Derivation of expression for $h(x_s)$ ($n \leq 4$)

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The heat flux convergence in the EBM model is given by

$$-\nabla \cdot \boldsymbol{\phi} \equiv h(x) = D\frac{\partial}{\partial x}\left(1-x^2\right)\frac{\partial}{\partial x}T(x, x_s) \tag{1}$$

where

$$T(x, x_s) = \sum_{n \text{ even}} P_n(x)T_n(x_s) = \sum_{n \text{ even}} P_n(x)\left(\frac{(2n+1)Q(x_s)}{n(n+1)D+B}\int_0^1 P_n(x)S(x)a(x, x_s)dx - \frac{\delta_{0n}A}{B}\right),$$

noting that for steady-state solutions $Q = Q(x_s)$. Expanding the Laplacian in equation (1):

$$\begin{aligned} h(x) &= -2Dx\frac{\partial}{\partial x}\sum_{n \text{ even}} P_n(x)T_n(x_s) + D(1-x^2)\frac{\partial^2}{\partial x^2}\sum_{n \text{ even}} P_n(x)T_n(x_s) \\ &= -2Dx\sum_{n \text{ even}} T_n(x_s)\frac{\partial P_n}{\partial x} + D(1-x^2)\sum_{n \text{ even}} T_n(x_s)\frac{\partial^2 P_n}{\partial x^2}. \end{aligned}$$

The order-m derivative of $P_n(x)$ with respect to x is given by

$$\frac{\partial^m P_n}{\partial x^m} = (-1)^m(1-x^2)^{-\frac{m}{2}}P_n^m(x) \tag{2}$$

where $P_n^m(x)$ is the associated Legendre polynomial[1]. Using this leads to

$$h(x_s) = \frac{2Dx_s}{\sqrt{1-x_s^2}}\sum_{\substack{n\geq 2,\\ \text{even}}} T_n(x_s)P_n^1(x_s) + D\sum_{\substack{n\geq 2,\\ \text{even}}} T_n(x_s)P_n^2(x_s). \tag{3}$$

Now expand for $n \leq 4$, collect $T_2(x_s)$ and $T_4(x_s)$ terms and use

$$\begin{aligned} P_0(x) &= 1 \\ P_2(x) &= \frac{1}{2}(3x^2-1) \\ P_4(x) &= \frac{1}{8}(35x^4-30x^2+3) \\ P_2^1(x) &= -3x\sqrt{1-x^2} \\ P_2^2(x) &= 3(1-x^2) \\ P_4^1(x) &= \frac{5}{2}x(3-7x^2)\sqrt{1-x^2} \\ P_4^2(x) &= \frac{15}{2}(7x^2-1)(1-x^2) \end{aligned}$$

[1] http://mathworld.wolfram.com/AssociatedLegendrePolynomial.html

to give

$$h(x_s) = -6DP_2(x_s)T_2(x_s) - 20DP_4(x_s)T_4(x_s), \tag{4}$$

which looks like $-\sum n(n+1)DP_n(x_s)T_n(x_s)$ and this is probably not a coincidence. Next, determine the forms of $T_2(x_s)$ and $T_4(x_s)$. For T_2,

$$T_2(x_s) = \frac{5Q(x_s)}{6D+B}\left(a_f\int_0^{x_s} f_2(x)dx + a_i\int_{x_s}^1 f_2(x)dx\right) \tag{5}$$

where

$$\begin{aligned} f_2(x) &= \frac{1}{2}\left(3x^2-1\right)\left(1+\frac{S_2}{2}\left(3x^2-1\right)\right) \\ &= \frac{9S_2}{4}x^4 + \frac{3}{2}(1-S_2)x^2 - \frac{2-S_2}{4} \\ \implies \int f_2(x) \equiv F_2(x) &= \frac{9S_2}{20}x^5 + \frac{1}{2}(1-S_2)x^3 - \frac{2-S_2}{4}x. \end{aligned}$$

Using this in equation (5),

$$\begin{aligned} T_2 &= \frac{5Q(x_s)}{6D+B}\left(a_f(F_2(x_s)-F_2(0)) + a_i(F_2(1)-F_2(x_s))\right) \\ &= \frac{5Q(x_s)}{6D+B}\left(\delta a F_2(x_s) + a_i F_2(1)\right) \\ &= \frac{5Q(x_s)\delta a}{6D+B}\left(\frac{9S_2}{20}x_s^5 + \frac{1-S_2}{2}x_s^3 - \frac{2-S_2}{4}x_s + \frac{a_i}{\delta a}\frac{S_2}{5}\right), \end{aligned} \tag{6}$$

where $\delta a = a_f - a_i$. Similarly,

$$T_4(x_s) = \frac{9Q(x_s)}{20D+B}\left(a_f\int_0^{x_s} f_4(x)dx + a_i\int_{x_s}^1 f_4(x)dx\right) \tag{7}$$

where

$$\begin{aligned} f_4(x) &= \frac{1}{8}\left(35x^4-30x^2+3\right)\left(1+\frac{S_2}{2}\left(3x^2-1\right)\right) \\ &= \frac{1}{16}\left(105S_2x^6 + 5(14-25S_2)x^4 + 3(13S_2-20)x^2 + 3(2-S_2)\right) \\ \implies \int f_4(x) \equiv F_4(x) &= \frac{1}{16}\left(15S_2x^7 + (14-25S_2)x^5 + (13S_2-20)x^3 + 3(2-S_2)x\right). \end{aligned}$$

Using this in equation (7),

$$\begin{aligned} T_4 &= \frac{9Q(x_s)}{20D+B}\left(a_f(F_4(x_s)-F_4(0)) + a_i(F_4(1)-F_4(x_s))\right) \\ &= \frac{9Q(x_s)}{20D+B}\left(\delta a F_4(x_s) + a_i F_4(1)\right) \\ &= \frac{9Q(x_s)\delta a}{16(20D+B)}\left(15S_2x_s^7 + (14-25S_2)x_s^5 + (13S_2-20)x_s^3 + 3(2-S_2)x_s\right); \end{aligned} \tag{8}$$

note that $F_4(1) = 0$. Finally we require the form of $Q(x_s)$.

$$Q(x_s) = (A+BT_s)\left(B\sum_{n \text{ even}}\frac{2n+1}{n(n+1)D+B}P_n(x_s)\int_0^1 P_n(x)S(x)a(x,x_s)dx\right)^{-1}$$

From here, it can be seen that an explicit representation of h is not practical to derive. It is not a polynomial in x_s.