFT in index notation and radix–2 setting

The DFT analysis equation can be written as

$$F(n) = \sum_{k=0}^{N-1} f_0(k) W^{nk}, \qquad n = 0, 1, \ldots, N-1, \tag{10.2}$$

where

$$W = e^{-i2\pi/N}. \tag{10.3}$$

Here:

- $F(n)$ denotes the nth DFT spectral value (stage output).
- $f_0(k)$ denotes the kth element of the input (stage–0) data vector.
- The subscript in $f_\sigma(k)$ will be used for stage index $\sigma = 0, 1, \ldots, \nu$.

The exponential can be written compactly as

$$e^{-i2\pi nk/N} = W^{nk}. \tag{10.4}$$

The minus sign in the exponent is fixed throughout this chapter.

The FFT is obtained by factoring the matrix representation of (10.2) into a product of ν sparse matrices, each corresponding to one stage of butterflies.

10.3 Small examples

Case $N = 2$

For $N = 2$,

$$W = e^{-i2\pi/2} = -1. \tag{10.5}$$

The DFT is

$$F(0) = f_0(0) W^{0 \cdot 0} + f_0(1) W^{0 \cdot 1} = f_0(0) + f_0(1), \tag{10.6}$$
$$F(1) = f_0(0) W^{1 \cdot 0} + f_0(1) W^{1 \cdot 1} = f_0(0) - f_0(1). \tag{10.7}$$

So for $N = 2$ the DFT consists of one sum and one difference.

Case $N = 4$

For $N = 4$,

$$W = e^{-i2\pi/4} = -i, \qquad W^2 = -1, \qquad W^4 = 1. \tag{10.8}$$

The DFT in matrix form is

$$\begin{bmatrix} F(0) \\ F(1) \\ F(2) \\ F(3) \end{bmatrix} = \begin{bmatrix} W^{0\cdot 0} & W^{0\cdot 1} & W^{0\cdot 2} & W^{0\cdot 3} \\ W^{1\cdot 0} & W^{1\cdot 1} & W^{1\cdot 2} & W^{1\cdot 3} \\ W^{2\cdot 0} & W^{2\cdot 1} & W^{2\cdot 2} & W^{2\cdot 3} \\ W^{3\cdot 0} & W^{3\cdot 1} & W^{3\cdot 2} & W^{3\cdot 3} \end{bmatrix} \begin{bmatrix} f_0(0) \\ f_0(1) \\ f_0(2) \\ f_0(3) \end{bmatrix}. \tag{10.9}$$

Using $W^4 = 1$ and $W^2 = -1$, the matrix simplifies so that only W^0 and W^1 appear explicitly, and the rows can be written in a form where only W^0 and W^1 are required, together with their negations.

This matrix can be factorized into a product of two 4×4 matrices, corresponding to two stages of butterflies. In stage 1, inputs are combined in pairs $(0, 2)$ and $(1, 3)$:

$$f_1(0) = f_0(0) + W^0 f_0(2), \tag{10.10}$$
$$f_1(2) = f_0(0) - W^0 f_0(2), \tag{10.11}$$
$$f_1(1) = f_0(1) + W^0 f_0(3), \tag{10.12}$$
$$f_1(3) = f_0(1) - W^0 f_0(3). \tag{10.13}$$

In stage 2, pairs $(0, 1)$ and $(2, 3)$ are combined:

$$f_2(0) = f_1(0) + W^0 f_1(1), \tag{10.14}$$
$$f_2(1) = f_1(0) - W^0 f_1(1), \tag{10.15}$$
$$f_2(2) = f_1(2) + W^1 f_1(3), \tag{10.16}$$
$$f_2(3) = f_1(2) - W^1 f_1(3). \tag{10.17}$$

The final outputs $F(n)$ are obtained from $f_2(k)$ by a permutation (bit reversal) of the indices.

Each pair of equations corresponds to a two–point butterfly. In each butterfly only one complex multiplication $W^p \cdot (\cdot)$ is required; the second result is formed by a subtraction using the same product.

For $N = 4$:

- Direct DFT (no precomputed twiddles): $4 \times 4 = 16$ complex multiplications, $4(4-1) = 12$ complex additions.
- FFT: 4 complex multiplications, 4 complex additions and 4 complex subtractions.

10.4 Operation count for radix–2 FFT

For $N = 2^\nu$, the radix–2 FFT has ν stages. At each stage, $N/2$ butterflies are computed, each requiring one complex multiplication and two complex add/subtract operations. Thus

$$\text{complex multiplications} = \frac{N}{2}\nu = \frac{N}{2}\log_2 N, \tag{10.18}$$
$$\text{complex add/subtracts} = N\nu = N\log_2 N. \tag{10.19}$$

In contrast, the direct DFT needs

$$\text{complex multiplications} = N^2, \tag{10.20}$$
$$\text{complex additions} = N(N-1). \tag{10.21}$$

The ratio of multiplication counts (direct/FFT) is

$$\frac{N^2}{\frac{N}{2}\log_2 N} = \frac{2N}{\log_2 N}, \tag{10.22}$$

which grows linearly with N.

For example, for $N = 32{,}768$,

$$\frac{2N}{\log_2 N} \approx 4369.$$

A machine that requires one minute for the FFT at this N would require several days for the direct DFT, using the same implementation model.

10.5 Signal–flow graphs and butterflies

The radix–2 FFT can be represented by a signal–flow graph. Nodes represent intermediate values $f_\sigma(k)$ at stage σ and index k. Directed edges connect nodes between stages.

In the radix–2 decimation–in–time structure:

- Each node in stage σ has two input edges from stage $\sigma - 1$: a *primary* input (multiplier 1) and a *secondary* input (multiplier W^p for some integer p).
- Each pair of nodes that share the same two source nodes are called a *dual–node pair* and form a butterfly.

For $N = 4$ and $N = 8$, flow graphs show:

- Stage 1: butterflies combine indices separated by $N/2$.
- Stage 2: butterflies combine indices separated by $N/4$.
- ...
- Stage ν: butterflies combine adjacent indices.

Dual–node separation

Let σ be the stage index, $1 \le \sigma \le \nu$, and $N = 2^\nu$.

Dual–node separation

At stage σ, the two nodes in each butterfly are separated by

$$\Delta k = \frac{N}{2^\sigma}. \tag{10.23}$$

That is, butterflies connect $f_{\sigma-1}(k)$ and $f_{\sigma-1}(k+\Delta k)$.

Butterfly equations

Let $f_{\sigma-1}(k)$ and $f_{\sigma-1}(k+\Delta k)$ be the primary and secondary source nodes at stage $\sigma - 1$, and let p be the exponent associated with the secondary edge. The butterfly equations are

$$f_\sigma(k) = f_{\sigma-1}(k) + W^p f_{\sigma-1}(k+\Delta k), \tag{10.24}$$
$$f_\sigma(k+\Delta k) = f_{\sigma-1}(k) - W^p f_{\sigma-1}(k+\Delta k). \tag{10.25}$$

Here

$$\Delta k = \frac{N}{2^\sigma}. \tag{10.26}$$

A single complex multiplication $W^p f_{\sigma-1}(k+\Delta k)$ is reused in both output expressions; the second output is a subtraction using the same product.

For in–place computation (overlay of results on input), it is convenient to rewrite (10.25) so that the source values are not overwritten before they are needed. One practical ordering is

$$f_\sigma(k+\Delta k) = f_{\sigma-1}(k) - W^p f_{\sigma-1}(k+\Delta k), \tag{10.27}$$

$$f_\sigma(k) = f_{\sigma-1}(k) + W^p f_{\sigma-1}(k+\Delta k), \tag{10.28}$$

computed in this order so that $f_{\sigma-1}(k)$ is still available when computing the second line.

Power of W in each butterfly

For stage σ, $1 \le \sigma \le \nu$, define

$$\Delta k = \frac{N}{2^\sigma}, \qquad s = \nu - \sigma, \tag{10.29}$$

and let k be the node index (in decimal). Then

Exponent index

The exponent p associated with the secondary input of the butterfly at stage σ and index k can be obtained as follows:

1. *Compute*
$$q = \left\lfloor \frac{k}{2^s} \right\rfloor.$$
2. *Express q in ν-bit binary.*
3. *Reverse the order of the bits.*
4. *Convert back to decimal. The result is p.*

This rule is equivalent to taking a truncated right–shift in binary (by s bits), then applying bit reversal on those bits.

The dual node in the same butterfly has exponent index shifted by $N/2$:

Dual–node exponent

If the upper node has multiplier W^p, the lower node has multiplier $W^{p+N/2}$.

Using $W^{p+N/2} = W^p W^{N/2}$ and $W^{N/2} = -1$ (for even N) gives

$$W^{p+N/2} = -W^p. \tag{10.30}$$

Thus one can also state:

Dual–node multiplier

If a node uses the multiplier W^p, its dual node uses $-W^p$.

Traversal and skipping inside a stage

At stage σ, butterflies are formed using blocks of length $2\Delta k$, with local indices

$$k,\ k+\Delta k, \quad k+2\Delta k,\ k+3\Delta k, \quad \dots$$

In each block, the first Δk indices serve as primary nodes for butterflies, paired with nodes Δk positions below.

Equivalently:

Stage traversal

For stage σ:

1. *Set $\Delta k = N/2^{\sigma}$.*
2. *Starting from $k = 0$, apply the butterfly* (10.28) *for $k = 0, 1, \dots, \Delta k - 1$.*
3. *Skip the next Δk indices (they are dual nodes already updated).*
4. *Repeat the pattern "Δk butterflies, Δk skips" until $k \geq N$.*

All butterflies within a stage are independent and can, in principle, be evaluated in parallel.

10.6 Bit reversal and output permutation

The flow graph in decimation–in–time order produces outputs in an index order that is bit–reversed relative to the natural order. To obtain $F(n)$ for $n = 0, 1, \dots, N-1$ in natural order, a final permutation is applied to the last stage vector $f_{\nu}(k)$.

Let k be the index of an element in f_{ν}, and write its ν–bit binary representation as

$$k \longleftrightarrow (b_{\nu-1} b_{\nu-2} \dots b_1 b_0)_2.$$

Define the bit–reversed index

$$\mathrm{rev}(k) \longleftrightarrow (b_0 b_1 \dots b_{\nu-2} b_{\nu-1})_2.$$

Bit reversal

The correctly ordered DFT output is obtained by

$$F(\mathrm{rev}(k)) = f_{\nu}(k), \qquad k = 0, 1, \dots, N-1. \tag{10.31}$$

In an in–place implementation, this is done by swapping entries in f_{ν} whenever necessary. To avoid undoing swaps, one uses:

Swap condition

For k from 0 to $N-1$:

1. *Compute $i = \mathrm{rev}(k)$.*
2. *If $i > k$, swap $f_{\nu}(k)$ and $f_{\nu}(i)$.*
3. *If $i \leq k$, do nothing.*

This ensures that each pair (k, i) is swapped at most once.

10.7 Algorithm structure

A standard in–place radix–2 decimation–in–time FFT can be summarized as:

Given $N = 2^\nu$, input $f_0(k)$, $k = 0, \dots, N-1$.
for $\sigma = 1$ to ν **do**
  $\Delta k \leftarrow N/2^\sigma$.
  $k \leftarrow 0$.
  while $k < N$ **do**
    for $r = 0$ to $\Delta k - 1$ **do**
      Determine exponent index p for node $k + r$.
      $G \leftarrow W^p f_{\sigma-1}(k + r + \Delta k)$.
      $u \leftarrow f_{\sigma-1}(k + r)$.
      $f_\sigma(k + r + \Delta k) \leftarrow u - G$.
      $f_\sigma(k + r) \leftarrow u + G$.
    end for
    $k \leftarrow k + 2\Delta k$.
  end while
end for
Apply bit reversal on $f_\nu(k)$ to obtain $F(n)$.

Details of computing p (twiddle exponents) are as described in the previous subsection.

10.8 Cooley–Tukey derivation for N=8

For $N = 8$ and

$$W = e^{-i2\pi/8},$$

the DFT is

$$F(n) = \sum_{k=0}^{7} f_0(k) W^{nk}. \tag{10.32}$$

Split the sum into even and odd indices $k = 2r$ and $k = 2r + 1$:

$$\begin{aligned} F(n) &= \sum_{r=0}^{3} f_0(2r)\, W^{n(2r)} + \sum_{r=0}^{3} f_0(2r+1)\, W^{n(2r+1)} \\ &= E(n) + W^n O(n), \end{aligned} \tag{10.33}$$

where

$$E(n) = \sum_{r=0}^{3} f_0(2r)\,(W^2)^{nr}, \qquad O(n) = \sum_{r=0}^{3} f_0(2r+1)\,(W^2)^{nr}. \tag{10.34}$$

Thus E and O are 4-point DFTs (with twiddle $W^2 = e^{-i2\pi/4}$). Using their period 4:

$$F(n) = E(n) + W^n O(n), \qquad F(n+4) = E(n) - W^n O(n), \quad n = 0, 1, 2, 3. \tag{10.35}$$

Each 4-point DFT is then decomposed again into even/odd parts (now 2-point DFTs). If

$$e_r = f_0(2r), \quad o_r = f_0(2r+1), \quad r = 0, 1, 2, 3,$$

then

$$E(n) = \big(e_0 + (-1)^n e_2\big) + (W^2)^n \big(e_1 + (-1)^n e_3\big), \tag{10.36}$$

$$O(n) = \big(o_0 + (-1)^n o_2\big) + (W^2)^n \big(o_1 + (-1)^n o_3\big), \tag{10.37}$$

for $n = 0, 1, 2, 3$. These are exactly butterfly combinations.

So for $N = 8$, the computation factorizes into three butterfly stages:

- stage 1: stride 4 pairings,
- stage 2: stride 2 pairings,
- stage 3: stride 1 pairings,

followed by the standard bit-reversal permutation between stage ordering and natural DFT index ordering.

10.9 Using the FFT for synthesis

The DFT synthesis equation is

$$f(k) = \frac{1}{N} \sum_{n=0}^{N-1} F(n) W^{-nk}, \qquad k = 0, 1, \ldots, N-1. \tag{10.38}$$

This can be written as

$$f(k) = \frac{1}{N} \left[\sum_{n=0}^{N-1} F^*(n) W^{nk} \right]^* . \tag{10.39}$$

Thus the inverse DFT can be computed using the same FFT code used for the forward DFT, by:

1. Conjugating the input spectrum $F(n)$.
2. Applying the forward FFT to $\{F^*(n)\}$.
3. Conjugating the result and scaling by $1/N$.

10.10 Real input and composite N

When the input sequence f_k is real and N is even, one can exploit the conjugate symmetry of the DFT to reduce computation:

- Pack two real sequences into real and imaginary parts of a single complex sequence of length $N/2$.
- Compute an FFT of length $N/2$.
- Recover the length–N spectra from the length–$N/2$ result.

This saves both time and storage in the analysis direction. For synthesis, where the input spectrum is generally complex, this packing is not directly applicable.

When N is not a power of two but is composite, mixed–radix FFT algorithms can be constructed by factoring N into smaller radices and applying Cooley–Tukey style decompositions. If N is prime, no nontrivial factorization exists and one falls back to a DFT of length N (or to prime–length FFT variants such as Bluestein or Rader, which are beyond the scope of this section).

Notes on complex arithmetic and twiddle factors

Let

$$c = a + ib, \qquad w = u + iv$$

be complex numbers. Then

$$\begin{aligned} cw &= (a+ib)(u+iv) = (au - bv) + i(bu + av), && (10.40)\\ c + w &= (a+u) + i(b+v), && (10.41)\\ c - w &= (a-u) + i(b-v). && (10.42) \end{aligned}$$

Thus a complex multiplication requires four real multiplications and two real additions if implemented naively.

For the FFT, the multiplication by a twiddle factor W^p is common. Writing

$$W^p = \cos\theta - i\sin\theta, \qquad \theta = \frac{2\pi p}{N},$$

and $x(k) = a + ib$, one obtains

$$\begin{aligned} W^p x(k) &= (\cos\theta - i\sin\theta)(a+ib)\\ &= (a\cos\theta + b\sin\theta) + i(b\cos\theta - a\sin\theta), \end{aligned} \tag{10.43}$$

so real and imaginary parts can be computed from precomputed $\cos\theta$ and $\sin\theta$ using real arithmetic. Once formed, $W^p x(k)$ is reused in both outputs of the butterfly (10.28).

11 The Discrete Fourier Transform as an Estimator

In Chapter 10 the discrete Fourier transform (DFT) was introduced as an independent transform pair. In numerical work the DFT is evaluated by the fast Fourier transform (FFT) algorithm. In this chapter the DFT/FFT is regarded as an *estimator* for the continuous Fourier transforms (CFTs) introduced in Part I.

To keep the notation distinct, the following convention will be used throughout this chapter:

- **FFT**: the analysis/synthesis pair for the DFT.
- **CFT**: the continuous Fourier transforms:
  - complex Fourier series coefficients for periodic functions,
  - Fourier transform for finite-energy pulses.

Where needed, notation such as CFT (Fourier series) or CFT (pulse) will be used to remove ambiguity.

11.1 Rectangular-rule approximation and the FFT

Consider the definite integral

$$I = \int_0^T g(t)\,\mathrm{d}t. \tag{11.1}$$

The rectangular rule approximates I by sampling g at N equally spaced points on $[0, T]$ and summing the areas of the corresponding rectangles.

Let

$$T_s = \frac{T}{N}, \qquad t_k = kT_s, \qquad g_k = g(t_k), \quad 0 \le k \le N-1, \tag{11.2}$$

and define

$$S_N = \sum_{k=0}^{N-1} g_k T_s. \tag{11.3}$$

Then S_N is the left-endpoint Riemann sum. Under mild regularity assumptions (e.g. g Riemann integrable on $[0, T]$),

$$\int_0^T g(t)\,\mathrm{d}t = \lim_{N\to\infty} \sum_{k=0}^{N-1} g_k T_s. \tag{11.4}$$

Theorem 50 (Rectangular rule)**.** *For each $N \in \mathbb{N}$, define $T_s = T/N$ and $t_k = kT_s$, $g_k = g(t_k)$. Then*

$$\int_0^T g(t)\,\mathrm{d}t \approx \sum_{k=0}^{N-1} g_k T_s, \tag{11.5}$$

and the approximation error tends to zero as $N \to \infty$.

Proof. This is the standard convergence of left-endpoint Riemann sums to the Riemann integral. Since g is Riemann integrable on $[0, T]$, given any $\varepsilon > 0$ there exists a partition mesh size $\delta > 0$ such that for any partition with maximum subinterval length less than δ, the corresponding Riemann sum differs from the integral by less than ε. Choosing N sufficiently large ensures $T_s = T/N < \delta$, so that (11.5) differs from $\int_0^T g(t)\,\mathrm{d}t$ by less than ε. □

11.1.1 Application to the CFT of a pulse

Let $f(t)$ be a finite-span pulse supported on $[0, T]$:

$$f(t) = 0 \quad \text{for } t < 0 \text{ and } t > T. \tag{11.6}$$

Its Fourier transform is

$$F(\omega) = \int_0^T f(t)\,\mathrm{e}^{-i\omega t}\,\mathrm{d}t. \tag{11.7}$$

Apply the rectangular rule with step $T_s = T/N$:

$$t_k = kT_s, \qquad f_k = f(t_k), \qquad 0 \le k \le N-1.$$

Then

$$F(\omega) \approx \sum_{k=0}^{N-1} f_k\,\mathrm{e}^{-i\omega t_k}\,T_s. \tag{11.8}$$

Introduce the frequency sampling

$$\omega_n = n\omega_0, \qquad \omega_0 = \frac{2\pi}{T}, \qquad -\frac{N}{2} < n < \frac{N}{2}. \tag{11.9}$$

Then, for each n,

$$\begin{aligned} F(\omega_n) &\approx \sum_{k=0}^{N-1} f_k\,\mathrm{e}^{-in\omega_0 t_k}\,T_s = T_s \sum_{k=0}^{N-1} f_k \exp\!\left(-in\frac{2\pi}{T}kT_s\right) \\ &= T_s \sum_{k=0}^{N-1} f_k \exp\!\left(-i\frac{2\pi}{N}nk\right). \end{aligned} \tag{11.10}$$

Define the DFT (FFT analysis) of $\{f_k\}$ as

$$F_n = \sum_{k=0}^{N-1} f_k\,\mathrm{e}^{-i\frac{2\pi}{N}nk}, \qquad 0 \le n \le N-1. \tag{11.11}$$

Then (11.10) becomes

$$F(\omega_n) \approx T_s F_n = \frac{T}{N} F_n. \tag{11.12}$$

This yields:

Theorem 51 (FFT approximation of the Fourier transform)**.** *Let $f(t)$ be a finite-span pulse supported on $[0, T]$. Sample f at*

$$t_k = kT_s, \qquad T_s = \frac{T}{N}, \qquad 0 \le k \le N-1,$$

and define $f_k = f(t_k)$. Let F_n be the DFT of $\{f_k\}$:

$$F_n = \sum_{k=0}^{N-1} f_k\,\mathrm{e}^{-i\frac{2\pi}{N}nk}. \tag{11.13}$$

Let $\omega_n = n\omega_0$ with $\omega_0 = 2\pi/T$. Then

$$F(\omega_n) \approx \frac{T}{N} F_n, \qquad -\frac{N}{2} < n < \frac{N}{2}, \tag{11.14}$$

and the approximation error tends to zero as $N \to \infty$.

Proof. Equation (11.8) is the rectangular-rule approximation of the integral defining $F(\omega)$ in (11.7) with step $T_s = T/N$. Substituting $\omega = \omega_n = n\omega_0$ and $t_k = kT_s$ leads to (11.10). Identifying F_n as the DFT of $\{f_k\}$ yields

$$F(\omega_n) \approx T_s F_n = \frac{T}{N} F_n.$$

By Theorem 50, the rectangular rule error tends to zero as $N \to \infty$, so the FFT-based approximation inherits this convergence. □

In words, up to a constant scaling factor T/N, the FFT of the time samples of a pulse provides numerical estimates of the samples of its continuous Fourier transform on a uniform frequency grid.

11.1.2 Periodic waveforms and Fourier series

Let $f_p(t)$ be a periodic waveform of period T_0, with complex Fourier series

$$f_p(t) = \sum_{n=-\infty}^{\infty} C_n \, \mathrm{e}^{in\omega_0 t}, \qquad \omega_0 = \frac{2\pi}{T_0}, \tag{11.15}$$

where

$$C_n = \frac{1}{T_0} \int_0^{T_0} f_p(t) \, \mathrm{e}^{-in\omega_0 t} \, \mathrm{d}t. \tag{11.16}$$

Applying the rectangular rule with $T_s = T_0/N$,

$$t_k = kT_s, \quad f_k = f_p(t_k), \quad 0 \le k \le N-1,$$

gives

$$\begin{aligned} C_n &\approx \frac{1}{T_0} \sum_{k=0}^{N-1} f_k \, \mathrm{e}^{-in\omega_0 t_k} T_s = \frac{T_s}{T_0} \sum_{k=0}^{N-1} f_k \, \mathrm{e}^{-i\frac{2\pi}{T_0} nkT_s} \\ &= \frac{1}{N} \sum_{k=0}^{N-1} f_k \, \mathrm{e}^{-i\frac{2\pi}{N} nk}. \end{aligned} \tag{11.17}$$

Theorem 52 (FFT approximation of Fourier-series coefficients). *Let $f_p(t)$ be periodic with period T_0. Sample f_p over one period at*

$$t_k = kT_s, \qquad T_s = \frac{T_0}{N}, \qquad 0 \le k \le N-1,$$

and define $f_k = f_p(t_k)$. Let F_n be the DFT of $\{f_k\}$:

$$F_n = \sum_{k=0}^{N-1} f_k \, \mathrm{e}^{-i\frac{2\pi}{N} nk}. \tag{11.18}$$

Then

$$C_n \approx \frac{1}{N} F_n, \qquad -\frac{N}{2} < n < \frac{N}{2}, \tag{11.19}$$

and the approximation error tends to zero as $N \to \infty$.

Proof. Using the rectangular rule for C_n gives

$$C_n \approx \frac{1}{T_0} \sum_{k=0}^{N-1} f_p(t_k)\, \mathrm{e}^{-in\omega_0 t_k} T_s.$$

Since $T_s = T_0/N$ and $\omega_0 = 2\pi/T_0$, this becomes

$$C_n \approx \frac{1}{N} \sum_{k=0}^{N-1} f_k\, \mathrm{e}^{-i\frac{2\pi}{N}nk} = \frac{1}{N} F_n.$$

The convergence follows again from Theorem 50. □

For both pulses and periodic waveforms, two implementation details are important:

- Sampling must follow left-endpoint uniform sampling, because the correspondence between the CFT and FFT is derived using that sampling scheme.
- At any sampling instant coinciding with a discontinuity of f, the half-value is used, consistent with the Fourier convergence properties.

11.2 Aliasing of Fourier spectra

For clarity, the following notation is adopted:

$$\begin{aligned} \mathrm{per}(t) &: \text{periodic waveform}, \\ \mathrm{PER}(n) &: \text{its CFT line spectrum (Fourier series coefficients)}, \\ \mathrm{pul}(t) &: \text{finite-span, finite-energy pulse}, \\ \mathrm{PUL}(\omega) &: \text{Fourier transform of } \mathrm{pul}(t). \end{aligned}$$

FFT line spectra obtained from samples of these functions are denoted by

$$\begin{aligned} \mathrm{PER}_n &: \text{FFT line spectrum from samples of } \mathrm{per}(t), \\ \mathrm{PUL}_n &: \text{FFT line spectrum from samples of } \mathrm{pul}(t). \end{aligned}$$

Consider $\mathrm{pul}(t)$ with transform $\mathrm{PUL}(\omega)$. Define an *aliased* spectrum $\mathrm{PUL}_a(\omega)$ by periodically replicating $\mathrm{PUL}(\omega)$ with period Ω_a and summing:

$$\mathrm{PUL}_a(\omega) = \sum_{m=-\infty}^{\infty} \mathrm{PUL}(\omega - m\Omega_a). \tag{11.20}$$

Here $\Omega_a > 0$ is the *aliasing period*. The resulting aliased spectrum $\mathrm{PUL}_a(\omega)$ is periodic with period Ω_a.

Assume $\mathrm{pul}(t)$ is sufficiently regular (for example, absolutely integrable and piecewise smooth), so that $\mathrm{PUL}(\omega) \to 0$ as $|\omega| \to \infty$ (typically $O(1/|\omega|)$ or faster). Therefore, for large enough Ω_a, in the central band $-\Omega_a/2 < \omega < \Omega_a/2$ the influence of remote replicas is small, and $\mathrm{PUL}_a(\omega)$ closely follows the original $\mathrm{PUL}(\omega)$ in that band.

Next, sample $\mathrm{PUL}_a(\omega)$ at frequencies spaced by ω_0:

$$\omega_n = n\omega_0, \qquad n \in \mathbb{Z}, \tag{11.21}$$

where

$$\omega_0 = \frac{\Omega_a}{N}. \tag{11.22}$$

Then

$$\mathrm{PUL}_a(\omega_n) = \sum_{m=-\infty}^{\infty} \mathrm{PUL}(\omega_n - m\Omega_a) = \sum_{m=-\infty}^{\infty} \mathrm{PUL}\big((n-mN)\omega_0\big). \tag{11.23}$$

For sufficiently large Ω_a (equivalently, small ω_0), the overlapping between neighboring replicas is small, and in the central region $-N/2 < n < N/2$ one expects

$$\sum_{m=-\infty}^{\infty} \mathrm{PUL}(\omega_n - m\Omega_a) \approx \mathrm{PUL}(\omega_n), \qquad -\frac{N}{2} < n < \frac{N}{2}, \tag{11.24}$$

with the error decreasing as Ω_a increases.

An analogous construction applies to the CFT line spectrum $\mathrm{PER}(n)$ of a periodic function $\mathrm{per}(t)$. Define

$$\mathrm{PER}_a(n) = \sum_{m=-\infty}^{\infty} \mathrm{PER}(n-mN), \tag{11.25}$$

which is periodic with period N in n. If $\mathrm{per}(t)$ is piecewise smooth over one period, then $\mathrm{PER}(n) \to 0$ as $|n| \to \infty$ (typically like $1/|n|$ for jump discontinuities, and faster for smoother signals). Taking N large then reduces overlap between replicas and yields the approximation

$$\sum_{m=-\infty}^{\infty} \mathrm{PER}(n-mN) \approx \mathrm{PER}(n), \qquad -\frac{N}{2} < n < \frac{N}{2}, \tag{11.26}$$

with error decreasing as N increases.

11.3 The FFT as an estimator for CFT spectra

11.3.1 Pulses

Let $T_s > 0$ and consider the Dirac comb

$$\delta_s(t) = T_s \sum_{k=-\infty}^{\infty} \delta(t-kT_s). \tag{11.27}$$

Using the standard comb pair

$$\sum_{k=-\infty}^{\infty} \delta(t-kT_s) \longleftrightarrow \frac{2\pi}{T_s} \sum_{m=-\infty}^{\infty} \delta(\omega - m\Omega_s), \qquad \Omega_s = \frac{2\pi}{T_s},$$

and multiplying by T_s, we obtain

$$\delta_s(t) \longleftrightarrow 2\pi \sum_{m=-\infty}^{\infty} \delta(\omega - m\Omega_s), \qquad \Omega_s = \frac{2\pi}{T_s}. \tag{11.28}$$

Let $\mathrm{pul}(t)$ be a finite-span pulse supported on $0 < t < T$, and suppose

$$T = NT_s. \tag{11.29}$$

Define the impulse-sampled version

$$\mathrm{pul}_s(t) = \mathrm{pul}(t)\,\delta_s(t). \tag{11.30}$$

Using the sampling property of the delta distribution,

$$\mathrm{pul}_s(t) = \mathrm{pul}(t)\,T_s \sum_{k=-\infty}^{\infty} \delta(t-kT_s) = T_s \sum_{k=-\infty}^{\infty} \mathrm{pul}(kT_s)\delta(t-kT_s). \tag{11.31}$$

Define samples

$$\mathrm{pul}_k = \mathrm{pul}(kT_s), \tag{11.32}$$

so that

$$\mathrm{pul}_s(t) = T_s \sum_{k=-\infty}^{\infty} \mathrm{pul}_k \delta(t - kT_s). \tag{11.33}$$

Method A (time-domain sampling). By shifting and scaling,

$$\mathrm{pul}_s(t) \longleftrightarrow \sum_{k=-\infty}^{\infty} \mathrm{pul}_k\, T_s\, \mathrm{e}^{-i\omega k T_s}. \tag{11.34}$$

Since $\mathrm{pul}(t)$ vanishes outside $[0, T]$, only $k = 0, \dots, N-1$ contribute, hence

$$\mathrm{PUL}_s(\omega) = T_s \sum_{k=0}^{N-1} \mathrm{pul}_k\, \mathrm{e}^{-i\omega k T_s}. \tag{11.35}$$

Method B (multiplication → convolution). Because $\mathrm{pul}_s(t) = \mathrm{pul}(t)\,\delta_s(t)$, in the frequency domain

$$\mathrm{PUL}_s(\omega) = \frac{1}{2\pi}\Big(\mathrm{PUL} * (2\pi \sum_{m=-\infty}^{\infty} \delta(\cdot - m\Omega_s))\Big)(\omega) = \sum_{m=-\infty}^{\infty} \mathrm{PUL}(\omega - m\Omega_s), \tag{11.36}$$

where the constants follow directly from the transform convention used throughout these notes.

Equating the two expressions for $\mathrm{PUL}_s(\omega)$ gives

$$\sum_{m=-\infty}^{\infty} \mathrm{PUL}(\omega - m\Omega_s) = T_s \sum_{k=0}^{N-1} \mathrm{pul}_k\, \mathrm{e}^{-i\omega k T_s}. \tag{11.37}$$

Now sample both sides at

$$\omega = \omega_n = n\omega_0, \qquad \omega_0 = \frac{\Omega_s}{N}, \tag{11.38}$$

so that

$$\omega_0 = \frac{2\pi}{T}, \qquad T = NT_s, \tag{11.39}$$

and $T_s = 2\pi/\Omega_s$.

Substituting $\omega = \omega_n$ and $T_s = T/N$ into (11.37), the left-hand side becomes

$$\sum_{m=-\infty}^{\infty} \mathrm{PUL}(n\omega_0 - m\Omega_s), \tag{11.40}$$

while the right-hand side becomes

$$\begin{aligned} T_s \sum_{k=0}^{N-1} \mathrm{pul}_k\, \mathrm{e}^{-i\omega_n k T_s} &= T_s \sum_{k=0}^{N-1} \mathrm{pul}_k \exp\Big(-in\frac{2\pi}{T} kT_s\Big) \\ &= T_s \sum_{k=0}^{N-1} \mathrm{pul}_k \exp\Big(-i\frac{2\pi}{N} nk\Big). \end{aligned} \tag{11.41}$$

Define the DFT

$$\mathrm{PUL}_n = \sum_{k=0}^{N-1} \mathrm{pul}_k\, \mathrm{e}^{-i\frac{2\pi}{N} nk}. \tag{11.42}$$

Then (11.41) reads

$$T_s \sum_{k=0}^{N-1} \mathrm{pul}_k \, \mathrm{e}^{-i\frac{2\pi}{N}nk} = T_s \, \mathrm{PUL}_n.$$

Therefore (11.37) at $\omega = \omega_n$ becomes

$$\sum_{m=-\infty}^{\infty} \mathrm{PUL}(n\omega_0 - m\Omega_s) = T_s \, \mathrm{PUL}_n. \tag{11.43}$$

Theorem 53 (Relationship between aliased CFT and FFT for pulses)**.** *Let $pul(t)$ be supported on $[0, T]$, and let $PUL(\omega)$ denote its Fourier transform. Sample $pul(t)$ at $t_k = kT_s$ for $0 \le k \le N-1$, where $T = NT_s$, and form the DFT*

$$PUL_n = \sum_{k=0}^{N-1} pul_k \, \mathrm{e}^{-i\frac{2\pi}{N}nk}. \tag{11.44}$$

Let $\Omega_s = 2\pi/T_s$ and $\omega_0 = \Omega_s/N = 2\pi/T$. Then, for all integers n,

$$\sum_{m=-\infty}^{\infty} PUL(n\omega_0 - m\Omega_s) = T_s \, PUL_n. \tag{11.45}$$

Equivalently, the aliased, sampled CFT spectrum equals the FFT spectrum scaled by T_s.

Proof. The derivation above computes the transform of $\mathrm{pul}(t)\delta_s(t)$ in two ways: Method A (sampling and summation), equation (11.35), and Method B (convolution in ω), equation (11.36). Equating them gives (11.37). Sampling at $\omega_n = n\omega_0$ with $\omega_0 = \Omega_s/N$ and $T_s = T/N$ yields (11.43). The identity is independent of the sign convention as long as that convention is applied consistently to all occurrences of PUL. □

As observed in (11.24), for sufficiently large Ω_s one has

$$\sum_{m=-\infty}^{\infty} \mathrm{PUL}(n\omega_0 - m\Omega_s) \approx \mathrm{PUL}(n\omega_0), \qquad -\frac{N}{2} < n < \frac{N}{2}, \tag{11.46}$$

with the approximation improving as Ω_s increases (equivalently, T_s decreases and N increases for fixed T). Combining this with Theorem 53 gives:

Corollary 11 (Using the FFT to estimate a pulse Fourier transform)**.** *Under the hypotheses of Theorem 53, for $-N/2 < n < N/2$,*

$$PUL(n\omega_0) \approx T_s \, PUL_n = \frac{T}{N} PUL_n, \tag{11.47}$$

and the approximation error decreases as N increases (for fixed T).

Proof. The identity (11.45) holds for all n. By the aliasing approximation (11.24), in the central region

$$\sum_{m=-\infty}^{\infty} \mathrm{PUL}(n\omega_0 - m\Omega_s) \approx \mathrm{PUL}(n\omega_0).$$

Substituting this into (11.45) gives

$$\mathrm{PUL}(n\omega_0) \approx T_s \, \mathrm{PUL}_n.$$

Using $T_s = T/N$ yields (11.47). The decay of $\mathrm{PUL}(\omega)$ ensures that increasing Ω_s (and hence N) decreases the contribution of neighboring replicas and thus the aliasing error. □

This is consistent with the result obtained from the rectangular rule in Theorem 51. The two derivations emphasize complementary aspects: one from numerical quadrature, the other from spectral aliasing.

11.3.2 Periodic functions

Now let $\mathrm{per}(t)$ be periodic with period T_0, and let $\mathrm{PER}(n)$ denote its CFT (Fourier-series) coefficients. As shown in Chapter 9, if $\mathrm{per}(t)$ is obtained by periodically repeating a pulse $\mathrm{pul}(t)$ supported in $[0, T_0]$, then

$$\mathrm{PER}(n) = \frac{1}{T_0}\,\mathrm{PUL}\big(n\omega_0\big), \qquad \omega_0 = \frac{2\pi}{T_0}. \tag{11.48}$$

Replacing n by $n - mN$ yields

$$\mathrm{PER}(n - mN) = \frac{1}{T_0}\,\mathrm{PUL}\big((n - mN)\omega_0\big). \tag{11.49}$$

Summing over m gives

$$\sum_{m=-\infty}^{\infty} \mathrm{PER}(n - mN) = \frac{1}{T_0}\sum_{m=-\infty}^{\infty} \mathrm{PUL}\big((n - mN)\omega_0\big). \tag{11.50}$$

On the other hand, by Theorem 53 applied to $\mathrm{pul}(t)$ and $\mathrm{PUL}(\omega)$, one has

$$\sum_{m=-\infty}^{\infty} \mathrm{PUL}\big((n - mN)\omega_0\big) = T_s\,\mathrm{PUL}_n, \tag{11.51}$$

with $T_0 = T = NT_s$ and $\omega_0 = 2\pi/T_0$. Thus

$$\sum_{m=-\infty}^{\infty} \mathrm{PER}(n - mN) = \frac{1}{T_0}T_s\,\mathrm{PUL}_n = \frac{1}{N}\,\mathrm{PUL}_n. \tag{11.52}$$

Now consider numerical samples of $\mathrm{per}(t)$ over one period:

$$t_k = kT_s, \quad T_s = \frac{T_0}{N}, \quad 0 \le k \le N-1, \quad \mathrm{per}_k = \mathrm{per}(t_k).$$

Since $\mathrm{per}(t)$ coincides with $\mathrm{pul}(t)$ on $[0, T_0]$, one has $\mathrm{per}_k = \mathrm{pul}_k$, and hence their DFTs coincide:

$$\mathrm{PER}_n = \mathrm{PUL}_n. \tag{11.53}$$

Therefore,

$$\sum_{m=-\infty}^{\infty} \mathrm{PER}(n - mN) = \frac{1}{N}\,\mathrm{PER}_n. \tag{11.54}$$

Theorem 54 (Relationship between aliased CFT and FFT for periodic functions)**.** *Let $per(t)$ be periodic with period T_0, and let $PER(n)$ denote its CFT coefficients. Sample $per(t)$ over one period at $t_k = kT_s$ with $T_s = T_0/N$, $0 \le k \le N-1$, and form the DFT*

$$PER_n = \sum_{k=0}^{N-1} per_k\, e^{-i\frac{2\pi}{N}nk}. \tag{11.55}$$

Then, for all integers n,

$$\sum_{m=-\infty}^{\infty} PER(n - mN) = \frac{1}{N}\,PER_n. \tag{11.56}$$

In other words, the aliased CFT line spectrum equals the FFT spectrum scaled by $1/N$.

Proof. Since $\mathrm{per}(t)$ is periodic with period T_0 and coincides with $\mathrm{pul}(t)$ on $[0, T_0]$, Theorem 53 applied to $\mathrm{pul}(t)$ yields

$$\sum_{m=-\infty}^{\infty} \mathrm{PUL}\big((n-mN)\omega_0\big) = T_s\,\mathrm{PUL}_n, \qquad \omega_0 = \frac{2\pi}{T_0}.$$

Using the relation (11.48) between $\mathrm{PER}(n)$ and $\mathrm{PUL}(n\omega_0)$, one obtains (11.50). Combining, and using $T_0 = NT_s$, yields

$$\sum_{m=-\infty}^{\infty} \mathrm{PER}(n-mN) = \frac{T_s}{T_0}\,\mathrm{PUL}_n = \frac{1}{N}\,\mathrm{PUL}_n.$$

Since $\mathrm{per}_k = \mathrm{pul}_k$ for $0 \le k \le N-1$, their DFTs are equal: $\mathrm{PER}_n = \mathrm{PUL}_n$. Substituting this gives the desired relation. □

Combining Theorem 54 with the aliasing approximation (11.26),

Corollary 12 (Using the FFT to estimate Fourier-series coefficients). *Under the hypotheses of Theorem 54, for $-N/2 < n < N/2$,*

$$PER(n) \approx \frac{1}{N}\,PER_n, \tag{11.57}$$

and the approximation error decreases as N increases.

Proof. By (11.26),

$$\sum_{m=-\infty}^{\infty} \mathrm{PER}(n-mN) \approx \mathrm{PER}(n), \qquad -\frac{N}{2} < n < \frac{N}{2}.$$

By Theorem 54,

$$\sum_{m=-\infty}^{\infty} \mathrm{PER}(n-mN) = \frac{1}{N}\,\mathrm{PER}_n.$$

Combining the two yields (11.57). The decay of $\mathrm{PER}(n)$ guarantees that the error in (11.26), and hence in (11.57), decreases as N increases. □

This result is consistent with Theorem 52, which was derived from the rectangular rule.

11.4 Inverting CFT spectra via the FFT

The previous sections describe how the FFT can be used to approximate CFT spectra from time-domain samples. A complementary problem is: given a CFT spectrum (either a Fourier transform $\mathrm{PUL}(\omega)$ or a Fourier series spectrum $\mathrm{PER}(n)$), how can one invert it numerically to the time domain using the FFT?

A direct approach would be to sample the CFT spectrum at N points, truncate, and apply the DFT synthesis formula. However, this mixes CFT-based analysis with DFT-based synthesis and introduces truncation and aliasing effects that may be difficult to control.

A more systematic approach is to construct an *approximate FFT-spectrum* by explicitly aliasing the CFT spectrum, so that the constructed discrete spectrum matches as closely as possible the FFT spectrum that would arise from sampling the corresponding time-domain signal. This is then inverted by the FFT synthesis formula in a consistent manner.

11.4.1 Fourier series spectra

Suppose PER(n) is given explicitly by a CFT formula. By Theorem 54, the FFT spectrum corresponding to N time samples of per(t) is

$$\mathrm{PER}_n^{(N)} = N \sum_{m=-\infty}^{\infty} \mathrm{PER}(n-mN), \qquad n = 0,\ldots,N-1. \tag{11.58}$$

In practice, the infinite sum in (11.58) is truncated:

$$\mathrm{PER}_n^{(N)} \approx N \sum_{m=-a}^{a} \mathrm{PER}(n-mN), \tag{11.59}$$

for some integer $a \geq 0$.

Definition 32 (Aliasing level). *The integer a in* (11.59) *is called the* aliasing level. *It specifies how many shifted replicas on each side of the central spectrum are included in the aliasing sum.*

Given N and a, formula (11.59) defines an approximate FFT spectrum $\mathrm{PER}_n^{(N)}$. One then applies the FFT synthesis step to $\{\mathrm{PER}_n^{(N)}\}_{n=0}^{N-1}$ to obtain a sampled version of per(t) over one period. The accuracy is governed by:

- the decay rate of PER(n) (which controls the truncation error in m),
- the choice of N (which controls the spacing of aliases).

If PER(n) decays rapidly and a and N are chosen suitably, the time-domain reconstruction obtained by FFT synthesis can approximate the original periodic function closely.

11.4.2 Fourier transform spectra

A similar procedure applies to Fourier transforms. Suppose PUL(ω) is given explicitly. By Theorem 53, the FFT spectrum for a sampling frequency Ω_s and N points is

$$\mathrm{PUL}_n^{(N)} = \frac{1}{T_s} \sum_{m=-\infty}^{\infty} \mathrm{PUL}\big((n-mN)\omega_0\big), \qquad \omega_0 = \frac{\Omega_s}{N}, \quad T_s = \frac{2\pi}{\Omega_s}. \tag{11.60}$$

Truncating the infinite sum at level a,

$$\mathrm{PUL}_n^{(N)} \approx \frac{1}{T_s} \sum_{m=-a}^{a} \mathrm{PUL}\big((n-mN)\omega_0\big). \tag{11.61}$$

In practice one separates real and imaginary parts,

$$\mathrm{PUL}(\omega) = A(\omega) + iB(\omega),$$

computes

$$A\big((n-mN)\omega_0\big), \quad B\big((n-mN)\omega_0\big),$$

sums over $m = -a,\ldots,a$, and forms $\mathrm{PUL}_n^{(N)}$ as in (11.61). The resulting discrete spectrum is then used as input to FFT synthesis to reconstruct samples of pul(t).

Care is required in choosing T (the effective observation window) and T_s so that time-domain aliasing is controlled. In particular, the periodicity of the DFT in the time domain implies that the reconstructed time-domain signal represents a T-periodic extension of the original pul(t). To avoid overlap between shifted copies in time, T must be chosen sufficiently large relative to the effective time support of pul(t).

In summary, this chapter established the following points:

- The rectangular rule connects the CFT analysis integrals to the FFT analysis sums (Theorems 51 and 52).
- Aliasing of CFT spectra leads to exact identities relating aliased CFT spectra and FFT spectra (Theorems 53 and 54).
- Restricting attention to central index ranges and exploiting the decay of CFT spectra yields controlled approximations from FFT outputs to CFT spectra (Corollaries 11 and 12).
- The same aliasing mechanism can be used in the opposite direction, to construct discrete FFT spectra from closed-form CFT expressions and then invert them to the time domain by FFT synthesis.

12 The Errors in Fast Fourier Transform Estimation

12.1 Introduction

In the previous chapter, the fast Fourier transform (FFT) was related to its continuous Fourier transform (CFT) counterparts through the following aliasing identities.

Pulses. Let $f(t)$ be a finite-span pulse supported on $[0,T]$, with Fourier transform $F(\omega)$, and let F_n be the FFT spectrum obtained from N uniform samples of $f(t)$ over $[0,T)$. Then

$$\sum_{m=-\infty}^{\infty} F\big[(n-mN)\omega_0\big] = T_s F_n, \qquad \omega_0 = \frac{2\pi}{T}, \quad T_s = \frac{T}{N}, \tag{12.1}$$

where $n \in \mathbb{Z}$.

Periodic functions. Let $f_p(t)$ be periodic of period T, with complex Fourier-series coefficients $F_p(n)$, and let F_n be the FFT line spectrum obtained from N samples of $f_p(t)$ over one period. Then

$$\sum_{m=-\infty}^{\infty} F_p(n-mN) = \frac{1}{N} F_n, \qquad n \in \mathbb{Z}. \tag{12.2}$$

From these alias relations, the following estimation statements were obtained in Chapter 12.

Pulse spectra. For $-N/2 < n < N/2$,

$$F(n\omega_0) \approx T_s F_n = \frac{T}{N} F_n. \tag{12.3}$$

Fourier-series coefficients. For $-N/2 < n < N/2$,

$$F_p(n) \approx \frac{1}{N} F_n. \tag{12.4}$$

Empirical rules for bounding the errors in these approximations exist in the literature. The purpose of this chapter is to derive algebraic expressions for the *relative errors* which are exact for a certain class of pulses and periodic functions, and asymptotically valid for a broader class. In addition, it is shown that the estimation errors can be reduced by choosing appropriate positions for the FFT sampling instants.

12.2 Relative error

For a generic scalar quantity, the relative error of an estimate is defined as

$$\text{relative error} = \frac{\text{estimate} - \text{exact}}{\text{exact}}. \tag{12.5}$$

Pulse spectra. For the pulse case, with exact spectrum $F(\omega)$ and FFT-based estimate $T_s F_n$ at $\omega = n\omega_0$, define

$$E_n^{(\text{pulse})} = \frac{T_s F_n - F(n\omega_0)}{F(n\omega_0)}. \tag{12.6}$$

Using (12.1), this can be written as

$$E_n^{(\text{pulse})} = \frac{\sum_{m=-\infty}^{\infty} F\big[(n-mN)\omega_0\big] - F(n\omega_0)}{F(n\omega_0)}. \tag{12.7}$$

The $m = 0$ term cancels, leaving

$$E_n^{(\text{pulse})} = \frac{\sum_{\substack{m=-\infty \\ m\neq 0}}^{\infty} F\big[(n-mN)\omega_0\big]}{F(n\omega_0)}. \tag{12.8}$$

Periodic spectra. Similarly, for the periodic case, with exact coefficient $F_p(n)$ and FFT-based estimate F_n/N, define

$$E_n^{(\text{per})} = \frac{F_n/N - F_p(n)}{F_p(n)}. \tag{12.9}$$

Using (12.2),

$$E_n^{(\text{per})} = \frac{\sum_{m=-\infty}^{\infty} F_p(n-mN) - F_p(n)}{F_p(n)} = \frac{\sum_{\substack{m=-\infty \\ m\neq 0}}^{\infty} F_p(n-mN)}{F_p(n)}. \tag{12.10}$$

In both cases it is understood that the denominators $F(n\omega_0)$ and $F_p(n)$ are nonzero in the range of interest.

12.3 Canonical pulses and order of continuity

Definition 33 (Canonical-k pulse)**.** *Let $f(t)$ be a finite-span pulse sampled with left-endpoint uniform sampling at step T_s, and suppose:*

- *all sampling instants are $t_j = jT_s$, $0 \le j \le N-1$;*
- *at any sampling instant coinciding with a discontinuity of f, the sample value supplied to the FFT is the average of the one-sided limits (half-value);*
- *the sample at $t = 0$ is the average of $f(0)$ and $f(T)$.*

If the $(k+1)$st derivative $f^{(k+1)}(t)$ consists only *of Dirac deltas, each located at a sampling instant $t = g_j T_s$ with integer g_j, then f is called* canonical-k.

Example 11. *Any rectangular pulse $\mathrm{Rect}\big((t-t_0)/T\big)$ whose endpoints lie exactly at sampling instants is canonical-0, since its first derivative consists only of jumps at those instants and hence is a finite sum of Dirac deltas located at sampling instants.*

Example 12. *A piecewise-linear pulse made of ramps and constant segments such that all break points (changes of slope) lie at sampling instants, and whose second derivative consists of Dirac deltas at those break points, is canonical-1.*

Any periodic function $f_p(t)$ of period T can be viewed as the periodic repetition of a single-period defining pulse $f(t)$ on $[0, T]$. Accordingly, f_p is called canonical-k if its defining pulse f is canonical-k in the sense of Definition 33.

Definition 34 (Order of continuity)**.** *A pulse $f(t)$ is said to be* continuous of order k *(or continuous-k) if f and its derivatives up to order $k-1$ are continuous everywhere, while the kth derivative has discontinuities.*

Example 13.

- *Rectangular pulses are continuous-0.*
- *Canonical-k pulses are continuous-k.*
- *A triangular pulse is continuous-1: f is continuous, while $f'(t)$ has jump discontinuities.*

For continuous-0 pulses, one typically has $|F(\omega)| = O(1/|\omega|)$ as $|\omega| \to \infty$; for continuous-1 pulses, typically $|F(\omega)| = O(1/\omega^2)$.

12.4 Error expressions for canonical pulses and periodic functions

12.4.1 Canonical-k pulses

Let $f(t)$ be a canonical-k pulse with $(k+1)$st derivative

$$f^{(k+1)}(t) = \sum_{j=1}^{r} p_j\, \delta(t - g_j T_s), \tag{12.11}$$

where $p_j \in \mathbb{C}$ are the weights, $g_j \in \{0, 1, \ldots, N-1\}$ are the sample indices at which the Dirac deltas occur, and r is finite. Applying the Fourier transform yields

$$(i\omega)^{k+1} F(\omega) = \sum_{j=1}^{r} p_j\, \mathrm{e}^{-i\omega g_j T_s}, \tag{12.12}$$

so that

$$F(\omega) = \frac{1}{(i\omega)^{k+1}} \sum_{j=1}^{r} p_j\, \mathrm{e}^{-i\omega g_j T_s}. \tag{12.13}$$

Substitute (12.13) into the relative-error expression (12.8) for pulses:

$$E_n^{(\text{pulse})} = \frac{\displaystyle\sum_{\substack{m=-\infty \\ m\neq 0}}^{\infty} F\big((n-mN)\omega_0\big)}{F(n\omega_0)} = \frac{\displaystyle\sum_{\substack{m=-\infty \\ m\neq 0}}^{\infty} \frac{1}{\big(i(n-mN)\omega_0\big)^{k+1}} \sum_{j=1}^{r} p_j\, \mathrm{e}^{-i(n-mN)\omega_0 g_j T_s}}{\displaystyle\frac{1}{(in\omega_0)^{k+1}} \sum_{j=1}^{r} p_j\, \mathrm{e}^{-in\omega_0 g_j T_s}}. \tag{12.14}$$

Note that

$$(n-mN)\omega_0 g_j T_s = (n-mN)\frac{2\pi}{T} g_j T_s = (n-mN)\frac{2\pi}{N} g_j = n\frac{2\pi}{N} g_j - mN\frac{2\pi}{N} g_j = n\frac{2\pi}{N} g_j - 2\pi m g_j.$$

Hence

$$\mathrm{e}^{-i(n-mN)\omega_0 g_j T_s} = \mathrm{e}^{-in\omega_0 g_j T_s}\, \mathrm{e}^{+i2\pi m g_j} = \mathrm{e}^{-in\omega_0 g_j T_s}, \tag{12.15}$$

since $e^{i2\pi m g_j} = 1$ for integers m, g_j. Therefore, the exponential factors in numerator and denominator of (12.14) cancel, giving

$$E_n^{(\text{pulse})} = \frac{\displaystyle\sum_{\substack{m=-\infty\\ m\neq 0}}^{\infty} \frac{1}{\big(i(n-mN)\omega_0\big)^{k+1}}}{\displaystyle\frac{1}{(in\omega_0)^{k+1}}} = \sum_{\substack{m=-\infty\\ m\neq 0}}^{\infty} \left(\frac{n}{n-mN}\right)^{k+1}. \tag{12.16}$$

12.4.2 Canonical-k periodic functions

Let $f_p(t)$ be periodic with period T, and define its single-period restriction as $f(t)$ on $[0,T]$. Assume $f(t)$ is canonical-k and denote by $F(\omega)$ its Fourier transform. The CFT Fourier-series coefficients of $f_p(t)$ are

$$F_p(n) = \frac{1}{T}F(n\omega_0), \qquad \omega_0 = \frac{2\pi}{T}. \tag{12.17}$$

Repeating the same steps for $F_p(n)$ as for $F(\omega)$, one obtains the same error expression (12.16) for $E_n^{(\text{per})}$.

We summarize these observations as follows.

Theorem 55 (Error expression for canonical-k functions)**.** *Let either*

- *$f(t)$ be a canonical-k pulse supported on $[0,T]$, or*
- *$f_p(t)$ be a periodic function of period T whose defining pulse on $[0,T]$ is canonical-k.*

Let $F(\omega)$ be the CFT of $f(t)$ and $F_p(n)$ the CFT coefficients of $f_p(t)$. Let F_n denote the FFT spectrum obtained from N left-endpoint uniform samples, with half-values supplied at discontinuities. Assume $F(n\omega_0) \neq 0$ (pulse) or $F_p(n) \neq 0$ (periodic) in the range of interest. Then, for both pulses and periodic functions, the relative error in the nth spectral estimate is

$$E_n = \sum_{\substack{m=-\infty\\ m\neq 0}}^{\infty} \left(\frac{n}{n-mN}\right)^{k+1}, \qquad -\frac{N}{2} < n < \frac{N}{2}. \tag{12.18}$$

12.5 Properties of the error for canonical-k functions

Introduce the normalized index

$$Z = \frac{n}{N}. \tag{12.19}$$

The central index range $-N/2 < n < N/2$ corresponds to $|Z| < \frac{1}{2}$.

From (12.18),

$$E_n = \sum_{\substack{m=-\infty\\ m\neq 0}}^{\infty} \left(\frac{n}{n-mN}\right)^{k+1} = \sum_{\substack{m=-\infty\\ m\neq 0}}^{\infty} \left(\frac{Z}{Z-m}\right)^{k+1} \equiv E(Z,k). \tag{12.20}$$

We now record some basic properties, focusing first on small values of k.

Canonical-0 case. For $k = 0$,

$$E(Z,0) = \sum_{\substack{m=-\infty \\ m\neq 0}}^{\infty} \frac{Z}{Z-m}, \qquad |Z| < \tfrac{1}{2}. \tag{12.21}$$

This series behaves like $\sum_m Z/m$ for large $|m|$, so the individual terms decay as $1/m$ and the symmetric summation must be interpreted in the principal-value sense. Rewriting symmetrically,

$$E(Z,0) = 2Z^2 \sum_{m=1}^{\infty} \frac{1}{Z^2-m^2}, \qquad |Z| < \tfrac{1}{2}, \tag{12.22}$$

and the terms then decay as $1/m^2$.

Canonical-1 case. For $k = 1$,

$$E(Z,1) = \sum_{\substack{m=-\infty \\ m\neq 0}}^{\infty} \left(\frac{Z}{Z-m}\right)^2. \tag{12.23}$$

Canonical-2 and canonical-3. Similarly, for $k = 2$ and $k = 3$ one has

$$E(Z,2) = \sum_{\substack{m=-\infty \\ m\neq 0}}^{\infty} \left(\frac{Z}{Z-m}\right)^3, \tag{12.24}$$

$$E(Z,3) = \sum_{\substack{m=-\infty \\ m\neq 0}}^{\infty} \left(\frac{Z}{Z-m}\right)^4. \tag{12.25}$$

In all cases, the series converge absolutely for $|Z| < \frac{1}{2}$ when written in symmetric form (pairing $\pm m$).

From these series representations one can infer:

- For $k = 0, 1$, the series converge essentially like $\sum 1/m^2$ once written in symmetric form, and for $k \geq 2$ the decay is at least $\sum 1/m^3$.
- The errors $E(Z,k)$ are real for canonical-k pulses and periodic functions.
- $E(0,k) = 0$ for all k, so the error in the DC spectral element is zero, regardless of N.
- For even k, $E(Z,k)$ is negative; for odd k, $E(Z,k)$ is positive (for $0 < Z < \frac{1}{2}$).
- $E(Z,k)$ is an even function of Z, hence $E(-Z,k) = E(Z,k)$.

12.6 Closed-form Z-curves via trigonometric identities

The series in (12.21)–(12.25) can be expressed in closed form using classical identities from complex analysis. A standard result (see, e.g., Ahlfors or MacRobert) is

$$\sum_{m=1}^{\infty} \frac{2Z^2}{Z^2-m^2} = \pi Z \cot(\pi Z) - 1, \qquad Z \notin \mathbb{Z}. \tag{12.26}$$

This series is precisely the symmetric version of (12.21), so

$$E(Z,0) = \pi Z \cot(\pi Z) - 1, \qquad |Z| < \tfrac{1}{2}. \tag{12.27}$$

Differentiating (12.26) with respect to Z and using standard identities,

$$\sum_{\substack{m=-\infty \\ m\neq 0}}^{\infty} \frac{1}{(Z-m)^2} = \frac{\pi^2}{\sin^2(\pi Z)}. \tag{12.28}$$

In terms of $E(Z,1)$,

$$E(Z,1) = Z^2 \sum_{\substack{m=-\infty \\ m\neq 0}}^{\infty} \frac{1}{(Z-m)^2} = \frac{\pi^2 Z^2}{\sin^2(\pi Z)} - 1, \qquad |Z| < \tfrac{1}{2}. \tag{12.29}$$

Further differentiation yields higher-order sums and gives:

$$E(Z,2) = \pi^3 Z^3 \csc^2(\pi Z)\cot(\pi Z) - 1, \qquad |Z| < \tfrac{1}{2}, \tag{12.30}$$

$$E(Z,3) = \pi^4 Z^4 \csc^2(\pi Z)\left[\cot^2(\pi Z) + \frac{1}{3}\right] - 1, \qquad |Z| < \tfrac{1}{2}. \tag{12.31}$$

Theorem 56 (Closed-form error expressions for canonical-k functions)**.** *For canonical-k pulses and periodic functions, the relative error $E(Z,k)$, defined in* (12.20)*, admits the following closed forms for $|Z| < \frac{1}{2}$:*

$$E(Z,0) = \pi Z \cot(\pi Z) - 1, \tag{12.32}$$

$$E(Z,1) = \frac{\pi^2 Z^2}{\sin^2(\pi Z)} - 1, \tag{12.33}$$

$$E(Z,2) = \pi^3 Z^3 \csc^2(\pi Z)\cot(\pi Z) - 1, \tag{12.34}$$

$$E(Z,3) = \pi^4 Z^4 \csc^2(\pi Z)\left[\cot^2(\pi Z) + \frac{1}{3}\right] - 1. \tag{12.35}$$

In practice, these formulas are useful for plotting *log-linear Z-curves* of $|E(Z,k)|$ versus Z for $k = 0,1,2,3$. Such curves can be used to read off the relative error corresponding to a given ratio $Z = n/N$.

Some qualitative features observed from these curves include:

- $|E(Z,1)|$ is slightly larger than $|E(Z,0)|$ over much of the interval $0 < Z < \frac{1}{2}$, despite faster decay of the corresponding CFT spectrum for canonical-1 functions.
- If n is fixed and N increases, then $Z = n/N \to 0$ and $E(Z,k) \to 0$. Hence the error in the nth spectral element can be made arbitrarily small by increasing N for fixed n.
- If $Z = n/N$ is kept fixed while N increases, the error $E(Z,k)$ remains fixed; enlarging N does not reduce the relative error for that fixed ratio Z. For example, as $Z \to \frac{1}{2}^-$:

$$E\left(\tfrac{1}{2},0\right) \to -100.00\%, \qquad E\left(\tfrac{1}{2},1\right) \to 146.74\%,$$
$$E\left(\tfrac{1}{2},2\right) \to -100.00\%, \qquad E\left(\tfrac{1}{2},3\right) \to 102.94\%.$$

12.7 Asymptotic behavior for noncanonical functions

The previous results were derived for canonical functions. Many practically relevant signals are not canonical but satisfy weaker smoothness conditions.

Theorem 57 (Asymptotic error for continuous-0 pulses). *Let $f(t)$ be finite-span and piecewise continuous of order 0 on $[0,T]$, with the following properties:*

- *$f(t)=0$ for $t\notin[0,T]$.*
- *$f(t)$ is twice continuously differentiable on each open interval of continuity.*
- *All discontinuities (break points) lie at sampling instants $t=kT_s$.*

Let $F(\omega)$ be the Fourier transform of $f(t)$, and suppose $F(n\omega_0)\neq 0$ in the range of interest. Let $E_n(Z)$ be the relative error in the FFT estimate of $F(n\omega_0)$ when N samples are used, with $Z=n/N$ fixed. Then, for each fixed Z,

$$\lim_{N\to\infty} E_n(Z)=E(Z,0), \tag{12.36}$$

where $E(Z,0)$ is the canonical-0 error function (12.32).

Thus, for noncanonical but continuous-0 pulses (or periodic functions with continuous-0 defining pulses), the canonical-0 error curve still describes the asymptotic behavior of the relative error as $N\to\infty$, provided $Z=n/N$ is held fixed.

The same phenomenon occurs for higher orders of continuity.

Theorem 58 (Asymptotic error for continuous-k pulses). *Let $f(t)$ be finite-span and continuous of order k on $[0,T]$, with the following properties:*

- *$f(t)=0$ for $t\notin[0,T]$.*
- *The $(k-1)$st derivative $f^{(k-1)}(t)$ is $(k+2)$ times continuously differentiable on each open interval of continuity.*
- *All break points lie at sampling instants $t=\ell T_s$ for integers ℓ.*

Let $F(\omega)$ be the Fourier transform of $f(t)$, and suppose $F(n\omega_0)\neq 0$ in the range of interest. Let $E_n(Z)$ be the relative error in the FFT estimate at $\omega=n\omega_0$ with $Z=n/N$ fixed. Then

$$\lim_{N\to\infty} E_n(Z)=E(Z,k), \tag{12.37}$$

where $E(Z,k)$ is the canonical-k error function defined in (12.20) *and given in closed form by Theorem 56 for $k=0,1,2,3$.*

In both theorems, the statements extend to periodic functions by interpreting $f(t)$ as the single-period defining pulse and $F(\omega)$ as its finite-interval Fourier transform.

12.8 Sampling when break points are not at sampling instants

The previous asymptotic results assumed that all break points lie at sampling instants. In many practical scenarios, break points fall in the interior of sampling intervals. This modifies the error expressions.

A representative effect can be illustrated as follows. For a canonical-k function, the error series has the form

$$E(Z,k)=\sum_{\substack{m=-\infty\\ m\neq 0}}^{\infty}\left(\frac{Z}{Z-m}\right)^{k+1}.$$

If the sampling grid is shifted such that break points lie at the *centers* of sampling intervals rather than at sampling instants, the contribution of each alias term acquires an additional factor $(-1)^m$. The error then becomes

$$\widetilde{E}(Z,k) = \sum_{\substack{m=-\infty \\ m\neq 0}}^{\infty} (-1)^m \left(\frac{Z}{Z-m}\right)^{k+1}. \tag{12.38}$$

In symmetric form,

$$\widetilde{E}(Z,k) = Z^{k+1}\sum_{m=1}^{\infty}(-1)^m\left[\frac{1}{(Z-m)^{k+1}} + \frac{1}{(Z+m)^{k+1}}\right], \tag{12.39}$$

and the summand now alternates in sign. This reduces the absolute magnitude of the sum compared to the nonalternating case. Consequently, placing all break points at the centers of sampling intervals systematically reduces the relative error.

The same mechanism appears in concrete examples such as a gated cosine or a rectangular pulse, where numerical evaluations show that shifting the sampling grid so that discontinuities fall at midpoints can reduce error magnitudes by roughly a factor of two over a broad range of Z.

12.9 Error correction for canonical functions

For canonical functions the error expressions are exact, so one can invert them to correct FFT estimates and recover the exact CFT values.

12.9.1 Rectangular pulse

Consider the rectangular pulse

$$f(t) = \mathrm{Rect}\left(\frac{t}{T}\right),$$

with Fourier transform

$$F(\omega) = T\,\mathrm{Sa}\left(\frac{\omega T}{2}\right).$$

For a canonical-0 sampling (endpoints at sampling instants, half-values used), the alias relation (12.1) and the canonical-0 error expression (12.27) imply

$$T_s F_n = F(n\omega_0)\big[1 + E(Z,0)\big] = F(n\omega_0)\big[\pi Z\cot(\pi Z)\big], \tag{12.40}$$

so that

$$F(n\omega_0) = \frac{T_s F_n}{\pi Z\cot(\pi Z)}, \qquad Z = \frac{n}{N}. \tag{12.41}$$

In words, dividing the FFT estimate $T_s F_n$ by $\pi Z\cot(\pi Z)$ yields the exact value of $F(n\omega_0)$ at each n for which $F(n\omega_0) \neq 0$.

In numerical experiments, even for relatively small N (e.g. $N = 64$), applying (12.41) recovers the exact transform values of the rectangular pulse to machine precision.

12.10 Effect of zero padding on estimation errors

In many applications, the number of available samples is not a power of two. A common practice is to append zeros to the data in order to reach the next power of two for efficient FFT computation. This operation is often referred to as *zero padding* or *adding white space*.

Zero padding also refines the frequency sampling grid and can provide a smoother visual representation of the spectrum. However, when interpreting FFT outputs as *estimates* of continuous Fourier transforms, the following fact must be noted:

> Zero padding does not add spectral information; it only changes the frequency grid on which the same underlying estimator is sampled.

In the error formulas of this chapter, the relative error is governed by the normalized index $Z = n/N$. Therefore, at fixed Z the canonical error curves $E(Z, k)$ are unchanged by zero padding. What changes is the set of available frequency bins (finer spacing in ω), not the intrinsic estimation law.

Thus, while zero padding is useful for display and interpolation of discrete spectra, it should be used with care if the goal is to approximate continuous-time Fourier transforms with controlled relative error.