PubH 7440: Introduction to Bayesian Analysis
Spring 2020, Homework 1
Due: February 6, 2020

1. (Carlin and Louis, Chapter 2, Problem 8): Suppose that $f(x|\theta,\sigma)$ is normal with mean θ and standard deviation σ.
    a. Suppose that σ is known, and show that the Jeffreys prior for θ is given by:
    $$p(\theta) = 1, \theta \in R$$
    b. Next, suppose that θ is known, and show that the Jeffreys prior for σ is given by:
    $$p(\sigma) = \frac{1}{\sigma}, \sigma > 0$$

2. (Gelman, Chapter 2, Problem 5): Let y be the number of heads in n spins of a coin, whose probability of heads if θ
    a. If your prior distribution for θ is uniform on the range $[0,1]$, derive your pior predictive distribution for y,
    $$\Pr(y = k) = \int_0^1 \Pr(y = k|\theta)\, d\theta$$
    For each $k = 0,1, \ldots, n$.
    b. Suppose you assign a $Beta(\alpha, \beta)$ prior distribution for θ, and then you observe y heads out of n spins. Show algebraically that your posterior mean of θ always lies between your prior mean, $\frac{\alpha}{\alpha+\beta}$, and the observed relative frequency of heads, $\frac{y}{n}$.
    c. Show that, if the prior distribution on θ is uniform, the posterior variance of θ is always less than the prior variance.

3. (Gelman, Chapter 2, Problem 8): A random sample of *n* students is drawn from a large population, and their weights are measured. The average weight of the *n* sampled students is $\bar{y} = 150$ pounds. Assume the weights in the population are normally distributed with unknown mean θ and known standard deviation 20 pounds. Suppose your prior distribution for θ is normal with mean 180 and standard deviation 40.
    a. Give your posterior distribution for θ.
    b. A new student is sampled at random from the same population and has a weight $\tilde{y}$ pounds. Give a posterior predictive distribution for $\tilde{y}$.
    c. For n = 10, give a 95% posterior interval for θ and a 95% posterior predictive interval for $\tilde{y}$.
    d. Do the same for n = 100.

4. (Gelman, Chapter 2, Problem 9): Suppose your prior distribution for θ, the proportion of Californians who support the death penalty, is beta with mean 0.6 and standard deviation of 0.3.
    a. Determine the parameters α and β of your prior distributions. Sketch the prior density function.

b. A random sample of 1000 Californians is taken, and 65% support the death penalty. What are your posterior mean and variance for θ? Draw the posterior density function.

c. Examine the sensitivity of the posterior distribution to different prior means and widths including a non-informative prior.

5. (Gelman, Chapter 2, Problem 14b): Consider n iid observations $\boldsymbol{y} = (y_1, \dots, y_n)$ from the normal distribution $N(\theta, \sigma^2)$, and the prior $\pi(\theta) = N(\mu, \tau^2)$. We know from the lecture that

$$p(\theta|\boldsymbol{y}) = p(\theta|\bar{y}) = N\left(\frac{\frac{\sigma^2}{n}\mu + \tau^2\bar{y}}{\frac{\sigma^2}{n} + \tau^2}, \frac{\sigma^2\tau^2}{\sigma^2 + n\tau^2}\right).$$

Derive the above posterior by staring with the $N(\mu, \tau^2)$ prior distribution and adding data points one at a time, using the posterior distribution at each step as the prior distribution for the next.