Homework 2 Solution
PubH 7440: Introduction to Bayesian Analysis
Spring 2020

1. (Gelman, Chapter 3, Problem 3):

(a).

Since
$p(\mu_c, \sigma_c^2|y_c) \propto p(\mu_c, \sigma_c^2) \prod_{i=1}^{32} p(y_{ci}|\mu_c, \sigma_c^2)$ and
$p(\mu_t, \sigma_t^2|y_t) \propto p(\mu_t, \sigma_t^2) \prod_{i=1}^{36} p(y_{ti}|\mu_t, \sigma_t^2)$.
From the derivations in the lecture notes, the marginal posterior distribution for μ_c and μ_t follow t-distribution with location-scale parameters to be $(1.013,\ 0.025/\sqrt{32.31})$ and $(1.173,\ 0.2/6)$ respectively, and degree of freedom to be 31 and 35 respectively.

(b).

Example R code:

```
mu.c <- 1.013 + (0.025/sqrt(32))*rt(1000,31)
mu.t <- 1.173 + (0.20/sqrt(36))*rt(1000,35)
dif <- mu.t - mu.c
hist (dif, xlab="mu_t - mu_c", yaxt="n",
breaks=seq(-.1,.4,.02), cex=2)
print (sort(dif)[c(25,976)])
```

2. (Gelman, Chapter 3, Problem 9):

Let $\bar{y}$ and s^2 be the sample mean and sample variance:
Then,

$$p(\mu, \sigma^2|y) \propto p(y|\mu, \sigma^2)p(\mu|\sigma^2)p(\sigma^2)$$

$$\propto \sigma^{-n} \exp\left(-\frac{(n-1)s^2 + n(\mu - \bar{y})^2}{2\sigma^2}\right) \sigma^{-1} \exp\left(-\frac{k(\mu - \mu_0)^2}{2\sigma^2}\right) (\sigma^2)^{-\nu_0 - 1} \exp\left(-\frac{\sigma_0^2}{\sigma^2}\right)$$

$$\propto \sigma^{-1} (\sigma^2)^{-\left(\frac{n}{2} + \nu_0 + 1\right)} \exp\left(-\frac{(n+k)\left(\mu - \frac{n\bar{y} + k\mu_0}{n+k}\right)^2 + \frac{nk(\mu_0 - \bar{y})^2}{n+k} + 2\sigma_0^2 + (n-1)s^2}{2\sigma^2}\right)$$

Hence

$$\mu, \sigma^2|y \sim N - Inv\left(\frac{\mu_0 k_0 + n\bar{y}}{n+k}, n+k; \frac{n}{2} + v_0, 2\sigma_0^2 + (n-1)s^2 + \frac{nk(\mu_0 - \bar{y})^2}{n+k}\right)$$

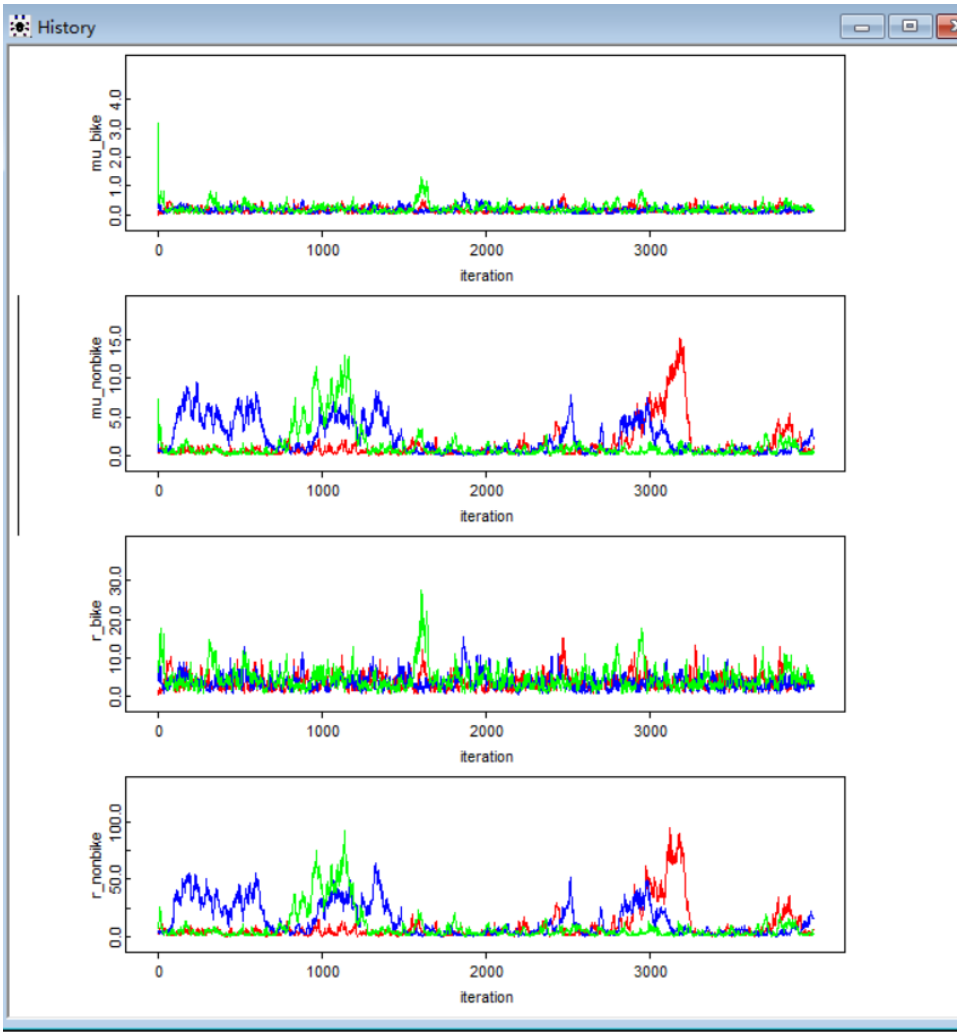
3.

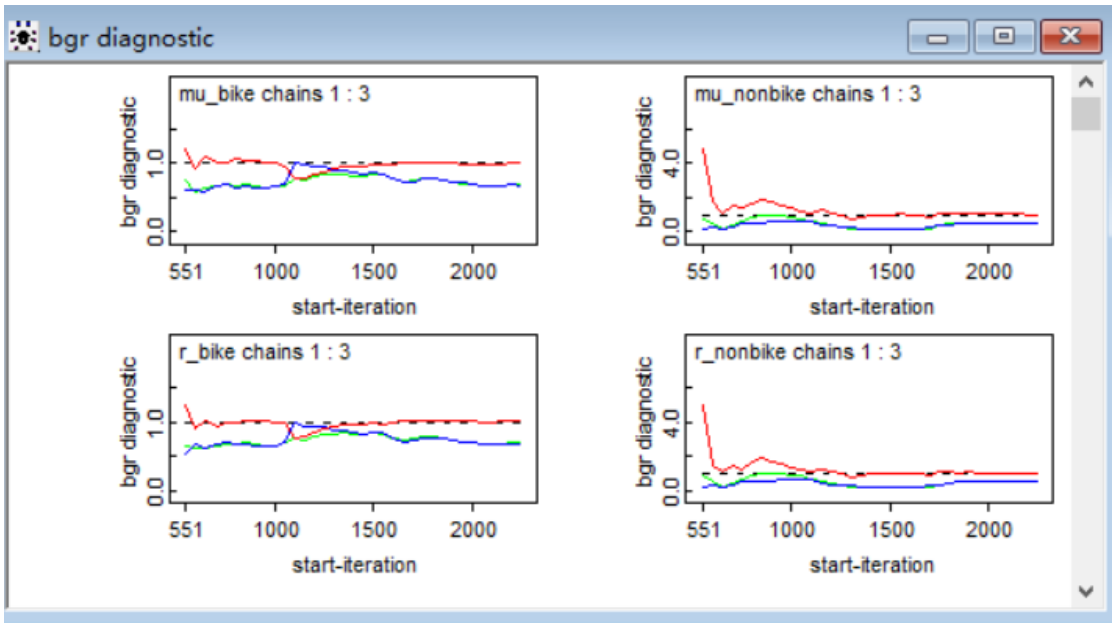
(a).

```
model
{
 for( i in 1 : k_bike ) {
```

```
y_bike[i] ~ dpois(lambda_bike[i])
lambda_bike[i] ~ dgamma(r_bike, mu_bike)
}
for( i in 1 : k_nonbike ) {
y_nonbike[i] ~ dpois(lambda_nonbike[i])
lambda_nonbike[i] ~ dgamma(r_nonbike, mu_nonbike)
}
r_bike ~ dgamma(0.01, 0.01)
mu_bike ~ dgamma(0.01, 0.01)
r_nonbike ~ dgamma(0.01, 0.01)
mu_nonbike ~ dgamma(0.01, 0.01)
}
```

(b).





The trace plots and Gelman-Rubin statistic for the bike intersections show clear convergence. It is less clear for the non-bike data. The trace plots suggest that the chains may not have converged, whereas the Gelman-Rubin statistic is consistent with convergence.

(c).

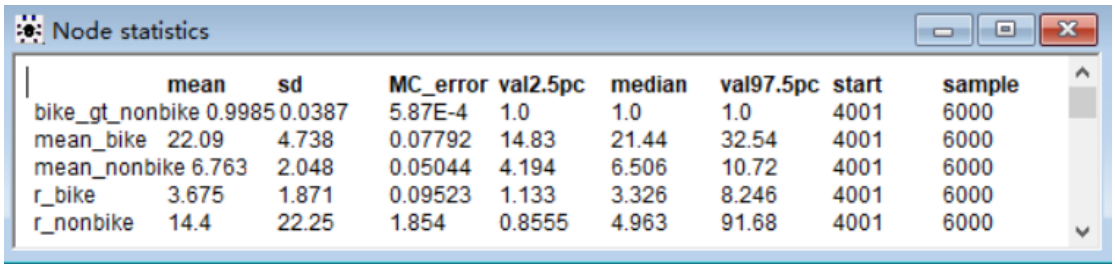
Node statistics

	mean	sd	MC_error	val2.5pc	median	val97.5pc	start	sample
lambda_bike[1]	16.8	3.821	0.04567	10.11	16.49	25.09	4001	6000
lambda_bike[2]	10.8	3.101	0.05455	5.57	10.51	17.58	4001	6000
lambda_bike[3]	11.6	3.236	0.04985	6.198	11.26	18.82	4001	6000
lambda_bike[4]	14.23	3.518	0.0509	8.069	13.93	21.88	4001	6000
lambda_bike[5]	19.36	4.058	0.05995	12.26	19.0	28.0	4001	6000
lambda_bike[6]	20.15	4.158	0.05826	12.79	19.89	29.07	4001	6000
lambda_bike[7]	18.51	4.005	0.05725	11.48	18.27	26.93	4001	6000
lambda_bike[8]	17.6	3.867	0.04935	10.85	17.3	25.9	4001	6000
lambda_bike[9]	33.0	5.42	0.08499	23.31	32.72	44.52	4001	6000
lambda_bike[10]	50.22	6.932	0.1526	37.59	49.86	64.87	4001	6000
lambda_nonbike[1]	9.297	2.712	0.1153	5.323	8.87	15.67	4001	6000
lambda_nonbike[2]	3.669	1.878	0.1069	0.6373	3.522	7.512	4001	6000
lambda_nonbike[3]	4.191	1.826	0.08966	1.052	4.094	7.979	4001	6000
lambda_nonbike[4]	5.216	1.758	0.05202	2.082	5.153	8.93	4001	6000
lambda_nonbike[5]	7.779	2.216	0.06465	4.357	7.464	12.98	4001	6000
lambda_nonbike[6]	6.753	1.98	0.03814	3.538	6.528	11.47	4001	6000
lambda_nonbike[7]	7.776	2.268	0.06609	4.298	7.449	13.22	4001	6000
lambda_nonbike[8]	7.241	2.082	0.04642	3.834	6.98	12.08	4001	6000
mu_bike	0.173	0.09305	0.004732	0.0451	0.1557	0.3952	4001	6000
mu_nonbike	2.256	3.544	0.2961	0.1075	0.7772	15.07	4001	6000
r_bike	3.675	1.871	0.09523	1.133	3.326	8.246	4001	6000
r_nonbike	14.4	22.25	1.854	0.8555	4.963	91.68	4001	6000

(d).

```
model
{
for( i in 1 : k_bike ) {
```

```
y_bike[i] ~ dpois(lambda_bike[i])
lambda_bike[i] ~ dgamma(r_bike, mu_bike)
}
for( i in 1 : k_nonbike ) {
y_nonbike[i] ~ dpois(lambda_nonbike[i])
lambda_nonbike[i] ~ dgamma(r_nonbike, mu_nonbike)
}
r_bike ~ dgamma(0.01, 0.01)
mu_bike ~ dgamma(0.01, 0.01)
r_nonbike ~ dgamma(0.01, 0.01)
mu_nonbike ~ dgamma(0.01, 0.01)

mean_bike<-r_bike/mu_bike
mean_nonbike<-r_nonbike/mu_nonbike
bike_gt_nonbike <- step(mean_bike-mean_nonbike)
}
```



	mean	sd	MC_error	val2.5pc	median	val97.5pc	start	sample
bike_gt_nonbike	0.9985	0.0387	5.87E-4	1.0	1.0	1.0	4001	6000
mean_bike	22.09	4.738	0.07792	14.83	21.44	32.54	4001	6000
mean_nonbike	6.763	2.048	0.05044	4.194	6.506	10.72	4001	6000
r_bike	3.675	1.871	0.09523	1.133	3.326	8.246	4001	6000
r_nonbike	14.4	22.25	1.854	0.8555	4.963	91.68	4001	6000

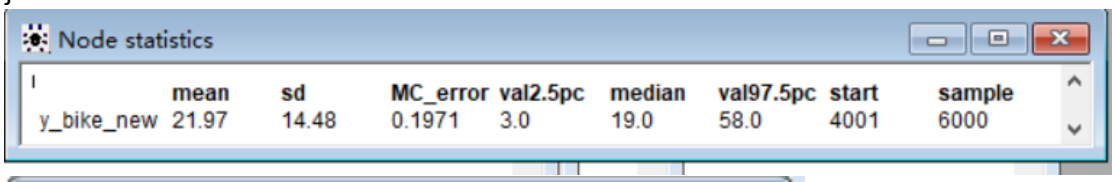
(e).

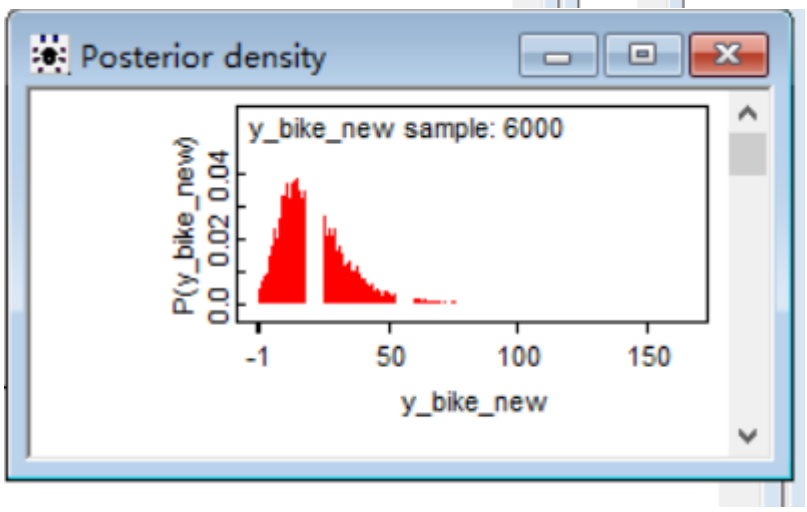
```
model
{
for( i in 1 : k_bike ) {

y_bike[i] ~ dpois(lambda_bike[i])
lambda_bike[i] ~ dgamma(r_bike, mu_bike)
}
for( i in 1 : k_nonbike ) {
y_nonbike[i] ~ dpois(lambda_nonbike[i])
lambda_nonbike[i] ~ dgamma(r_nonbike, mu_nonbike)
}
r_bike ~ dgamma(0.01, 0.01)
mu_bike ~ dgamma(0.01, 0.01)
r_nonbike ~ dgamma(0.01, 0.01)
mu_nonbike ~ dgamma(0.01, 0.01)

mean_bike<-r_bike/mu_bike
mean_nonbike<-r_nonbike/mu_nonbike
```

bike_gt_nonbike <- step(mean_bike-mean_nonbike)

lambda_bike_new~dgamma(r_bike,mu_bike)
y_bike_new~dpois(lambda_bike_new)

}





4.

Example R code:

```
####define the target density###
f<-function(x){
   out <- (1/3) * dnorm(x, mean = -5) + (1/3) * dnorm(x) + (1/3) * dnorm(x, mean = 5)
   return(out)
}
####importance sampling###
set.seed(100)
## sampling from candidate density
n <- 200
x<-rnorm(n,0,sqrt(20))
####calculate the weights###
w<-f(x)/dnorm(x, sd = sqrt(20))
round(weighted.mean(x, w),2)
round(weighted.mean(x^2, w) - weighted.mean(x,w)^2,2)
####rejection sampling###
rej_algorithm <- function(xxx) {
   accept <- 0
   while(accept == 0) {
      M <- 3
```

```
    theta <- rnorm(n=1, sd = 5)
    U <- runif(n=1,min = 0, max = 1)
    accept <- 1*(U < f(theta) / (M * dnorm(theta, sd = 5)))
  }
  return(theta)
}
set.seed(100)
rej_samples <- sapply(1:200, rej_algorithm)
round(mean(rej_samples),2)
round(var(rej_samples),2)
```