Electromagnetic Theory

Chapter 1 – Maxwell Equations

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1.1 Formulation of Maxwell Equations

1.1.1 Differential and Integral Form

Faraday's law

$$\vec{\nabla} \times \vec{E} = -\partial_t \vec{B} - \vec{M} \tag{1.1.1}$$

Maxwell-Ampere's Law

$$\vec{\nabla} \times \vec{H} = \partial_t \vec{D} + \vec{J} \tag{1.1.2}$$

Continuity relation

$$\vec{\nabla} \cdot \vec{J} = -\partial_t \rho, \quad \vec{\nabla} \cdot \vec{M} = -\partial_t \varrho \tag{1.1.3}$$

*Lorentz equation of force

$$\vec{F} = q\left(\vec{E} + \vec{v} \times \vec{B}\right) \tag{1.1.4}$$

1.1.1 Differential and Integral Form

- $\vec{E}$: electric field intensity (V/m)
- $\vec{H}$: magnetic field intensity (A/m)
- $\vec{D}$: electric flux density (C/m^2)
- $\vec{B}$: magnetic flux density (T)
- $\vec{J}$: volumetric electric current density (A/m^2)
- ρ: electric charge density (C/m^3)
- $\vec{M}$: volumetric magnetic current density (V/m^2)
- ϱ: magnetic charge density (Wb/m^3)

1.1.1 Differential and Integral Form

Taking divergence of (1.1.2)

$$\vec{\nabla} \cdot (\vec{\nabla} \times \vec{H}) = \partial_t \vec{\nabla} \cdot \vec{D} + \vec{\nabla} \cdot \vec{J} \equiv 0 \tag{1.1.5}$$

And from (1.1.3) we get

$$\partial_t (\vec{\nabla} \cdot \vec{D} - \rho) \equiv 0 \tag{1.1.6}$$

This implies that

$$\vec{\nabla} \cdot \vec{D} - \rho = C(x, y, z) \tag{1.1.7}$$

similarly

$$\vec{\nabla} \cdot \vec{B} - \varrho = C'(x, y, z) \tag{1.1.8}$$

1.1.1 Differential and Integral Form

In (1.1.7), if $C \neq 0$, it can be absorbed into ρ. The case is similar to (1.1.8). Thus, we can set $C = C' = 0$.

We get the Gauss's law

$$\vec{\nabla} \cdot \vec{D} = \rho \tag{1.1.9}$$

$$\vec{\nabla} \cdot \vec{B} = \varrho \tag{1.1.10}$$

Note that (1.1.9) and (1.1.10) are not independent of (1.1.1)-(1.1.3).

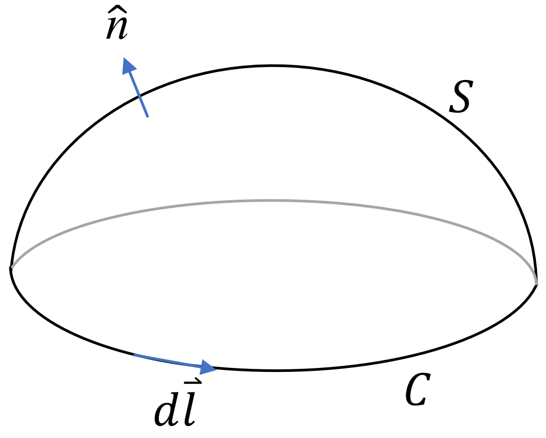
1.1.1 Differential and Integral Form

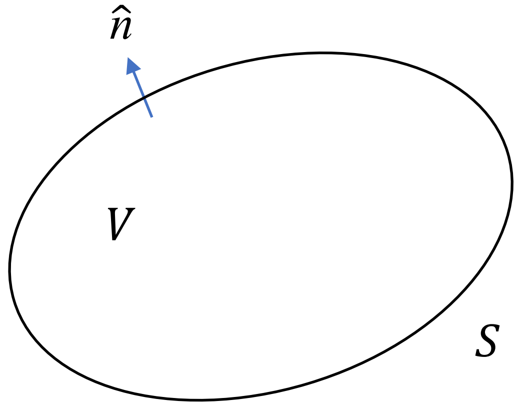
Stokes' theorem

$$\int_S \vec{\nabla} \times \vec{\Psi} \cdot d\vec{s} = \oint_C \vec{\Psi} \cdot d\vec{l} \qquad (1.1.11)$$

Divergence theorem

$$\int_V \vec{\nabla} \cdot \vec{\Psi} \, dv = \oint_S \vec{\Psi} \cdot d\vec{s} \qquad (1.1.12)$$





$d\vec{s} = \hat{n} ds$

1.1.1 Differential and Integral Form

Applying (1.1.11) to (1.1.1) and (1.1.2), we get

$$\oint_C \vec{E} \cdot d\vec{l} = -\int_S (\partial_t \vec{B} + \vec{M}) \cdot d\vec{s} \qquad (1.1.13)$$

$$\oint_C \vec{H} \cdot d\vec{l} = \int_S (\partial_t \vec{D} + \vec{J}\,) \cdot d\vec{s} \qquad (1.1.14)$$

Applying (1.1.12) to (1.1.3), (1.1.9) and (1.1.10), we get

$$\oint_S \vec{J} \cdot d\vec{s} = -\partial_t \int_V \rho \, dv \qquad (1.1.15)$$

$$\oint_S \vec{M} \cdot d\vec{s} = -\partial_t \int_V \varrho \, dv \qquad (1.1.16)$$

$$\oint_S \vec{D} \cdot d\vec{s} = \int_V \rho \, dv \qquad (1.1.17)$$

$$\oint_S \vec{B} \cdot d\vec{s} = \int_V \varrho \, dv \qquad (1.1.18)$$

1.1.1 Differential and Integral Form

Another theorem for the curl operator

$$\int_V \vec{\nabla} \times \vec{\Psi} dv = \oint_S (\hat{n} \times \vec{\Psi}) ds \qquad (1.1.19)$$

Applying (1.1.19) to (1.1.1) and (1.1.2), we get

$$\oint_S (\hat{n} \times \vec{E}) ds = -\int_V (\partial_t \vec{B} + \vec{M}) dv \qquad (1.1.20)$$

$$\oint_S (\hat{n} \times \vec{H}) ds = \int_V (\partial_t \vec{D} + \vec{J}) dv \qquad (1.1.21)$$

1.1.2 Constitutive Relations

Excluding $\vec{M}$ and ϱ (non-physical), there are 5 vectors and 1 scalar, resulting in 16 unknowns.

From (1.1.1)-(1.1.3), there are 7 scalar equations, which means extra 9 equations are needed to make the system determinate.

The constitutive relations relates $\vec{D}, \vec{B}, \vec{J}$ with $\vec{E}, \vec{H}$ by

$$\begin{cases} \vec{D} = \bar{\bar{C}}_1(\vec{E}, \partial_t\vec{E}, \partial_t^2\vec{E}, \ldots \vec{H}, \partial_t\vec{H}, \partial_t^2\vec{H}, \ldots) \\ \vec{B} = \bar{\bar{C}}_2(\vec{E}, \partial_t\vec{E}, \partial_t^2\vec{E}, \ldots \vec{H}, \partial_t\vec{H}, \partial_t^2\vec{H}, \ldots) \\ \vec{J} = \bar{\bar{C}}_3(\vec{E}, \partial_t\vec{E}, \partial_t^2\vec{E}, \ldots \vec{H}, \partial_t\vec{H}, \partial_t^2\vec{H}, \ldots) \end{cases} \tag{1.1.22}$$

In general, $\bar{\bar{C}}_j$ $(j = 1, 2, 3)$ are tensor functions of time.

1.1.2 Constitutive Relations

We confine our exposition on the material properties by the following restrictions :

- Stationary: $\bar{\bar{C}}_j$ are not functions of time
- Non-chiral: $\bar{\bar{C}}_1, \bar{\bar{C}}_3$ are related to $\vec{E}$ and $\bar{\bar{C}}_2$ is related to $\vec{H}$
- Linear: $\bar{\bar{C}}_j$ are related to $\vec{E}$ and $\vec{H}$ only (no higher derivatives)

Often simplifications can be made if the medium is

- Isotropic: $\bar{\bar{C}}_j$ are scalars, i.e., C_j (otherwise they are called anisotropic)
- Homogeneous: $\bar{\bar{C}}_j$ are not functions of space
- In this note, we call a medium simple if it is linear, isotropic and homogeneous.

1.1.2 Constitutive Relations

When a linear dielectric medium is perturbed by an electric field, the constitutive relation for $\vec{D}$ and $\vec{E}$ is

$$\vec{D} = \epsilon_0 \vec{E} + \vec{\mathcal{P}} \tag{1.1.23}$$

ϵ_0 is the electric permittivity of the vacuum (8.854×10^{-12} F/m), and $\vec{\mathcal{P}}$ is the electric polarization defined as

$$\vec{\mathcal{P}} = \epsilon_0 \bar{\bar{\chi}}_e \cdot \vec{E} \tag{1.1.24}$$

$\bar{\bar{\chi}}_e$ is the electric susceptibility tensor. If the medium is isotropic, then $\bar{\bar{\chi}}_e$ becomes a scalar, and $\vec{\mathcal{P}}$ is parallel to $\vec{E}$. We define

$$\vec{D} = \bar{\bar{\epsilon}} \cdot \vec{E} \tag{1.1.25}$$

$$\bar{\bar{\epsilon}} = \epsilon_0 (\bar{\bar{I}} + \bar{\bar{\chi}}_e) \tag{1.1.26}$$

1.1.2 Constitutive Relations

Similarly in magnetic medium, we have

$$\vec{B} = \mu_0 \vec{H} + \vec{\mathcal{M}} \tag{1.1.27}$$

μ_0 is the magnetic permeability of the vacuum ($4\pi \times 10^{-7}$ H/m), and $\vec{\mathcal{M}}$ is the magnetic polarization defined as

$$\vec{\mathcal{M}} = \mu_0 \bar{\bar{\chi}}_m \cdot \vec{H} \tag{1.1.28}$$

where $\bar{\bar{\chi}}_m$ is the magnetic susceptibility tensor. We can define

$$\vec{B} = \bar{\bar{\mu}} \cdot \vec{H} \tag{1.1.29}$$

$$\bar{\bar{\mu}} = \mu_0 (\bar{\bar{I}} + \bar{\bar{\chi}}_m) \tag{1.1.30}$$

1.1.2 Constitutive Relations

The constitutive relation between $\vec{J}$ and $\vec{E}$ is

$$\vec{J} = \bar{\bar{\sigma}} \cdot \vec{E} \tag{1.1.31}$$

$\bar{\bar{\sigma}}$ is the conductivity of the medium. The conductivity in vacuum is 0 (S/m).

For a simple medium, we can simplify the tensors $\bar{\bar{\epsilon}}$, $\bar{\bar{\mu}}$ and $\bar{\bar{\sigma}}$ to ϵ, μ and σ, respectively. (1.1.1) and (1.1.2) can be rewritten as

$$\vec{\nabla} \times \vec{E} = -\mu \partial_t \vec{H} \tag{1.1.32}$$

$$\vec{\nabla} \times \vec{H} = (\sigma + \epsilon \partial_t) \vec{E} \tag{1.1.33}$$

1.1.3 Symmetry and Duality

It is noted that when fictitious magnetic sources ϱ and $\vec{M}$ are introduced, Maxwell equations become symmetric. By exploiting symmetry, Maxwell equations can be separated into one set involving only electric sources and another involving only magnetic sources.

$$\begin{cases} \vec{\nabla} \times \vec{E} = -\mu \partial_t \vec{H} \\ \vec{\nabla} \times \vec{H} = \epsilon \partial_t \vec{E} - \vec{J} \\ \vec{\nabla} \cdot \epsilon \vec{E} = \rho \\ \vec{\nabla} \cdot \mu \vec{H} = 0 \end{cases} \quad \& \quad \begin{cases} \vec{\nabla} \times \vec{E} = -\mu \partial_t \vec{H} - \vec{M} \\ \vec{\nabla} \times \vec{H} = \epsilon \partial_t \vec{E} \\ \vec{\nabla} \cdot \epsilon \vec{E} = 0 \\ \vec{\nabla} \cdot \mu \vec{H} = \varrho \end{cases}$$

(1.1.34a) & (1.1.34b)

1.1.3 Symmetry and Duality

When the following transformations are applied, equations (1.1.34a) and (1.1.34b) are interchanged:

$$\begin{aligned} \vec{E} &\rightarrow \vec{H}', & \vec{H} &\rightarrow -\vec{E}' \\ \vec{J} &\rightarrow \vec{M}', & \vec{M} &\rightarrow -\vec{J}' \\ \rho &\rightarrow \varrho', & \varrho &\rightarrow -\rho' \\ \epsilon &\rightarrow \mu', & \mu &\rightarrow \epsilon' \end{aligned} \qquad (1.1.35)$$

When applying duality to same medium ($\eta = \sqrt{\mu/\epsilon}$)

$$\begin{aligned} \vec{E} &\rightarrow \eta\vec{H}', & \vec{H} &\rightarrow -\vec{E}'/\eta \\ \vec{J} &\rightarrow \vec{M}'/\eta, & \vec{M} &\rightarrow -\eta\vec{J}' \\ \rho &\rightarrow \varrho'/\eta, & \varrho &\rightarrow -\eta\rho' \end{aligned} \qquad (1.1.36)$$

1.2 Matching Conditions

1.2 Matching Conditions

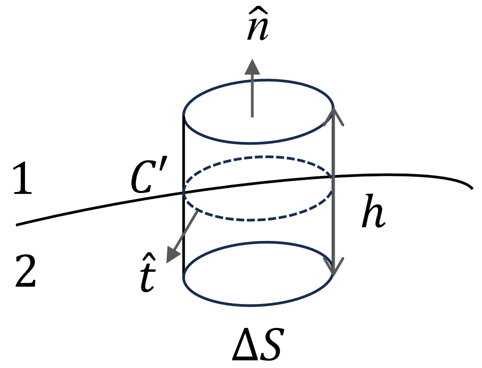
To derive boundary conditions, integral form of Maxwell equations are needed.

Applying (1.1.20) to the pillbox

$$\hat{n} \times \left(\vec{E}_1 - \vec{E}_2\right)\Delta S + \int_{side}\left(\hat{t} \times \vec{E}\right)ds = -\Delta S \int_{-h/2}^{h/2}(\partial_t \vec{B} + \vec{M})d\zeta \quad (1.2.1)$$

when $h \to 0$, we define the surface magnetic current density (V/m) as

$$\vec{M}_s = \lim_{h\to 0} \int_{-h/2}^{h/2} \vec{M} d\zeta \quad (1.2.2)$$



1.2 Matching Conditions

we get

$$\hat{n} \times \left(\vec{E}_1 - \vec{E}_2\right) = -\vec{M}_s \tag{1.2.3}$$

Similarly, applying (1.1.21) to the pillbox, when $h \to 0$, we define the surface electric current density (A/m) as

$$\vec{J}_s = \lim_{h \to 0} \int_{-h/2}^{h/2} \vec{J} d\zeta \tag{1.2.4}$$

we get

$$\hat{n} \times \left(\vec{H}_1 - \vec{H}_2\right) = \vec{J}_s \tag{1.2.5}$$

Note that $\vec{J}_s$ exists only when one of the medium's $\sigma \to \infty$ (PEC).

1.2 Matching Conditions

Applying (1.1.17) to the pillbox

$$\hat{n} \cdot \left(\vec{D}_1 - \vec{D}_2\right)\Delta S + \int_{side} \hat{t} \cdot \vec{D}\, ds = \Delta S \int_{-h/2}^{h/2} \rho d\zeta \qquad (1.2.6)$$

when $h \to 0$, we define the surface charge density (C/m²) as

$$\rho_s = \lim_{h \to 0} \int_{-h/2}^{h/2} \rho d\zeta \qquad (1.2.7)$$

we get

$$\hat{n} \cdot \left(\vec{D}_1 - \vec{D}_2\right) = \rho_s \qquad (1.2.8)$$

Similarly, from (1.1.18) we get

$$\hat{n} \cdot \left(\vec{B}_1 - \vec{B}_2\right) = \varrho_s \qquad (1.2.9)$$

1.2 Matching Conditions

Lastly, applying (1.1.15) to the pillbox

$$\hat{n}\cdot\left(\vec{J}_1-\vec{J}_2\right)\Delta S+\int_{side}\hat{t}\cdot\vec{J}\,ds=-\Delta S\int_{-h/2}^{h/2}\partial_t\rho d\zeta \qquad (1.2.10)$$

when $h\to 0$, we can express the side integral as

$$\oint_{C'}\lim_{h\to 0}\int_{-h/2}^{h/2}\vec{J}d\zeta\cdot\hat{t}dl=\oint_{C'}\vec{J}_s\cdot\hat{t}dl=\int_{S'}\vec{\nabla}_s\cdot\vec{J}_s\,ds \qquad (1.2.11)$$

we get

$$\left(\vec{J}_1-\vec{J}_2\right)+\vec{\nabla}_s\cdot\vec{J}_s=-\partial_t\rho_s \qquad (1.2.12)$$

Similarly, from (1.1.16), we get

$$\left(\vec{M}_1-\vec{M}_2\right)+\vec{\nabla}_s\cdot\vec{M}_s=-\partial_t\rho_s \qquad (1.2.13)$$

1.2 Matching Conditions

Special conditions arise when one of the media is a perfect electric conductor (PEC) or a perfect magnetic conductor (PMC). Inside a perfect conductor, electromagnetic fields vanish.

For an interface between two lossless dielectrics, the boundary conditions have no source terms.

1.3 Wave Equations

1.3 Wave Equations

Consider a lossless simple medium, taking the curl of (1.1.1)

$$\vec{\nabla} \times \vec{\nabla} \times \vec{E} = \vec{\nabla}(\vec{\nabla} \cdot \vec{E}) - \nabla^2 \vec{E} = -\partial_t(\vec{\nabla} \times \mu \vec{H}) - \vec{\nabla} \times \vec{M} \quad (1.3.1)$$

$$\nabla^2 \vec{E} - \mu\epsilon\partial_t^2 \vec{E} = \mu\partial_t \vec{J} + \vec{\nabla}(\rho/\epsilon) + \vec{\nabla} \times \vec{M} \quad (1.3.2)$$

(1.3.2) is the wave equation for electric field. We can obtain the wave equation for magnetic field by applying the duality transform (1.1.36) to (1.3.2)

$$\nabla^2 \vec{H} - \mu\epsilon\partial_t^2 \vec{H} = -\vec{\nabla} \times \vec{J} + \vec{\nabla}(\varrho/\mu) + \epsilon\partial_t \vec{M} \quad (1.3.3)$$

1.3 Wave Equations

(1.3.2) and (1.3.3) form a set of coupled inhomogeneous DE

$$\Box \begin{pmatrix} \vec{E} \\ \vec{H} \end{pmatrix} = \begin{pmatrix} \mu \partial_t \vec{J} + \vec{\nabla}\left(\frac{\rho}{\epsilon}\right) + \vec{\nabla} \times \vec{M} \\ -\vec{\nabla} \times \vec{J} + \vec{\nabla}\left(\frac{\varrho}{\mu}\right) + \epsilon \partial_t \vec{M} \end{pmatrix} \tag{1.3.4}$$

$\Box \equiv \nabla^2 - v^{-2}\partial_t^2$ is the d'Alembert operator with $v = (\mu\epsilon)^{-1/2}$. In source-free regions, we have

$$\Box \begin{pmatrix} \vec{E} \\ \vec{H} \end{pmatrix} = 0 \tag{1.3.5}$$

1.4 Time Harmonic Form

1.4 Time Harmonic Form

A time domain signal can be decomposed into a spectrum of time harmonic components

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} g(\omega)e^{i\omega t}d\omega \tag{1.4.1}$$

with

$$g(\omega) \; = \int_{-\infty}^{\infty} f(t)e^{-i\omega t}dt \tag{1.4.2}$$

The Fourier and inverse Fourier transform relationship is

$$g(\omega) = \mathfrak{F}[f(t)]$$

$$f(t) \; = \mathfrak{F}^{-1}[g(\omega)] \tag{1.4.3}$$

1.4 Time Harmonic Form

Now we consider the harmonic electric field at angular frequency ω. We consider its phasor form

$$\vec{E}(\vec{r},t) = \Re\left[\vec{E}(\vec{r},\omega)e^{i\omega t}\right] \tag{1.4.4}$$

Note that we use the same notation for both the time domain field and frequency domain field, and we will suppress its dependency when there is no ambiguity.

In general, the phasor field $\vec{E}(\vec{r},\omega)$ is a complex number.

1.4 Time Harmonic Form

The time harmonic Maxwell's equation takes the form

$$\vec{\nabla} \times \vec{E} = -i\omega\vec{B} - \vec{M} \tag{1.4.5}$$

$$\vec{\nabla} \times \vec{H} = i\omega\vec{D} + \vec{J} \tag{1.4.6}$$

$$\vec{\nabla} \cdot \vec{D} = \rho \tag{1.4.7}$$

$$\vec{\nabla} \cdot \vec{B} = \varrho \tag{1.4.8}$$

The method to translate the time-domain equation to frequency domain is to replace ∂_t with $i\omega$, and vice versa.

1.4 Time Harmonic Form

Analyzing (1.4.6), with $\vec{D} = \epsilon \vec{E}$ and $\vec{J} = \vec{J}_c + \vec{J}_i = \sigma \vec{E} + \vec{J}_i$, we get

$$\vec{\nabla} \times \vec{H} = i\omega \left(\epsilon - i\frac{\sigma}{\omega} \right) \vec{E} + \vec{J}_i \tag{1.4.9}$$

We define the complex permittivity as

$$\epsilon_c = \epsilon - i\frac{\sigma}{\omega} = \epsilon' - i\,\epsilon'', \quad \epsilon', \epsilon'' \in \mathbb{R} \tag{1.4.10}$$

In general $\epsilon = \epsilon_R - i\epsilon_I$. We define the loss tangent of the medium as

$$\tan\delta = \frac{\epsilon''}{\epsilon'} = \frac{\epsilon_I}{\epsilon_R} + \frac{\sigma}{\omega\epsilon_R} \tag{1.4.11}$$

Medium with $\tan\delta \ll 1$ is characterized as good dielectric, and with $\tan\delta \gg 1$ is characterized as good conductor.

1.4 Time Harmonic Form

The curl part of Maxwell's equations can be further simplified as

$$\vec{\nabla} \times \vec{E} = -\mathcal{Z}\vec{H} - \vec{M}_i \tag{1.4.12}$$

$$\vec{\nabla} \times \vec{H} = \mathcal{Y}\vec{E} + \vec{J}_i \tag{1.4.13}$$

with

$$\mathcal{Z} = i\omega\mu \tag{1.4.14}$$

$$\mathcal{Y} = i\omega\epsilon_c \tag{1.4.15}$$

$\mathcal{Z}\vec{H}$ has the same unit of $\vec{M}_i$ and $\mathcal{Y}\vec{E}$ has the same unit of $\vec{J}_i$. In this representation, the impressed currents are separated from the induced ones.

1.4 Time Harmonic Form

The d'Alembert operator in (1.3.4) in frequency domain becomes

$$\nabla^2 - \frac{(i\omega)^2}{1/\mu\epsilon_c} = \nabla^2 + k^2 \qquad (1.4.16)$$

where

$$k = \omega\sqrt{\mu\epsilon_c} = k_R + ik_I \qquad (1.4.17)$$

is the wavenumber. The square root of ϵ_c is chosen so that k_I relates to the physical attenuation of the wave propagation.

Several literatures use $\gamma^2 = (\alpha + i\beta)^2 = -\omega^2\mu\epsilon_c = -k^2$. The relationship between the real and imaginary parts are $\alpha = -k_I$ and $\beta = k_R$. Thus $k = \beta - i\alpha$ and $\gamma = ik$.

1.4 Time Harmonic Form

In a source-free region, Maxwell equations reduce to wave equations for the electric and magnetic field components. Each Cartesian component of the fields satisfies the homogeneous Helmholtz equation

$$\nabla^2\psi + k^2\psi = 0 \tag{1.4.18}$$

In cylindrical and spherical coordinates, the field components are generally coupled and described by vector wave equations, except in separable cases where they can still satisfy the scalar Helmholtz equation.

1.5 Fundamental Properties

1.5.1 Poynting Theorem

Considering the identity

$$\vec{\nabla} \cdot \left(\vec{E} \times \vec{H}\right) \equiv \vec{H} \cdot \vec{\nabla} \times \vec{E} - \vec{E} \cdot \vec{\nabla} \times \vec{H} \qquad (1.5.1)$$

We define the instantaneous Poynting vector $\vec{S} = \vec{E} \times \vec{H}$ (W/m^2)

$$\vec{\nabla} \cdot \vec{S} = \vec{H} \cdot \left(-\partial_t \vec{B} - \vec{M}\right) - \vec{E} \cdot \left(\partial_t \vec{D} + \vec{J}\right) =$$

$$-\partial_t \left(\frac{\mu}{2} \vec{H} \cdot \vec{H}\right) - \partial_t \left(\frac{\epsilon}{2} \vec{E} \cdot \vec{E}\right) - \vec{H} \cdot \vec{M} - \vec{E} \cdot \vec{J} \qquad (1.5.2)$$

1.5.1 Poynting Theorem

The term $\frac{\epsilon}{2}\vec{E}\cdot\vec{E} = w_e$ and $\frac{\mu}{2}\vec{H}\cdot\vec{H} = w_m$ denote the electric and magnetic energy density, and $\vec{H}\cdot\vec{M} + \vec{E}\cdot\vec{J} = p_l$ denotes the power loss/supply per unit volume. We rewrite the Poynting theorem as

$$\vec{\nabla}\cdot\vec{S} = -\partial_t(w_e + w_m) - p_l \quad (1.5.3)$$

and conduct integration over a finite volume V by applying the divergence theorem:

$$\oint_S \hat{n}\cdot\vec{S}ds = -\partial_t\int_V(w_e + w_m)dv - \int_V p_l dv \quad (1.5.4)$$

1.5.1 Poynting Theorem

From $\vec{E}(\vec{r},t) = \Re\left[\vec{E}(\vec{r})e^{i\omega t}\right] = \left(\vec{E}(\vec{r})e^{i\omega t} + \vec{E}^*(\vec{r})e^{-i\omega t}\right)/2$, and

$$\vec{S}(\vec{r},t) = \left(\Re\left[\vec{E}(\vec{r}) \times \vec{H}^*(\vec{r})\right] + \Re\left[\vec{E}(\vec{r}) \times \vec{H}(\vec{r})e^{i2\omega t}\right]\right)/2 \quad (1.5.5)$$

we have

$$\vec{S}_{\mathrm{av}} = \frac{1}{T}\int_0^T \vec{E}(\vec{r},t) \times \vec{H}(\vec{r},t)dt = \frac{1}{2}\Re\left(\vec{E} \times \vec{H}^*\right) = \Re\left(\vec{S}_C\right) \quad (1.5.6)$$

where

$$\vec{S}_C = \frac{1}{2}\vec{E} \times \vec{H}^* \quad (1.5.7)$$

is defined as the complex Poynting vector.

1.5.1 Poynting Theorem

From the identity (1.5.1)

$$\vec{\nabla}\cdot\vec{S}_C=\vec{\nabla}\cdot\left(\frac{\vec{E}\times\vec{H}^*}{2}\right)=-i2\omega\left(\frac{\mu\left|\vec{H}\right|^2}{4}-\frac{\epsilon\left|\vec{E}\right|^2}{4}\right)-\frac{\vec{E}\cdot\vec{J}^*}{2}-\frac{\vec{H}^*\cdot\vec{M}}{2}=$$
$$-i2\omega(w_m-w_e)-p_l \qquad (1.5.8)$$

is the complex Poynting theorem, where $w_m=\mu\left|\vec{H}\right|^2/4$, $w_e=\epsilon\left|\vec{E}\right|^2/4$ and $p_l=\left(\vec{E}\cdot\vec{J}^*+\vec{H}^*\cdot\vec{M}\right)/2$ are the time average electric and magnetic energy density and the time average power loss/supply.

1.5.2 Uniqueness Theorem

In a volume V, there exists sources $\vec{J}$ and $\vec{M}$. Assume the medium is simple but may be lossy, i.e, (ε, μ) are in gerneral complex. Our goal is to determine the conditions such that the electromagnetic field inside the volume V is unique.

Suppose there are two sets of solutions exist in V, say, $\left(\vec{E}_1, \vec{H}_1\right)$ and $\left(\vec{E}_2, \vec{H}_2\right)$. We want to find the conditions for $\delta\vec{E} = \vec{E}_1 - \vec{E}_2 = 0$ and $\delta\vec{H} = \vec{H}_1 - \vec{H}_2 = 0$ inside V.

1.5.2 Uniqueness Theorem

From (1.5.5) and (1.5.6), we have

$$\begin{cases} \vec{\nabla} \times \vec{E}_1 = -i\omega\mu\vec{H}_1 - \vec{M} \\ \vec{\nabla} \times \vec{H}_1 = i\omega\varepsilon\vec{E}_1 + \vec{J} \end{cases}, \begin{cases} \vec{\nabla} \times \vec{E}_2 = -i\omega\mu\vec{H}_2 - \vec{M} \\ \vec{\nabla} \times \vec{H}_2 = i\omega\varepsilon\vec{E}_2 + \vec{J} \end{cases} \qquad (16.9)$$

inside V. Substracting the two sets of equations, we get

$$\begin{cases} \vec{\nabla} \times \delta\vec{E} = -i\omega\mu\delta\vec{H} \\ \vec{\nabla} \times \delta\vec{H} = i\omega\varepsilon\delta\vec{E} \end{cases} \qquad (1.6.10)$$

1.5.2 Uniqueness Theorem

Dot the first equation in (1.5.10) with $\delta\vec{H}^*$ and the second equation with $\delta\vec{E}^*$, and subtract the two:

$$\vec{\nabla}\times\delta\vec{E}\cdot\delta\vec{H}^* - \vec{\nabla}\times\delta\vec{H}\cdot\delta\vec{E}^* = \vec{\nabla}\cdot\left(\delta\vec{E}\times\delta\vec{H}^*\right)$$

$$= i\omega\left(\mu\left|\delta\vec{H}\right|^2 - \varepsilon^*\left|\delta\vec{E}\right|^2\right) \qquad (1.5.11)$$

Conducting integration of (1.5.11) over V

$$\oint_S \delta\vec{E}\times\delta\vec{H}^*\cdot d\vec{s} = \int_V i\omega\left(\mu\left|\delta\vec{H}\right|^2 - \varepsilon^*\left|\delta\vec{E}\right|^2\right)dv \qquad (1.5.12)$$

1.5.2 Uniqueness Theorem

The key point is, if the surface integral in (1.5.12) equals to zero, so does the volume integral, which implies

$$\begin{cases} \int_V \left(\Re[\mu] \left| \delta \vec{H} \right|^2 - \Re[\varepsilon] \left| \delta \vec{E} \right|^2 \right) dv = 0 \\ \int_V \left(\Im[\mu] \left| \delta \vec{H} \right|^2 + \Im[\varepsilon] \left| \delta \vec{E} \right|^2 \right) dv = 0 \end{cases} \tag{1.5.13}$$

For disipative media, both $\Im[\varepsilon]$ and $\Im[\mu] < 0$, implying $\delta \vec{E} = \delta \vec{H} = 0$ everywhere inside V.

1.5.2 Uniqueness Theorem

For the surface integral in (1.5.12) to be zero, that is,

$$\oint_S \delta\vec{E} \times \delta\vec{H}^* \cdot d\vec{s} = \oint_S \hat{n} \times \delta\vec{E} \cdot \delta\vec{H}^* ds = \oint_S \delta\vec{H}^* \times \hat{n} \cdot \delta\vec{E} ds = 0 \tag{1.5.14}$$

The possible conditions are:

i. tangential $\vec{E}$ on S is specified, i.e., $\hat{n} \times \delta\vec{E} = 0$, or

ii. tangential $\vec{H}$ on S is specified, i.e., $\hat{n} \times \delta\vec{H} = 0$, or

iii. tangential $\vec{E}$ is specified over part of S, and tangential $\vec{H}$ is specified the rest of S.

1.5.2 Uniqueness Theorem

Uniqueness is established for general lossy media. For lossless media, from the first equation in (1.6.13) shows that there can exist infinite resonant solutions if the electric field energy stored inside V is equal to the magnetic field energy.

The principle extends to non-homogeneous media by applying it locally to small homogeneous regions. The method is limited to linear media.

1.5.3 Reciprocity Theorem

Suppose we have a set of sources $\left(\vec{J}_1, \vec{M}_1\right)$ which generate the electromagnetic field $\left(\vec{E}_1, \vec{H}_1\right)$. In the same medium, another set of sources $\left(\vec{J}_2, \vec{M}_2\right)$ generate the electromagnetic field $\left(\vec{E}_2, \vec{H}_2\right)$.

We define the reaction of field 1 on source 2 and the reaction of field 2 on source 1 as

$$\langle 1, 2\rangle = \int_V \left(\vec{E}_1 \cdot \vec{J}_2 - \vec{H}_1 \cdot \vec{M}_2\right) dv \qquad (1.5.15a)$$

$$\langle 2, 1\rangle = \int_V \left(\vec{E}_2 \cdot \vec{J}_1 - \vec{H}_2 \cdot \vec{M}_1\right) dv \qquad (1.5.15b)$$

We are finding the relation between $\langle 1, 2\rangle$ and $\langle 2, 1\rangle$.

1.5.3 Reciprocity Theorem

From (1.4.12) and (1.4.13) we can get the following set of eqautions

$$\vec{\nabla} \times \vec{E}_1 = -\mathcal{Z}\vec{H}_1 - \vec{M}_1 \quad (1.5.16a)$$

$$\vec{\nabla} \times \vec{H}_1 = \mathcal{Y}\vec{E}_1 + \vec{J}_1 \quad (1.5.16b)$$

$$\vec{\nabla} \times \vec{E}_2 = -\mathcal{Z}\vec{H}_2 - \vec{M}_2 \quad (1.5.16c)$$

$$\vec{\nabla} \times \vec{H}_2 = \mathcal{Y}\vec{E}_2 + \vec{J}_2 \quad (1.\ 5.16d)$$

Subtract (1.5.16a) dot $\vec{H}_2$ with (1.5.16d) dot $\vec{E}_1$:

$$-\vec{\nabla} \cdot \left(\vec{E}_1 \times \vec{H}_2\right) = \mathcal{Z}\vec{H}_1 \cdot \vec{H}_2 + \vec{H}_2 \cdot \vec{M}_1 + \mathcal{Y}\vec{E}_1 \cdot \vec{E}_2 + \vec{E}_1 \cdot \vec{J}_2$$

(1.5.17a)

1.5.3 Reciprocity Theorem

Interchanging 1 and 2 in (1.5.17a) and we get

$$-\vec{\nabla}\cdot\left(\vec{E}_2\times\vec{H}_1\right)=\mathcal{Z}\vec{H}_1\cdot\vec{H}_2+\vec{H}_1\cdot\vec{M}_2+\mathcal{Y}\vec{E}_1\cdot\vec{E}_2+\vec{E}_2\cdot\vec{J}_1$$

(1.5.17b)

Then subtract (1.5.17a) with (1.5.17b)

$$\vec{\nabla}\cdot\left(\vec{E}_2\times\vec{H}_1-\vec{E}_1\times\vec{H}_2\right)=\left(\vec{E}_1\cdot\vec{J}_2-\vec{H}_1\cdot\vec{M}_2\right)-\left(\vec{E}_2\cdot\vec{J}_1-\vec{H}_2\cdot\vec{M}_1\right)$$

(1.5.18)

This is the Lorentz reciprocity theorem in differential form. Integrating (1.5.18) and we obtain the integral form

$$\oint_S\left(\vec{E}_2\times\vec{H}_1-\vec{E}_1\times\vec{H}_2\right)\cdot d\vec{s}=\langle 1,2\rangle-\langle 2,1\rangle \qquad (1.5.19)$$

1.5.3 Reciprocity Theorem

Extend S to infinity, we get the Rayleigh-Carson reciprocity theorem

$$\langle 1,2\rangle = \langle 2,1\rangle \tag{1.5.20}$$

For a source-free region, we have

$$\oint_S \left(\vec{E}_2 \times \vec{H}_1 - \vec{E}_1 \times \vec{H}_2\right) \cdot d\vec{s} = 0 \tag{1.5.21}$$

For example, if $\left(\vec{E}_1, \vec{H}_1\right)$ and $\left(\vec{E}_2, \vec{H}_2\right)$ represent two different modes in a section of a hollow waveguide. These electromagnetic field pairs must satisfy equation (1.5.21).

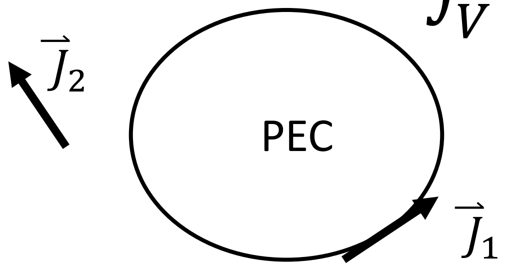
Our derivation assumes a simple medium but applies to general media, excluding nonreciprocal anisotropic materials with a nonsymmetric tensor.

1.5.3 Reciprocity Theorem

As a corollary, we can prove that an impressed electric current sources located on PEC do not radiate by Rayleigh-Carson reciprocity theorem.

Assume a source current $\vec{J}_1$ is placed on a PEC, and there exists an arbitrary source $\vec{J}_2 = Idl\hat{a}$ outside the PEC. Form the reciprocity theorem, we have

$$\langle 1,2\rangle = \int_V \vec{E}_1 \cdot \vec{J}_2 dv = Idl\vec{E}_1(\vec{r}) \cdot \hat{a} = \langle 2,1\rangle = \int_V \vec{E}_2 \cdot \vec{J}_1 dv = 0 \quad (1.5.22)$$



1.5.3 Reciprocity Theorem

The last equality holds because that $\vec{E}_2$ generated by $\vec{J}_2$ should satisfy the boundary condition that the tangential component of $\vec{E}_2$ on the PEC is equal to zero. From (1.5.22), we have $\vec{E}_1(\vec{r}) \cdot \hat{a} = 0$. Since $\vec{r}$ and $\hat{a}$ are arbitrary, we cen deduce that $\vec{E}_1 = 0$ everywhere, indicating that when an electric current element is placed tangentially on a conductor's surface, it does not radiate.

A similar proof applies when an impressed magnetic current is placed on a PMC, showing that it does not radiate.

1.5.4 Equivalence Principle

The equivalence principle states that the fields outside a given region can be replicated by appropriate equivalent sources, replacing the actual sources inside.

The equivalence principle has multiple forms. We will first discuss the volume equivalence principle, followed by the surface equivalence principle.

1.5.4 Equivalence Principle

A. Volume Equivalence Principle

Conisder sources $\left(\vec{J}, \vec{M}\right)$ radiate in vacuum

$$\begin{cases} \vec{\nabla} \times \vec{E}_0 = -i\omega\mu_0\vec{H}_0 - \vec{M} \\ \vec{\nabla} \times \vec{H}_0 = i\omega\epsilon_0\vec{E}_0 + \vec{J} \end{cases} \tag{1.5.23}$$

And also consider the fields when obstacles are presented

$$\begin{cases} \vec{\nabla} \times \vec{E} = -i\omega\mu\vec{H} - \vec{M} \\ \vec{\nabla} \times \vec{H} = i\omega\epsilon\vec{E} + \vec{J} \end{cases} \tag{1.5.24}$$

1.5.4 Equivalence Principle

Subtracting (1.5.24) with (1.5.23), we get the equations for the scattered field

$$\begin{cases} \vec{\nabla} \times (\vec{E} - \vec{E}_0) = \vec{\nabla} \times \vec{E}_s = -i\omega(\mu\vec{H} - \mu_0\vec{H}_0) \\ \vec{\nabla} \times (\vec{H} - \vec{H}_0) = \vec{\nabla} \times \vec{H}_s = i\omega(\epsilon\vec{E} - \epsilon_0\vec{E}_0) \end{cases} \quad (1.5.25)$$

With further simplification

$$\begin{cases} \vec{\nabla} \times \vec{E}_s = -i\omega\mu_0\vec{H}_s - \vec{M}_{eq} \\ \vec{\nabla} \times \vec{H}_s = i\omega\epsilon_0\vec{E}_s + \vec{J}_{eq} \end{cases} \quad (1.5.26)$$

1.5.4 Equivalence Principle

where

$$\begin{cases} \vec{M}_{eq} = i\omega(\mu - \mu_0)\vec{H} \\ \vec{J}_{eq} = i\omega(\epsilon - \epsilon_0)\vec{E} \end{cases} \tag{1.5.27}$$

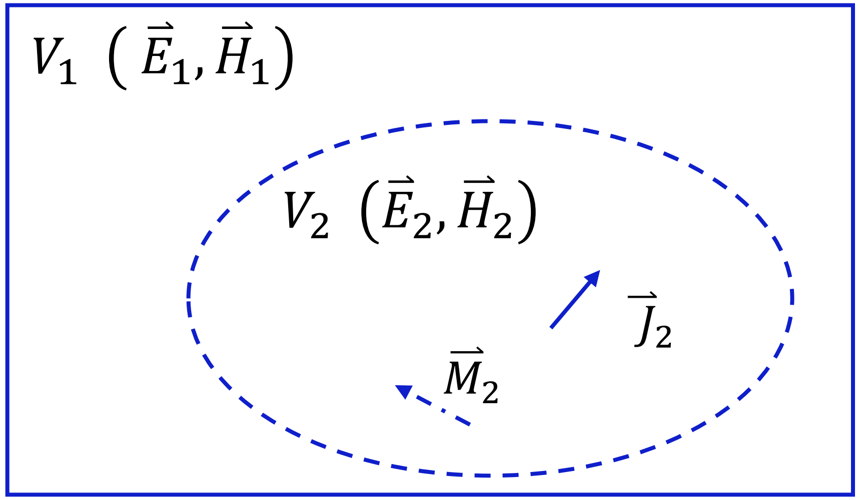
are the equivalent volume currents. Noted that the equivalent volume currents have non-zero value only when $\mu \neq \mu_0$ or $\epsilon \neq \epsilon_0$, that is, the position of the obstacle.

1.5.4 Equivalence Principle

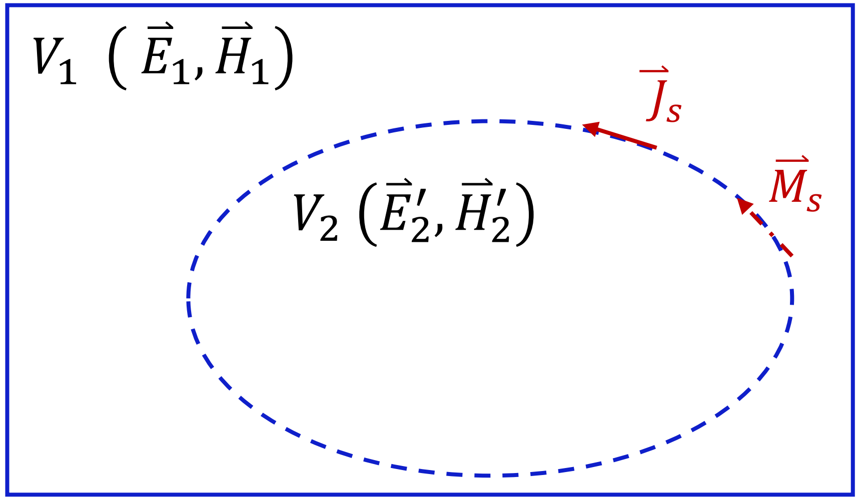
B. Surface Equiavalence Principle

The following figure is a geneal depiction of surface equivalence principle. If we are only interested in the fields in V_1, we can replace the sources in V_2 with equivalent surface currents on the boundary.

Physical Problem



Equivalent Problem



1.5.4 Equivalence Principle

From the matching conditions (1.2.3) and (1.2.5), the sources in V_2 are replaced by surface currents on the boundary with

$$\begin{cases} \vec{J}_s = \hat{n} \times \left(\vec{H}_1 - \vec{H}_2'\right) \\ \vec{M}_s = -\hat{n} \times \left(\vec{E}_1 - \vec{E}_2'\right) \end{cases} \tag{1.5.28}$$

Three special cases are considered:

i. The first is simply setting $\left(\vec{E}_2', \vec{H}_2'\right)$ to zero, then we get

$$\begin{cases} \vec{J}_s = \hat{n} \times \vec{H}_1 \\ \vec{M}_s = -\hat{n} \times \vec{E}_1 \end{cases} \tag{1.5.29a}$$

1.5.4 Equivalence Principle

ii. Further set V_2 as PEC, as we have shown in sectyion 1.5.3 that electric surface currents do not radiate on PEC, we get

$$\begin{cases} \vec{J}_s = 0 \\ \vec{M}_s = -\hat{n} \times \vec{E}_1 \end{cases} \tag{1.5.29b}$$

iii. Or, further set V_2 as PMC, since that magnetic surface currents do not radiate on PMC, we get

$$\begin{cases} \vec{J}_s = \hat{n} \times \vec{H}_1 \\ \vec{M}_s = 0 \end{cases} \tag{1.5.29c}$$

Problems

1. Show (1.1.19) by using the divergence theorem.
2. Show that $-\hat{r} \times (\hat{r} \times \vec{u}) = \hat{\theta}(\hat{\theta} \cdot \vec{u}) + \hat{\phi}(\hat{\phi} \cdot \vec{u}) = (\hat{\theta}\hat{\theta} + \hat{\phi}\hat{\phi}) \cdot \vec{u}$ where $\vec{u}$ is an arbitrary vector. We use the dyadic notation in the last expression.
3. Show that $\vec{\nabla}\left(\frac{1}{R}\right) = -\vec{\nabla}'\left(\frac{1}{R}\right) = -\frac{\vec{R}}{R^3}$ with $\vec{R} = \vec{r} - \vec{r}'$ and $|\vec{R}| = R$. $\vec{\nabla}$ denotes operation with respect to $\vec{r}$, and $\vec{\nabla}'$ denotes operation with respect to $\vec{r}'$.

Problems

4. Show that $\vec{\nabla} \cdot \left(\frac{\vec{R}}{R^3}\right) = -\nabla^2 \left(\frac{1}{R}\right) = 4\pi\delta\left(\vec{R}\right)$ with $\vec{R} = \vec{r} - \vec{r}'$.
5. Consider the vector Helmholtz quation $(\nabla^2 + k^2)\vec{E} = 0$. Suppose $\vec{\nabla} \cdot \vec{E} = 0$, show that $\vec{E} = \vec{\nabla} \times (\psi\vec{u})$ is a solution with $\vec{u}$ being a constant vector and ψ satisfies the scalar Helmholtz equation (1.4.18).
6. Show that with inhonogenous $\epsilon(\vec{r})$, in source-free region the electric field satisfies the equation $(\nabla^2 + k^2)\vec{E} = -\vec{\nabla}\left(\vec{E} \cdot \frac{\vec{\nabla}\epsilon}{\epsilon}\right)$.