**Homework 3 Solution**
**PubH 7440: Introduction to Bayesian Analysis**
**Spring 2020**

1. **(Carlin and Louis, Chapter 5, Problem 9):**

**(a).**

Since

$$P(\theta|y,\lambda,b_1,b_2) \propto P(\theta,\lambda,b_1,b_2|y) \propto \theta^{a_1-1}e^{-\frac{\theta}{b_1}}\prod_{i=1}^{k}\theta^{Y_i}e^{-\theta}$$

Hence $\theta|y,\lambda,b_1,b_2 \sim Gamma(\sum_{i=1}^{k} Y_i + a_1, rate = \frac{1}{b_1} + k)$.

$$P(\lambda|y,\theta,b_1,b_2) \propto P(\theta,\lambda,b_1,b_2|y) \propto \lambda^{a_2-1}e^{-\frac{\lambda}{b_2}}\prod_{i=k+1}^{n}\lambda^{Y_i}e^{-\lambda}$$

Hence $\lambda|y,\theta,b_1,b_2 \sim Gamma(\sum_{i=k+1}^{n} Y_i + a_2, rate = \frac{1}{b_2} + n - k)$.

$$P(b_1|y,\theta,\lambda,b_2) \propto P(\theta,\lambda,b_1,b_2|y) \propto b_1^{-a_1}e^{-\frac{\theta}{b_1}}b_1^{-c_1-1}e^{-\frac{d_1}{b_1}}$$

Hence $b_1|y,\theta,\lambda,b_2 \sim InvGamma(a_1 + c_1, \theta + d_1)$

$$P(b_2|y,\theta,\lambda,b_1) \propto P(\theta,\lambda,b_1,b_2|y) \propto b_2^{-a_2}e^{-\frac{\lambda}{b_2}}b_2^{-c_2-1}e^{-\frac{d_2}{b_2}}$$

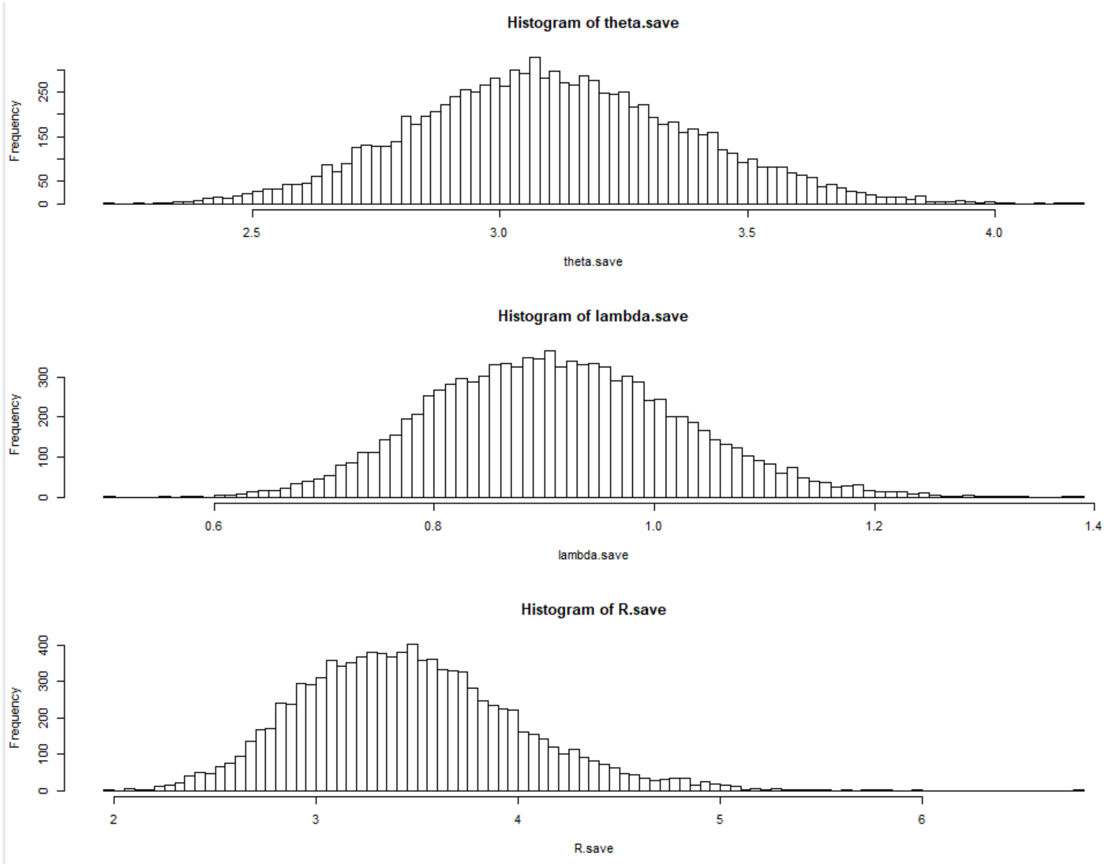
Hence $b_2|y,\theta,\lambda,b_1 \sim InvGamma(a_2 + c_2, \lambda + d_2)$

Example R code:

```
library(MCMCpack)

set.seed(2020)

# data loading
coal = read.csv("C:\\Users\\liujiawei\\Desktop\\7440\\coalminingdisaster.csv")
y = coal$Count
# parameters
k = 40
n = dim(coal)[1]
a1 = a2 = .5
c1 = c2 = d1 = d2 = 1

# burnin
runs = 10000
burn = 1000
```

```r
theta.save = rep(NA,runs)
lambda.save = rep(NA,runs)
b1.save = rep(NA,runs)
b2.save = rep(NA,runs)
R.save = rep(NA,runs)

# Gibbs Sampling

theta = lambda = b1 = b2 = 1

for (it in 1:(runs+burn)) {

  theta = rgamma(1,sum(y[1:k])+a1, rate=1/b1+k)
  lambda = rgamma(1,sum(y[(k+1):n])+a2, rate=1/b2+n-k)
  b1 = rinvgamma(1,a1+c1,scale=theta+d1)
  b2 = rinvgamma(1,a2+c2,scale=lambda+d2)
  R = theta/lambda

  if(it>burn) {
    theta.save[it-burn] <- theta
    lambda.save[it-burn] <- lambda
    b1.save[it-burn] <- b1
    b2.save[it-burn] <- b2
    R.save[it-burn] <- R
  }

}
mean(theta.save)
mean(lambda.save)
mean(R.save)

par(mfrow=c(3,1))
hist(theta.save)
hist(lambda.save)
hist(R.save)
```

```
> mean(theta.save)
[1] 3.106531
> mean(lambda.save)
[1] 0.9116761
> mean(R.save)
[1] 3.459783
```




**(b).**

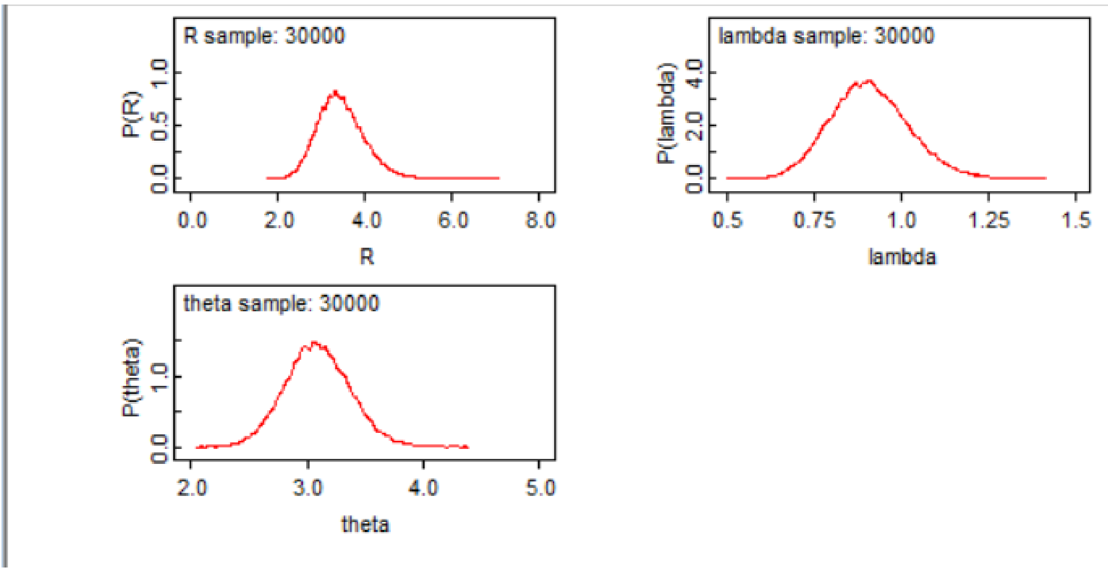
Example WinBUGS code:

```
# data
list(count=c(4,5,4,1,0,4,3,4,0,6,3,3,4,0,2,6,3,3,5,4,5,3,1,4,4,1,5,5,3,4,2,5,2,2,3,4,2,1,3,2,2,1,1,
1,1,3,0,0,1,0,1,1,0,0,3,1,0,3,2,2,0,1,1,1,0,1,0,1,0,0,0,2,1,0,0,0,1,1,0,2,3,3,1,1,2,1,1,1,1,2,4,2,0
,0,0,1,4,0,0,0,1,0,0,0,0,0,1,0,0,1,0,1), k=40, n=112)


model
{
for( i in 1:k){
count[i] ~ dpois(theta)
}
for(i in (k+1):n){
count[i]~dpois(lambda)
}
theta~dgamma(0.5,b1_inv)
lambda~dgamma(0.5,b2_inv)
b1_inv<-1/b1
b2_inv<-1/b2
b1~dgamma(1,1)
b2~dgamma(1,1)
```

```
R <- theta/lambda
}
```

Node statistics

| | mean | sd | MC_error | val2.5pc | median | val97.5pc | start | sample |
|---|---|---|---|---|---|---|---|---|
| R | 3.45 | 0.5296 | 0.002939 | 2.536 | 3.404 | 4.607 | 1001 | 30000 |
| lambda | 0.9104 | 0.1121 | 6.506E-4 | 0.7038 | 0.9059 | 1.143 | 1001 | 30000 |
| theta | 3.093 | 0.2774 | 0.001573 | 2.569 | 3.086 | 3.661 | 1001 | 30000 |




Outputs from two approaches are approximately equal.

**(c).**

Calculate the unnormalized conditional posterior for $k$, to be $P_k^*$

At iteration $m$,

Draw $\theta$ from $Gamma(\sum_{i=1}^{k} Y_i + a_1, rate = \frac{1}{b_1} + k)$ and

Draw $\lambda$ from $Gamma(\sum_{i=k+1}^{n} Y_i + a_2, rate = \frac{1}{b_2} + n - k)$

Draw $b_1$ from $InvGamma(a_1 + c_1, \theta + d_1)$ and $b_2$ from $InvGamma(a_2 + c_2, \lambda + d_2)$

Draw $k_m^*$ from $Discrete\ Unif\ (1, \dots, n)$ (Might be of low acceptance rate. Could use truncated random walk, adjust $\alpha$ accordingly)

Compute $\alpha = \frac{P_k^*(k_m^*)}{P_k^*(k_{m-1}^*)}$, let $k_m = k_m^*$ with probability $\min(\alpha, 1)$, otherwise $k_m = k_{m-1}$