Derivation of steady-state solution for $D(x_s)$ ($n \leq 4$)

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The steady state solutions are given by

$$L_n T_n + \delta_{0n} A = Q H_n \tag{1}$$

where $L_n = n(n+1)D + B$ and

$$H_n = H_n(x_s) = (2n+1)\int_0^1 P_n(x) a(x, x_s) S(x) dx \equiv (2n+1) J_n(x_s). \tag{2}$$

Rearranging (1) and multiplying by $P_n(x)$:

$$T_n P_n = \frac{QH_n - \delta_{0n}A}{L_n} P_n(x) \tag{3}$$

$$\implies T(x) = \sum_{n \text{ even}} T_n P_n(x) = \sum_{n \text{ even}} \frac{QH_n - \delta_{0n}A}{L_n} P_n(x) = \sum_{n \text{ even}} \frac{QH_n P_n(x)}{L_n} - \frac{A}{B}. \tag{4}$$

Truncating at $n = 4$:

$$\begin{aligned} T(x) &= Q\left(\frac{H_0 P_0(x)}{L_0} + \frac{H_2 P_2(x)}{L_2} + \frac{H_4 P_4(x)}{L_4}\right) - \frac{A}{B} \\ &= Q\left(\frac{J_0}{B} + \frac{5J_2 P_2(x)}{6D + B} + \frac{9J_4 P_4}{20D + B}\right) - \frac{A}{B} \end{aligned}$$

Evaluating at $x = x_s$:

$$T_s = Q\frac{J_0(6D+B)(20D+B) + 5J_2P_2B(20D+B) + 9J_4P_4B(6D+B)}{B(6D+B)(20D+B)} - \frac{A}{B}$$

where $P_m = P_m(x_s)$ and $J_m = J_m(x_s)$ is implied. Rearranging gives

$$\frac{A + BT_s}{Q}(20D+B)(6D+B) = J_0(6D+B)(20D+B) + 5J_2P_2B(20D+B) + 9J_4P_4B(6D+B),$$

which is a quadratic in D. Using $(20D + B)(6D + B) \equiv 120D^2 + 26BD + B^2$ and collecting terms with common factors of D^2, D and D^0:

$$\begin{aligned} 120\left(\frac{A + BT_s}{Q} - J_0\right) D^2 + \left(\frac{26B(A + BT_s)}{Q} - 26BJ_0 - 100BJ_2P_2 - 54BJ_4P_4\right) D + \ldots \\ \ldots + \left(\frac{B^2(A + BT_s)}{Q} - B^2J_0 - 5B^2J_2P_2 - 9B^2J_4P_4\right) = 0 \end{aligned}$$

This gives the solution for $D(x_s)$ as

$$D(x_s) = \frac{-\beta \pm \sqrt{\beta^2 - 4\alpha\gamma}}{2\alpha}, \tag{5}$$

where

$$\begin{aligned}
\alpha &= 120\left(\frac{A + BT_s}{Q} - J_0\right) \\
\beta &= 2B\left(13\left(\frac{A + BT_s}{Q} - J_0\right) - 50J_2P_2 - 27J_4P_4\right) \\
\gamma &= B^2\left(\frac{A + BT_s}{Q} - J_0 - 5J_2P_2 - 9J_4P_4\right).
\end{aligned}$$