Decision Theory, Generativive / discriminative models

- multiple points w/ different classes could be at the same point
- want probabilistic model

ex. 10% pop has cancer, 90% doesn't, calorie intake

$P(X \mid Y)$ X: calories
Y: -1/1 don't/do have cancer

$$P(X) = P(X \mid Y=1)\,P(Y=1) + P(X \mid Y=-1)\,P(Y=-1) = 0.14$$

Guess whether guy has cancer given 1400 calories/day

$$\underbrace{P(Y=1 \mid X)}_{\text{posterior}} = \frac{P(X \mid Y=1)\,\overbrace{P(Y=1)}^{\text{prior}}}{P(X \mid Y=1)\,P(Y=1) + P(X \mid Y=-1)\,P(Y=-1)} \Bigg\}\ \text{Bayes rule}$$

$$= \frac{0.05}{0.14} \approx 36\%$$

loss function $L(z, y)$ punishment for wrong classification

Defs: loss fn above is asymmetrical
The 0-1 loss function is 1 for incorrect, 0 for correct

Let $r: \mathbb{R}^d \to \pm 1$ be a decision rule, aka classifier:
1 = "in class", -1 = "not in class"
The risk is the expected loss over all values of x, y:

$$R(r) = \mathbb{E}[L(r(X), Y)] = \sum_{x}\Big(L(r(x), 1)\,P(Y=1 \mid X=x) + L(r(x), -1)\,P(Y=-1 \mid X=x)\Big)\,P(X=x)$$

$$= P(Y=1) \sum_x L(r(x), 1)\, P(X=x \mid Y=1)$$
$$+ P(Y=-1) \sum_x L(r(x), -1)\, P(X=x \mid Y=-1)$$

Bayes decision rule aka Bayes classifier is fn r^* that minimizes $R(r)$

$$L(z, y) = 0 \text{ for } z = y$$

$$r^*(x) = \begin{cases} 1 \text{ if } \overbrace{L(-1,1)\, P(Y=1 \mid X=x)}^{\text{false negative}} > \overbrace{L(1,-1)\, P(Y=-1 \mid X=x)}^{\text{false positive}} \\ -1 \text{ else} \end{cases}$$

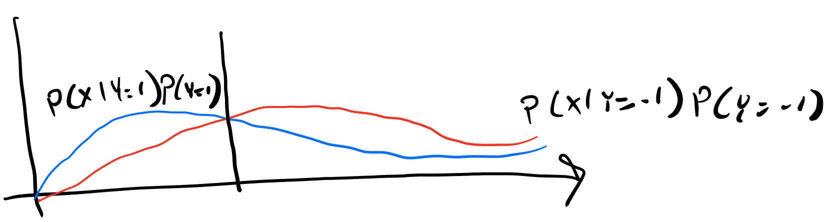
When L is symmetric, pick class w/ bigger posterior probability.

The Bayes risk, aka optimal risk = risk of Bayes classifier

$R(r^*) = 0.249$ No r gives a lower risk

Deriving fn r^* is called risk minimization

Continuous distribution



$$R(r) = \mathbb{E}[\, L(r(x), y)\,]$$

$$= P(Y=1) \int L(r(x), 1)\, f(X=x \mid Y=1)\, dx$$
$$+ P(Y=-1) \int L(r(x), -1)\, f(X=x \mid Y=-1)\, dx$$

Bayes risk is the area under the minimum of the two curves.

$$R(r^*) = \int \min_{y=\pm1} L(-y, y)\, f(X=x \mid Y=y)\, P(Y=y)\, dx$$

If L is 0-1 loss, risk is $P(r(x)$ is wrong$)$

Bayes optimal decision boundary is $\{x : P(Y=1 \mid X=x) = 0.5\}$

3 ways to build classifiers

① Generative models

- assume sample pts come from prob distribution, different for each class
- guess form
- for each class, fit dist parameters to c pts $f(X \mid Y=c)$
- for each c, estimate $P(Y=c)$
- Bayes gives $P(Y \mid X)$

- If 0-1 loss, pick c that maximizes

$$P(X=x \mid Y=c)\,P(Y=c)$$

(2) Discriminative models (logistic regression)

- Model $P(Y|x)$

(3) Find decision boundary (SVM)

- model $r(x)$ directly w/o posterior prob

(1) and (2): $P(Y|x)$ tells prob guess is wrong

(1): you can diagnose outliers: $P(x)$ small

(1): often hard to estimate distributions