Quantum Information Measures from Cech Cohomology

In this quick note we describe some scenarios where a Cech cohomology on a natural presheaf associated with a quantum system picks out subspaces of states in a physical system that has some quantum information theoretic significance.

Given the Hilbert space H for a multipartitie quantum system:

- We construct a natural presheaf over the discrete space of labels for the multi-partite system, by assigning to each open set a corresponding tensor product Hilbert space and using the partial trace as the restriction map.
- We then pick some open covers for the index set and compute the Cech cohomology, and suggest quantum information theoretic interpretations for the cohomology generators.

In particular this construction applies to a decomposition of a conformal field theory, it would be interesting to rephrase this in terms holographic duals of CFT states.

The Presheaf $\mathcal{F}$

The data given is a Hilbert space H that decmposes as a tensor product of subspaces labelled by some index set I

$$H \simeq \otimes_{\alpha \in I} H_\alpha$$

Give I the discrete topology, we define a presheaf $\mathcal{F}$ over the topological space I as follows.

$$\mathcal{H}(J) = \otimes_{\alpha \in J} H_\alpha$$

for subsets $J \subset I$. For $K \subset J$, we define the restriction map as

$$res_{J \to K} : \mathcal{F}(J) \to \mathcal{F}(K)$$

$$\rho \mapsto Tr_{J-K} \rho$$

where Tr_S is the partial trace tracing out the spaces $\mathcal{F}(S)$. Functoriality of the presheaf follows from composition properties of the partial trace - all simple to check.

3-party example

We give a first example of the construction above and compute the cech cohomology wrt a simple cover.

Consider a hilbert space that decomposes into a tensor product of 3 subspaces:

$$H = H_0 \otimes H_1 \otimes H_2, \quad I \equiv \{0, 1, 2\}$$

Our sheaf over open sets looks like

$$\mathcal{F}(\{\alpha\}) = H_\alpha, \mathcal{F}(\{\alpha, \beta\}) = H_\alpha \otimes H_\beta, \mathcal{F}(\{0, 1, 2\}) = H$$

Now consider an open cover

$$\mathcal{U} = \{U_0 \equiv \{0, 1\}, U_1 \equiv \{1, 2\}\}$$

Consider the following sequence:

$$0 \to \mathcal{F}(\{0, 1, 2\}) \xrightarrow{r} \mathcal{F}(\{0, 1\}) \oplus \mathcal{F}(\{1, 2\}) \xrightarrow{\delta} \mathcal{F}(\{1\}) \to 0$$

where

$$r \equiv res_{I \to \{0,1\}} \oplus res_{I \to \{1,2\}} = Tr_2 \oplus Tr_0$$

$$\delta \equiv res_{\{0,1\} \to 1} - res_{\{1,2\} \to 1} = Tr_0 - Tr_2$$

Which is the Cech cohomology $C^*(\mathcal{U}, \mathcal{F})$ extended on the left by the restriction of the whole sheaf (one could call this the Mayer Vietoris sequence for this sheaf wrt to this open cover).

This diagram is exact except at $\xrightarrow{r} \cdot \xrightarrow{\delta}$. To see δ is surjective, simply note every vector $v \in H_1$ has an preimage under T_2 in $H_1 \otimes H_2$, given by a tensor product $v \otimes w$ for any $w \in H_2$. The difference between a preimage of v and preimage of 0 gives back v under δ.

It's clear from the partial traces formulas above that $r \circ \delta = 0$.
So at the only non-exact point in this sequence we can compute the cohomology.

We can start discussing physics at this point. Write $\rho \in \mathcal{F}(\{0, 1\}) \oplus \mathcal{F}(\{1, 2\})$ in its components, with a normalization for convenience

$$\rho = \frac{1}{\sqrt{2}}(\rho_a + \rho_b)$$

for some $\rho_a \in \mathcal{F}(\{0, 1\})$, $\rho_b \in \mathcal{F}(\{1, 2\})$. This says the state is a superposition of two states each having support in 2 of the 3 spaces in the system.

Consider a closed element in C^0, closedness reads

$$\delta\rho = 0 \implies Tr_2\rho = Tr_0\rho$$

Physically, the above says the state ρ is “maximally mixed” (TODO: ask someone about the correct terminology here) at the Hilbert space H_1, in the following sense: the expectation value of any operator $\mathcal{O}$ that only has support in H_1 has property

$$tr(\rho O) = tr(\rho_a \mathcal{O}) = tr(\rho_b \mathcal{O})$$

Pretty sure this type of thing shows up in quantum communications somewhere. If we interpret each hilbert space as a qubit, the above says someone (say, Eve) who only has access to the qubit on H_1 cannot distinguish ρ_a, ρ_b and the superposition ρ. If ρ_a, ρ_b are eigenstates of some observable, Alice and Bob can communicate by making measurements on the superposition state at H_0 and H_2, collapsing it to one of ρ_a and ρ_b, and Eve would be left completely out of the loop.

States in the same cohomology class $\rho' \in [\rho]$ differ by an exact form. Physically the difference is a state that has a representation as a sum of partial traces on the full Hilbert space H.

More intuition on what the cohomology quotient measures: there’s a simple way for a state to be maximally mixed in the above sense - namely if they’re a superposition of two partial traces of the same global state ρ. In this case the vanishing of difference in expectation values follows from commutativity of the partial trace, physically this says this matching of expectation values has a dumb explanation that these all come from the same global operator. So physically the cohomology H^1 is exactly the set of the maximally mixed states that don’t have this dumb origin.

Generalizations

The above example generalizes, in both simple and not-so simple ways.

Simple generalizations

Consider the index set

$$\{0, 1, 2, 3\}$$

and consider the open cover

$$\{0, 1, 2\}, \{1, 2, 3\}$$

It's easy to follow the steps above and see everything carries through, and we get a statement about the Hilbert space supported on $\{1, 2\}$. This is a dumb example because it reduces to the above example by adding a bracket.

$$H_0 \otimes H_1 \otimes H_2 \otimes H_3 = H_0 \otimes (H_1 \otimes H_2) \otimes H_3$$

The point then is that the index set isn't crucial to our construction, the construction actually only depends on the nerve of the cover. Hence this generalizes to quantum field theory Hilbert spaces (in cases where you believe they exist) easily: modify our presheaf to assign Hilbert spaces to open sets in a manifold, define the partial trace the usual way that field theorists like to define it, and if you're comfortable with everything in this so far our computation in the presheaf over discrete space carries over to any open cover off the manifold with the same nerve complex.

Less trivial generalizations

The example above is basically a Mayer-Vietoris argument, our cech complex is 1-level deep. We need to consider examples with more complicated covers, which then yields deeper complexes. I have yet to come up with any simple physical explanation of cohomology in these cases. WIP.

TODO:

- The physics arguments can be made clearer if we consider an example with just pure states, I think.
- More complicated covers
- Draw pictures