Electromagnetic Theory

Chapter 2 –Propagation and Transmission

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2.1 Plane Waves

2.1.1 Plane Wave Solution

Plane waves are solutions to scalar homogeneous Helmholtz equation (1.4.18) in rectangular coordninate system with

$$\nabla^2 = \partial_x^2 + \partial_y^2 + \partial_z^2 \tag{2.1.1}$$

Using separation of variables

$$\psi = X(x)Y(y)Z(z) \tag{2.1.2}$$

And substituting back in to (2.1.1), we get

$$\frac{1}{X}\frac{d^2X}{dx^2} + \frac{1}{Y}\frac{d^2Y}{dy^2} + \frac{1}{Z}\frac{d^2Z}{dz^2} + k^2 = 0 \tag{2.1.3}$$

2.1.1 Plane Wave Solution

We can now write the following since the functions are independent of each other

$$\frac{d^2\Xi}{d\xi^2} + k_\xi^2 \Xi = 0 \tag{2.1.4}$$

where $\Xi \coloneqq X, Y, Z$ and $\xi \coloneqq x, y, z$. The seperation constants are related by

$$k_x^2 + k_y^2 + k_z^2 = k^2 \tag{2.1.5}$$

We define the wavevector accordingly

$$\vec{k} = k\hat{k} = k_x\hat{x} + k_y\hat{y} + k_z\hat{z} \tag{2.1.6}$$

2.1.1 Plane Wave Solution

From (2.1.4), we get the elementary plane wave solution

$$\psi = e^{\pm i(k_x x + k_y y + k_z z)} = e^{\pm i\vec{k}\cdot\vec{r}} \tag{2.1.7}$$

Transforming the phasor into time domain (with $e^{i\omega t}$ convention), we can show that $e^{i\vec{k}\cdot\vec{r}}$ is plane wave propagating in $-\vec{k}$ direction and $e^{-i\vec{k}\cdot\vec{r}}$ is plane wave propagating in $+\vec{k}$ direction. We choose the $+\vec{k}$ direction in our exposition, then electric field can be expressed as

$$\vec{E} = \vec{E}_0 \psi = \vec{E}_0 e^{-i\vec{k}\cdot\vec{r}} \tag{2.1.8}$$

where $\vec{E}_0$ is a constant vector.

2.1.1 Plane Wave Solution

Noted that for plane wave solutions the del operator can be replaced by

$$\vec{\nabla} \Rightarrow -i\vec{k} \tag{2.1.9}$$

Then from $\vec{\nabla} \cdot \vec{E} = 0$ we get

$$\vec{\nabla} \cdot \left(\vec{E}_0 e^{-i\vec{k}\cdot\vec{r}}\right) = -ie^{-i\vec{k}\cdot\vec{r}}\left(\vec{k} \cdot \vec{E}_0\right) = 0 \tag{2.1.10}$$

For the magnetic field, we have

$$\vec{H} = \frac{i}{\omega\mu}\vec{\nabla} \times \vec{E} = \frac{1}{\omega\mu}\vec{k} \times \vec{E} = \frac{1}{\eta}\hat{k} \times \vec{E} = \vec{H}_0 e^{-i\vec{k}\cdot\vec{r}} \tag{2.1.11}$$

with $\vec{H}_0 = \frac{1}{\eta}\hat{k} \times \vec{E}_0$ and $\eta = \sqrt{\mu/\epsilon}$. (2.1.10) and (2.1.11) imply that $\vec{k}$, $\vec{E}_0$ and $\vec{H}_0$ are penedicular to each other, and the direction of the $\vec{E}_0 \times \vec{H}_0$ aligning with the direction of $\hat{k}$.

2.1.1 Plane Wave Solution

Plugging (2.1.8) and (2.1.11) into the source-free Maxwell equations, we get a general relationship between $\vec{E}$, $\overrightarrow{H}$ and $\vec{k}$

$$\vec{k} \times \vec{E} = \omega\mu\vec{H} \tag{2.1.12}$$

$$\vec{k} \times \vec{H} = -\omega\epsilon\vec{E} \tag{2.1.13}$$

$$\vec{k} \cdot \vec{E} = 0 \tag{2.1.14}$$

$$\vec{k} \cdot \vec{H} = 0 \tag{2.1.15}$$

Note that the above equations also apply to $(\vec{E}_0, \vec{H}_0)$.

2.1.2 Angular Spectrum Representation

The angular spectrum representation is a mathematical technique to describe EM fields as superposition of plane waves.

To expand an EM field, we can select an arbitrary *z*-axis and evaulate the following 2D inverse Fourier transform on the plane perpendicular to the *z*-axis and acquire the spectral field

$$\vec{\mathcal{E}}\left(k_x, k_y, z\right) = \frac{1}{4\pi^2}\iint_{-\infty}^{\infty} \vec{E}(x, y, z) e^{i\left(k_x x + k_y y\right)} dxdy \qquad (2.1.16)$$

where x, y are the transverse position components in Cartesian coordinates and k_x, k_y are the corresponding spatial frequencies in *k* domain.

2.1.2 Angular Spectrum Representation

Then, the EM field can be written as the 2D Fourier transform of the spectral field

$$\vec{E}(x,y,z) = \iint_{-\infty}^{\infty} \vec{\mathcal{E}}\left(k_x,k_y,z\right) e^{-i\left(k_x x + k_y y\right)} dk_x dk_y \qquad (2.1.17)$$

By inserting (2.1.17) into the homogeneous vector Helmholtz equation $(\nabla^2 + k^2)\vec{E} = 0$ and define

$$k_z \equiv \begin{cases} \sqrt{k^2 - k_x^2 - k_y^2}, & k_x^2 + k_y^2 \le k^2 \\ -i\sqrt{k_x^2 + k_y^2 - k^2}, & k_x^2 + k_y^2 > k^2 \end{cases} \qquad (2.1.18)$$

2.1.2 Angular Spectrum Representation

We can get scalar Helmholtz equation for each Cartesian component of $\vec{\mathcal{E}}$ ($\xi \coloneqq x, y, z$)

$$\left(\frac{d^2}{dz^2} + k_z^2\right) \mathcal{E}_\xi = 0 \tag{2.1.19}$$

Thus, the spectral field $\vec{\mathcal{E}}$ propagates along the z-axis as

$$\vec{\mathcal{E}}\left(k_x, k_y, z\right) = \vec{\mathcal{E}}\left(k_x, k_y, 0\right) e^{\mp i k_z z} \tag{2.1.20}$$

The minus sign in (2.1.18) when $k_x^2 + k_y^2 > k^2$ ensures that the spectral field remains finite when $z \to \pm\infty$.

2.1.2 Angular Spectrum Representation

By inserting (2.1.20) into (2.1.17) we finally get the angular spectrum representation for the electric field

$$\vec{E}(x,y,z) = \iint_{-\infty}^{\infty} \vec{\mathcal{E}}\left(k_x, k_y, 0\right) e^{-i\left(k_x x + k_y y \pm i k_z z\right)} dk_x dk_y \quad (2.1.21)$$

We can get the magnetic field by (2.1.12). Note that the angular spectrum representation satisfies the Helmholtz equation but not the full set of Maxwell equations necessarily. To ensure consistency with Maxwell equations, (2.1.14) and (2.1.15) impose the condition that the wavevector $\vec{k}$ is perpendicular to the spectral amplitudes.

2.1.2 Angular Spectrum Representation

A further analysis of the plane wave factor $e^{-i(k_x x+k_y y\pm ik_z z)}$ gives two characteristic solutions

$$\begin{cases} e^{-i(k_x x+k_y y\pm ik_z z)}, & k_x^2+k_y^2 \le k^2 \\ e^{-i(k_x x+k_y y)}e^{-|k_z z|}, & k_x^2+k_y^2 > k^2 \end{cases} \tag{2.1.22}$$

The first term in (2.1.22) corresponds to the propagating wave and the second term corresponds to the evanesent wave.

2.1.3 Polarization

Consider a +*z*-propagated plane wave

$$\vec{E} = \left(E_x \hat{x} + E_y \hat{y}\right) e^{-ikz} \tag{2.1.23}$$

where E_x, E_y are complex-valued with the following expression

$$\begin{cases} E_x = |E_x| e^{i\delta_x} \\ E_y = \left|E_y\right| e^{i\delta_y} \end{cases} \tag{2.1.24}$$

Note that the amplitudes $|E_x|, \left|E_y\right|$ and phases δ_x, δ_y can be arbitrary. To simplify analysis, we considered three types of polarizations:

2.1.3 Polarization

A. Linear Polarization

For linearly polarized field we have

$$\delta_x - \delta_y = n\pi \tag{2.1.25}$$

with $n \in \mathbb{Z}$.

B. Circular Polarization

For circularly polarized field we have

$$\begin{cases} \delta_x - \delta_y = \dfrac{n\pi}{2} \\ |E_x| = |E_y| \end{cases} \tag{2.1.26}$$

2.1.3 Polarization

To characterize the direction of rotation, we considered the following expression with $E_0 \in \mathbb{R}$

$$\vec{E}_{RC} = (E_0\hat{x} - iE_0\hat{y})e^{-ikz} \tag{2.1.27a}$$

$$\vec{E}_{LC} = (E_0\hat{x} + iE_0\hat{y})e^{-ikz} \tag{2.1.27b}$$

To illustrate that $\vec{E}_{RC}$ corresponds to the right-hand circular polarization (RHCP) and $\vec{E}_{LC}$ the left-hand circular polarization (LHCP), we can transform (2.1.27) back to time domain

$$\vec{E}_{RC}(z,t) = E_0[\cos(\omega t - kz)\,\hat{x} + \sin(\omega t - kz)\,\hat{y}] \tag{2.1.28a}$$

$$\vec{E}_{LC}(z,t) = E_0[\cos(\omega t - kz)\,\hat{x} - \sin(\omega t - kz)\,\hat{y}] \tag{2.1.28b}$$

2.1.3 Polarization

By letting $z = 0$, as t progresses, we can show that $\vec{E}_{RC}$ rotates counterclockwise and $\vec{E}_{LC}$ rotates clockwise on the *x*-*y* plane when seen from the +*z* axis. Note that the optics community follows the opposite convention.

C. Elliptical Polarization

Elliptical polarization is the general case that occurs when conditions (2.1.25) and (2.1.26) are not met. However, a wave that is neither linearly nor circularly polarized is not necessarily elliptically polarized. Most natural electromagnetic waves, such as daylight, are unpolarized.

2.1.4 Wave Velocities

Consider a plane wave propagating in the z-direction, it's time-domain form can be expressed as

$$\vec{E}(z,t) = \Re\left[\vec{E}_0(z)e^{i(\omega t - kz)}\right] \tag{2.1.29}$$

The constant phase plane is determined by

$$\omega t - kz = C \tag{2.1.30}$$

where C is a constant. Take derivative respect to t for both sides of (2.1.30), we get the phase velocity

$$v_p = \frac{dz}{dt} = \frac{\omega}{k} = \frac{1}{\sqrt{\mu\epsilon}} \tag{2.1.31}$$

v_p remains constant if ϵ and μ do not vary with frequency, in which case the medium is termed non-dispersive.

2.1.4 Wave Velocities

A simple harmonic field carries no information. Communication requires modulating a carrier, introducing a frequency spread. Group velocity is relevant when this spread is narrow.

Consider a narrow-band signal $s(t)$ modulated on a high frequency $e^{i\omega_0 t}$. We are analyzing the propagation of the packet $s(t)e^{i\omega_0 t}$. Let the Fourier transform of $s(t)$ be $S(\omega)$, then the Fourier transform of $s(t)e^{i\omega_0 t}$ will be $S(\omega - \omega_0)$. Suppose the signal went through a distance r and was received by a receiver, then the field at the receiver can be expressed as

$$S_r(\omega, k) = S(\omega - \omega_0)e^{-ikr} \tag{2.1.32}$$

2.1.4 Wave Velocities

Note that the wavenumber k is also a function of ω. For narrow-band cases, we can expand k with

$$k \approx k(\omega_0) + \frac{dk}{d\omega}\Big|_{\omega=\omega_0}(\omega - \omega_0) = k_0 + k'(\omega - \omega_0) \qquad (2.1.33)$$

Plug (2.1.33) into (2.1.32), we have

$$S_r(\omega, k) = S(\omega - \omega_0)e^{-ik_0 r}e^{-ik'(\omega-\omega_0)r} \qquad (2.1.34)$$

Inverse Fourier transform (2.1.34) we get the time-domain signal

$$s_r(t, k) = \mathfrak{F}^{-1}[S_r(\omega, k)] = s(t - k'r)e^{i(\omega_0 t - k_0 r)} \qquad (2.1.35)$$

which means that the wave packet travels with the group velocity of

$$v_g = \frac{1}{k'} = \frac{d\omega}{dk} \qquad (2.1.36)$$

2.1.4 Wave Velocities

Lastly, let us find the relation between v_p and v_g

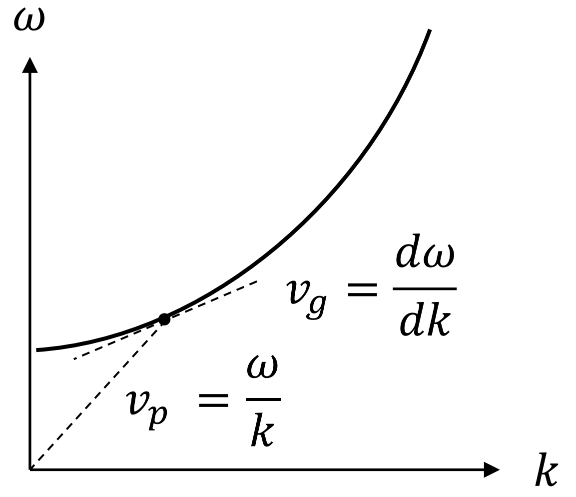
$$v_g = \frac{1}{dk/d\omega} = \frac{1}{d\left(\frac{\omega}{v_p}\right)/d\omega} = \frac{v_p}{1-\frac{\omega}{v_p}\frac{dv_p}{d\omega}} \tag{2.1.37}$$

Three possible cases are characterized:

i. $\frac{dv_p}{d\omega} = 0$ and $v_g = v_p$: no dispersion

ii. $\frac{dv_p}{d\omega} < 0$ and $v_g < v_p$: normal dispersion

iii. $\frac{dv_p}{d\omega} > 0$ and $v_g > v_p$: anomalous dispersion



2.2 Plane Wave Propagation

2.2.1 Propagation in Unbounded Medium

If the medium is lossless, we have a real wavenumber $k = \omega\sqrt{\mu\epsilon}$. However, the unit vector $\hat{k}$ can be complex. Starting from $\hat{k} \cdot \hat{k} = 1$, assume that

$$\hat{k} = \vec{k}' - i\vec{k}'' \qquad (2.2.1)$$

we have

$$\hat{k} \cdot \hat{k} = k'^2 - k''^2 - 2i\vec{k}' \cdot \vec{k}'' = 1 \qquad (2.2.2)$$

where $|\vec{k}'| = k'$ and $|\vec{k}''| = k''$. (2.2.2) implies that

$$k'^2 - k''^2 = 1 \qquad (2.2.3a)$$

$$\vec{k}' \cdot \vec{k}'' = 0 \qquad (2.2.3b)$$

2.2.1 Propagation in Unbounded Medium

If

$$k' = 1, k'' = 0 \tag{2.2.4}$$

Then (2.2.3b) is also satisfied. In this case, we have a uniform plane wave solution.

For general cases when $|\vec{k}'| \neq 0$ and $|\vec{k}''| \neq 0$, the solution (2.1.8) becomes

$$\vec{E} = \vec{E}_0 e^{-i\vec{k}\cdot\vec{r}} = \vec{E}_0 e^{-\vec{k}''\cdot\vec{r}}\, e^{-i\vec{k}'\cdot\vec{r}} \tag{2.2.5}$$

denoting a non-uniform plane wave with the constant phase plane and the constant amplitude plane perpendicular to each other.

2.2.1 Propagation in Unbounded Medium

For lossy medium, the wavenumber $k = \omega\sqrt{\mu\epsilon_c} = \beta - i\alpha$ is a complex number as in (1.4.16). For cases when $\hat{k}$ is real, the solution (2.1.8) takes the form of

$$\vec{E} = \vec{E}_0 e^{-i\vec{k}\cdot\vec{r}} = \vec{E}_0 e^{-\alpha\hat{k}\cdot\vec{r}}\, e^{-i\beta\hat{k}\cdot\vec{r}} \tag{2.2.6}$$

denoting an evanecent plane wave with its amplitude decreasing in the propagation direction.

Using the expression (1.4.10) $\epsilon_c = \epsilon' - i\,\epsilon''$, we can express propagation constant β and attenuation constant α as

2.2.1 Propagation in Unbounded Medium

$$\beta = \omega\left\{\frac{\mu\epsilon'}{2}\left[\sqrt{1+\left(\frac{\epsilon''}{\epsilon'}\right)^2}+1\right]\right\}^{1/2} \tag{2.2.7}$$

$$\alpha = \omega\left\{\frac{\mu\epsilon'}{2}\left[\sqrt{1+\left(\frac{\epsilon''}{\epsilon'}\right)^2}-1\right]\right\}^{1/2} \tag{2.2.8}$$

The intrinsic impedance and phase velocity are

$$\eta_c = \sqrt{\frac{\mu}{\epsilon_c}} = \sqrt{\frac{\mu}{\epsilon'}}\left(1 - i\,\frac{\epsilon''}{\epsilon'}\right)^{-1/2} \tag{2.2.9}$$

$$v_p = \frac{\omega}{\beta} = \left\{\frac{\mu\epsilon'}{2}\left[\sqrt{1+\left(\frac{\epsilon''}{\epsilon'}\right)^2}-1\right]\right\}^{-1/2} \tag{2.2.10}$$

2.2.2 Oblique Incidence on an Interface

In this section, we examine the behavior of a plane wave incident on a planar dielectric boundary, deriving the general case for oblique incidence.

Special cases can be easily derived from the following analysis:

i. For cases of normal incidence, we simply set the incident angle $\theta_i = 0$.

ii. When the incident medium is a PEC, the analysis simplifies by setting $\epsilon_2 = \epsilon_c \rightarrow i\infty$ and $\eta_2 \rightarrow 0$.

2.2.2 Oblique Incidence on an Interface

Let the incident electric field amplitude be E_i, and define the reflection and transmission coefficients as

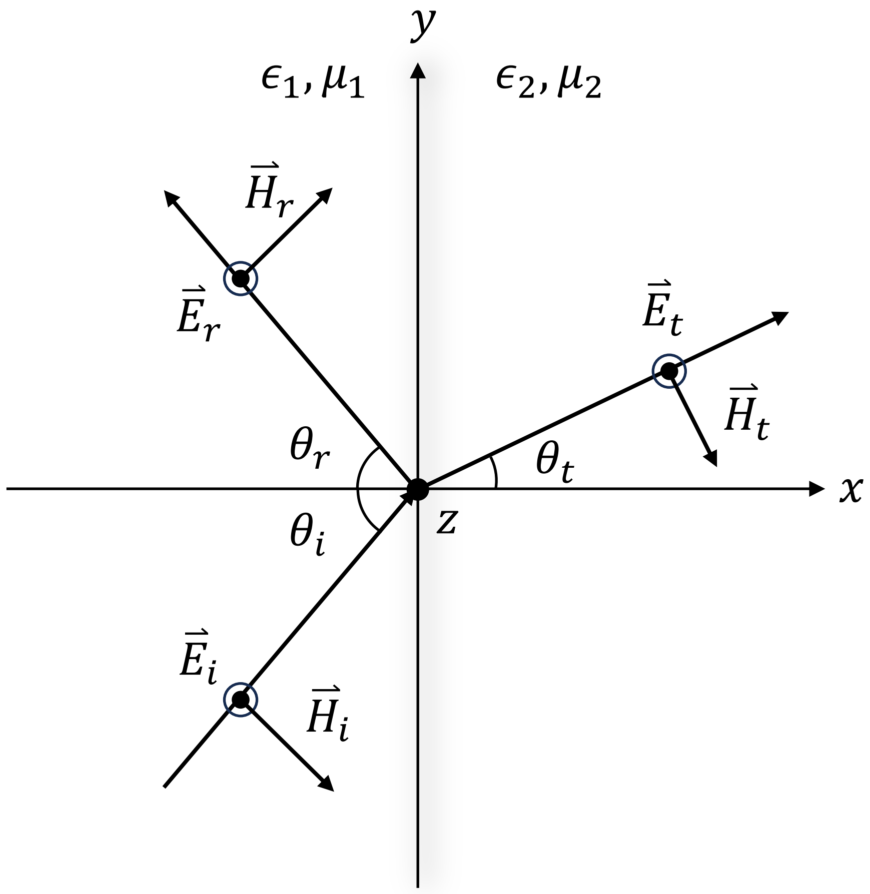
$$\Gamma = E_r / E_i \tag{2.2.11}$$

$$\mathrm{T} = E_t / E_i \tag{2.2.12}$$

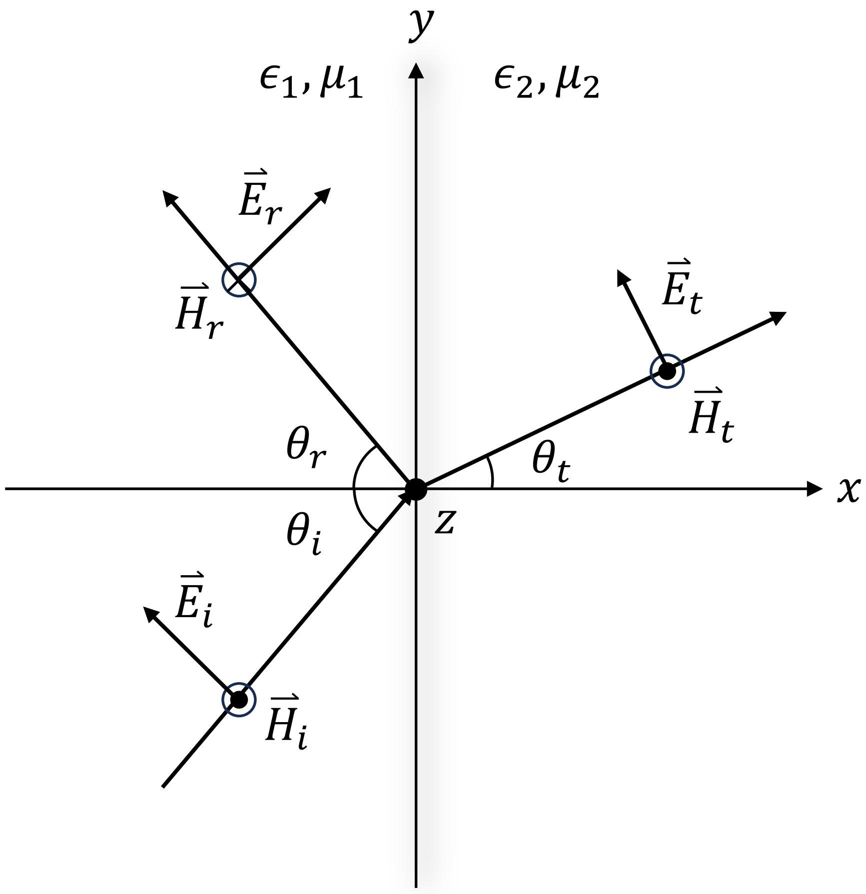
where E_r is the amplitude of the reflected wave and E_t is the amplitude of the transmitted wave.

We consider two cases: perpendicular and parallel polarization. Since any arbitrarily polarized plane wave can be represented as a combination of the two, analyzing them separately allows for a comprehensive understanding of interactions with the boundaries.

2.2.2 Oblique Incidence on an Interface



Perpendicular Polarization



Parallel Polarization

2.2.2.1 Perpendicular Polarization

Perpendicular polarization occurs when the electric field is normal to the plane of incidence, with the magnetic field lying within the plane.

From (2.1.8) and (2.1.11) we can express the incident field as

$$\vec{E}_i = E_i e^{-i\vec{k}_i\cdot\vec{r}}\hat{z} = E_i e^{-ik_1(x\cos\theta_i + y\sin\theta_i)}\ \hat{z} \tag{2.2.13a}$$

$$\begin{aligned}\vec{H}_i &= \frac{1}{\eta_1}\hat{k}_i \times \vec{E}_i = \frac{1}{\eta_1}(\cos\theta_i\,\hat{x} + \sin\theta_i\,\hat{y}) \times E_i e^{-ik_1(x\cos\theta_i + y\sin\theta_i)}\ \hat{z} \\ &= \frac{E_i}{\eta_1} e^{-ik_1(x\cos\theta_i + y\sin\theta_i)}(\sin\theta_i\,\hat{x} - \cos\theta_i\,\hat{y})\end{aligned} \tag{2.2.13b}$$

2.2.2.1 Perpendicular Polarization

The reflected and transmitted fields can be written accordingly

$$\vec{E}_r = \Gamma_\perp E_i e^{-ik_1(-x\cos\theta_r + y\sin\theta_r)}\,\hat{z} \tag{2.2.14a}$$

$$\vec{H}_r = \frac{\Gamma_\perp E_i}{\eta_1} e^{-ik_1(-x\cos\theta_r + y\sin\theta_r)}(\sin\theta_r\,\hat{x} + \cos\theta_r\,\hat{y}) \tag{2.2.14b}$$

$$\vec{E}_t = \mathrm{T}_\perp E_i e^{-ik_2(x\cos\theta_t + y\sin\theta_t)}\,\hat{z} \tag{2.2.15a}$$

$$\vec{H}_t = \frac{\mathrm{T}_\perp E_i}{\eta_2} e^{-ik_2(x\cos\theta_t + y\sin\theta_t)}(\sin\theta_t\,\hat{x} - \cos\theta_t\,\hat{y}) \tag{2.2.15b}$$

Applying the following matching conditions:

$$\left(\vec{E}_i + \vec{E}_r\right)\Big|_{\mathrm{tan},\,x=0} = \left(\vec{E}_t\right)\Big|_{\mathrm{tan},\,x=0} \tag{2.2.16a}$$

$$\left(\vec{H}_i + \vec{H}_r\right)\Big|_{\mathrm{tan},\,x=0} = \left(\vec{H}_t\right)\Big|_{\mathrm{tan},\,x=0} \tag{2.2.16b}$$

2.2.2.1 Perpendicular Polarization

we get

$$e^{-ik_1 y \sin\theta_i} + \Gamma_\perp e^{-ik_1 y \sin\theta_r} = \mathrm{T}_\perp e^{-ik_2 y \sin\theta_t} \qquad (2.2.17a)$$

$$\frac{1}{\eta_1}\left(-\cos\theta_i\, e^{-ik_1 y \sin\theta_i} + \Gamma_\perp \cos\theta_r\, e^{-ik_1 y \sin\theta_r}\right)$$

$$= \frac{-1}{\eta_2}\mathrm{T}_\perp \cos\theta_t\, e^{-ik_2 y \sin\theta_t} \qquad (2.2.17b)$$

Since (2.2.17) holds for all values of y, the phase matching condition, also known as the Snell's law, is satisfied

$$k_1 \sin\theta_i = k_1 \sin\theta_r = k_2 \sin\theta_t \qquad (2.2.18)$$

For the relection angle, we have $\theta_i = \theta_r$.

2.2.2.1 Perpendicular Polarization

By (2.2.18), (2.2.17) is further reduced to

$$1 + \Gamma_{\perp} = \mathrm{T}_{\perp} \tag{2.2.19a}$$

$$\frac{\cos\theta_i}{\eta_1}(-1 + \Gamma_{\perp}) = \frac{-\cos\theta_t}{\eta_2}\mathrm{T}_{\perp} \tag{2.2.19b}$$

Solving (2.2.19), we get

$$\Gamma_{\perp} = \frac{\eta_2/\cos\theta_t - \eta_1/\cos\theta_i}{\eta_2/\cos\theta_t + \eta_1/\cos\theta_i} \tag{2.2.20a}$$

$$\mathrm{T}_{\perp} = \frac{2\eta_2/\cos\theta_t}{\eta_2/\cos\theta_t + \eta_1/\cos\theta_i} \tag{2.2.20b}$$

Note that $\eta_1/\cos\theta_i$ and $\eta_2/\cos\theta_t$ represent the ratio of the tangential electric field to the tangential magnetic field.

2.2.2.2 Parallel Polarization

Parallel polarization occurs when the electric field lies in the plane of incidence. Similarly, the fields can be expressed as

$$\vec{E}_i = E_i e^{-ik_1(x\cos\theta_i + y\sin\theta_i)}(-\sin\theta_i\,\hat{x} + \cos\theta_i\,\hat{y}) \qquad (2.2.21a)$$

$$\vec{H}_i = \frac{E_i}{\eta_1} e^{-ik_1(x\cos\theta_i + y\sin\theta_i)}\,\hat{z} \qquad (2.2.21b)$$

$$\vec{E}_r = \Gamma_\parallel E_i e^{-ik_1(-x\cos\theta_r + y\sin\theta_r)}(\sin\theta_r\,\hat{x} + \cos\theta_r\,\hat{y}) \qquad (2.2.22a)$$

$$\vec{H}_r = -\frac{\Gamma_\parallel E_i}{\eta_1} e^{-ik_1(-x\cos\theta_r + y\sin\theta_r)}\,\hat{z} \qquad (2.2.22b)$$

$$\vec{E}_t = \mathrm{T}_\parallel E_i e^{-ik_2(x\cos\theta_t + y\sin\theta_t)}(-\sin\theta_t\,\hat{x} + \cos\theta_t\,\hat{y}) \qquad (2.2.23a)$$

$$\vec{H}_t = \frac{\mathrm{T}_\parallel E_i}{\eta_2} e^{-ik_2(x\cos\theta_t + y\sin\theta_t)}\,\hat{z} \qquad (2.2.23b)$$

2.2.2.2 Parallel Polarization

Applying the matching conditions (2.2.16) and phase matching relation (2.2.18), we get

$$\Gamma_{\parallel} = \frac{\eta_2 \cos\theta_t - \eta_1 \cos\theta_i}{\eta_2 \cos\theta_t + \eta_1 \cos\theta_i} \tag{2.2.24a}$$

$$\mathrm{T}_{\parallel} = \frac{2\eta_2 \cos\theta_t}{\eta_2 \cos\theta_t + \eta_1 \cos\theta_i}\left(\frac{\cos\theta_i}{\cos\theta_t}\right) \tag{2.2.24b}$$

The factor $\left(\frac{\cos\theta_i}{\cos\theta_t}\right)$ in (2.2.24b) comes from the definition of $\mathrm{T}_{\parallel}$ being the ratio of the total electric field $\mathrm{T}_{\parallel} = \frac{E_t}{E_i} = \frac{E_{yt} / \cos\theta_t}{E_{yi} / \cos\theta_i} = \frac{E_{yt}}{E_{yi}}\left(\frac{\cos\theta_i}{\cos\theta_t}\right)$.

2.2.2.3 Critical Angle

Consider the case when $\theta_t = 90°$. From (2.2.18), we have

$$\sqrt{\epsilon_1\mu_1}\sin\theta_c = \sqrt{\epsilon_2\mu_2} \tag{2.2.25}$$

or,

$$\theta_c = \sin^{-1}\left(\sqrt{\epsilon_2\mu_2/\epsilon_1\mu_1}\right) \tag{2.2.26}$$

where θ_c is called the critical angle. When $\theta_i \geq \theta_c$, total reflection occurs. Take perpendicular polarization for example, examining (2.2.15a), we can get the transmitted field

$$E_t = \mathrm{T}_\perp E_i e^{-ik_2(x\cos\theta_t + y\sin\theta_t)} = \mathrm{T}_\perp E_i e^{-ik_1 y\sin\theta_i} e^{-ik_2 x\cos\theta_t} \tag{2.2.27}$$

2.2.2.3 Critical Angle

where

$$\cos\theta_t = \sqrt{1 - \sin^2\theta_t} = \sqrt{1 - \frac{\epsilon_1\mu_1}{\epsilon_2\mu_2}\sin^2\theta_i} \tag{2.2.28}$$

When $\theta_i > \theta_c = \sin^{-1}\left(\sqrt{\epsilon_2\mu_2/\epsilon_1\mu_1}\right)$, (2.2.28) is purely imaginary, and can be expressed as

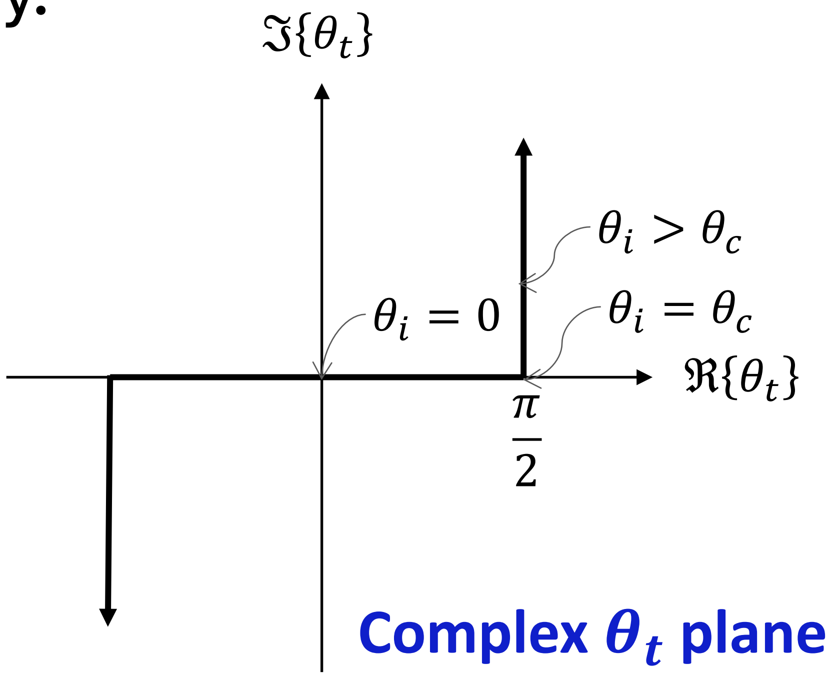
$$\cos\theta_t = \cos\left(\frac{\pi}{2} + i\delta_t\right) = -i\sinh\delta_t = -i\sqrt{\frac{\epsilon_1\mu_1}{\epsilon_2\mu_2}\sin^2\theta_i - 1} \tag{2.2.29}$$

with $\sinh\delta_t > 0$. then (2.2.27) becomes

$$E_t = \mathrm{T}_\perp E_i e^{-\sinh\delta_t k_2 x} e^{-ik_1 y\sin\theta_i} \tag{2.2.30}$$

2.2.2.3 Critical Angle

This is clearly a non-uniform plane wave solution as shown in (2.2.5). The transmitted wave propagates along the planar boundary, and decays exponentially in the direction perpendicular to the planar boundary.



Complex θ_t plane

2.2.2.4 Brewster's Angle

Zero reflection occurs when (2.2.20a) or (2.2.24a) equal to zero. For perpendicular polarization, we need $\eta_2/\cos\theta_t = \eta_1/\cos\theta_i$, or

$$(1-\sin^2\theta_i) = \frac{\mu_1\epsilon_2}{\mu_2\epsilon_1}(1-\sin^2\theta_t) \qquad (2.2.31)$$

From (2.2.18), we have $\sin^2\theta_t = \frac{\mu_1\epsilon_1}{\mu_2\epsilon_2}\sin^2\theta_i$. Thus,

$$\sin\theta_i = \sqrt{\frac{\epsilon_2/\epsilon_1-\mu_2/\mu_1}{\mu_1/\mu_2-\mu_2/\mu_1}} \qquad (2.2.32)$$

For θ_i to have real solution, it is required that $\sqrt{\frac{\epsilon_2/\epsilon_1-\mu_2/\mu_1}{\mu_1/\mu_2-\mu_2/\mu_1}} \leq 1$, or

$$\left|\frac{\epsilon_2}{\epsilon_1}-\frac{\mu_2}{\mu_1}\right| \leq \left|\frac{\mu_1}{\mu_2}-\frac{\mu_2}{\mu_1}\right| \qquad (2.2.33)$$

2.2.2.4 Brewster's Angle

However, for most non-magnetic materials ($\mu_1 = \mu_2 = \mu_0$), the denominator of (2.2.32) is zero, which implies that there exists no zero reflection for perpendicular polarizations.

For the parallel polarization we need $\eta_2 \cos\theta_t = \eta_1 \cos\theta_i$, or

$$(1 - \sin^2\theta_i) = \frac{\mu_2\epsilon_1}{\mu_1\epsilon_2}(1 - \sin^2\theta_t) \qquad (2.2.34)$$

Similar to previous analysis, from (2.2.18) we have

$$\sin\theta_i = \sqrt{\frac{\epsilon_2/\epsilon_1 - \mu_2/\mu_1}{\epsilon_2/\epsilon_1 - \epsilon_1/\epsilon_2}} \qquad (2.2.35)$$

For θ_i to have real solution, it is required that

$$\left|\frac{\epsilon_2}{\epsilon_1} - \frac{\mu_2}{\mu_1}\right| \leq \left|\frac{\epsilon_2}{\epsilon_1} - \frac{\epsilon_1}{\epsilon_2}\right| \qquad (2.2.36)$$

2.2.2.4 Brewster's Angle

For most non-magnetic materials, (2.2.35) reduces to

$$\theta_b = \sin^{-1}\left(\sqrt{\frac{\epsilon_2}{\epsilon_1+\epsilon_2}}\right) = \cos^{-1}\left(\sqrt{\frac{\epsilon_1}{\epsilon_1+\epsilon_2}}\right) = \tan^{-1}\left(\sqrt{\frac{\epsilon_2}{\epsilon_1}}\right) \quad (2.2.37)$$

where θ_b is called the Brewster's angle.

It is noted that since $\sqrt{\epsilon_2}\sin\theta_t = \sqrt{\epsilon_1}\sin\theta_b = \sqrt{\frac{\epsilon_1\epsilon_2}{\epsilon_1+\epsilon_2}}$, we have

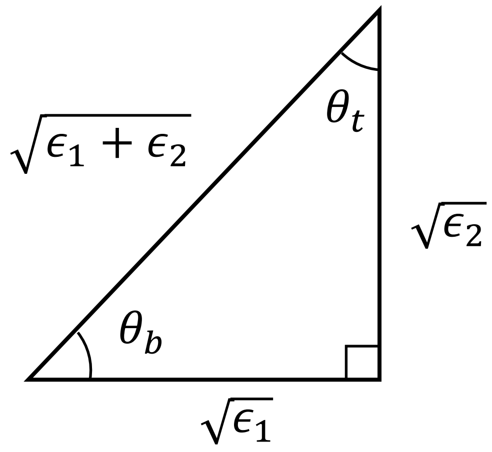
$$\theta_t = \sin^{-1}\left(\sqrt{\frac{\epsilon_1}{\epsilon_1+\epsilon_2}}\right) = \cos^{-1}\left(\sqrt{\frac{\epsilon_2}{\epsilon_1+\epsilon_2}}\right) = \tan^{-1}\left(\sqrt{\frac{\epsilon_1}{\epsilon_2}}\right) \quad (2.2.38)$$

Thus,

$$\theta_b + \theta_t = \frac{\pi}{2} \quad (2.2.39)$$

2.2.2.4 Brewster's Angle

The electric field induces electron oscillations along the direction of $\vec{E}_t$ when the EM wave is transmitted. These oscillations emit a reflected wave back into the first medium. However, since no radiation occurs along the direction of $\vec{E}_t$, there is no reflected wave when the reflected and transmitted waves are perpendicular—as at Brewster's angle.



2.3 Waveguide Transmission

2.3.1 Solution to Waveguide Problems

Waveguides are transmission structures that guide electromagnetic waves and support higher-order modes beyond TEM waves.

Assume the waveguide is extended in the *z*-direction, A general approach to solving waveguide problems is to first determine the *z*-components of the fields and then derive the corresponding transverse components. Assume the fields have the form of

$$\vec{E}(x, y, z) = \vec{E}_0(x, y)e^{-ik_z z} \tag{2.3.1a}$$

$$\vec{H}(x, y, z) = \vec{H}_0(x, y)e^{-ik_z z} \tag{2.3.1b}$$

2.3.1 Solution to Waveguide Problems

By separating the field into its transverse and longitudidal components $\vec{E}_0 = E_{\perp}^0 \hat{t} + E_z^0 \hat{z}$ ($\vec{H}_0 = H_{\perp}^0 \hat{t} + H_z^0 \hat{z}$), and put (2.3.1) into the source-free curl-set Maxwell equations with $\partial_z \to -ik_z$, we can express the transverse components in terms of E_z^0 and H_z^0 as

$$\vec{E}_{\perp}^0 = -\frac{i}{k_c^2}\left(k_z \vec{\nabla}_{\perp} E_z^0 - \omega\mu\hat{z} \times \vec{\nabla}_{\perp} H_z^0\right) \quad (2.3.2a)$$

$$\vec{H}_{\perp}^0 = -\frac{i}{k_c^2}\left(k_z \vec{\nabla}_{\perp} H_z^0 + \omega\epsilon\hat{z} \times \vec{\nabla}_{\perp} E_z^0\right) \quad (2.3.2b)$$

with

$$k_c^2 = k^2 - k_z^2 \quad (2.3.3)$$

2.3.1 Solution to Waveguide Problems

(2.3.2) can be simplified under specific assumptions about the field components. If $H_z^0 = 0$ while $E_z^0 \neq 0$, the resulting mode is called a transverse magnetic (TM) mode. Conversely, if the $E_z^0 = 0$ and $H_z^0 \neq 0$, the mode is known as a transverse electric (TE) mode.

TM Mode

$$H_z^0 = 0 \tag{2.3.4a}$$

$$\vec{E}_\perp^0 = -\frac{i}{k_c^2} k_z \vec{\nabla}_\perp E_z^0 \tag{2.3.4b}$$

$$\vec{H}_\perp^0 = -\frac{i}{k_c^2} \omega\epsilon \hat{z} \times \vec{\nabla}_\perp E_z^0 \tag{2.3.4c}$$

TE Mode

$$E_z^0 = 0 \tag{2.3.5a}$$

$$\vec{E}_\perp^0 = \frac{i}{k_c^2} \omega\mu \hat{z} \times \vec{\nabla}_\perp H_z^0 \tag{2.3.5b}$$

$$\vec{H}_\perp^0 = -\frac{i}{k_c^2} k_z \vec{\nabla}_\perp H_z^0 \tag{2.3.5c}$$

2.3.1 Solution to Waveguide Problems

Note that the fields also satisfy the homogeneous vector Helmholtz equation $(\nabla^2 + k^2)\vec{F} = 0$. Put (2.3.1) into the Helmholtz equation and separate the Laplacian operator into the longitudinal part and the transverse part. Extracting the z-components, we get

$$(\nabla_\perp^2 + k_c^2)E_z^0 = 0 \tag{2.3.6a}$$

$$(\nabla_\perp^2 + k_c^2)H_z^0 = 0 \tag{2.3.6b}$$

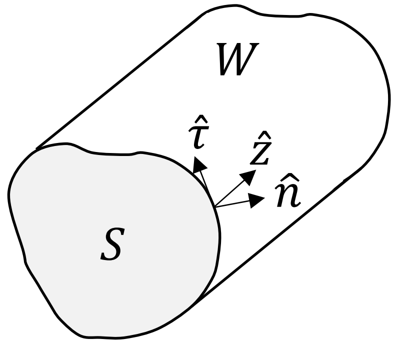
From (2.3.6), the longitudinal fields can be determined based on the specific boundary conditions, after which the transverse fields can be obtained using (2.3.2).

2.3.1 Solution to Waveguide Problems

If the waveguide is composed of a single connected (tube-like) PEC, the boundary condition is

$$\hat{n} \times \vec{E}\Big|_W = 0 \tag{2.3.7}$$

where W denotes the surface of the waveguide, $\hat{n}$ is the normal unit vector pointing outwards, and $\hat{\tau} = \hat{z} \times \hat{n}$. By expending $\vec{E}_\perp^0 = E_\tau^0 \hat{\tau} + E_n^0 \hat{n}$, (2.3.7) requires that both the the tangential components E_τ^0 and E_z^0 vanish on the PEC.



2.3.1 Solution to Waveguide Problems

For TM waves, we can impose the Dirichlet boundary condition

$$E_z^0|_W = 0 \quad \text{(TM waves)} \tag{2.3.8}$$

For TE waves, From (2.3.5b), considering the τ-component, we have

$$E_\tau^0|_W = \frac{i}{k_c^2}\omega\mu\partial_n H_z^0|_W = 0$$

which implies the Neumann boundary condition

$$\partial_n H_z^0|_W = 0 \text{ (TE waves)} \tag{2.3.9}$$

Imposing the boundary conditions yields solutions only for specific discrete values of k_c, known as eigenvalues. Each eigenvalue characterizes a distinct waveguide mode.

2.3.1 Solution to Waveguide Problems

Let

$$k_c = \omega_c\sqrt{\mu\epsilon} \tag{2.3.10}$$

and from (2.3.3), we can get

$$k_z = k_c\sqrt{(f/f_c)^2 - 1} = k\sqrt{1 - (f_c/f)^2} \tag{2.3.11}$$

where

$$f_c = \omega_c/2\pi = k_c/(2\pi\sqrt{\mu\epsilon}) \tag{2.3.12}$$

is called the cut-off frequency. From (2.3.1), a propagating solution requires k_z to have a non-zero real part. This condition is satisfied only when $f > f_c$. Therefore, a waveguide functions as a high-pass filter.

2.3.1 Solution to Waveguide Problems

Generally, $k_z = \beta - i\alpha$. If the waveguide is lossless, then $\beta = k_z$. Consider the case when $f > f_c$, for a lossless waveguide we can calculate the guided wavelength in *z*-direction by

$$\lambda_z = 2\pi/\beta = \lambda/\sqrt{1-(f_c/f)^2} > \lambda \qquad (2.3.13)$$

where $\lambda = 2\pi/k$ is the plane wave wavelength a in an unbounded region with the same material as in the waveguide.

We then calculate the phase velocity

$$v_p = \omega/\beta = v/\sqrt{1-(f_c/f)^2} > v \qquad (2.3.14)$$

and the group velocity

2.3.1 Solution to Waveguide Problems

$$v_g = \frac{d\omega}{d\beta} = \frac{1}{d\beta/d\omega} = \frac{1}{d\left[(\omega/v)\sqrt{1-(\omega_c/\omega)^2}\right]/d\omega}$$

$$= \frac{v}{d\sqrt{\omega^2-\omega_c^2}/d\omega} = \frac{v}{\omega/\sqrt{\omega^2-\omega_c^2}} = v\sqrt{1-(f_c/f)^2} < v$$

(2.3.15)

From (2.3.14), the phase velocity in a waveguide is always higher than in an unbounded medium and varies with frequency, indicating dispersion. Its relation to group velocity is given by:

$$v_p v_g = v^2 \tag{2.3.16}$$

2.3.2 Characteristics of Waveguide Modes

In this section, we introduce the orthogonality properties of waveguide modes. We establish the modal orthogonality between different waveguide modes in the sense of linear vector space theory. This involves showing that distinct modes form an orthogonal set of solutions with respect to an inner product, which we prove using Green’s identities.

2.3.2.1 Green's Identities

A. Green's first identity

Consider two scalar field f and g. From divergence theorem, we can get

$$\int_V \vec{\nabla} \cdot \left(g\vec{\nabla} f\right) dv = \oint_S g\vec{\nabla} f \cdot d\vec{s} \qquad (2.3.17)$$

Expand the integrand on the RHS we get the Green's first identity

$$\int_V \left(g\nabla^2 f + \vec{\nabla} g \cdot \vec{\nabla} f\right) dv = \oint_S g\vec{\nabla} f \cdot d\vec{s} = \oint_S g\partial_n f ds \qquad (2.3.18)$$

In 2D we have

$$\int_S \left(g\nabla_\perp^2 f + \vec{\nabla}_\perp g \cdot \vec{\nabla}_\perp f\right) ds = \oint_L g\vec{\nabla}_\perp f \cdot d\vec{l} = \oint_L g\partial_n f dl \qquad (2.3.19)$$

2.3.2.1 Green’s Identities

B. Green’s second identity

Interchange g and f in (2.3.18) and substract the two, we get the Green’s second identity

$$\int_V (g\nabla^2 f - f\nabla^2 g)dv = \oint_S \left(g\vec{\nabla}f - f\vec{\nabla}g\right)\cdot d\vec{s} = \oint_S (g\partial_n f - f\partial_n g)ds \tag{2.3.20}$$

In 2D we have

$$\int_S (g\nabla_\perp^2 f - g\nabla_\perp^2 f)ds = \oint_L \left(g\vec{\nabla}_\perp f - f\vec{\nabla}_\perp g\right)\cdot d\vec{l} = \oint_L (g\partial_n f - f\partial_n g)dl \tag{2.3.21}$$

2.3.2.1 Green's Identities

There are other useful identities derived from Green's identity. For example, interchange g and f in (2.3.18) and add the two, we get

$$\int_V \vec{\nabla} g \cdot \vec{\nabla} f dv = \frac{1}{2}\left[\oint_S (g\partial_n f + f\partial_n g) ds - \int_V (f\nabla^2 g + g\nabla^2 f) dv\right] \tag{2.3.22}$$

In 2D we have

$$\int_S \vec{\nabla}_\perp g \cdot \vec{\nabla}_\perp f ds = \frac{1}{2}\left[\oint_L (g\partial_n f + f\partial_n g) dl - \int_S (f\nabla_\perp^2 g + g\nabla_\perp^2 f) ds\right] \tag{2.3.23}$$

2.3.2.2 Modal Orthogonality

From (2.3.6), we see that waveguide problems reduce to two-dimensional eigenfunction problems, requiring analysis only over the waveguide cross-section. Let us represent the solution of either E_z^0 or H_z^0 as a doubly infinite set of eigenfunctions

$$\psi_{pq}, \quad p, q \in \mathbb{N} \tag{2.3.24}$$

which satisfies the Helmholtz equation

$$\left(\nabla_\perp^2 + k_{pq}^2\right)\psi_{pq} = 0 \tag{2.3.25}$$

and either the Dirichlet or the Neumann boundary conditions

$$\psi_{pq} = 0 \tag{2.3.26a}$$

$$\partial_n \psi_{pq} = 0 \tag{2.3.26b}$$

2.3.2.2 Modal Orthogonality

First let us consider two distinct eigenfunctions ψ_{pq} and $\psi_{p'q'}$. Plug them into (2.3.25) and substract the two, we get

$$\psi_{p'q'}\nabla_\perp^2\psi_{pq} - \psi_{pq}\nabla_\perp^2\psi_{p'q'} = \left(k_{p'q'}^2 - k_{pq}^2\right)\psi_{pq}\psi_{p'q'} \quad (2.3.27)$$

Applying (2.3.21) we have

$$\oint_L \left(\psi_{p'q'}\partial_n\psi_{pq} - \psi_{pq}\partial_n\psi_{p'q'}\right)dl = \int_S \left(k_{p'q'}^2 - k_{pq}^2\right)\psi_{pq}\psi_{p'q'}\,ds \quad (2.3.28)$$

Since ψ_{pq} and $\psi_{p'q'}$ satisfy either (2.3.26a) or (2.3.26b), the LHS of (2.3.28) vanishes.

2.3.2.2 Modal Orthogonality

For $k_{p'q'}^2 \neq k_{pq}^2$ (non-degenerate distinct modes), we have the orthogonality relation

$$\int_S \psi_{pq}\psi_{p'q'}ds = 0 \tag{2.3.29}$$

Similarly, using (2.3.23), we have

$$\int_S \vec{\nabla}_\perp \psi_{pq} \cdot \vec{\nabla}_\perp \psi_{p'q'} ds = 0 \tag{2.3.30}$$

Now, let us consider three cases when calculating $\int_S \vec{E}_\perp^0 \times \vec{H}_\perp^0 \cdot d\vec{s}$

i. Both ψ_{pq} and $\psi_{p'q'}$ are TM modes

ii. ψ_{pq} is TM mode and $\psi_{p'q'}$ is TE mode

iii. Both ψ_{pq} and $\psi_{p'q'}$ are TE modes

2.3.2.2 Modal Orthogonality

From (2.3.4-5), we will encounter the following integrands

$$\vec{\nabla}_{\perp}\psi_{pq} \times \left(\hat{z} \times \vec{\nabla}_{\perp}\psi_{p'q'}\right) \tag{2.3.31a}$$

$$\vec{\nabla}_{\perp}\psi_{pq} \times \vec{\nabla}_{\perp}\psi_{p'q'} \cdot \hat{z} \tag{2.3.31b}$$

$$\left(\hat{z} \times \vec{\nabla}_{\perp}\psi_{pq}\right) \times \vec{\nabla}_{\perp}\psi_{p'q'} \tag{2.3.31c}$$

From vector identities and (2.3.29-30) we can show that (2.3.31) are identically zero. Thus, we have in gerenral

$$\int_S \vec{E}^0_{\perp(pq)} \times \vec{H}^0_{\perp(p'q')} \cdot d\vec{s} = 0 \tag{2.3.32}$$

Hence, different modes inside the waveguide do not couple.

Problems

1. Verify (2.1.9) with gradient, divegence and curl in Cartesian coordinate system.
2. Derive (2.2.7) and (2.2.8).
3. Show that the current sheet $\vec{J} = J_0\hat{x}$ on the z-plane generates the plane wave with $E_x = -\eta J_0 e^{-ik|z|}/2$ in an infitinte homogeneous region.
4. Derive (2.3.2) and express them in Cartesian coordinates.

Problems

5. Show that For TM mode, we have

$$\vec{E}_\perp^0 = -\frac{k_z}{\omega\epsilon}\hat{z}\times\vec{H}_\perp^0, \qquad \vec{H}_\perp^0 = \frac{\omega\epsilon}{k_z}\hat{z}\times\vec{E}_\perp^0$$

and for TE mode, we have

$$\vec{E}_\perp^0 = -\frac{\omega\mu}{k_z}\hat{z}\times\vec{H}_\perp^0, \qquad \vec{H}_\perp^0 = \frac{k_z}{\omega\mu}\hat{z}\times\vec{E}_\perp^0$$

6. Show that TEM mode cannot exist in a hollow waveguide with arbitrary cross section.