Precalc Note Shells

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Chapter 1

Functions and Their Graphs

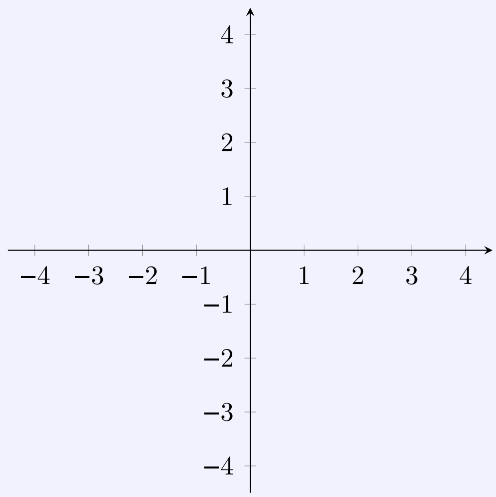
1.1 Rectangular Coordinates

Definition: Rectangular Coordinates

We plot points on a pair of axes. They form the **Cartesian Plane**.

The **x-axis** runs ____________________. The **y-axis** runs ____________________.

The points where the x and y axes intersect is called the ________________.
The x and y axes divide the plane into four **quadrants.**



We represent a point in the Cartesian Plane as an **ordered pair** (x, y). The first number is called the **x-coordinate** and the second number is called the **y-coordinate**.

To plot a point (x, y), move along the x-axis until you reach the x-coordinate, and then move along the y-axis until you reach the y-coordinate.

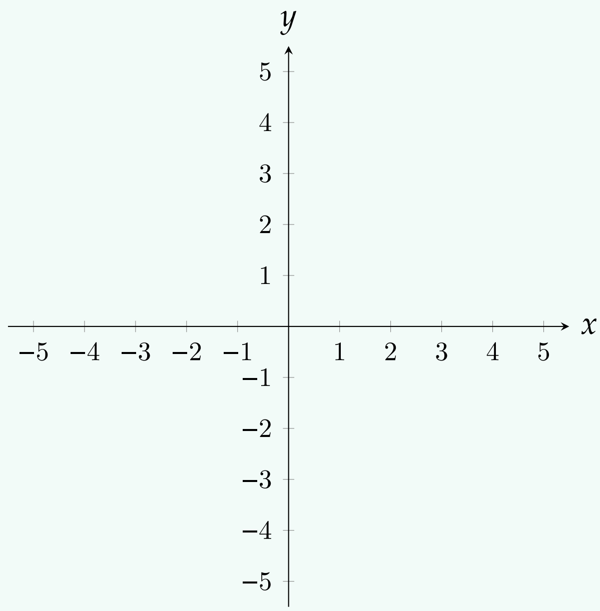
Formula 1.1.1 Distance Formula

For a right triangle with side lengths a and b, and a hypotenuse with length c, the **Pythagorean Theorem** says:

To find the distance between two points (x_1, y_1) and (x_2, y_2), we use the **distance formula:**

Example 1.1.1

Plot the points $(4, 0)$, $(2, 1)$ and $(-1, -5)$ below. Calculate the distance between each pair of points. Do these points form a right triangle?



Distance between $(4, 0)$ and $(2, 1)$:

Distance between $(4, 0)$ and $(-1, -5)$:

Distance between $(2, 1)$ and $(-1, -5)$:

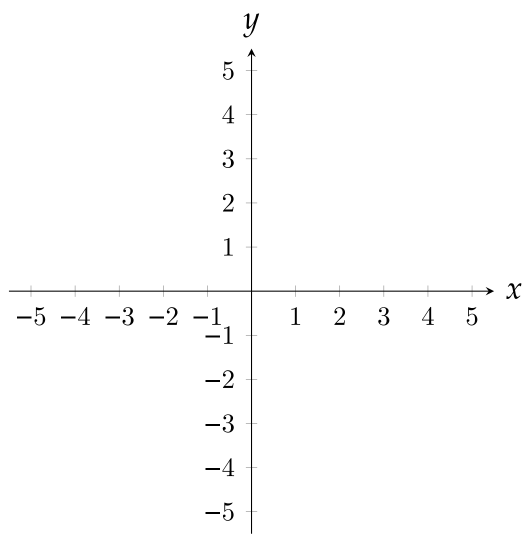
To check if these points form a right triangle, we plug the distances into the Pythagorean Theorem:

Note:

When using the distance formula, always leave your answer in exact form, unless told otherwise. For example, $\sqrt{10}$ should be left as $\sqrt{10}$, **not** approximated as a decimal like 3.16.

Classwork Question 1.1.1

Plot the points $(4, 4)$, $(3, 5)$ and $(-1, -1)$ below. Calculate the distance between each pair of points. Do these points form a right triangle?



Formula 1.1.2 Midpoint Formula

The **midpoint** between two points (x_1, y_1) and (x_2, y_2) is the point exactly halfway along the line segment between those two points. To find it, we use the **midpoint formula:**

Example 1.1.2

Find the midpoint between the points $(5, 3)$ and $(-2, 1)$.

From the first point: $x_1 =$ _____ and $y_1 =$ _____.

From the second point: $x_2 =$ _____ and $y_2 =$ _____.

Now we plug in to the midpoint formula and simplify:

Classwork Question 1.1.2

Find the midpoint between the points $(3, -2)$ and $(-1, 4)$.

Definition: Translation

A **translation** of a point consists of a horizontal shift, a vertical shift, or both.

To translate a point (x, y):

To the right by a units, we ______________ to the _____ coordinate.

To the left by a units, we ___________________ from the _____ coordinate.

Up by a units, we ______________ to the _____ coordinate.

Down by a units, we ___________________ from the _____ coordinate.

Example 1.1.3

Translate the point $(4, -2)$ left by 3 units and up by 1 unit.

To translate left by 3 units, ________________________ from the _____ coordinate.

To translate up by 1 unit, ________________________ to the _____ coordinate:

Classwork Question 1.1.3

Translate the point $(-2, 5)$ to the right by 4 units and down by 7 units.

1.2 Graphs of Equations

Given an equation, we can plot it on a graph. To do so, we use the following steps:

1.

2.

3.

Example 1.2.1

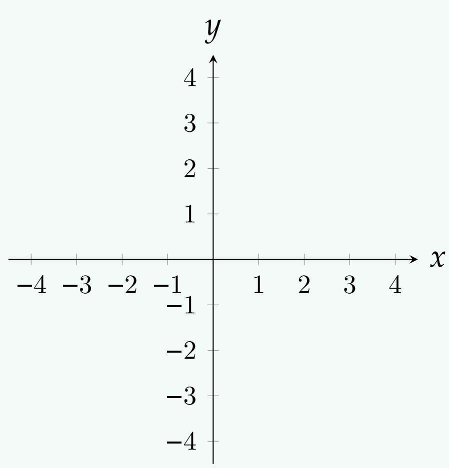
Check if the point $(-1, 3)$ is on the graph of $x^2+y=4$. Then sketch a graph of the equation.

To check if $(-1, 3)$ is on the graph, we plug in $x =$ _____ and $y =$ _____, and check if the equation is true:

To sketch the graph, we will solve the equation for y:

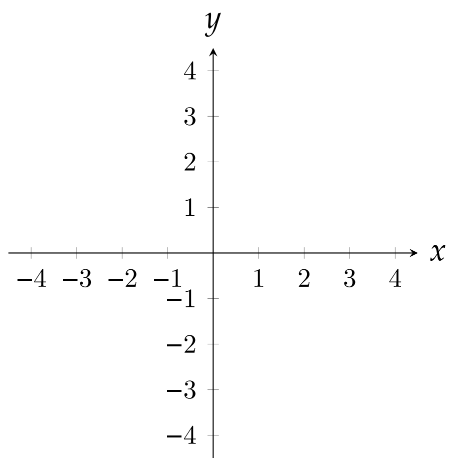
Now let's plug in some points. We can organize them into a table:

Finally, we plot the points and sketch the graph:



Classwork Question 1.2.1

Check if the point $(1, 4)$ is on the graph of the equation $y - x^2 = -3$. Then sketch a graph.



Definition: Intercepts

An **x-intercept** of a graph is any point on the graph where __________.

A **y-intercept** of a graph is any point on the graph where __________.

To find the x-intercepts, set __________ and solve for _____.

To find the y-intercepts, set __________ and solve for _____.

Example 1.2.2

Find the x- and y- intercepts of the equation $y = x^3 - x$.

To find the x-intercepts, we set _____ = _____ and solve for _____:

To find the y-intercepts, we set _____ = _____ and solve for _____:

Note:

Remember that x- and y- intercepts are **points**, not just numbers!

Classwork Question 1.2.2

Find the x- and y- intercepts of the equation $y = x^2 - 3x - 4$.

Definition: Symmetry

A graph is said to be **symmetric** with respect to:

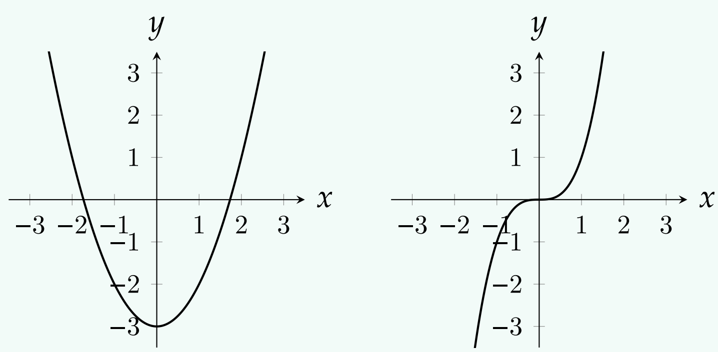
The x-axis if it looks the same when flipped ________________________.

The y-axis if it looks the same when flipped ________________________.

The origin it looks the same when rotated ________________________.

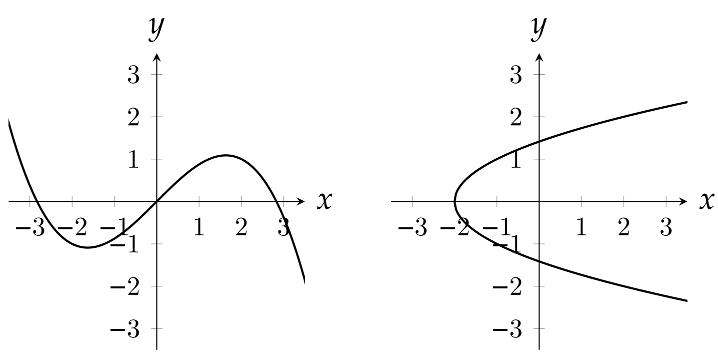
Example 1.2.3

Describe the symmetry of each graph, if any.



Classwork Question 1.2.3

Describe the symmetry of each graph, if any.



Algebra Reminders

Let's remind ourselves of some algebra rules:

A negative number to an even exponent ________________________________.

A negative number to an odd exponent ________________________________.

Example 1.2.4

Simplify each of the following:

(a) $(-3)^3 =$

(b) $(-2)^4 =$

(c) $(-x)^{10} =$

(d) $(-y)^7 =$

When solving for a variable, we only add $\pm$ when taking an **even** root.

Example 1.2.5

Solve for y in each equation below:

(a) $y^2 = x^2 - 1$

(b) $y^5 = x^3 + 7$

(c) $y^3 = x - 3$

(d) $y^6 = x^5 - 6$

We should also remember that squares and square roos **do not distribute!**

Is $(a+b)^2 = a^2 + b^2$? Yes No

The proper expansion is: ______________________________

Is $\sqrt{a+b} = \sqrt{a} + \sqrt{b}$? Yes No

Addition/subtraction under square roots must be left as is.

We will sometimes need to get square roots out of the denominator of a fraction. This is called **rationalizing the denominator**. To rationalize the denominator, multiply the numerator and denominator by the square root.

Example 1.2.6

Rationalize the denominator each of the following:

(a) $\dfrac{2}{\sqrt{3}} \cdot \underline{\qquad\qquad} = \underline{\qquad\qquad}$

(b) $\dfrac{5}{2\sqrt{10}} \cdot \underline{\qquad\qquad} = \underline{\qquad\qquad} = \underline{\qquad\qquad}$

Let's refresh (some of) our exponent rules.

Formula 1.2.1 (Some) Exponent Rules

$x^a \cdot x^b = \underline{\qquad\qquad}$ $\dfrac{x^a}{x^b} = \underline{\qquad\qquad}$ $(x^a)^b = \underline{\qquad\qquad}$ $x^{-a} = \underline{\qquad\qquad}$ $(xy)^a = \underline{\qquad\qquad}$

Example 1.2.7

Simplify each of the following:

(a) $x^3 \cdot x^5 =$

(b) $\dfrac{y^7}{y^2} =$

(c) $(t^4)^8 =$

(d) $s^{-3} =$

(e) $(2z)^4 =$

1.3 Linear Equations in Two Variables

Definition: Linear Equation in Two Variables

A **linear equation in two variables** has the form

_____ is called the **slope**, which corresponds with the _______________ of the graph.

_______ is the _____-intercept.

This way of writing a linear equation is called **slope-intercept** form.

A line that goes up as you go to the right has ____________________ slope.

A line that goes down as you go to the right has ____________________ slope.

The slope of a horizontal line is _____. The equation of a horizontal line is _______.

The slope of a vertical line is ___________. The equation of a vertical line is _______.

Formula 1.3.1 Slope Formula

The slope of the line passing through two points (x_1, y_1) and (x_2, y_2) is

Example 1.3.1

Find the slope of the line passing through the points $(1, -4)$ and $(3, 2)$.

From the first point: $x_1 =$ _____ and $y_1 =$ _____.

From the second point: $x_2 =$ _____ and $y_2 =$ _____.

Plugging into the slope formula gives:

Note:

Be careful not to mix up the order of subtraction! It doesn't matter which point you label as 1 or 2, but make sure you are consistent throughout your calculation.

Classwork Question 1.3.1

Find the slope of the line passing through the points $(5, -2)$ and $(1, 4)$.

Definition: Point-Slope Form

If a linear equation has slope m, and the point (x_1, y_1) lies on the graph of the equation, we can write the linear equation in **point-slope** form:

Point-slope form is helpful for finding the equation of a line. We can then convert the point-slope form into slope-intercept form by just doing a little algebra.

Example 1.3.2

Find the slope-intercept form of the line through the points $(-2, -1)$ and $(3, 9)$.

First we need to find the slope:

We then create an equation using point-slope form:

The question asks for slope-intercept form, so we solve using some algebra:

Classwork Question 1.3.2

Find the slope-intercept form of the line through the points $(-2, 7)$ and $(1, 4)$.

Definition: Parallel and Perpendicular Lines

Two lines are **parallel** if they have slopes that are ____________________.

Two lines are **perpendicular** if they have slopes that are _________________________.

To find the negative reciprocal of a number, flip the number as a fraction and change its sign (+ to − or − to +)

Example 1.3.3

Find the negative reciprocal of the following:

$3 \rightarrow$ $\qquad$ $\frac{4}{7} \rightarrow$ $\qquad$ $-\frac{6}{5} \rightarrow$ $\qquad$ $-\frac{1}{9} \rightarrow$

Example 1.3.4

Write the equations of the lines that pass through the point $(2, -5)$ and are (a) parallel to and (b) perpendicular to the line $3x - y = -2$.

First we will solve the given line for y:

For (a), we need a parallel line, so its slope will be _____. We write it in point-slope form:

For (b), we need a perpendicular line, so its slope will be the negative reciprocal, _____. We write it in point-slope form:

Classwork Question 1.3.3

Write the equations of the lines that pass through the point $(-1, 3)$ and are (a) parallel to and (b) perpendicular to the line $4x + 2y = 6$.

Example 1.3.5

Find the slope of the line passing through the points. Write an equation for the line.

(a) $(1,3),\ (3,9),\ (10,30)$ (b) $(1.3,4.5),\ (1.3,7.8),\ (1.3,12.2)$

For (a), we pick two points and find the slope between them:

An equation for this line would be ________________.

For (b), we pick two points and find the slope between them:

An equation for this line would be ________________.

Classwork Question 1.3.4

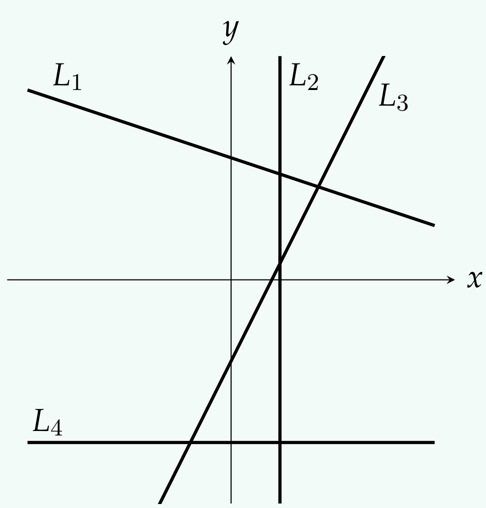
Find the slope of the line passing through the points. Write an equation for the line.

(a) $(0,10),\ (2,0),\ (5,-15)$ (b) $(-2.2,9.3),\ (1.7,9.3),\ (7.2,9.3)$

Even if we aren't given exact numbers, we can pair lines with their slopes by checking whether the line goes up/down as we go to the right, or whether the line is horizontal/vertical.

Example 1.3.6

Drawn below are four lines. Their slopes, in no particular order, are $-\frac{1}{3}, 0, 2,$ and undefined. Pair each line with its slope.



Interval Notation

When talking about domains and ranges of functions, we will write our answers using **interval notation**.

Definition: Interval Notation

If we have two numbers a and b where $a < b$, then:

The set of numbers x such that $a < x < b$ is written as _______.
This is called an **open interval**.

The set of numbers x such that $a \leq x \leq b$ is written as _______.
This is called a **closed interval**.

The set of numbers x such that $a \leq x < b$ is written as _______.

The set of numbers x such that $a < x \leq b$ is written as _______.

The important thing to remember is that square brackets [ ] **include** that endpoint, and parentheses ( ) **exclude** that endpoint.

Example 1.3.7

Write each of the following sets of numbers using interval notation:
(a) The set of numbers x such that $0 < x \leq 5$.
(b) The set of numbers x such that $-3 \leq x \leq 9$.
(c) All numbers between 3 and 9, including 3, but excluding 9.
(d) All numbers between -5 and -1, excluding both -5 and -1.

(a) We want to ______________ 0 but ______________ 5, so the answer is _______.

(b) We want to ______________ both -3 and 9, so the answer is _______

(c) This describes the numbers x such that ______________, so the answer is _______

(d) This describes the numbers x such that ______________, so the answer is _______

We can also use interval notation for intervals that extend infinitely in one direction or the other. In that case, we will have ∞ or $-\infty$ as one (or both) of our endpoints. Whenever this happens, we **always** use ( ) with ∞ or $-\infty$.

Example 1.3.8

Write each of the following sets of numbers using interval notation:
(a) The set of numbers greater than or equal to 5.
(b) The set of numbers less than -2.
(c) The set of all numbers.

(a) We want to __________ 5 and go to _______, so the answer is _______.

(b) We want to __________ -2 and go to _______, so the answer is _______

(c) We want to go all the way from _______ to _______, so the answer is _______.

If we need to put two or more intervals together, we use the **union** symbol, $\cup$.

Example 1.3.9

Write each of the following sets of numbers using interval notation:
(a) The set of numbers such that $0 \leq x \leq 1$ or $4 < x < 5$.
(b) The set of numbers less than -2 or greater than or equal to 2.
(c) The set of numbers that are not -1 or 3.

(a) Our two intervals in question are _______ and _______, so the answer is

____

(b) Our two intervals in question are _______ and _______, so the answer is

______________.

(c) To solve questions like these, we create three intervals:

(i) the numbers less than -1,
(ii) the numbers between -1 and 3, excluding both,
(iii) the numbers greater than 3.

Those intervals are _______, _______, and _______, respectively, so our answer is

____

1.4 Functions

In mathematics, a **set** is any group or collection of objects. The objects inside a set are called **elements**. For our course, we will focus on sets whose elements are numbers.

Definition: Function

A **function** between two sets A and B is a rule that sends each element in A to

__________________________ element in B.

We can think of the set A as the inputs, and the set B as the possible outputs. The set A is called the **domain** of the function, and the set B is called the **range**.

It's helpful to draw a picture to represent a function:

It is OKAY if:

One of more elements in A get sent to the same element in B:

One or more elements in B are not sent to by any element in A:

It is NOT OKAY if:

One of more elements in A do not get sent to some element in B:

An element of A gets sent to two or more elements in B:

To test whether or not an algebraic equation is a function, solve for y and check that each input gives exactly one output.

Example 1.4.1

Determine whether or not each of the following expresses y as a function of x:
(a) $y^2 - 5x = 9$
(b) $x^2 + 7y = 14$

For (a):

For (b):

Classwork Question 1.4.1

Determine whether or not each of the following expresses y as a function of x:
(a) $y^3 + x^2 = 1$
(b) $y^2 - x^5 = -3$

When we write a function, we typically use a letter to represent it. For example, we can represent the equation $y = x^2 - x + 1$ as $f(x) = x^2 - x + 1$.

Replacing y by $f(x)$ is helpful as shorthand. Instead of saying "the value of y when $x = 3$," we simply write $f(3)$. To evaluate $f(3)$, just substitute 3 in for x:

We can also put expressions into a function. For example, to evaluate $f(x+1)$ we substitute $x + 1$ in everywhere we see an x, and then simplify:

While $f(x)$ is the most common, we can use other letters for the function and variable. For example, if we have the function $h(t) = 4t + 3$, to evaluate $h(5)$, we just substitute 5 in for t.

Classwork Question 1.4.2

Consider the function $g(x) = 2x^2 - 3x + 4$. Evaluate each of the following:
(a) $g(4)$
(b) $g(-1)$
(c) $g(x-2)$

A **piecewise** function is a function that is defined by two or more different equations. To evaluate a piecewise function, we need to check which equation to use before we plug in.

Example 1.4.2

Consider the function $f(x) = \begin{cases} x^2 + 2, & x \leq 0 \\ 3x - 2, & x > 0 \end{cases}$. Evaluate each of the following:

(a) $f(-2)$ (b) $f(4)$ (c) $f(0)$

For (a), since -2 _____ 0, we will plug into _______________:

For (b), since 4 _____ 0, we will plug into _______________:

For (c), since 0 _____ 0, we will plug into _______________:

Classwork Question 1.4.3

Consider the function $f(x) = \begin{cases} 4x - 7, & x < 2 \\ 8 - x^2, & x \geq 2 \end{cases}$. Evaluate each of the following:

(a) $f(1)$ (b) $f(4)$ (c) $f(2)$

If we want to find where a function $f(x)$ is equal to some value, we set $f(x)$ equal to that value and solve for x (or whichever letter is being used).

Example 1.4.3

Let $f(x) = x^2 - 8x + 4$. Find all values of x where $f(x) = -3$.

To solve this, we set $f(x) =$ _____ and solve using some algebra:

We can also set two different functions equal to one another.

Example 1.4.4

Let $f(x) = x^2 + 1$ and $g(x) = 3x - x^2$. Find all values of x where $f(x) = g(x)$.

To solve this, we set $f(x) =$ _______ and solve for x:

Classwork Question 1.4.4

Let $f(x) = x^2 + 6x - 24$ and $g(x) = 4x - x^2$. Find all values of x where $f(x) = g(x)$.

Recall that the **domain** is the set of all valid inputs for a function. If we are just given a function and not told its domain, we need to find it ourselves.

Definition: Implied Domain

The **implied domain** of a function is

There are two common ways for a function to be undefined:

1.

2.

Example 1.4.5

Find the implied domain of each of the following functions:

(a) $f(x) = \dfrac{3}{x^2 - 9}$ (b) $g(x) = \sqrt{2x - 5}$ (c) $h(x) = \sqrt[3]{x^3 - 4}$

For (a), we cannot ______________________________, so

For (b), we cannot ____, so

For (c), we can ____, so

Classwork Question 1.4.5

Find the implied domain of each of the following functions:

(a) $f(x) = x^2 + x^4 + x^6$ (b) $g(x) = \sqrt[4]{x - 16}$ (a) $h(x) = \dfrac{12}{25 - x^2}$

Formula 1.4.1 Difference Quotient

Given a function f, its **difference quotient** is

Example 1.4.6

Find the difference quotient $\dfrac{f(x+h)-f(x)}{h}$ of $f(x) = x^2 - 4x + 7$.

It's good to break the calculation into steps. First we will calculate $f(x+h)$:

Next we calculate $f(x+h) - f(x)$ by canceling:

Finally, we divide everything by h:

Note:

Sometimes the difference quotient is written as $\dfrac{f(x)-f(a)}{x-a}$, where a is some number. In that case, calculate $f(a)$ first, then calculate $f(x) - f(a)$, and finally divide everything by $x - a$.

Classwork Question 1.4.6

Find the difference quotient $\dfrac{f(x+h)-f(x)}{h}$ of $f(x) = x^2 + 5x - 3$.

1.5 Analyzing Graphs of Functions

Definition: Reading Graphs

The **graph** of a function consists of the ordered pairs $(x, f(x))$, where x is in the domain of f.

To evaluate $f(x)$ using its graph, ______________________________________.

A **closed dot** • on a graph means that point is __________________________.

An **open dot** ◦ on a graph means that point is __________________________.

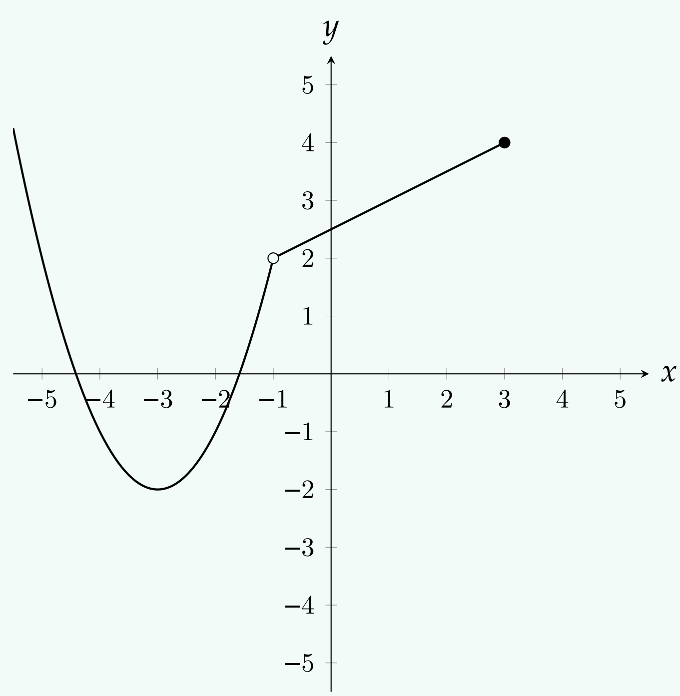
An open/closed dot at the end of a graph means the graph ___________________.

If a graph continues off one of the sides without a dot, we assume it

__________________________.

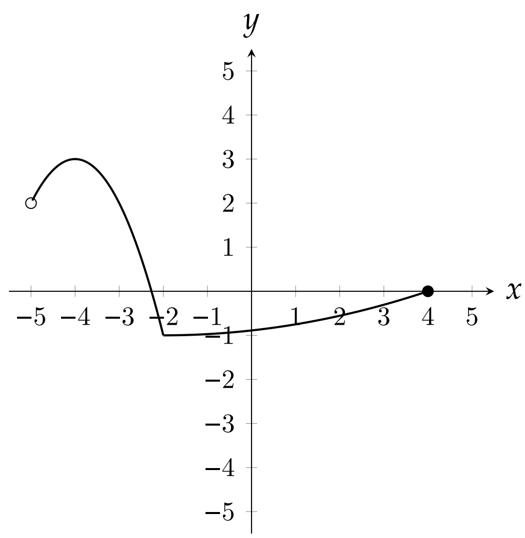
Example 1.5.1

State the domain and range of the function $f(x)$ graphed below. Evaluate $f(1)$, $f(3)$, and $f(-1)$.



Classwork Question 1.5.1

State the domain and range of the function $f(x)$ graphed below. Evaluate $f(-2)$, $f(4)$, and $f(-5)$.



Remember that a function can have only one output for each input. To test whether or not a graph depicts a function, we use the **vertical line test.**

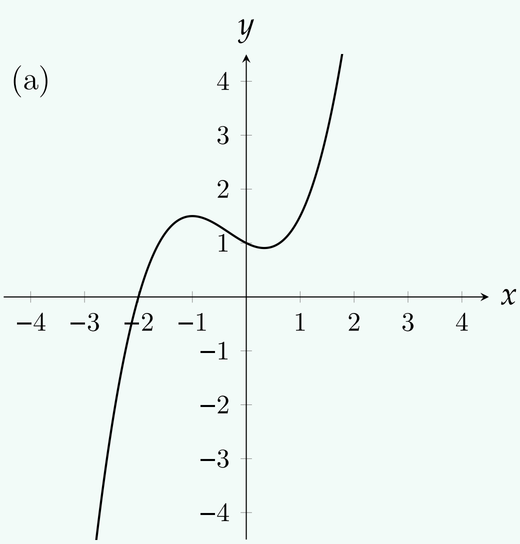
Formula 1.5.1 Vertical Line Test

A graph depicts a function as long as

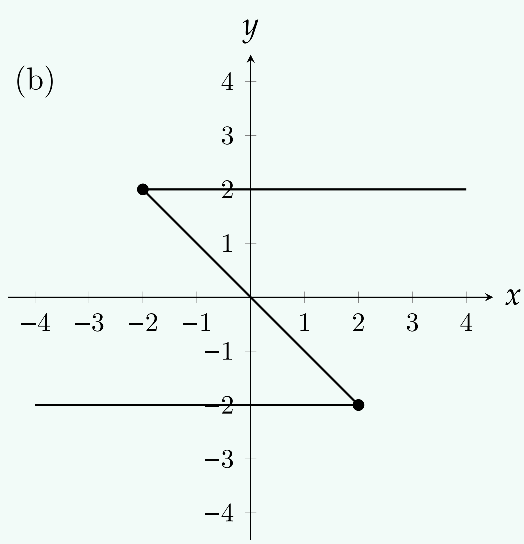
Example 1.5.2

Use the vertical line test to determine if each of the following is the graph a function.

(a)



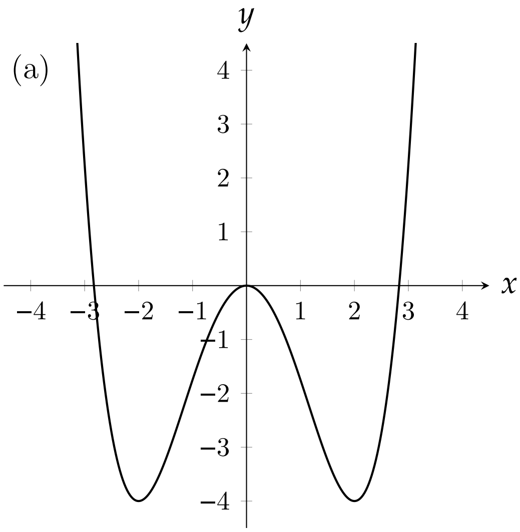
(b)

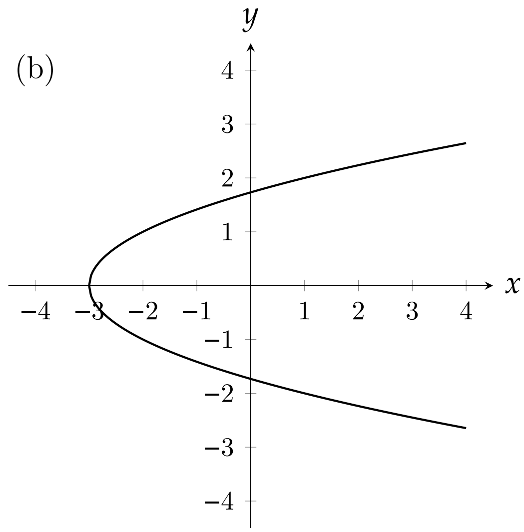


Note:
It is okay if a vertical line does not intersect the graph at all. That just means that x value is not in the domain of the function.

Classwork Question 1.5.2

Use the vertical line test to determine if each of the following is the graph a function.





Definition: Zeroes of a Function

The **zeroes** of a function $f(x)$ are the x-values where ______________________________.

Note:
The zeroes of a function are the x-coordinates of its x-intercepts. On a graph, they are where the function touches the x-axis.

Example 1.5.3

Find the zeroes of the following functions:

(a) $f(x) = x^2 - 5x + 6$ (b) $g(x) = \dfrac{x^2 - 2}{x + 1}$

For (a), we set $f(x) =$ _____ and solve for x:

For (b), we again set $g(x) =$ _____ and solve for x:

Definition: Increasing, Decreasing, and Constant

A function is said to be **increasing** on an interval if

A function is said to be **decreasing** on an interval if

A function is said to be **constant** on an interval if

Note:
For increasing, decreasing, or constant intervals, we always use **open intervals**.

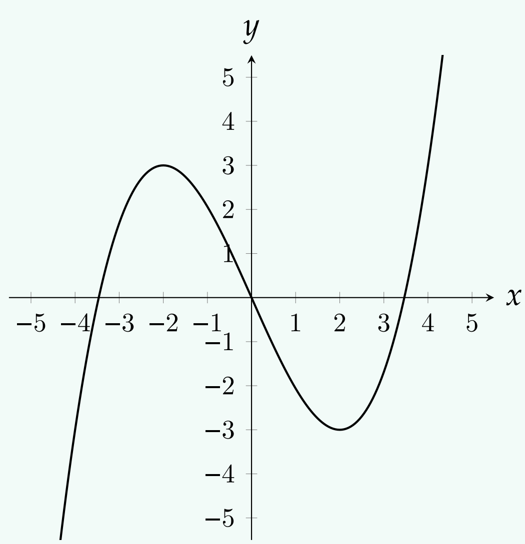
Definition: Relative Maximum and Minimum

A function value $f(a)$ is said to be a **local maximum** or **relative maximum** if

A function value $f(a)$ is said to be a **local minimum** or **relative minimum** if

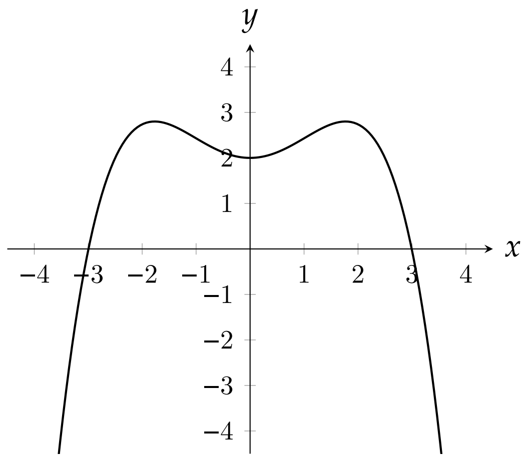
Example 1.5.4

State the intervals where the function below is increasing, decreasing, and constant. Estimate the coordinates of any relative maxima or minima.



Classwork Question 1.5.3

State the intervals where the function below is increasing, decreasing, and constant. Estimate the coordinates of any relative maxima or minima.



Formula 1.5.2 Average Rate of Change

The **average rate of change** of a function $f(x)$ is

Example 1.5.5

Compute the average rate of change of $f(x) = x^2 - 2x + 1$ from 1 to 3.

We will take $x_1 =$ _____ and $x_2 =$ _____, and plug them into f:

The average rate of change is therefore

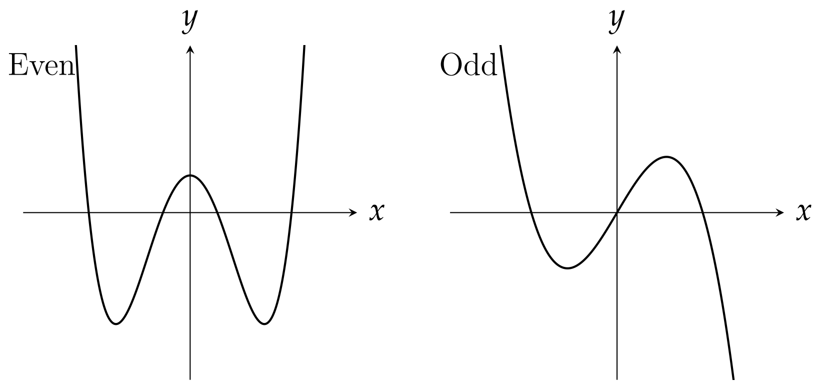
Classwork Question 1.5.4

Compute the average rate of change of $f(x) = x^3 - 3$ from -2 to -1.

Definition: Even and Odd Functions

A function $f(x)$ is called **even** if ____________________ Symmetry:

A function $f(x)$ is called **odd** if ____________________ Symmetry:



Example 1.5.6

Determine if each of the following is even, odd, or neither. Describe the symmetry:
(a) $f(x) = x^3 - x$ (b) $g(x) = x^4 - x^2 + 2$

For (a), we substitute in _____ and simplify:

For (b), we again substitute in _____ and simplify:

Classwork Question 1.5.5

Determine if each of the following is even, odd, or neither. Describe the symmetry:
(a) $f(x) = 3x^6 - x^2 - 4$ (b) $g(x) = 5x^3 + x + 2$

1.6 A Library of Parent Functions

In this section, we will learn about a variety of **parent functions**. We call them parent functions because we can build many other functions by putting them together in various ways.

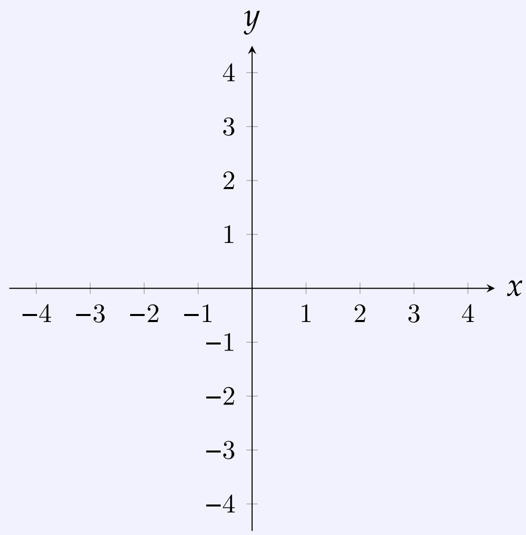
Definition: Constant Function

A **constant function** has the form

Domain: Range:

x-intercepts: y-intercept:

Even / Odd / Neither Increasing: Decreasing:



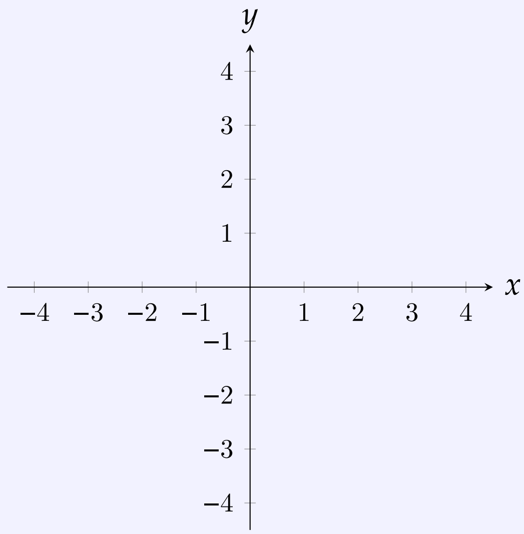
Definition: Identity Function

The **identity function** is

Domain: Range:

x-intercepts: y-intercept:

Even / Odd / Neither Increasing: Decreasing:



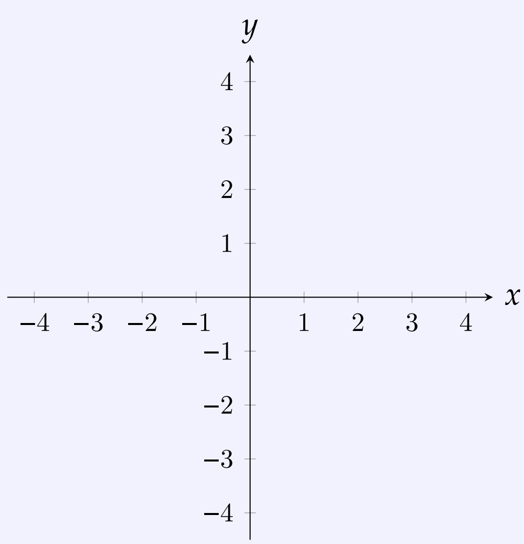
Definition: Squaring Function

The **squaring function** is

Domain: Range:

x-intercepts: y-intercept:

Even / Odd / Neither Increasing: Decreasing:



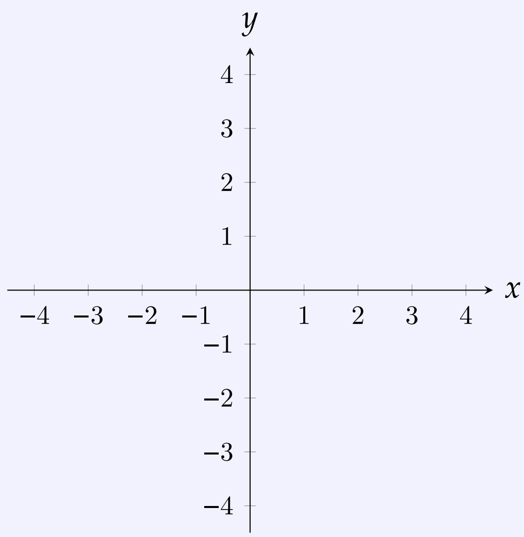
Definition: Cubic Function

The **cubic function** is

Domain: Range:

x-intercepts: y-intercept:

Even / Odd / Neither Increasing: Decreasing:



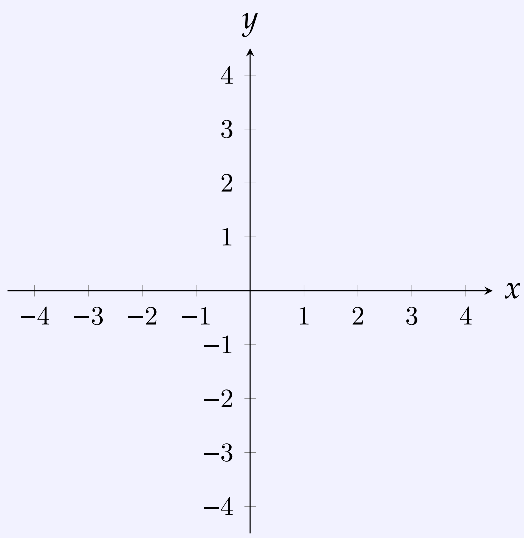
Definition: Square Root Function

The **square root function** is

Domain: Range:

x-intercepts: y-intercept:

Even / Odd / Neither Increasing: Decreasing:



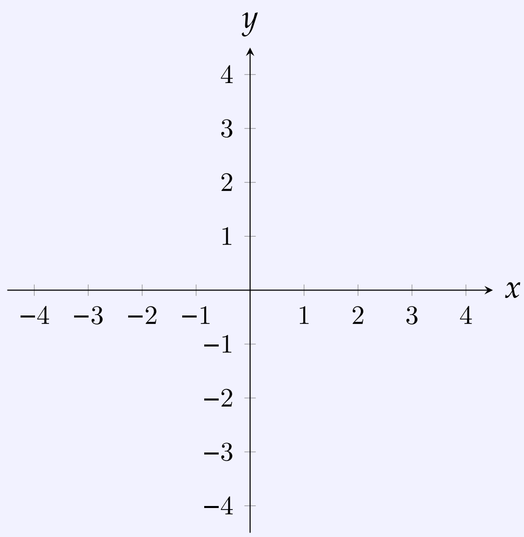
Definition: Absolute Value Function

The **absolute value function** is

Domain: Range:

x-intercepts: y-intercept:

Even / Odd / Neither Increasing: Decreasing:



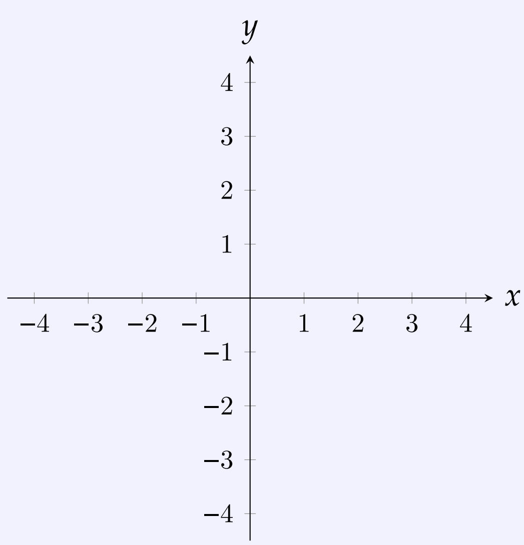
Definition: Reciprocal Function

The **reciprocal function** is

Domain: Range:

x-intercepts: y-intercept:

Even / Odd / Neither Increasing: Decreasing:



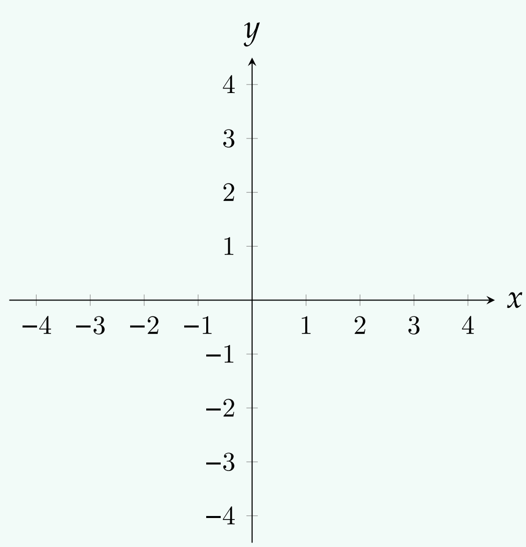
Recall that a **piecewise** function is defined by two or more equations. When we want to graph a piecewise function, we need to pay careful attention to the inequalities.

Example 1.6.1

Sketch a graph of the piecewise function $f(x) = \begin{cases} x+1 & x \leq -2 \\ x^2 - 1 & -2 < x \leq 1 \\ \frac{1}{2}x - 2 & x > 1 \end{cases}$

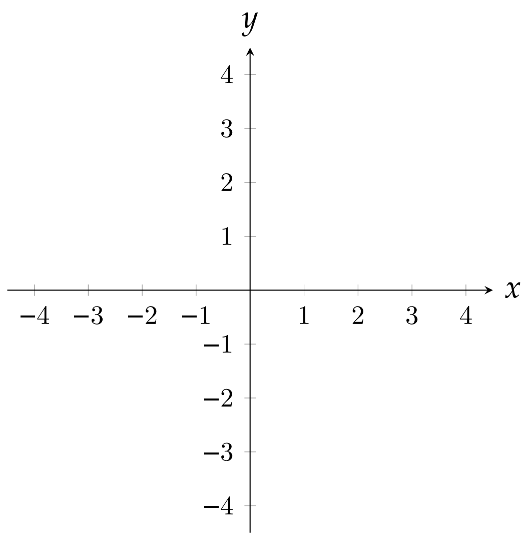
First, we want to plug in x values to get some points. To do this, we will make a table and plot those points:

For each inequality, we will also plug into the function with the < or > sign. We plot these points as open circles. Then we connect our graph like usual.



Classwork Question 1.6.1

Sketch a graph of the piecewise function $g(x) = \begin{cases} x+1 & x < -1 \\ 2x^2 - 3 & -1 \le x \le 1 \\ 3 - x & x > 1 \end{cases}$



1.7 Transformations of Functions

Given the graph of a function $f(x)$, we can transform it by shifting it up, down, left, and right.

Formula 1.7.1 Shifting a Function

Given a function $f(x)$, we can shift it using the following formulas:

Up by c units:

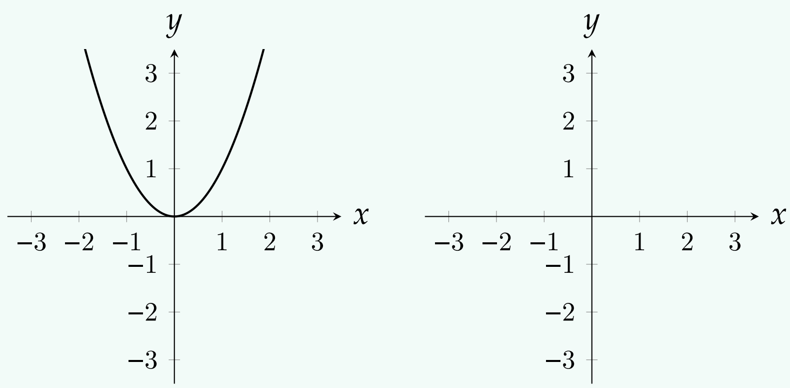
Down by c units:

Right by c units:

Left by c units:

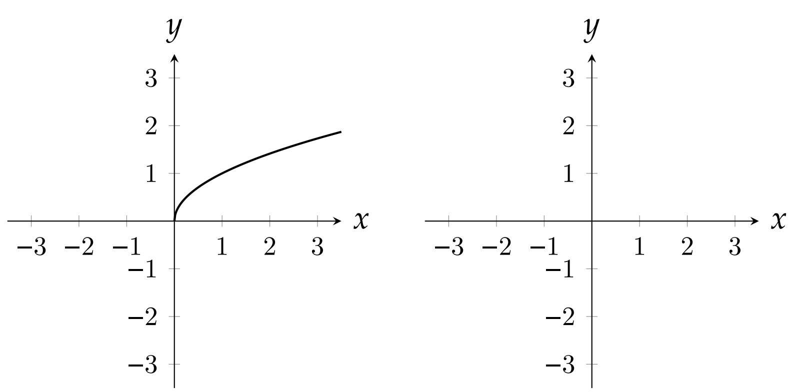
Example 1.7.1

Shown below is the graph of $f(x) = x^2$. Write the equation of the function obtained by shifting $f(x)$ right 2 units and up 1 unit. Sketch the shifted function.



Classwork Question 1.7.1

Shown below is the graph of $g(x) = \sqrt{x}$. Write the equation of the function obtained by shifting $g(x)$ left 3 units and down 2 units. Sketch the shifted function.



Another way we can transform a function is to reflect it over the x- or y- axis.

Formula 1.7.2 Reflections

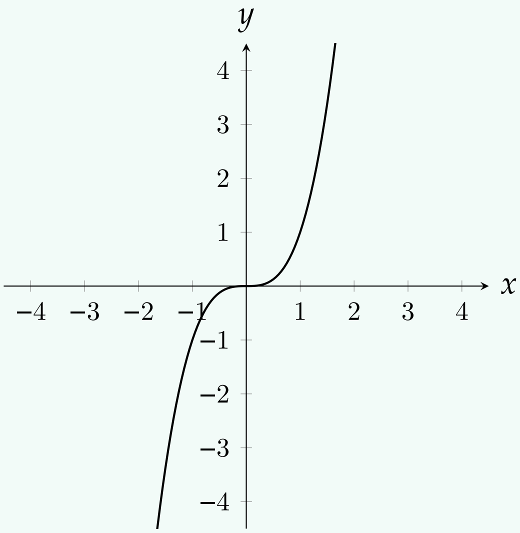
Given a function $f(x)$, we can reflect it over one of the axes using the following formulas:

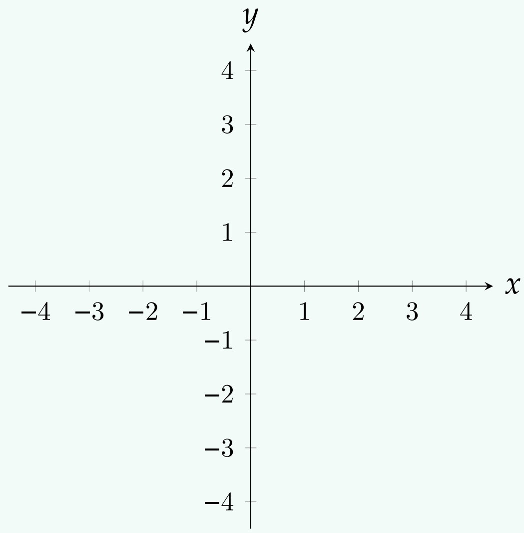
Reflection over x-axis:

Reflection over y-axis:

Example 1.7.2

Shown below is the graph of $f(x) = x^3$. Write the equation of the function obtained by reflecting $f(x)$ across the y-axis, and then shifting the function to the left by 2 units. Sketch the transformed function on the axes on the right.





Note:

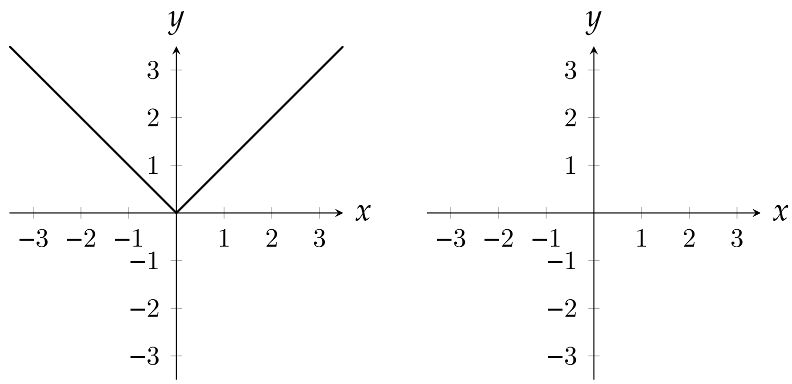
The order of the transformations is important! Changing the order can result in a different function at the end.

Note:

Reflecting over the y-axis will flip any horizontal shifts. For example, $\sqrt{-x+1}$ is a reflection over the y axis, followed by a shift **right** one unit.

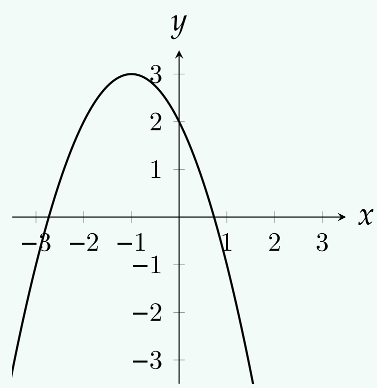
Classwork Question 1.7.2

Shown below is the graph of $g(x) = |x|$. Write the equation of the function obtained by reflecting $g(x)$ over the x-axis, and then shifting the graph up 1 unit. Sketch the shifted function on the axes on the right.



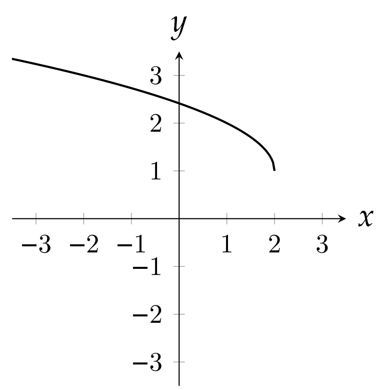
Example 1.7.3

Shown below is a transformed version of a parent function. Identify the parent function, describe what transformations were applied in words, and write the equation for the graph.



Classwork Question 1.7.3

Shown below is a transformed version of a parent function. Identify the parent function, describe what transformations were applied in words, and write the equation for the graph.



Formula 1.7.3 Vertical/Horizontal Stretches/Shrinks

To stretch or shrink a graph of a function $f(x)$ vertically:

It is a stretch when ______________ and a shrink when ______________.

To apply a vertical stretch/shrink, multiply the _____ coordinate by _____.

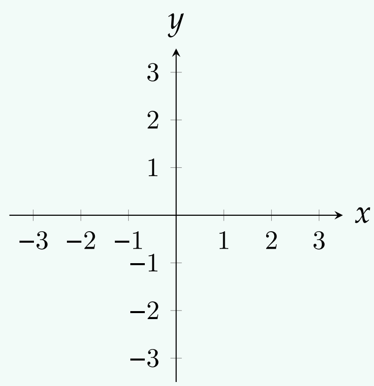
To stretch or shrink a graph of a function $f(x)$ horizontally:

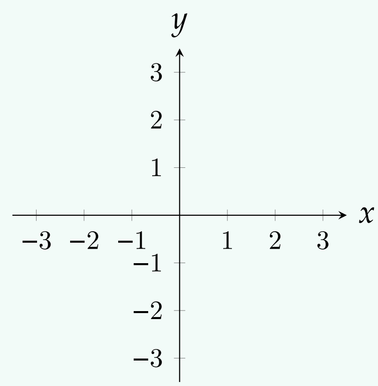
It is a stretch when ______________ and a shrink when ______________.

To apply a vertical stretch/shrink, multiply the _____ coordinate by _____.

Example 1.7.4

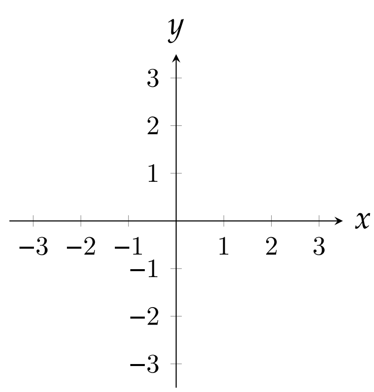
Consider the function $f(x) = x^2$. Sketch each of the following transformations:
(a) $g(x) = 2f(x) - 3$ (b) $h(x) = -f(\frac{1}{3}x)$

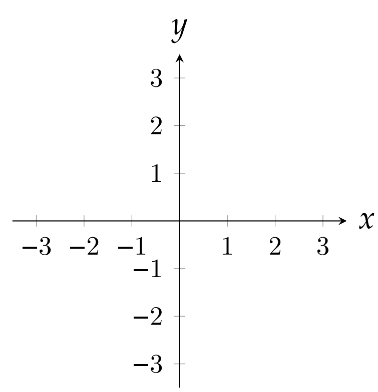




Classwork Question 1.7.4

Consider the function $f(x) = x^3$. Sketch each of the following transformations:
(a) $g(x) = \frac{1}{2}f(x - 1)$ (b) $h(x) = f(-2x)$





Example 1.7.5

A function $f(x)$ undergoes the following transformations, in the following order: Shift left by 1 unit, vertical stretch by a factor of 2, reflection over the y-axis. Write the function that represents the transformed version of $f(x)$.

Classwork Question 1.7.5

A function $f(x)$ undergoes the following transformations, in the following order: Reflection over the x-axis, horizontal shrink by a factor of $\frac{1}{3}$, shift up by 4 units. Write the function that represents the transformed version of $f(x)$.

Example 1.7.6

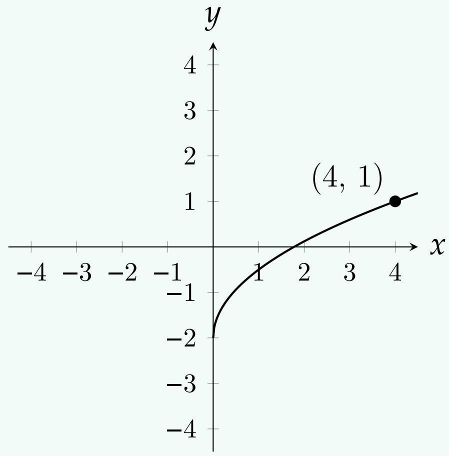
A function $f(x)$ undergoes some transformations, and the result is $f(-\frac{2}{3}x) + 1$. Describe the transformations applied, in order.

Classwork Question 1.7.6

A function $f(x)$ undergoes some transformations, and the result is $-3f(x + 2)$. Describe the transformations applied, in order.

Example 1.7.7

Shown below is a transformed version of $\sqrt{x}$. Find the equation of the transformed function.

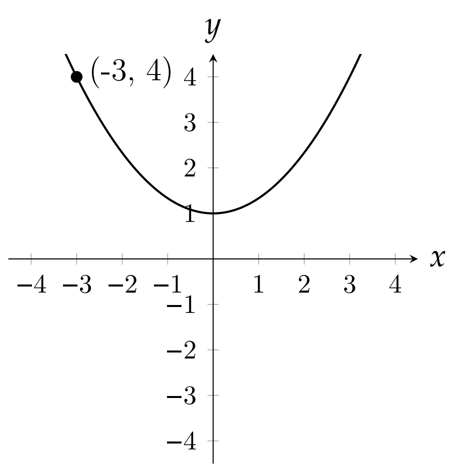


It appears that the transformations are: ____________________________________.

This transformed function has the form $f(x) =$ ____________. Now we plug in the given point and solve for c:

Classwork Question 1.7.7

Shown below is a transformed version of x^2. Find the equation of the transformed function.



1.8 Composite Functions

If we have two functions f and g, there are many ways to combine them. One way is to use an arithmetic combination.

Definition: Arithmetic Combinations of Functions

Given two functions f and g, we define:

Sum:

Difference:

Product:

Quotient:

Example 1.8.1

Given $f(x) = x^2 - 3$ and $g(x) = 2x + 1$, find each of the following:
(a) $(f - g)(2)$ (b) $(fg)(-1)$

For (a):

For (b):

Classwork Question 1.8.1

Given $f(x) = 2x - 5$ and $g(x) = x^3 + 2$, find each of the following:
(a) $(f + g)(3)$ (b) $\left(\frac{f}{g}\right)(0)$

Remember that we cannot divide by 0 or take the $\sqrt{\ }, \sqrt[4]{\ }, \sqrt[6]{\ }, \sqrt[8]{\ }, \ldots$ of a negative number. When we combine two functions, we need to make sure both of their original domains are respected, as well as the new domain of the combined function.

Example 1.8.2

State the domain of $\left(\frac{f}{g}\right)(x)$ for $f(x) = \sqrt{x}$ and $g(x) = 4 - x^2$.

The domain of $f(x)$ is __________

The domain of $g(x)$ is __________.

The combined function is:

We cannot ____________________, so we need to exclude __________.

Thus, our final domain is _________________________.

Classwork Question 1.8.2

State the domain of $\left(\frac{f}{g}\right)(x)$ for $f(x) = \sqrt{x+4}$ and $g(x) = 2x - 1$.

Another way we can combine functions is to **compose** them. This means we take one function and use it as the input to another function.

Definition: Composition of Functions

Given two functions f and g, the **composition** of f with g is

Example 1.8.3

Let $f(x) = x^2 - 4$ and $g(x) = 1 + x$. Find:
(a) $(f \circ g)(x)$ (b) $(g \circ f)(x)$

For (a), we will plug _____ into _____:

For (b), we will plug _____ into _____:

Note:
The example above shows that $(f \circ g)(x) \neq (g \circ f)(x)$ in general, so the order matters!

Classwork Question 1.8.3

Let $f(x) = 2x - 3$ and $g(x) = x^2 + 5$. Find:
(a) $(f \circ g)(x)$ (b) $(g \circ f)(x)$

When we compose functions, we need to be very careful with the domain. If we want to find the domain of $(f \circ g)(x)$, we need to check two things:

1.

2.

Note:
Make sure to pay attention to the order of the functions. For the domain of $(g \circ f)(x)$, we need to check that x is in the domain of $f(x)$, and the output of $f(x)$ is in the domain of $g(x)$.

Example 1.8.4

Let $f(x) = x + 2$ and $g(x) = \sqrt{x-4}$. Find $(f \circ g)(x)$ and $(g \circ f)(x)$, and state each of their domains.

First we'll find $(f \circ g)(x)$:

The domain of $g(x)$ is __________. We now need to make sure that the output of $g(x)$ is in the domain of $f(x)$.

The domain of $f(x)$ is __________. Therefore, $g(x)$ can be _______________.

Thus, the domain of $(f \circ g)(x)$ is __________.

Next we'll find $(g \circ f)(x)$:

The domain of $f(x)$ is __________. We now need to make sure that the output of $f(x)$ is in the domain of $g(x)$.

The domain of $g(x)$ is __________. Therefore, $f(x)$ has to be _______________:

Thus, the domain of $(g \circ f)(x)$ is _______.

Classwork Question 1.8.4

Let $f(x) = x - 6$ and $g(x) = \dfrac{1}{x+7}$. Find $(f \circ g)(x)$ and $(g \circ f)(x)$, and state each of their domains.

So far we have taken two functions and composed them into one function. But sometimes, we want to take one function and break it down as the composition of two simpler functions.

To do this, we will look for an "inner" function and an "outer" function. Look for the inner function first. It will typically be "inside" one of the parent functions we discussed in Section 1.6.

Once we have found our "inner" function, we replace it by x. The remaining function is then our "outer" function.

Example 1.8.5

Let $h(x) = \dfrac{\sqrt{x-6}}{10}$. Find two functions f and g such that $h(x) = (f \circ g)(x)$.

Note:

Sometimes, there can be multiple ways to decompose a function. For example, we could decompose $h(x) = \dfrac{1}{(x+1)^2}$ into $(f \circ g)(x)$ as either of the following:

(a) $g(x) = x + 1$ and $f(x) = \dfrac{1}{x^2}$

(b) $g(x) = (x+1)^2$ and $f(x) = \dfrac{1}{x}$

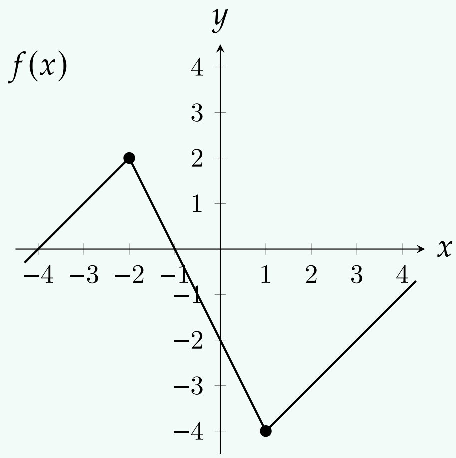
Classwork Question 1.8.5

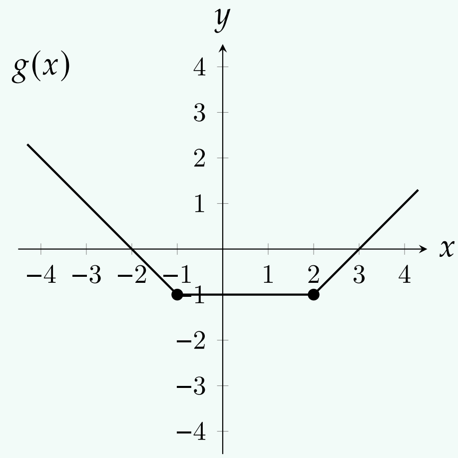
Let $h(x) = (x^2 - 4)^3$. Find two functions f and g such that $h(x) = (f \circ g)(x)$.

We can also compute compositions using graphs.

Example 1.8.6

Use the graphs of $f(x)$ and $g(x)$ below to compute the following compositions.

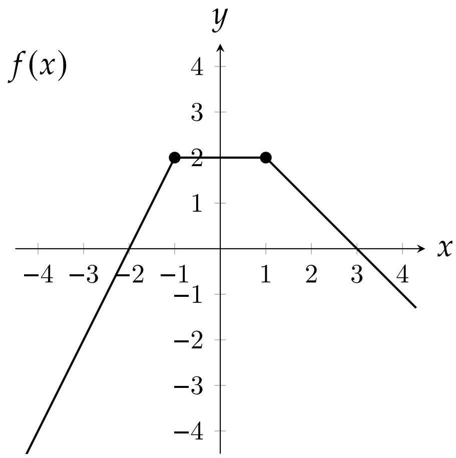


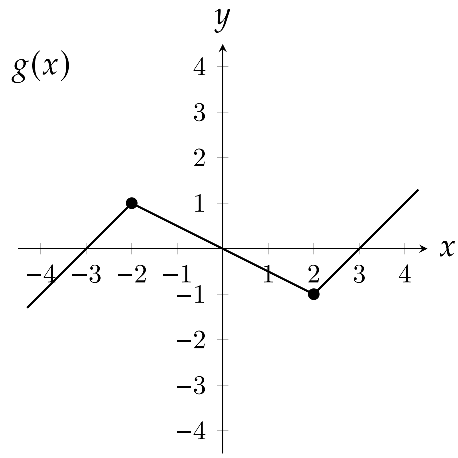


(a) $(f-g)(2)$ (b) $(fg)(-3)$ (c) $(f\circ g)(1)$

Classwork Question 1.8.6

Use the graphs of $f(x)$ and $g(x)$ below to compute the following compositions.





(a) $(f+g)(0)$ (b) $\left(\frac{f}{g}\right)(2)$ (c) $(g\circ f)(-1)$

1.9 Inverse Functions

Definition: Inverse Function

Given a function f, the inverse function (if it exists) is:

The inverse function has the property that if $f(a) = b$, then $f^{-1}(b) = a$. This means f^{-1} "undoes" f. Given the output of f for some input, f^{-1} takes the output and returns the input.

Note:

The notation f^{-1} does NOT mean $\frac{1}{f}$!

Example 1.9.1

Find the inverse function for the function in the table below.

x	-1	2	4	5	8
$f(x)$	0	1	2	4	8

x					
$f^{-1}(x)$					

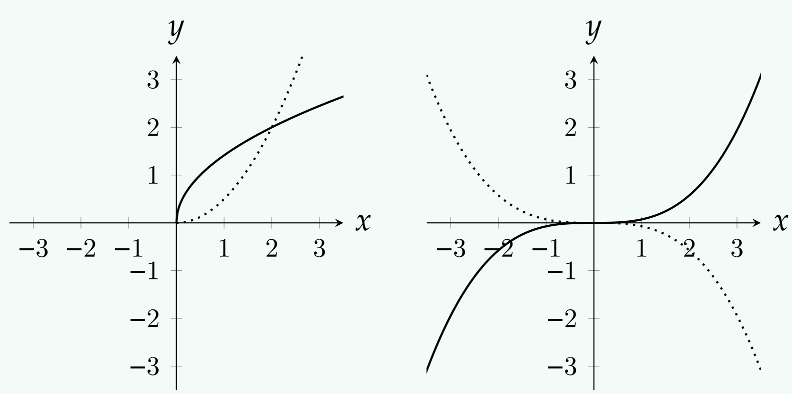
Formula 1.9.1 Points on Graphs of Inverse Functions

If the point (a, b) lies on the graph of f, then the point ________ lies on the graph of f^{-1}.

To verify if a graph is the inverse of some function:

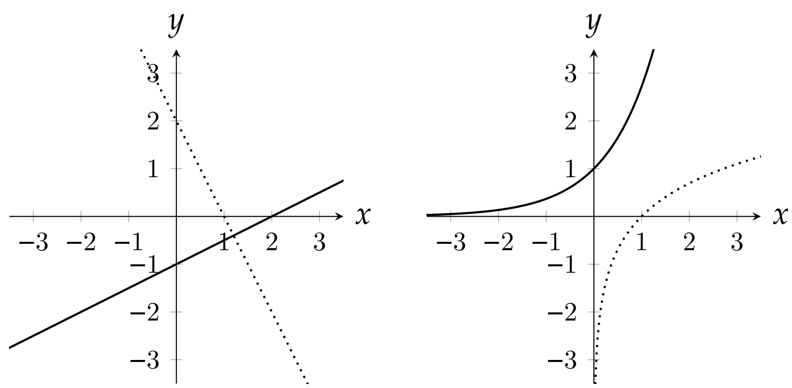
Example 1.9.2

On each of the axes below, a function is graphed as a solid line. Determine if the function graphed as a dotted line is the inverse function of the solid line function.



Classwork Question 1.9.1

On each of the axes below, a function is graphed as a solid line. Determine if the function graphed as a dotted line is the inverse function of the solid line function.



Not every function has an inverse function. If two different inputs give the same output, the function cannot have an inverse function.

Definition: One-to-One Function

A function is called **one-to-one** if

Example 1.9.3

Explain why the function given in the table below cannot have an inverse function.

x	0	1	2	3	4
$f(x)$	-2	9	4	3	9

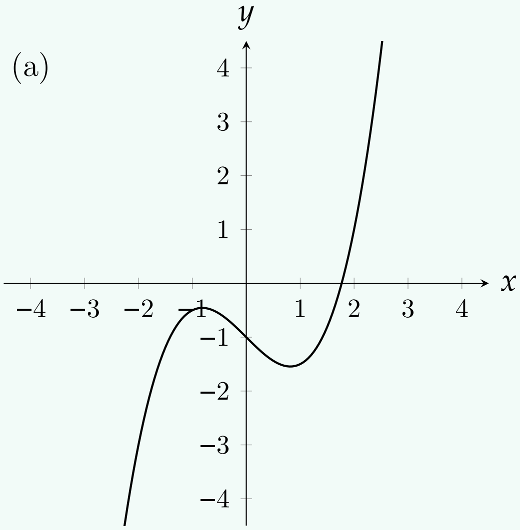
If we are given the graph of a function, we can check whether or not it has an inverse function by using the **horizontal line test**.

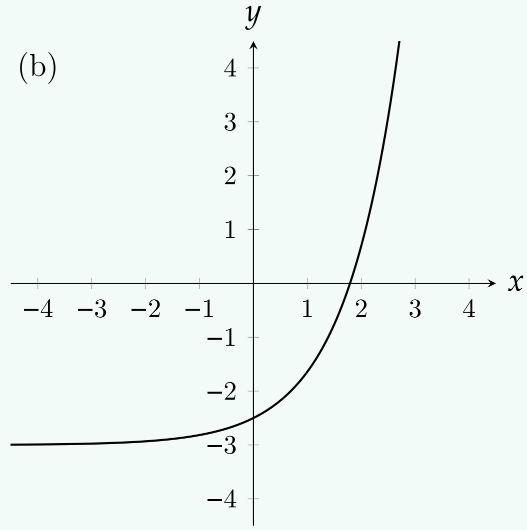
Formula 1.9.2 Horizontal Line Test

A function f has an inverse so long as

Example 1.9.4

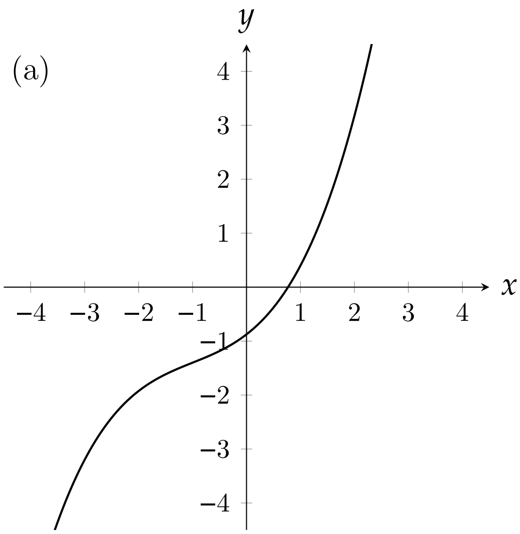
Use the horizontal line test to determine if each of the following functions has an inverse.

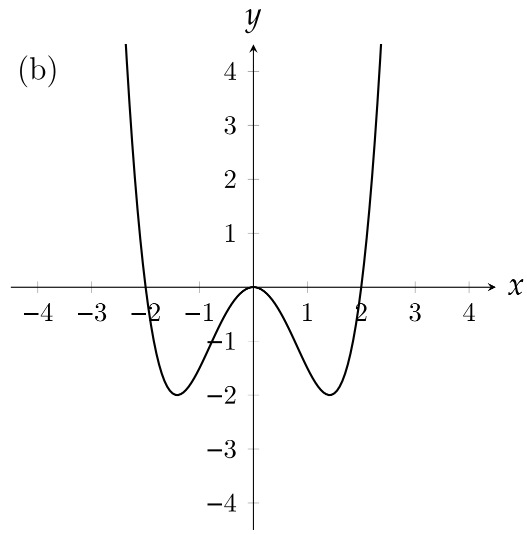




Classwork Question 1.9.2

Use the horizontal line test to determine if each of the following functions has an inverse.





If we are given a function, we can use some algebra to find its inverse.

Formula 1.9.3 Finding the Inverse of a Function

To find the inverse of a function f:

1.

2.

3.

4.

Example 1.9.5

Find the inverse of $f(x) = \dfrac{x+1}{x-2}$.

First we replace ________ by _____:

Then we swap _____ and _____:

Finally, we solve for _____:

Classwork Question 1.9.3

Find the inverse of $f(x) = \dfrac{3x-1}{x+4}$.

There is a relationship between the domains and ranges of f and f^{-1}. If we restrict the domain of a function, that affects the inverse function.

Formula 1.9.4 Domains and Ranges of a Function and its Inverse

The __________________ of f is the __________________ of f^{-1}.

The __________________ of f^{-1} is the __________________ of f.

Example 1.9.6

Consider the function $f(x) = x^2 - 2$ for $x \geq 0$. Find the inverse of this function and state the domains and ranges of f and f^{-1}.

Normally, $x^2 - 2$ does not have an inverse because it fails the horizontal line test. However, we have restricted the domain to $[0, \infty)$, which does pass the horizontal line test.

The inverse function is:

The domain of f is ______________, which is also the ______________ of f^{-1}.

The domain of f^{-1} is ______________, which is also the ______________ of f.

Classwork Question 1.9.4

Consider the function $f(x) = (x+3)^2$ for $x \geq -3$. Find the inverse of this function and state the domains and ranges of f and f^{-1}.

We can use inverse functions to solve certain word problems.

Example 1.9.7

The cost to manufacture computer chips is \$30 plus \$0.50 per chip produced. So the total cost y in terms of chips produced x is $y = 30 + 0.50x$.

(a) Find the inverse function.
(b) Explain what x and y represent in the inverse function.
(c) Determine the number of chips produced when the total cost is \$100.

For (a), we swap x and y and solve for y:

For (b), x is the ________________________ and y is the ________________________.

For (c), we plug 100 in for _____ in the _____________________________:

Classwork Question 1.9.5

The time required for a student to complete their math homework is 10 minutes, plus 5 minutes per problem. So the total time y in terms of number of problems x is $y = 10+5x$.

(a) Find the inverse function.
(b) Explain what x and y represent in the inverse function.
(c) Determine the number of problems when the total number of minutes is 85.

Chapter 2

Polynomials and Rational Functions

2.1 Quadratic Functions and Models

Definition: Polynomial Function

A **polynomial function** has the form

The **leading coefficient** of the polynomial is _____.

The **degree** of the polynomial is the ________________________.

A **constant** function has degree _____. A **linear** function has degree _____.

A **quadratic** function has degree _____. Quadratic functions have the form:

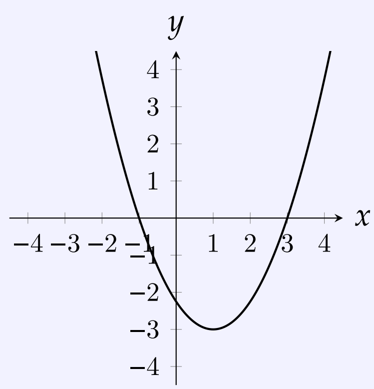
Definition: Graphs of Quadratic Functions

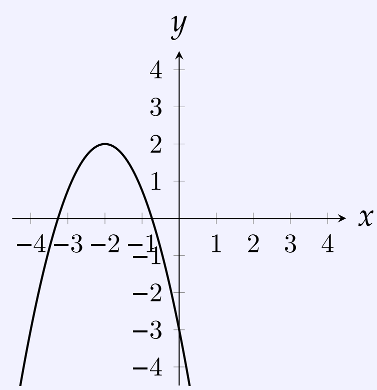
The graph of a quadratic function is called a **parabola**.

A parabola is symmetric over its ____________________________.

The point where the parabola intersects the axis of symmetry is called the ____________.

If $a > 0$, the parabola opens ____________. If $a < 0$, the parabola opens ____________.





Formula 2.1.1 Standard Form of a Quadratic Function

If the vertex of a parabola is at the point (h, k), the standard form of the parabola is

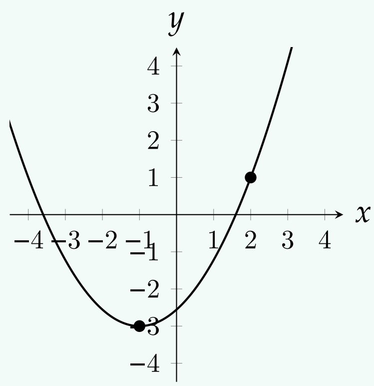
To write the standard form of a quadratic using its graph or its vertex and a point:

1.

2.

Example 2.1.1

The graph of a parabola is shown. Write the equation of the parabola in standard form.

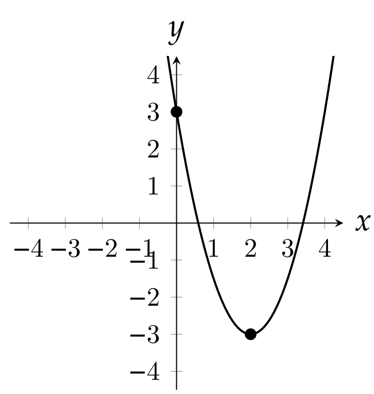


The vertex of the parabola is ________. Substituting this into the standard form equation:

Another point on the parabola is ________. We plug in $x =$ ____ and $f(x) =$ ____, and then solve for a:

Classwork Question 2.1.1

The graph of a parabola is shown. Write the equation of the parabola in standard form.



If we are given the equation of a parabola as $f(x) = ax^2 + bx + c$, we can convert it into standard form by completing the square.

Formula 2.1.2 Completing the Square

The steps to complete the square of a quadratic $ax^2 + bx + c$ are:

1.

2.

3.

4.

5.

Example 2.1.2

Write the equation of the parabola $f(x) = 2x^2 + 4x + 7$ in standard form.

To convert the parabola into vertex form, we complete the square.

First we factor out _____ from the whole quadratic:

The coefficient on x in the parentheses is _____. We divide it by 2 and square it to get _____.

Third, we add and subtract _____ inside the parentheses:

Fourth, we factor the trinomial inside the parenthesis:

Finally, we simplify the last two numbers and distribute _____:

Classwork Question 2.1.2

Write the equation of the parabola $f(x) = 3x^2 + 18x - 1$ in standard form.

To find the x-intercepts of a parabola, we have three methods:

1.

2.

3.

Formula 2.1.3 Quadratic Formula

The solutions of a quadratic equation $ax^2 + bx + c = 0$ are given by the quadratic formula:

Example 2.1.3

Find the x-intercepts of the parabola given by $f(x) = 2x^2 - 6x + 1$.

We plug into the quadratic formula:

Classwork Question 2.1.3

Find the x-intercepts of the parabola given by $f(x) = 3x^2 + 5x - 2$.

Many word problems can be solved by finding the maximum or minimum value of a quadratic.

Formula 2.1.4 Maximum/Minimum of Quadratic

To find the maximum or minimum of a quadratic:

1.

2.

If $a > 0$, it is a ______________________. If $a < 0$, it is a ______________________.

Example 2.1.4

The height of kickball after being kicked is given by $h(t) = -\frac{9}{8}t^2 + 6t + 1$, where h is in feet and t is in seconds. What is the maximum height of the kickball?

We first compute the t-coordinate of the vertex: $t =$ ________ = _____

To find the maximum height, we plug in $t =$ _____ and simplify:

Example 2.1.5

Find two positive numbers whose product is a maximum, where the sum of the first and 3 times the second is 18.

Let's say our two numbers are x and y. We want to maximize ________.

The second part of the question gives us the equation: ______________________.

We now need to solve this equation for either x or y. Our final answer will be the same no matter what, but solving for x looks easier:

In our $x \cdot y$ above, we replace x with ______________________ and distribute:

This quadratic is maximized when $y =$ __________, which simplifies to _____.

Finally, we get our value of x by plugging into our equation for x above:

Classwork Question 2.1.4

The cost for a factory to produce n gadgets per day is given by $C(n) = 0.15n^2 - 30n + 2500$. What daily number of gadgets minimizes the cost and what is the minimum daily cost?

Let's try a problem with a little bit of everything.

Example 2.1.6

Consider the quadratic $f(x) = x^2 - 6x + 10$. Write it in standard form. What is the vertex of this quadratic? What are its x and y intercepts?

First we put the quadratic in standard form:

From the standard form, we can see that the vertex is __________.

To find the x-intercepts, we set the standard form = _____ and solve:

To find the y-intercept, we plug in $x =$ _____:

Classwork Question 2.1.5

Consider the quadratic $f(x) = x^2 + 10x - 2$. Write it in standard form. What is the vertex of this quadratic? What are its x and y intercepts?

2.2 Polynomial Functions of a Higher Degree

In the previous section, we focused on quadratics, which are polynomials with degree 2. We can also talk about polynomials with degrees higher than 2.

Properties of Polynomials

1. Polynomials are **continuous**, which means they have no ________________________.

2. Polynomials are **smooth**, which means they have no ________________________.

3. Polynomials have 1 of 4 **end behaviors**. The end behavior of a function describes

____.

Formula 2.2.1 Leading Coefficient Test

If the degree of a polynomial $f(x)$ is **odd**, then:

If $a_n > 0$, then as $x \to \infty$, $f(x) \to$ _____ and as as $x \to -\infty$, $f(x) \to$ _____.

If $a_n < 0$, then as $x \to \infty$, $f(x) \to$ _____ and as as $x \to -\infty$, $f(x) \to$ _____.

If the degree of a polynomial $f(x)$ is **even**, then:

If $a_n > 0$, then as $x \to \infty$, $f(x) \to$ _____ and as as $x \to -\infty$, $f(x) \to$ _____.

If $a_n < 0$, then as $x \to \infty$, $f(x) \to$ _____ and as as $x \to -\infty$, $f(x) \to$ _____.

We can organize the end behaviors into a table:

	Odd Degree	Even Degree
$a_n > 0$		
$a_n < 0$		

If $f(x) \to \infty$, we say the graph **rises**.

If $f(x) \to -\infty$, we say the graph **falls**.

Example 2.2.1

Find the degree and leading coefficient of each polynomial below, and then describe the end behavior.

(a) $f(x) = 6x^4 - 3x^3 + 10x^2 - 20x + 8$
(b) $g(x) = -6x^2(x^5 - 10x^4 - 5)$
(c) $h(x) = (x^2 + 4)(3 + 3x)^3(9 + x^3)^2$

For (a), the degree is _____ and the leading coefficient is _____.

Therefore, as $x \to \infty$, $f(x) \to$ _____ and as as $x \to -\infty$, $f(x) \to$ _____.

For (b), we first want to distribute:

Now we can see the degree is _____ and the leading coefficient is _____.

Therefore, as $x \to \infty$, $g(x) \to$ _____ and as as $x \to -\infty$, $g(x) \to$ _____.

For (c), we want to think about how much each factor contributes to the degree of the expanded polynomial. To do this, we will pick out each x term in each factor (along with its coefficient), and raise it to the power outside the parentheses:

Next, we will multiply each of these terms together. This gives us the leading term of our polynomial:

Now we can see the degree is _____ and the leading coefficient is _____.

Therefore, as $x \to \infty$, $h(x) \to$ _____ and as as $x \to -\infty$, $h(x) \to$ _____.

Classwork Question 2.2.1

Find the degree and leading coefficient of each polynomial below, and then describe the end behavior.

(a) $f(x) = 3x^5 - 10x^4 + 30x - 19$
(b) $g(x) = 5x^2(x^6 - 2x^4 - 5x^3 - x + 10)$
(c) $h(x) = (x^5 - 3)(2x + 2)^4(x^2 + 9)^3$

Recall that the **zeroes** of a function f are the x-values where $f(x) = 0$.

Formula 2.2.2 Factor Theorem

If the x-value a is a zero of a polynomial, then __________ is a factor of the polynomial.

The Factor Theorem means that we can find factors of a polynomial by looking for its zeroes, and vice-versa. Sometimes, a factor can appear more than once in a polynomial.

Definition: Repeated Zeroes

If a polynomial has a factor of the form ____________ for $k > 1$, then we say a is a **repeated zero with multiplicity** k.

At a zero with odd multiplicity, the polynomial will ______________ the x-axis.

At a zero with even multiplicity, the polynomial will ___________________________ the x-axis. Another feature of a function are its **turning points**.

Definition: Turning Points

A **turning point** of a function is an x-value where

We can put a limit on the number of zeroes and turning points of a polynomial.

Formula 2.2.3 Fundamental Theorem of Algebra

A polynomial of degree n has at most _____ zeroes and __________ turning points.

Example 2.2.2

Find all of the real zeroes of $f(x) = x^4 - 25x^2$, and state each of their multiplicities. What is the maximum number of turning points this polynomial can have?

To find the zeroes of the polynomial, we factor it:

_____ has multiplicity _____, _____ has multiplicity _____, and _____ has multiplicity _____.

The polynomial has degree ____ so the maximum number of turning points is ____.

Classwork Question 2.2.2

Find all of the real zeroes of $f(x) = x^3 - 4x^2 + 4x$, and state each of their multiplicities. What is the maximum number of turning points this polynomial can have?

Example 2.2.3

Construct a polynomial of degree 5 that has zeroes only at the points $x = -3$ and $x = 2$.

In order for -3 and 2 to be the only zeroes, we need __________ and __________ to be the only factors of our polynomial.

Since we need to polynomial to be degree 5, we the multiplicities to ____________________. Thus, one possible answer is:

Classwork Question 2.2.3

Construct a polynomial of degree 7 that has zeroes only at the points $x = -4$, $x = 1$, and $x = 3$.

2.6 Rational Functions

Definition: Rational Function

A **rational function** is a function of the form

We want to talk about the behavior of a rational function near certain values. To do this, we have some special notation.

$x \longrightarrow a^+$ means we consider the behavior as we approach a from the ______________.

$x \longrightarrow a^-$ means we consider the behavior as we approach a from the ______________.

To find the near a value not in the domain of a rational function, we check the sign of the numerator and denominator as we approach the point from the right/left. This will tell us whether the function goes to ∞ or $-\infty$.

Example 2.6.1

Find the domain of $f(x) = \dfrac{5}{x-1}$ and describe the behavior of f near any excluded values.

The domain is ______________. The only excluded value is _____.

As $x \longrightarrow 1^+$, x _____ 1, so the sign of the denominator is _____.

As $x \longrightarrow 1^-$, x _____ 1, so the sign of the denominator is _____.

At $x = 1$, the numerator is _____, which is _____.

Thus, the overall sign is _____ as as $x \longrightarrow 1^+$, and _____ as $x \longrightarrow 1^-$.

Therefore, as $x \longrightarrow 1^+$, $f(x) \rightarrow$ _______, and as $x \longrightarrow 1^-$, $f(x) \rightarrow$ _______.

Classwork Question 2.6.1

Find the domain of $f(x) = \dfrac{9}{x-7}$ and describe the behavior of f near any excluded values.

Example 2.6.2

Find the domain of $f(x) = \dfrac{5x}{x^2 - 9}$ and describe the behavior of f near any excluded values.

The domain is ______________________. The excluded values are ___________.

As $x \to 3^+$, x ___ 3, meaning x^2 ___ 9, so the sign of the denominator is _____.

As $x \to 3^-$, x ___ 3, meaning x^2 ___ 9, so the sign of the denominator is _____.

At $x = 3$, the numerator is _____, which is _____.

Thus, the overall sign is _____ as as $x \to 3^+$, and _____ as $x \to 3^-$.

Therefore, as $x \to 3^+$, $f(x) \to$ _______, and as $x \to 3^-$, $f(x) \to$ _______.

As $x \to -3^+$, x ___ -3, meaning x^2 ___ 9, so the sign of the denominator is _____.

As $x \to -3^-$, x ___ -3, meaning x^2 ___ 9, so the sign of the denominator is _____.

At $x = -3$, the numerator is _____, which is _____.

Thus, the overall sign is _____ as as $x \to -3^+$, and _____ as $x \to -3^-$.

Therefore, as $x \to -3^+$, $f(x) \to$ _______, and as $x \to -3^-$, $f(x) \to$ _______.

Classwork Question 2.6.2

Find the domain of $f(x) = \dfrac{x^2}{x^2 - 1}$ and describe the behavior of f near any excluded values.

Definition: Vertical and Horizontal Asymptotes

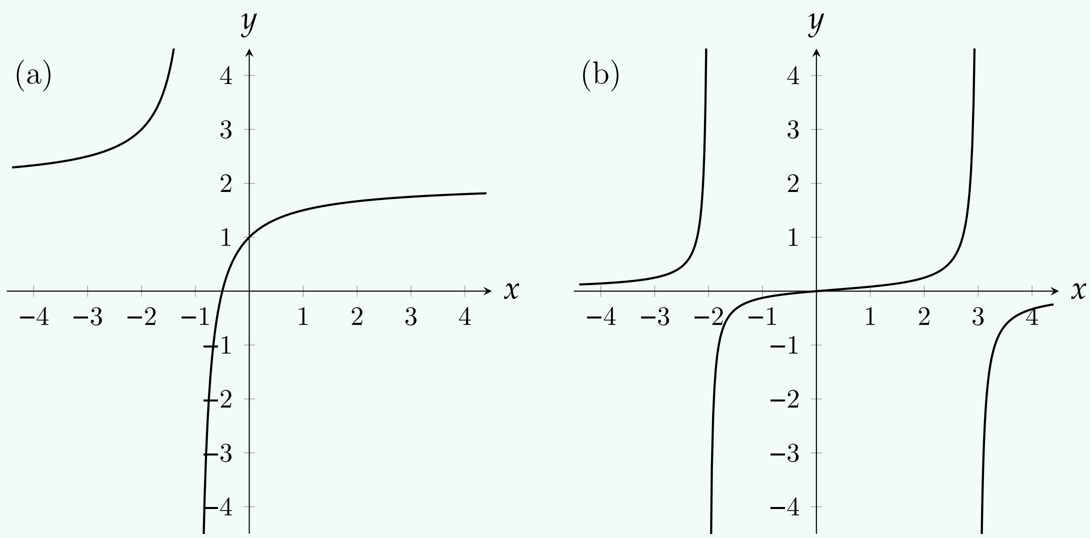
Let f be any rational function.

The vertical line $x = a$ is a **vertical asymptote** of f when

The horizontal line $y = b$ is a **horizontal asymptote** of f when

Example 2.6.3

Draw dotted lines for each vertical and horizontal asymptote of the graphs below. Also write the equation for each asymptote.



Note:

A rational function can cross a horizontal asymptote. The horizontal asymptote describes where $f(x)$ goes as $x \to \infty$ or $x \to -\infty$.

A rational function never crosses a vertical asymptote.

Definition: Hole

A rational function f has a **hole** at $x = a$ if

Formula 2.6.1 Finding Vertical/Horizontal Asymptotes and Holes

To find the vertical/horizontal asymptotes and holes of a rational function:

1.

2.

2a.

2b.

3.

3a.

3b.

3c.

Example 2.6.4

Write the equation of each vertical/horizontal asymptote of $f(x) = \dfrac{x^2+x-2}{x^2-x-6}$. Also list the points where f has a hole. What value does $f(x)$ approach as $x \to \infty$?

First, we factor the numerator and denominator:

Since _______ is a common factor, there is a _______ at $x =$ ____. Its y-coordinate is:

Thus, there is a _______ at _______ and a _____________________ at _______.

The degree of the numerator is ____, and the degree of the denominator is ____.

Therefore, we have a ____________________________ at _________.

As $x \to \infty$, $f(x)$ approaches its ________________. Thus, $f(x) \to$ _____ as $x \to \infty$.

Note:

Vertical asymptotes **must** be written as an $x =$ and horizontal asymptotes **must** be written as a $y =$ equation. For example, a vertical asymptote is written as $x = 3$, not just 3 or VA = 3.

Classwork Question 2.6.3

Write the equation of each vertical/horizontal asymptote of $f(x) = \dfrac{-3x}{x^2 + x}$. Also list the points where f has a hole. What value does $f(x)$ approach as $x \longrightarrow \infty$?

Formula 2.6.2 Sketching Graphs of Rational Functions

To sketch the graph of a rational function, use the following steps:

1.

2.

3.

4.

5.

Example 2.6.5

Sketch a graph of $f(x) = \dfrac{x-1}{x^2+2x-3}$.

First we factor the denominator and cancel common factors:

We can see that we have:

A hole at $x =$ _____, with y-coordinate $y =$ _____.

A vertical asymptote at $x =$ _____.

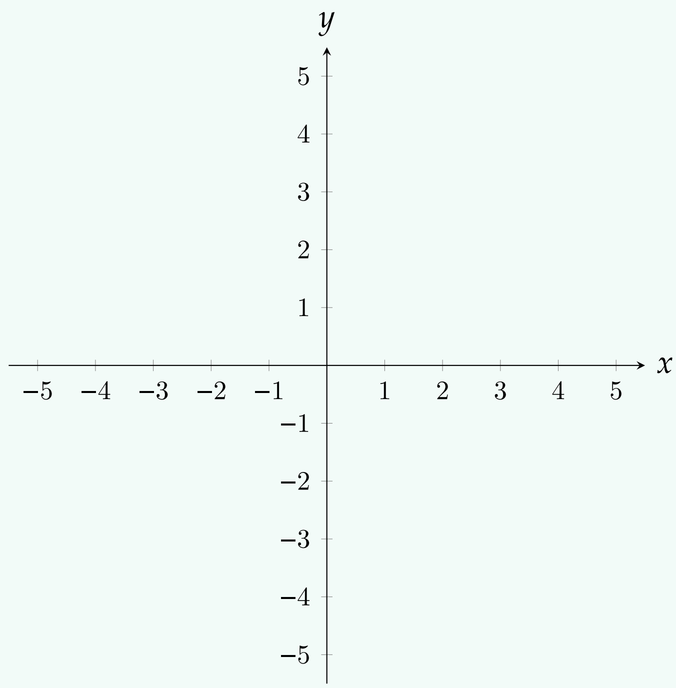
A horizontal asymptote at $y =$ _____.

Do we have a y-intercept? ____________________.

Do we have any x-intercepts? ____________________.

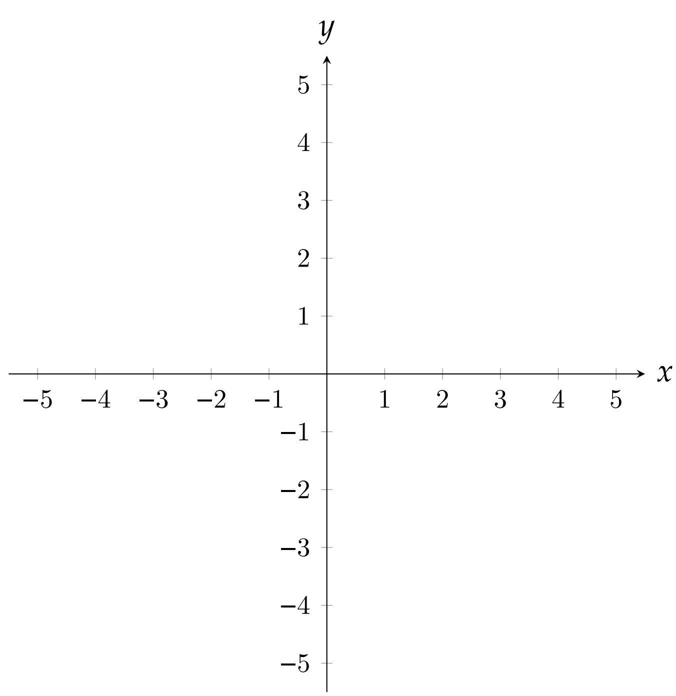
We need to plot additional points ______________________________:

Finally, we sketch smooth curves to complete the graph.



Classwork Question 2.6.4

Sketch a graph of $f(x) = \dfrac{2x-1}{x}$.



Example 2.6.6

Consider the function $f(x) = \dfrac{6(x+2)}{(x-1)^2(x+3)}$. The following table gives some of the points on the graph of f:

x	-4	-2.5	-1	3
$f(x)$	0.48	-0.49	0.75	1.25

Sketch a graph of f.

We first want to find the holes and vertical/horizontal asymptotes.

Are there any holes? ______________________________.

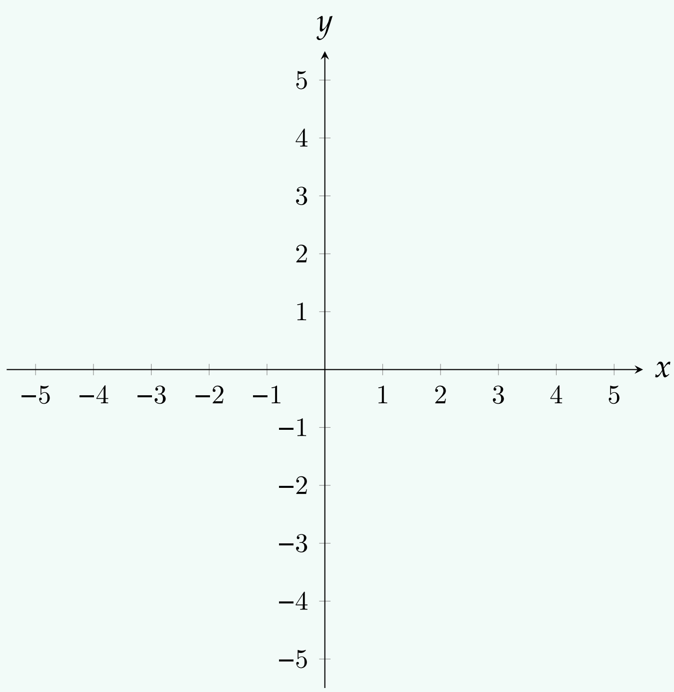
Are there any vertical asymptotes? ______________________________.

Is there a horizontal asymptote? ______________________________.

Is there a y-intercept? ______________________________.

Are there any x-intercepts? ______________________________.

Plotting these, along with the points in the table:

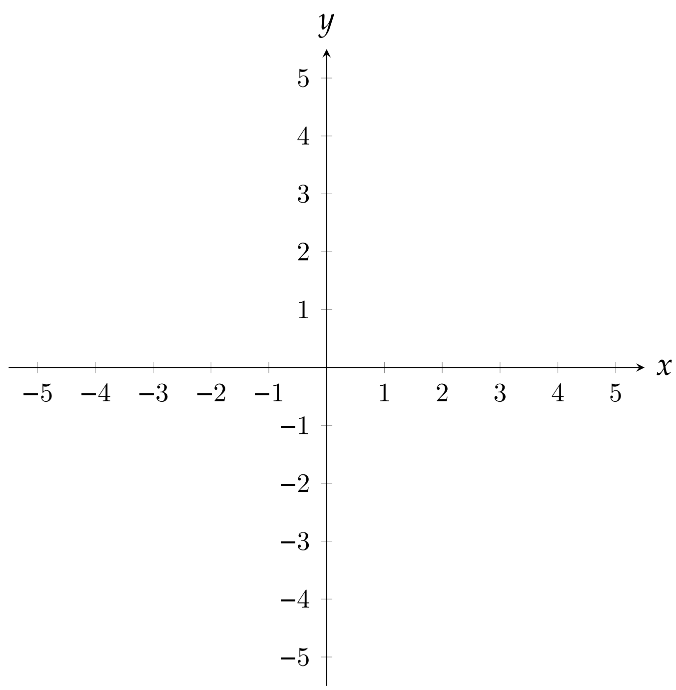


Classwork Question 2.6.5

Consider the function $f(x) = \dfrac{4(x-1)}{(x+3)^2(x-2)}$. The following table gives some of the points on the graph of f:

x	-4	-2	1.5	3
$f(x)$	3.33	3	-0.20	0.22

Sketch a graph of f.



Example 2.6.7

The cost C in dollars to remove $p\%$ of pollutants from a lake is given by

$$C = \frac{80,000p}{100-p}, \quad 0 \leq p < 100$$

(a) How much would it cost to remove 90% of the pollutants from the lake?
(b) What happens to the cost as we approach 100% removal?
(c) Is it possible to reach 100% removal? Explain why or why not.

For (a), we plug in $p =$ _____ and simplify:

Thus, the total cost is ____________.

For (b), our function has a ______________________________ at $p =$ _______.

Therefore, as $p \to 100^-$, $C \to$ _______.

For (c), we ____________ reach 100% removal because:

Classwork Question 2.6.6

The number of items sold N in terms of the product's price p (in dollars) is given by

$$N = \frac{20,000p}{p^2+100}, \quad p > 0$$

(a) How many products are sold when the price is \$20?
(b) What is the horizontal asymptote of this rational function?
(c) Describe what happens to the number of products sold as $p \to \infty$.

Chapter 3

Exponential and Logarithmic Functions

3.1 Exponential Functions and Their Graphs

So far, we have only dealt with **algebraic** functions. These are functions involving polynomials, roots, and rational functions. Functions not of these types are called **transcendental** functions. One example of a transcendental function is the **exponential** function.

Before we start working with exponential functions, let's remind ourselves of our exponent rules:

Formula 3.1.1 Exponent Rules

For the following, x and y are non-zero real numbers, a and b are any real numbers.

$x^a \cdot x^b =$ ________

$\dfrac{x^a}{x^b} =$ ________

$(x^a)^b =$ ________

$(xy)^a =$ ________

$\left(\dfrac{x}{y}\right)^a =$ ________

$x^{a/b} =$ ________

$x^{-a} =$ ________

$x^0 =$ ________

Example 3.1.1

Simplify each of the following. Write your answer no negative or fractional exponents:

(a) $x^3 \cdot x^{1/2} \cdot x^0$ (b) $\dfrac{(xy)^5}{x^4(y^2)^3}$ (c) $x^{-3} \cdot x^{-1/3} \cdot \left(\dfrac{y^2}{x^3}\right)^2 \cdot y^{3/2}$

(a):

(b):

(c):

Definition: Exponential Function

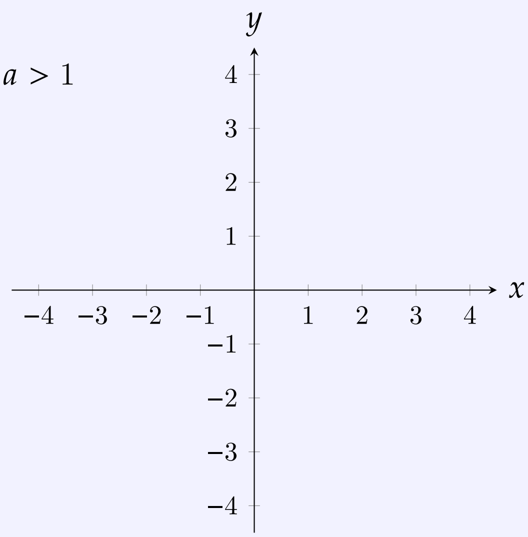
An **exponential function** is a function of the form

The number a is called the **base** of the exponential function.

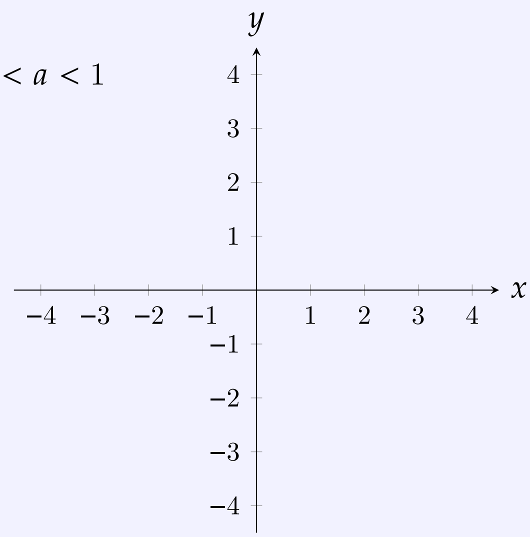
Domain: Range: Horizontal Asymptote:

x-intercepts: y-intercept: Even / Odd / Neither

$a > 1$

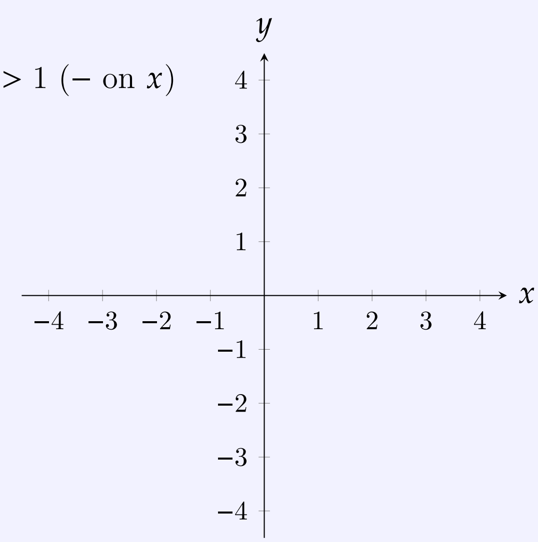


$0 < a < 1$

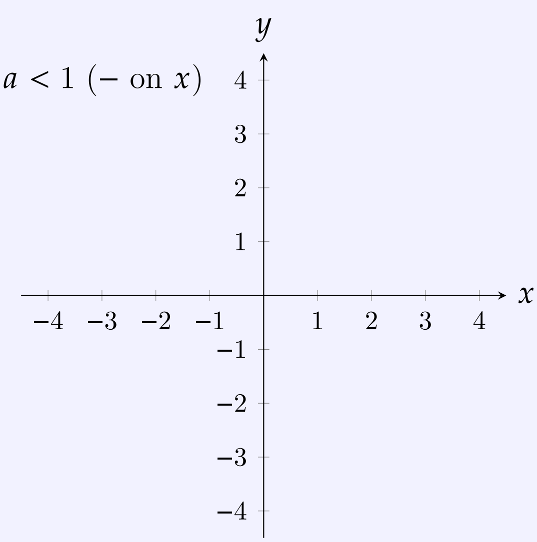


If there is a $-$ on x in the exponential function, then the graphs flip:

$a > 1$ ($-$ on x)



$0 < a < 1$ ($-$ on x)



Example 3.1.2

Fill in the table of values for the exponential function $f(x) = 3^x$

x	-2	-1	0	1	2	3
$f(x)$						

Classwork Question 3.1.1

Fill in the table of values for the exponential function $f(x) = 2^x$

x	-2	-1	0	1	2	3
$f(x)$						

Example 3.1.3

Classify each of the following functions as polynomial, exponential, or neither.

(a) $x^3 - 5$ (b) 3^{x+4} (c) $3x^{1/3}$

(a) This function is ____________________ because:

(b) This function is ____________________ because:

(c) This function is ____________________ because:

Classwork Question 3.1.2

Classify each of the following functions as polynomial, exponential, or neither.

(a) $10^x - 8$ (b) $|10x| + 2$ (c) $4x^{10} - 7$

Example 3.1.4

For each exponential function, describe the transformations and state the y-intercept.

(a) $f(x) = 3^x + 7$ (b) $g(x) = 0.5^{x+2}$ (c) $h(x) = 7^{-x+3}$

For (a), the transformations are: ______________________________.

Plugging in $x =$ _____ gives ______________________, so the y-intercept is _______.

For (b), the transformations are: ______________________________.

Plugging in $x =$ _____ gives ______________________, so the y-intercept is _______.

For (c), the transformations are: ______________________________.

Plugging in $x =$ _____ gives ______________________, so the y-intercept is _______.

Classwork Question 3.1.3

For each exponential function below, describe the transformations applied and state the y-intercept.

(a) $f(x) = 2^{-x}$ (b) $g(x) = 5^{x-6}$ (c) $h(x) = 0.2^x - 3$

While we can define an exponential function for any $a > 0$ such that $a \neq 1$, there is one number in particular we use more than the others.

Definition: Natural Exponential Function

The **natural exponential function** is

The number $e = 2.718281828\ldots$ is called the **natural base**.

When we are solving equations involving exponential functions with the same base, we set the two exponents equal to each other and solve that equation instead.

Formula 3.1.2 One-to-One Property

If $a > 0$ and $a \neq 1$, then $a^x = a^y$ means __________.

Example 3.1.5

Use the One-to-One Property to solve each equation below.

(a) $e^{x^2-3x} = e^{x+5}$ (b) $8^x = 2^{12-x}$ (c) $6^{x+2} = (\sqrt{6})^x$

For (a), the One-to-One Property means that __________ = __________. Solving this equation gives:

For (b), we don't have the same base on both sides, so we can't apply the One-to-One Property right away. However, we can get them into the same base.

We know that $8 = 2$^_____, so $8^x =$ _______ = __________.

The One-to-One Property means that __________ = __________. Solving this equation gives:

For (c), we know that $\sqrt{6} = 6$^_____, so $(\sqrt{6})^x =$ _______ = __________.

The One-to-One Property means that __________ = __________. Solving this equation gives:

Classwork Question 3.1.4

Use the One-to-One Property to solve each equation below.

(a) $e^{2x+6} = e^{x^2-2}$ (b) $9^x = 3^{4x+4}$ $5^{x-1} = (\frac{1}{25})^x$

Example 3.1.6

Use exponent properties to determine if any of the following functions are equal:

$f(x) = 3^{x-2}$ $\qquad g(x) = \frac{1}{9}(3^x)$ $\qquad h(x) = 3^x - 9$

A good strategy is to put every function into the same form and compare. Let's try to make every function look like $f(x)$.

For $g(x)$, we can rewrite $\frac{1}{9} =$ _____ $=$ _____. Using exponent properties, we get:

Therefore, $g(x) =$ __________.

For $h(x)$, we can rewrite $9 =$ _____ $=$ _____. Can we go any further? _____.

Therefore, $h(x) =$ __________.

Classwork Question 3.1.5

Use exponent properties to determine if any of the following functions are equal:

$f(x) = 2^{1-x}$ $\qquad g(x) = 2^{2x-2}$ $\qquad h(x) = (\frac{1}{2})^{x-1}$

3.2 Logarithmic Functions and Their Graphs

In the previous section, we learned about exponential functions of the form $f(x) = a^x$. The inverse of an exponential function is a **logarithmic** function.

Definition: Logarithmic Function

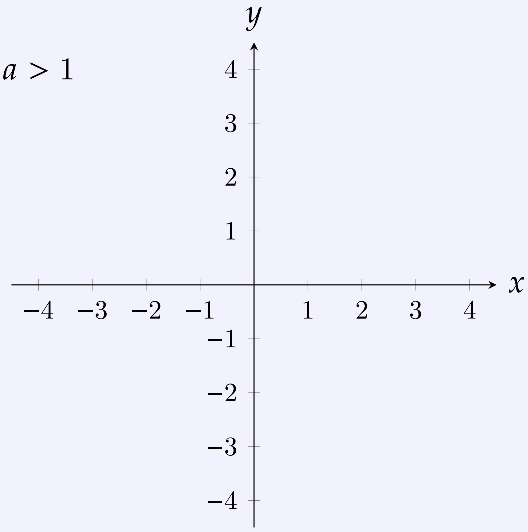
A **logarithmic function** is a function of the form

The number a is called the **base** of the logarithmic function.

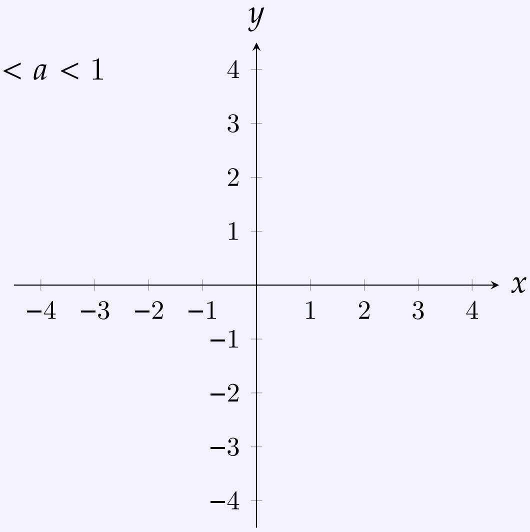
Domain: Range: Vertical Asymptote:

x-intercepts: y-intercept: Even / Odd / Neither

$a > 1$



$0 < a < 1$



To evaluate a logarithm $\log_a x$, ask yourself: "a raised to what number gives me x?"

Example 3.2.1

Evaluate each of the following, if possible:

(a) $\log_2 8$ (b) $\log_9 3$ (c) $\log_4 \frac{1}{16}$ (d) $\log_5(-5)$

(a) Ask: ____ raised to what number gives ____? 2^____ = ____, so $\log_2 8 =$ ____.

(b) Ask: ____ raised to what number gives ____? 9^____ = ____, so $\log_9 3 =$ ____.

(c) Ask: ____ raised to what number gives ____? 4^____ = ____, so $\log_4 \frac{1}{16} =$ ____.

(d) The expression is __________ because ____________________________________.

Classwork Question 3.2.1

Evaluate each of the following, if possible:

(a) $\log_3 81$ (b) $\log_6 0$ (c) $\log_7 \frac{1}{7}$ (d) $\log_8 2$

Given an equation involving exponents or logarithms, we can convert from one to the other. The equations $x = a^y$ and $y = \log_a x$ are equivalent, so keep track where $a, x,$ and y go in each.

Example 3.2.2

Write each exponential equation in logarithmic form.

(a) $2^5 = 32$ (b) $12^{-2} = \frac{1}{144}$

(a) We have $a =$ ____, $x =$ ____, and $y =$ ____. So the logarithmic form is __________.

(b) We have $a =$ ____, $x =$ ____, and $y =$ ____. So the logarithmic form is __________.

Example 3.2.3

Write each logarithmic equation in exponential form.

(a) $\log_9 81 = 2$ (b) $\log_{11} \frac{1}{1331} = -3$

(a) We have $a =$ ____, $x =$ ____, and $y =$ ____. So the exponential form is __________.

(b) We have $a =$ ____, $x =$ ____, and $y =$ ____. So the exponential form is __________.

Classwork Question 3.2.2

Convert each exponential/logarithmic equation into logarithmic/exponential form.

(a) $4^3 = 64$ (b) $\log_{15} 225 = 2$ (c) $9^{3/2} = 27$ (d) $\log_6 \frac{1}{36} = -2$

Definition: Common Logarithm and Natural Logarithm

The **common logarithm** is the logarithmic function with base _____. We write:

The **natural logarithm** is the logarithmic function with base _____. We write:

Formula 3.2.1 Properties of Logarithms

For any base $a > 0$ with $a \neq 1$, and $x > 0$:

1. $\log_a 1 =$ _____
2. $\log_a a =$ _____
3. $\log_a a^x =$ _____
4. $a^{\log_a x} =$ _____
5. If $\log_a x = \log_a y$, then __________. (One-to-One Property)

Example 3.2.4

Use the properties of logarithms to evaluate the following:

(a) $\log_{\sqrt{5}} \sqrt{5}$ (b) $\ln e^3$ (c) $10^{\log 4.5}$

For (a), we use property _____, so $\log_{\sqrt{5}} \sqrt{5} =$ _____.

For (b), we use property _____, so $\ln e^3 =$ _____.

For (c), we use property _____, so $10^{\log 4.5} =$ _____.

Classwork Question 3.2.3

Use the properties of logarithms to evaluate the following:

(a) $e^{\ln 6}$ (b) $\log_{7.3} 1$ (c) $\log 10,000$

Just like with exponential functions, the One-to-One Property of logarithmic functions is helpful for solving equations with logarithms that have the same base.

Example 3.2.5

Use the One-to-One Property to solve each equation below:

(a) $\log_3(x^2 + 4x) = \log_3 5$ (b) $\ln(2x - 3) = \ln(-3x + 7)$

For (a), the One-to-One Property means _________ = _________. Solving this gives:

For (b), the One-to-One Property means __________ = __________. Solving this gives:

Classwork Question 3.2.4

Use the One-to-One Property to solve each equation below:

(a) $\log_8(x^2 - 6x) = \log_8 7$ (b) $\ln(-x - 1) = \ln(4x + 9)$

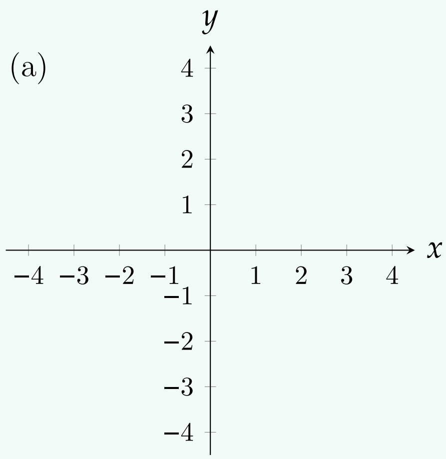
Example 3.2.6

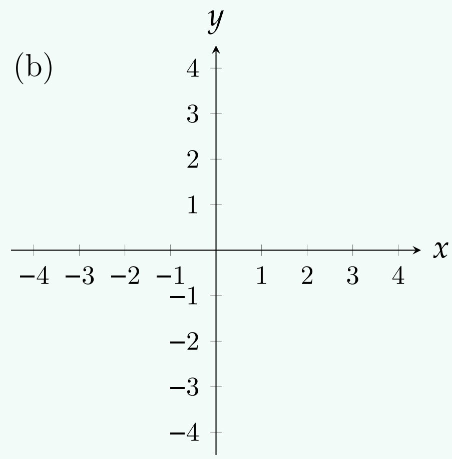
For each function below, describe the transformations and sketch the graph of the function.

(a) $\log_2(x-3)$ (b) $\log_5(-x)$

For (a), the transformation is __________________________________.

For (b), the transformation is __________________________________.

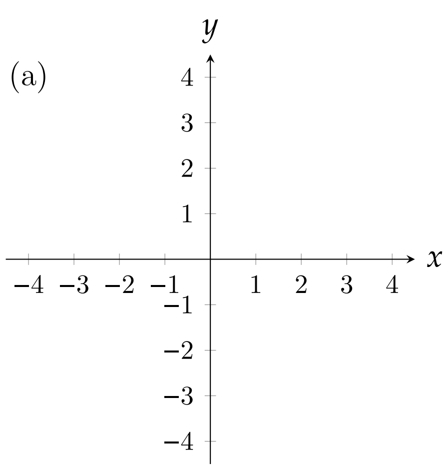


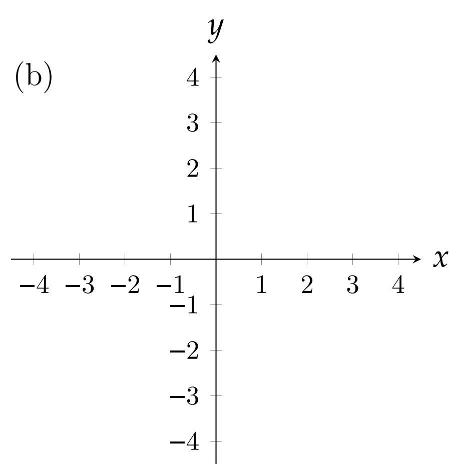


Classwork Question 3.2.5

For each function below, describe the transformations and sketch the graph of the function.

(a) $\log_4(x)+2$ (b) $-\log_7 x$





The domain of $\log_a x$ is __________. To find the domain of a transformed function:

____.

The x-intercept of $\log_a x$ is _____. To find the x-intercept of a transformed function:

____.

The vertical asymptote of $\log_a x$ is __________. To find the vertical asymptote of a transformed function:

____.

Example 3.2.7

Find the domain, x-intercept, and vertical asymptote of $f(x) = \log_2(x + 9) - 3$.

For the domain, we need __________. We solve the inequality:

For the x-intercept, we set __________ and solve:

For the vertical asymptote, we set __________ and solve:

Classwork Question 3.2.6

Find the domain, x-intercept, and vertical asymptote of $f(x) = \log_7(x - 7) - 2$.

3.2B Applications of Exponents and Logarithms

Certain real-world phenomena can be modeled using exponential and logarithmic functions.

Definition: Exponential Model

An **exponential model** takes the form:

a represents ________________________________, and is called the **growth factor**.

P_0 represents ________________________________, and is called the **initial value**.

The units of t will depend on the problem (hours, days, months, etc.)

Example 3.2.8

A petri dish has 100 bacteria in it. Every hour, the number of bacteria in the petri dish triples. Write a function to model the number of bacteria in the petri dish after t hours. How many bacteria will be in the petri dish after 3 hours? How long will it take for there to be 900 bacteria? Remember to put units on your answers.

We will use an exponential function as our model, $f(t) =$ _______.

Since the number of bacteria triples every hour, we will set $a =$ _____.

The initial number of bacteria is 100, so we will set $P_0 =$ _____.

Therefore, our model is $f(t) =$ _________. To get the number of bacteria after 3 hours, we plug in $t =$ _____:

To find the time needed to reach 900 bacteria, we set ______________ and solve:

Classwork Question 3.2.7

50 people in Storrs are infected with the flu. Each day, the number of people infected with the flu doubles. Write a function to model the number of people infected with the flu after t days. How many people will be infected after 4 days? How long will it take for 1600 people to be infected? Remember to put units on your answers.

We can also use exponential models for phenomena that decay over time.

Example 3.2.9

You drink a cup of coffee containing 100 mg of caffeine. Every hour, the amount of caffeine in your bloodstream is 75% of what it was the previous hour. Write a function to model the amount of caffeine in your bloodstream after t hours. How much caffeine will be in your bloodstream after 8 hours? Remember to put units.

We will use an exponential function as our model, $f(t) =$ ________.

Since only 75% remains after each hour, $a =$ _____.

The initial amount of caffeine is 100 mg, so we will set $P_0 =$ _____.

Therefore, our model is $f(t) =$ __________. To get the amount of caffeine after 8 hours, we plug in $t =$ _____:

Classwork Question 3.2.8

Your car is currently worth \$50,000. Every year, its value is 90% of what it was the year before. Write a function to model the value of your car after t years. How much will your car be worth after 5 years? Remember to put units.

We can also use logarithmic functions to model real-world phenomena.

Example 3.2.10

A **human memory model** describes how much a person remembers about a topic over time. As time passes, we tend to forget more and more about the topic.

Students in a history class were given a test at the end of the semester. Each month after that, they were re-tested to see how much they remembered. The average score on the test is given by $f(t) = 85 - 8\ln(t+1)$, where t is the number of months after the end of the semester.

(a) What is the original average score (ie, at $t = 0$)?
(b) What are the average scores at 1 month and 2 months? (You'll need a calculator)
(c) Find the drop between 0 and 1 months, and between 1 and 2 months. What does this suggest about human memory?

(a) For the original score, we plug in $t =$ _____:

(b) For the scores at 1 and 2 months, we plug in $t =$ _____ and $t =$ _____:

(c) From 0 to 1, the drop is _______. From 1 to 2, the drop is _______. This suggests:

Classwork Question 3.2.9

Frechner's law describes how loud a human perceives a sound, compared to its intensity. The greater the intensity, the louder we perceive a sound to be.

Subjects in a study were played sounds at various intensities. They were asked to rate each intensity by how loud they perceived the sound to be. The average rating is given by $L(i) = 3\log\left(1 + \frac{i}{2}\right)$, where i represents the intensity.

(a) What is the average rating for silence (ie, at $i = 0$)?
(b) What are the average ratings for sounds with intensities of 10 and 20?
(c) Find the change between intensities 0 and 10, and between 10 and 20. What does this suggest about how humans perceive sound?

3.3 Properties of Logarithms

The logarithm function we discussed in section 3.2 has many useful properties.

Formula 3.3.1 Properties of Logarithms

For any base $a > 0$ with $a \neq 1$, $x > 0$, $y > 0$, and any real number b:

1. $\log_a(x \cdot y) =$ ____________________ (Product rule)

2. $\log_a\left(\frac{x}{y}\right) =$ ____________________ (Quotient rule)

3. $\log_a\left(x^b\right) =$ ____________________ (Power rule)

Using these properties, we can compute certain tricky logarithms by hand.

Example 3.3.1

Compute each of the following, if possible:

(a) $\log_9 27$ (b) $\log_4 \sqrt[7]{64}$ (c) $\log_8 0$ (d) $\ln(7e^5)$

For (a), $27 =$ _____ $\cdot$ _____. Using the __________ rule, we get $\log_9 27 =$ __________:

For (b), $\sqrt[7]{64} =$ _____ ^ _____. Using the __________ rule, we get $\log_4 \sqrt[7]{64} =$ __________:

For (c), we cannot evaluate it because ____.

For (d), using the __________ rule, we get $\ln(7e^5) =$ _______________:

Note:
You should leave your answer in terms of ln, not a decimal, unless specifically asked otherwise.

Classwork Question 3.3.1

Compute each of the following, if possible:

(a) $\log_4 128$ (b) $\log_3 \sqrt[6]{27}$ (c) $\ln\left(\frac{e^8}{4}\right)$ (d) $\log_2(-4)$

We can also use the rules in reverse to evaluate certain expressions involving logarithms.

Example 3.3.2

Compute each of the following:

(a) $\log_6 4 + \log_6 9$ (b) $\log_3 54 - \log_3 2$ (c) $2\log 5 + \frac{1}{2}\log 16$

For (a), we can use the ______________ rule:

For (b), we can use the ______________ rule:

For (c), we first use the ______________ rule, then the ______________ rule:

Classwork Question 3.3.2

Compute each of the following:

(a) $\log_5 75 - \log_5 3$ (b) $\log_4 \frac{1}{3} + \log_4 48$ (c) $2\log_6 3 + \frac{1}{3}\log_6 64$

We can use the logarithm properties to expand a logarithm that contains several variables.

Example 3.3.3

Use the properties of logarithms to expand each of the following:

(a) $\log_2(5x^2y\sqrt[3]{z})$ (b) $\ln\left(\frac{(xz)^4}{y+1}\right)$ (c) $\log_5\left(y^3\sqrt{\frac{z}{x^5}}\right)$

(a) $\log_2(5x^2y\sqrt[3]{z})$ = ______________________ ________ rule

= ______________________ ________ rule

(b) $\ln\left(\frac{(xz)^4}{y+1}\right)$ = ______________________ ________ rule

= ______________________ ________ rule

= ______________________ ________ rule

(c) $\log_5\left(y^3\sqrt{\frac{z}{x^5}}\right)$ = ______________________ ________ rule

= ______________________ ________ rule

= ______________________ ________ rule

= ______________________ ________ rule

Classwork Question 3.3.3

Use the properties of logarithms to expand each of the following:

(a) $\log_7\left(\dfrac{8x^3}{y^6z}\right)$ (b) $\log_3\left(\sqrt[4]{xy^2z^3}\right)$ (c) $\log\left(z^2\dfrac{\sqrt[3]{x-4}}{\sqrt[5]{y}}\right)$

We can also use the logarithm properties to condense an expression into a single logarithm.

Example 3.3.4

Condense each of the following into a single logarithm:

(a) $\ln 6 - \ln y + 9\ln z$ (b) $\log x - 2\log y - 5\log(z+7)$

(a) $\ln 6 - \ln y + 9\ln z$ = ______________________ ___________ rule

= ______________________ ___________ rule

= ______________________ ___________ rule

(b) $\log x - 2\log y - 5\log(z+7)$ = ______________________ ___________ rule

= ______________________ ___________ rule

= ______________________ ___________ rule

Classwork Question 3.3.4

Condense each of the following into a single logarithm:

(a) $\log 15 + 3\log y - 8\log z$ (b) $-2\ln x + \frac{1}{5}\ln(y+3) + \ln z$

Most calculators are only equipped with buttons for log and ln. In order to evaluate logarithms in a different base, we need to use the **change of base formula**.

Formula 3.3.2 Change of Base Formula

Given a logarithm $\log_a x$, we can convert it into base b with the **change of base formula**:

Example 3.3.5

Write an expression equal to the given logarithm, using logarithms with the specified base.

(a) $\log_3 8$, base $= 5$ (b) $\log_\pi 23$, base $= e$

(a)

(b)

Note:

The change of base formula is necessary if your calculator only has the log and ln buttons.

To evaluate $\log_4 22$, for example, you should type log(22)/log(4) or ln(22)/ln(4).

Classwork Question 3.3.5

Write an expression equal to the given logarithm, using logarithms with the specified base.

(a) $\log_9 13$, base = 2 (b) $\log_{14.2} 3$, base = 10

In this next example, we will go over some common mistakes with logarithms.

Example 3.3.6

For each of the following, explain what is wrong:

(a) $\log(x+y) = \log x + \log y$ (b) $\ln x - \ln y = \dfrac{\ln x}{\ln y}$ (c) $(\log x)^a = a \log x$

For (a), the mistake is:

For (b), the mistake is:

For (c), the mistake is:

3.4 Exponential and Logarithmic Equations

To solve an exponential equation:

1.

2.

Example 3.4.1

Solve each exponential equation:
(a) $3(2^x) - 11 = 13$ (b) $4e^{3x} + 2 = 10$

For (a), we first isolate the exponential:

We can see $x =$ _____ by the ______________________________.

For (b), we first isolate the exponential:

We cancel out the base using _____, and then solve for x:

Note:
You cannot multiply a coefficient into an exponential! For example, $3(2^x) \neq 6^x$.

Classwork Question 3.4.1

Solve each exponential equation:
(a) $5(4^x) - 62 = 18$ (b) $7(10^{2x}) + 8 = 29$

If an exponential equation has two different bases, use a logarithm to cancel out one base, and then use log properties to solve the rest of the equation.

Example 3.4.2

Solve the equation: $3^{x+1} = 7^{-x+2}$

It doesn't matter which base we pick to cancel. Let's cancel out base 3 using __________:

We can move the exponent out of the log using the _______________ rule:

Now we can solve the rest of the equation:

Classwork Question 3.4.2

Solve the equation: $e^{x-1} = 4^{-x-3}$

Example 3.4.3

Solve the equation: $e^{2x} - 3e^{x} + 2 = 0$

First, we will make the substitution $y =$ _____.

Then we replace e^{x} with _____ and e^{2x} with _____ to get _______________. Now we solve the quadratic like usual:

Next, we substitute back out for y and solve each equation:

Classwork Question 3.4.3

Solve the equation: $e^{2x} - 7e^x + 12 = 0$

Example 3.4.4

Solve the equation: $x^2e^{-x} - 2xe^{-x} - 8e^{-x} = 0$

Since _____ appears in each term, we can factor it out:

Thus, we need to solve __________ = 0 and ____________________ = 0.

Since an exponential function _______________, we get no solutions from __________.

Now we solve the other equation:

Classwork Question 3.4.4

Solve the equation: $x^2e^{2x} + 10xe^{2x} + 16e^{2x} = 0$

To solve a logarithmic equation, first condense each side and then exponentiate.

Example 3.4.5

Solve the equation: $\log_4(3x+13) - \log_4 x = 2$

First, we will condense the left hand side into one logarithm using the ____________ rule:

Next, we cancel out the $\log_4$ by ____________________:

Finally, we solve the remaining equation:

Classwork Question 3.4.5

Solve the equation: $\log 2 + \log(x-5) = 1$

Sometimes when we solve a logarithmic equation, we get an **extraneous solution**.

Example 3.4.6

Solve the equation and check for extraneous solutions: $\log_6 x + \log_6(x-5) = 1$.

We first condense the left hand side using the _______________ rule:

Next, we cancel out $\log_6$ by ____________________:

We solve the remaining equation:

To check for extraneous solutions, we plug each x value back into the original equation:

Classwork Question 3.4.6

Solve the equation and check for extraneous solutions: $\log_3(x-8) + \log_3 x = 2$

Example 3.4.7

Solve the equation and check for extraneous solutions: $\log_2(x-6) - \log_2(x-3) = \log_2(x+2)$

We first condense the left hand side using the ______________ rule:

Next, we cancel out $\log_2$ by ___________________:

We solve the remaining equation:

To check for extraneous solutions, we plug each x value back into the original equation:

The only values we found were extraneous, so this equation has ______________.

Classwork Question 3.4.7

Solve the equation and check for extraneous solutions: $\log_4(1-x)-\log_4(x-3)=\log_4(x-1)$

Certain word problems will involve solving an exponential or logarithmic function.

Example 3.4.8

The value V, in dollars, of an investment account grows according to the model $V=5000e^{0.025t}$, where t is the number of years since the initial deposit. Find the number of years required for the investment to reach \$6000.

To find the number of years, we want to solve the equation ______________________.

First we rearrange:

We cancel out e by using _________:

Finally, we solve the remaining equation:

3.5 Exponential and Logarithmic Models

Exponential functions can be used to model many things in the real world. One example has to do with interest on a bank account or loan.

Formula 3.5.1 Compound Interest Formula

The **compound interest formula** models the growth of a bank account or loan that accrues compound interest:

A is the balance in the account or loan.
P is the principal, which is the initial deposit or loan amount.
r is the annual interest rate, written as a decimal.
n is the compounding period, which is the number of times per year the interest is compounded.
t is the length of time, in years.

Common compounding periods are:
Annual - 1 time per year
Semiannual - 2 times per year
Quarterly - 4 times per year
Monthly - 12 times per year
Daily - 365 times per year

Example 3.5.1

You take out a \$3000 loan that earns 2% annual interest, compounded monthly. Write an equation to model the amount of money in you owe on your loan over time. How much money will you owe after 5 years?

The initial loan amount is the principal, so $P =$ _______.

The interest rate is 2%, so $r =$ _______.

Since the compounding is monthly, $n =$ _______. Therefore, our model is:

To get the amount you owe after five years, we plug in $t =$ _______:

Classwork Question 3.5.1

You deposit \$8000 into a bank account that earns 3% annual interest, compounded quarterly. Write an equation to model the amount of money in your bank account over time. How much money will be in your bank account after 9 years?

Example 3.5.2

A bank offers 4% annual interest, compounded daily. How long would it take for your money to double, if you deposited it into this bank? Remember to put units.

Our initial deposit is _____, and we want our final amount to be $A =$ _____.

The interest rate is 4%, so $r =$ _____.

Since the compounding is daily, $n =$ _____. Plugging everything in gives:

Now we solve for t:

Classwork Question 3.5.2

A bank offers 2% annual interest, compounded monthly. How long would it take for your money to double, if you deposited it into this bank? Remember to put units.

Another way for interest to compound is **continuously**.

Formula 3.5.2 Continuous Compounding Formula

The **continuous compounding formula** is:

A is the balance in the account.
P is the principal, which is the initial deposit.
r is the annual interest rate, written as a decimal.
t is the length of time, in years.

Example 3.5.3

The table below shows information about a bank account, with continuous compounding.

Initial Investment	Annual Interest Rate	Time to Double	Amount After 10 Years
?	3.1%	?	\$20,000

Find the missing entries.

The interest rate is 3.1%, so $r =$ _______.

We want to solve for P. From the table, we get $A=$ _______ when $t =$ _______. Plugging these in and solving for P:

To find the time to double, we set $A =$ _______ and solve for t:

Classwork Question 3.5.3

The table below shows information about a bank account, with continuous compounding.

Initial Investment	Annual Interest Rate	Time to Double	Amount After 10 Years
?	5.8%	?	\$70,000

Find the missing entries.

In chemistry, the **half-life** of an isotope of an element is the time required for half of the isotope to decay into a different atom. We can model this using an exponential function.

Formula 3.5.3 Half-life Equation

The **half-life** equation is

Q is the amount remaining
Q_0 is the initial quantity
H is the half-life
t is the amount of time

Example 3.5.4

The half-life of Radium-226 is 1600 years. You start with an initial quantity of 150 grams of Radium-226. Write an equation to model the amount of Radium-226 remaining over time. How much will remain after 2000 years? Remember to put units.

According to the given information, we have Q_0 = __________ and H = __________. Therefore, our model is:

To get the quantity remaining after 2000 years, we plug in t = __________:

Classwork Question 3.5.4

The half-life of Bismuth-210 is 5 days. You start with an initial quantity of 350 grams of Bismuth-210. Write an equation to model the amount of Bismuth-210 remaining over time. How much will remain after 12 days? Remember to put units.

Example 3.5.5

The half-life of Thorium-230 is 75,000 years. You have a sample of 8 grams of Thorium-230, which is estimated to be 200,000 years old. What was the initial quantity? Remember to put units.

According to the given information, we have $H =$ ____________.

We are told that $Q =$ ____________ when $t =$ ____________. Plugging these in and solving for Q_0:

Classwork Question 3.5.5

The half-life of Lead-210 is 22 years. You have a sample of 140 grams of Lead-210, which is estimated to be 50 years old. What was the initial quantity? Remember to put units.

Example 3.5.6

Researchers have modeled the population of a country P, in thousands, from 1971 to 2024 using the equation $P = 81.9e^{0.0381t}$, where t represents the year starting at 1971 (so $t = 1$ represents 1971, $t = 2$ represents 1972, etc).

(a) Use the model to find the population in 2000.
(b) According to this model, when will the population reach 500,000?
(c) Is this model valid for long-term population predictions? Explain.

2000 is represented by the input $t =$ _____, so we plug that in:

To find when the population reaches 500,000, we solve the equation ____________________:

This model __________ valid for long-term population predictions because:

Formula 3.5.4 Finding Exponential Curve Between Two Points

Given two points $(0, a)$ and (x_0, y_0), we can find an exponential curve of the form $f(x) = a \cdot b^x$ or $f(x) = ae^{bx}$ that passes through those two points:

1.

2.

3.

Example 3.5.7

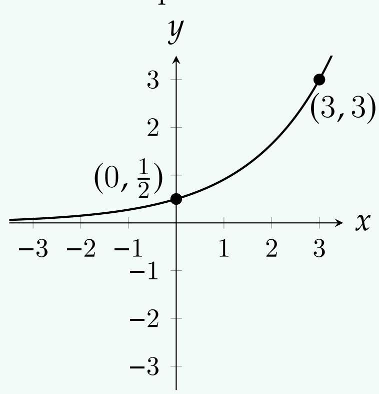
Find an exponential curve of the form $f(x) = a \cdot b^x$ that passes through the points $(0, 3)$ and $(4, 10)$.

We can see that $a =$ _____. Plugging in the other point gives _____ $=$ _______. Now we solve for b:

So our function is $f(x) =$ __________.

Example 3.5.8

Find the equation of the form $f(x) = ae^{bx}$ for the graph below:



Classwork Question 3.5.6

Find an exponential curve that passes through the points $(0, 9)$ and $(6, 2)$.

Chapter 4

Trigonometry

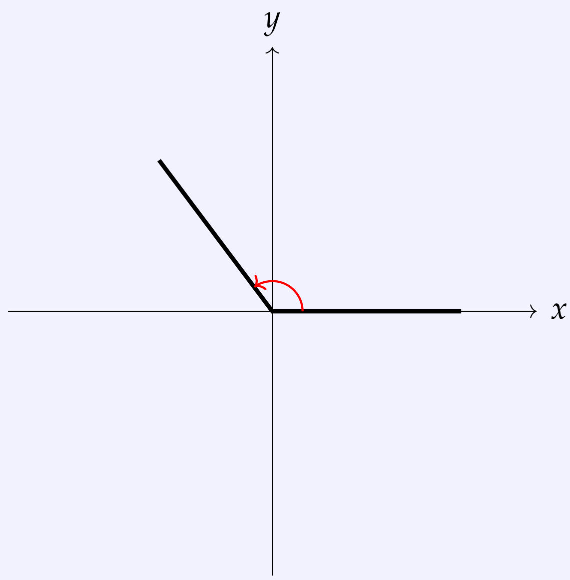
4.1 Radian and Degree Measure

Trigonometry comes from the Greek word that means "measurement of triangles." In modern times, trigonometry refers to the study of angles in general.

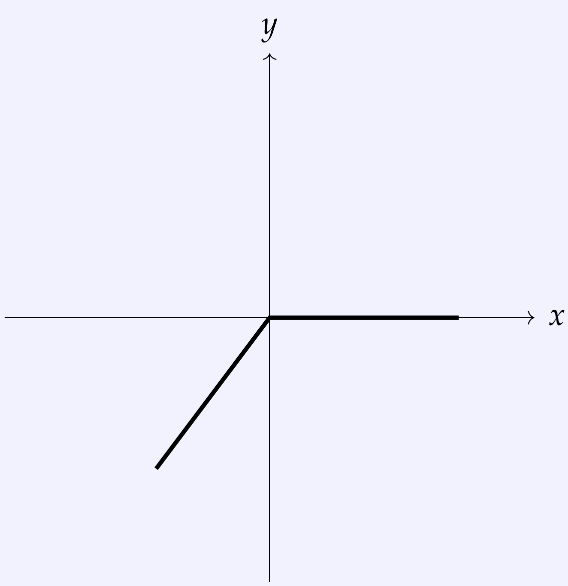
Definition: Angle Vocabulary

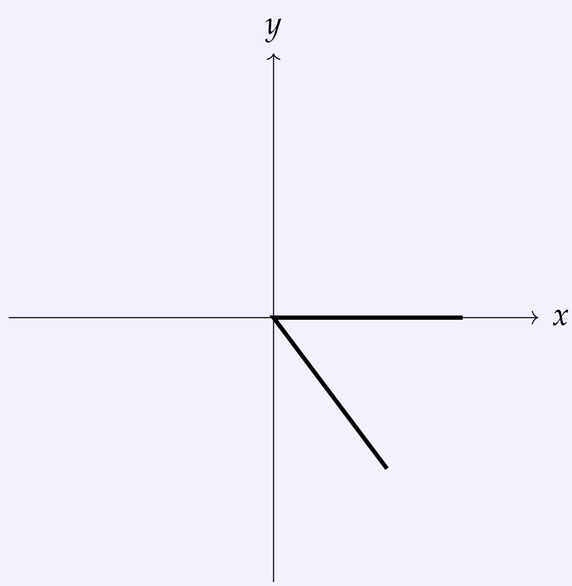
An **angle** consists of two rays with their endpoints at the origin, called the **vertex**. The **initial side** lies on the x-axis, pointing in the positive direction (to the right). The **terminal side** is the rotated ray that forms the angle. This is called **standard position**.

We label angles using Greek letters such at θ (theta), α (alpha), and β (beta).



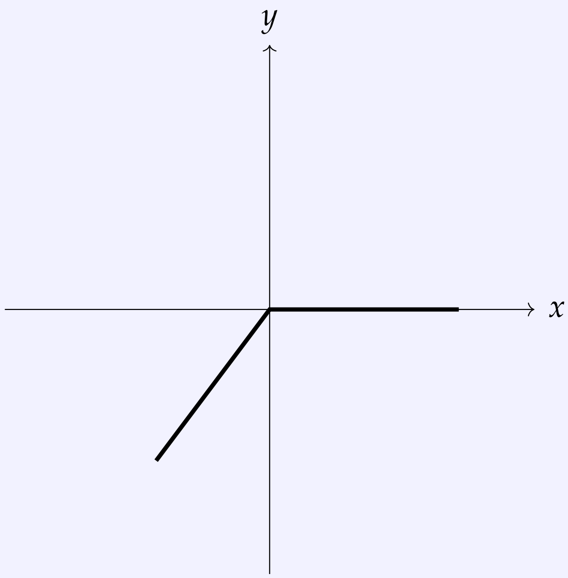
Positive angles go **counterclockwise**. Negative angles go **clockwise**.

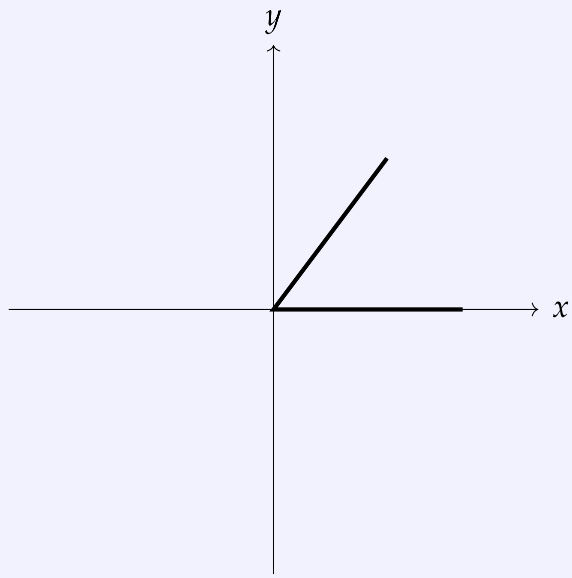




Definition: Angle Vocabulary Continued

Two angles that have the same terminal side are called **coterminal**. Coterminal angles can be made by going clockwise and counterclockwise, or by going a full loop around:

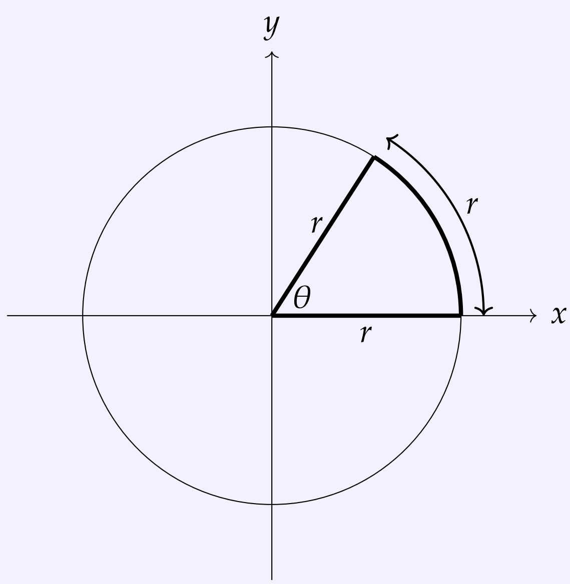




We have two different ways to talk about angles: **radians** and **degrees**.

Definition: Radian

One **radian** is the angle made by traveling the distance of the radius along the circumference of a circle:



Definition: Radian Measure of an Angle

The **radian measure** of an angle describes how far along a circle the angle lies.

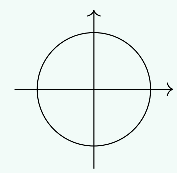
In a full circle, there are _____ radians.

To sketch a radian angle in standard position:

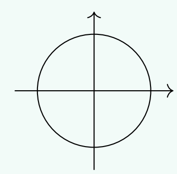
Example 4.1.1

Sketch each angle in standard position. Which quadrant does the angle lie in?

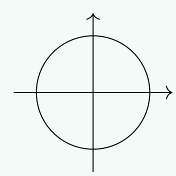
(a) $\dfrac{\pi}{3}$ (b) $-\dfrac{5\pi}{6}$ (c) $\dfrac{9\pi}{4}$



(a) Divide by _____ to get _____. Postive angle means we go __________:



(b) Divide by _____ to get _____. Negative angle means we go __________:

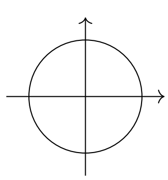


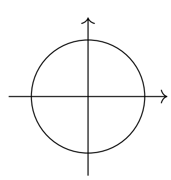
(c) Divide by _____ to get _____. Positive angle means we go __________:

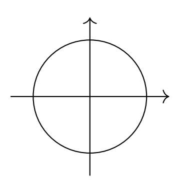
Classwork Question 4.1.1

Sketch each angle in standard position. Which quadrant does the angle lie in?

(a) $\dfrac{3\pi}{4}$ (b) $-\dfrac{2\pi}{3}$ (c) $\dfrac{8\pi}{3}$







Definition: Degree Measure

Degrees are another way to measure angles.

One **degree** is a rotation of ________ around the circle. We denote degrees with a circle $\circ$ in the upper right, such as 30°.

Formula 4.1.1 Radian-Degree Conversion

To convert from radians to degrees:

To convert from degrees to radians:

Example 4.1.2

Convert the following from radians to degrees:

(a) $\frac{2\pi}{3}$ (b) 3π (c) $-\frac{\pi}{2}$

(a)

(b)

(c)

Classwork Question 4.1.2

Convert the following from radians to degrees:

(a) $\frac{\pi}{4}$ (b) 6π (c) $-\frac{\pi}{6}$

Example 4.1.3

Convert the following from degrees to radians:
(a) $270°$ (b) $450°$ (c) $-60°$

(a)

(b)

(c)

Classwork Question 4.1.3

Convert the following from degrees to radians:
(a) $135°$ (b) $720°$ (c) $-120°$

Formula 4.1.2 Coterminal Angles

To obtain coterminal angles for radians:

To obtain coterminal angles for degrees:

Example 4.1.4

Find two coterminal angles for the given angle, one positive and one negative.

(a) $\dfrac{\pi}{3}$ (b) $150°$

For (a), __________ for the positive angle and __________ for the negative angle:

For (b), __________ for the positive angle and __________ for the negative angle:

Classwork Question 4.1.4

Find two coterminal angles for the given angle, one positive and one negative.

(a) $\dfrac{3\pi}{4}$ (b) $60°$

Definition: Complementary and Supplementary Angles

Given an angle α, its **complement** is

Given an angle α, its **supplement** is

If $\alpha >$ _____, it has no complement.

If $\alpha >$ _____, it has no complement and no supplement.

Example 4.1.5

Find the complement and supplement of each angle, if possible:

(a) $\dfrac{\pi}{3}$ (b) $\dfrac{4\pi}{7}$

For (a), to find the complement, we subtract _____ − _____ = _____.

For the supplement, we subtract _____ − _____ = _____.

For (b), is there a complement? ______________________________.

For the supplement, we subtract _____ − _____ = _____.

Classwork Question 4.1.5

Find the complement and supplement of each angle, if possible:

(a) $\frac{\pi}{12}$ (b) $\frac{6\pi}{5}$

Formula 4.1.3 Arc Length Formula

The length of the arc on a circle of radius r and angle θ (in radians) is:

Example 4.1.6

Find the length of the arc on a circle with radius 20 and angle $135°$.

First we convert $135°$ to radians:

Next, we plug into the arc length formula:

Classwork Question 4.1.6

Find the length of the arc on a circle with radius 12 and angle $210°$.

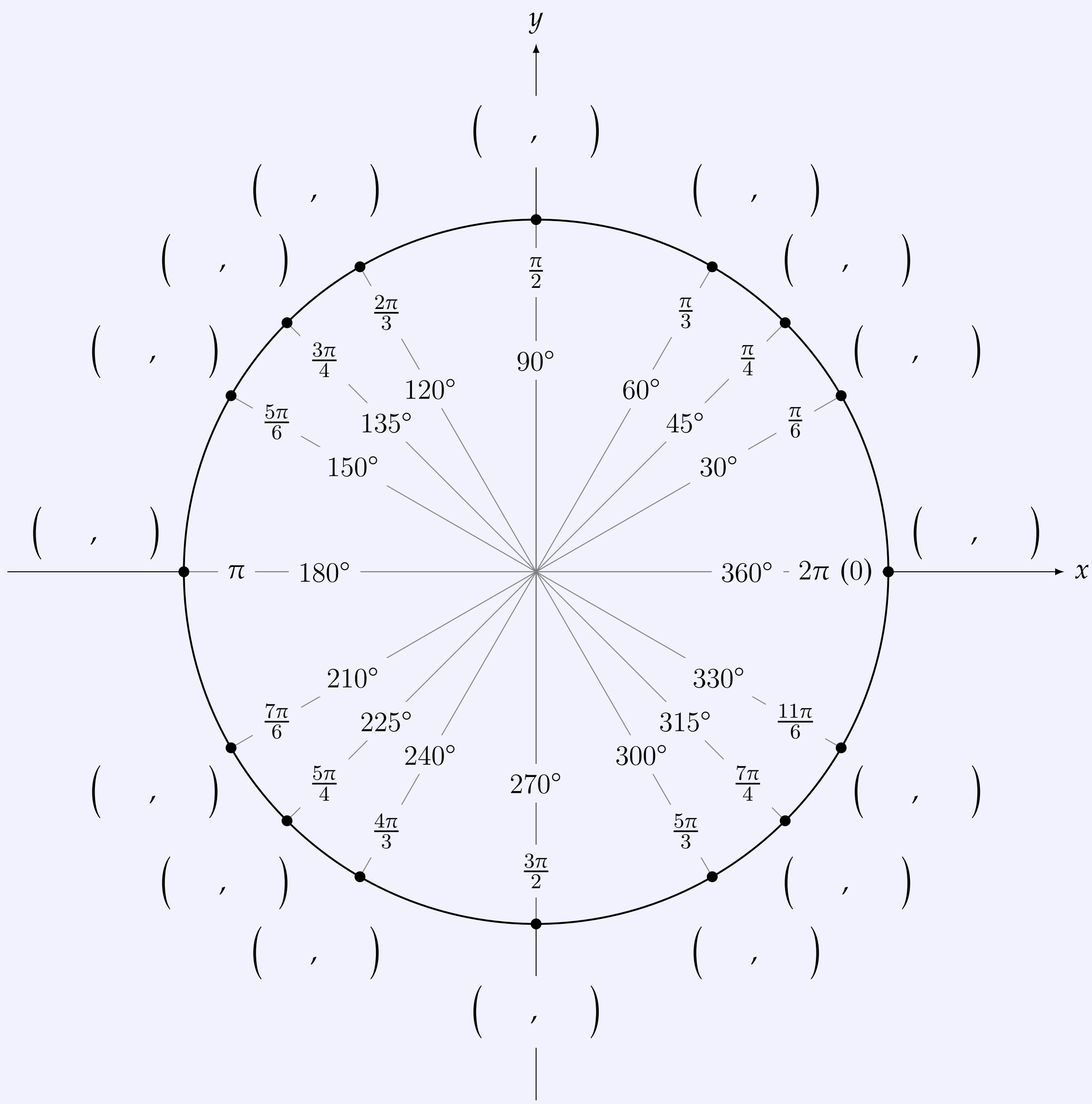
4.2 Trigonometric Functions: The Unit Circle

The unit circle is the key to understanding trigonometry. Almost every problem will reference or utilize the unit circle in some way.

Definition: Unit Circle

The **unit circle** is the circle of radius 1 centered at the origin. Its equation is:

There are some special angles and points that we will reference on the unit circle:



Using the unit circle, we can define the six trigonometric functions:

Definition: Trigonometric Functions

Given an angle θ with the corresponding point (x, y) on the unit circle, the six trigonometric functions are:

Sine: $\sin\theta =$

Cosine: $\cos\theta =$

Tangent: $\tan\theta =$

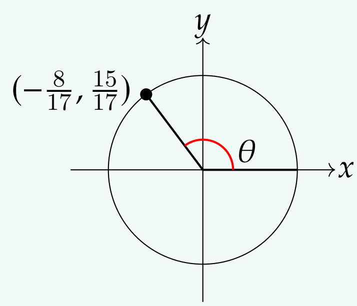
Cosecant: $\csc\theta =$

Secant: $\sec\theta =$

Cotangent: $\cot\theta =$

Example 4.2.1

Evaluate each trig function for the angle shown below:



We have $x =$ _____ and $y =$ _____, so:

$\sin\theta =$

$\cos\theta =$

$\tan\theta =$

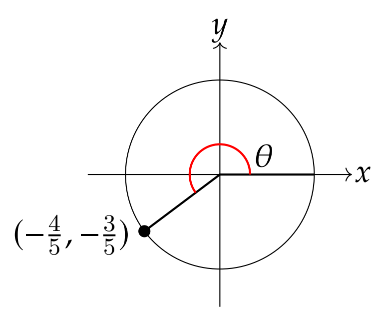
$\csc\theta =$

$\sec\theta =$

$\cot\theta =$

Classwork Question 4.2.1

Evaluate each trig function for the angle shown below:



To evaluate the trig functions without being given the point, we have to reference the unit circle.

Example 4.2.2

Evaluate the six trig functions for $\theta = \frac{\pi}{6}$.

Looking at the unit circle, $\frac{\pi}{6}$ is associated with the point ________. We can then evaluate each trig function using this point:

$\sin\theta =$ $\csc\theta =$

$\cos\theta =$ $\sec\theta =$

$\tan\theta =$ $\cot\theta =$

Classwork Question 4.2.2

Evaluate the six trig functions for $\theta = \frac{5\pi}{3}$.

Remember that we cannot divide by 0, so some trig functions are undefined at certain values.

Example 4.2.3

Evaluate the six trig functions for $\theta = \frac{3\pi}{2}$.

Looking at the unit circle, $\frac{3\pi}{2}$ is associated with the point ________. We can then evaluate each trig function using this point:

$\sin\theta =$ $\csc\theta =$

$\cos\theta =$ $\sec\theta =$

$\tan\theta =$ $\cot\theta =$

Classwork Question 4.2.3

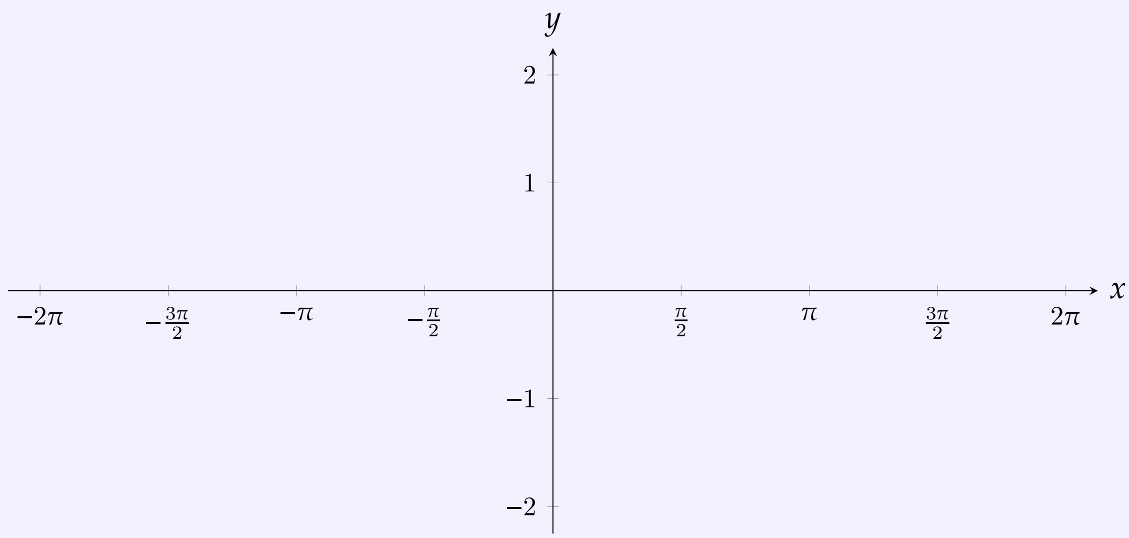
Evaluate the six trig functions for $\theta = \pi$.

Definition: Sine Function

The **sine function** is

Domain: Range:

x-intercepts: y-intercept: Even / Odd / Neither

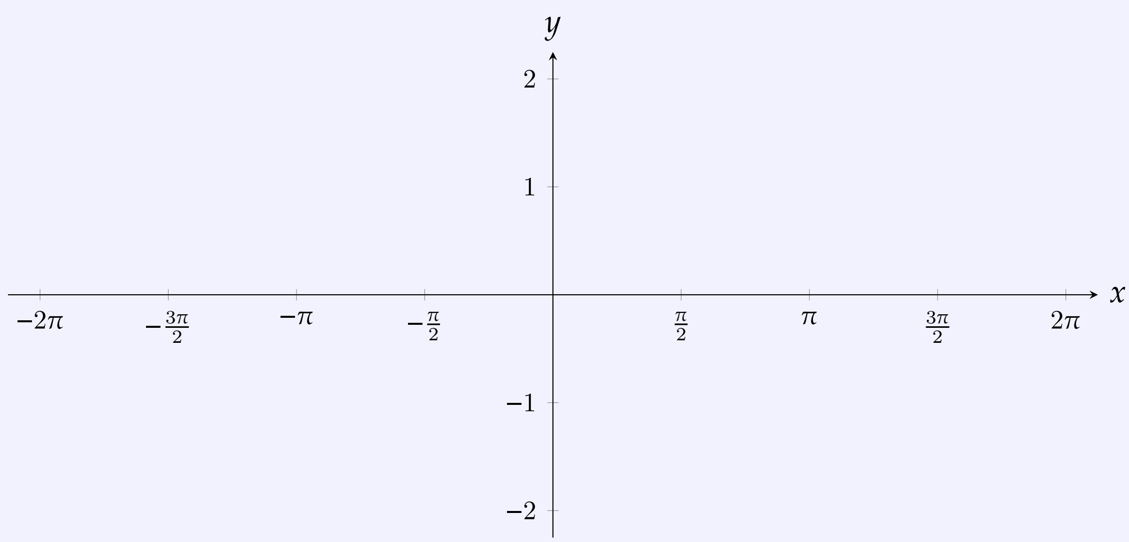


Definition: Cosine Function

The **cosine function** is

Domain: Range:

x-intercepts: y-intercept: Even / Odd / Neither



The graphs of sine and cosine are **periodic**, which means they repeat over and over with a regular pattern.

Definition: Periodic Function

A function $f(x)$ is called **periodic** if

The number T is called the **period**. This is how often the function repeats itself.

The period of sin and cos is _____.

Using the periodicity of sin and cos can help us evaluate the functions. We can keep adding/-subtracting 2π until our input angle is in the interval $[0, 2\pi]$.

Example 4.2.4

Evaluate each of the following:

(a) $\sin\left(\frac{11\pi}{4}\right)$ (b) $\cos\left(-\frac{13\pi}{2}\right)$

For (a), we will subtract _____ until our angle is between 0 and 2π:

The angle _____ corresponds to the point _____, so $\sin\left(\frac{11\pi}{4}\right) =$ _____

For (b), we will add _____ until our angle is between 0 and 2π:

The angle _____ corresponds to the point _____, so $\cos\left(-\frac{13\pi}{2}\right) =$ _____

Classwork Question 4.2.4

Evaluate each of the following:

(a) $\cos\left(\frac{25\pi}{6}\right)$ (b) $\sin\left(-\frac{17\pi}{3}\right)$

We will look at the properties and graphs of the other trig functions later, but for right now, we can define each of them in terms of sine and cosine.

Formula 4.2.1 Trig Functions in Terms of Sine and Cosine

We can define the four other trig functions in terms of sine and cosine as follows:

$\csc\theta =$

$\tan\theta =$

$\sec\theta =$

$\cot\theta =$

Example 4.2.5

Evaluate each of the following, if possible.

(a) $\csc\left(\dfrac{\pi}{4}\right)$ (b) $\sec(0)$ (c) $\tan\left(\dfrac{\pi}{2}\right)$ (d) $\cot\left(\dfrac{\pi}{2}\right)$

For (a), we use the fact that $\csc\theta =$ __________. Then we evaluate:

For (b), we use the fact that $\sec\theta =$ __________. Then we evaluate:

For (c), we use the fact that $\tan\theta =$ __________. Then we evaluate:

For (d), we use the fact that $\cot\theta =$ __________. Then we evaluate:

Classwork Question 4.2.5

Evaluate each of the following, if possible.

(a) $\cot\left(\dfrac{7\pi}{4}\right)$ (b) $\sec(\pi)$ (c) $\tan\left(\dfrac{\pi}{3}\right)$ (d) $\csc(\pi)$

Recall that sin is odd and cos is even. This can help us when we want to find certain values, given some information.

Example 4.2.6

(a) Suppose $\sin(-\theta) = -\frac{8}{15}$. Find $\sin\theta$ and $\csc\theta$.

(b) Suppose $\cos(-\theta) = \frac{14}{17}$. Find $\cos\theta$ and $\sec\theta$.

Since sin is _____, we know that $\sin(-\theta)$ = __________. Using this, we can find the value of $\sin\theta$:

Since $\sin\theta$ = _____, and $\csc\theta$ = __________, we know $\csc\theta$ = _____.

Since cos is _____, we know that $\cos(-\theta)$ = __________. Using this, we can find the value of $\cos\theta$:

Since $\cos\theta$ = _____, and $\sec\theta$ = __________, we know $\sec\theta$ = _____.

Classwork Question 4.2.6

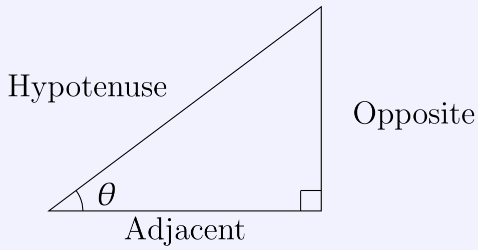
(a) Suppose $\sin(-\theta) = \frac{13}{44}$. Find $\sin\theta$ and $\csc\theta$

(b) Suppose $\cos(-\theta) = -\frac{4}{19}$. Find $\cos\theta$ and $\sec\theta$

4.3 Right Triangle Trigonometry

Definition: Trig Functions (Right Triangle)

Consider the right triangle:



The angle θ will always touch the hypotenuse (Hyp). The side also touching θ is the adjacent side (Adj). The side across from θ is the opposite side (Opp). The six trig functions defined for the right triangle are:

$\sin\theta =$

$\csc\theta =$

$\cos\theta =$

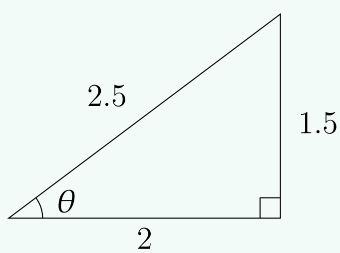
$\sec\theta =$

$\tan\theta =$

$\cot\theta =$

Example 4.3.1

Evaluate all six trig functions for the triangle below:



First we'll label each side. Then, we can evaluate:

$\sin\theta =$

$\csc\theta =$

$\cos\theta =$

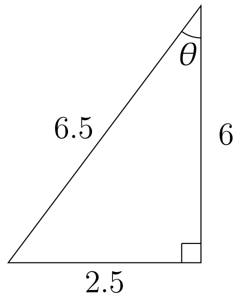
$\sec\theta =$

$\tan\theta =$

$\cot\theta =$

Classwork Question 4.3.1

Evaluate all six trig functions for the triangle below:



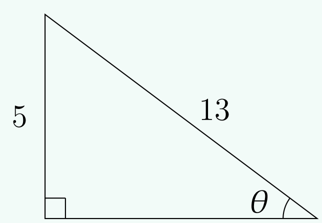
Note:

If we were to double the lengths of all the sides in the triangles above, the values of the trig functions won't change. This is because the doubled triangle is **similar** to the original triangle, meaning their side lengths are proportional.

Sometimes we need to use the Pythagorean Theorem to fill in a missing side of a right triangle.

Example 4.3.2

Evaluate all six trig functions for the right triangle shown below:



In the Pythagorean Theorem, we have $a =$ _____ and $c =$ _____. Plugging in and solving:

Now we can evaluate the trig functions:

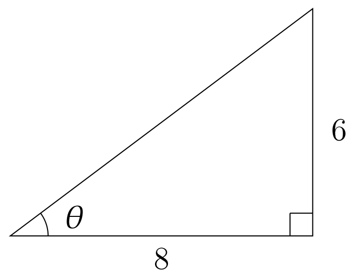
$\sin\theta =$ $\csc\theta =$

$\cos\theta =$ $\sec\theta =$

$\tan\theta =$ $\cot\theta =$

Classwork Question 4.3.2

Evaluate all six trig functions for the right triangle shown below:



Sometimes, you will need to draw the triangle yourself.

Example 4.3.3

Suppose for an acute angle θ, that $\cos\theta = \frac{8}{17}$. Sketch a right triangle and use it to evaluate all five other trig functions.

First we'll sketch a right triangle and label θ:

Since $\cos\theta = \frac{8}{17}$, we know the __________ side is _____, and the __________ is _____.

Now we plug $a =$ _____ and $c =$ _____ into the Pythagorean Theorem and solve:

Now we can evaluate the trig functions:

$\sin\theta =$	$\csc\theta =$
$\cos\theta =$	$\sec\theta =$
$\tan\theta =$	$\cot\theta =$

Classwork Question 4.3.3

Suppose for an acute angle θ, that $\sin\theta = \frac{12}{13}$. Sketch a right triangle and use it to evaluate all five other trig functions.

Example 4.3.4

Find the value of each acute angle (ie, $0 < \theta < \frac{\pi}{2}$).

(a) $\sin\theta = \dfrac{\sqrt{3}}{2}$ (b) $\sec\theta = \dfrac{2\sqrt{3}}{3}$ (c) $\tan\theta = 1$

Because our angle needs to be acute, we look for angles in Quadrant _____.

For (a), we need _____ = _____, which corresponds with $\theta =$ _____.

For (b), we use the fact that $\sec\theta =$ _____.

Thus, we need _____ = _____ = _____, which corresponds with $\theta =$ _____.

For (c) we use the fact that $\tan\theta =$ _____.

Thus, we need _____ = _____, which corresponds with $\theta =$ _____.

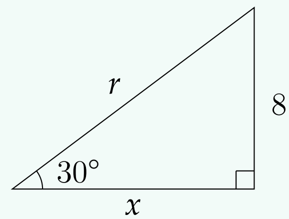
Classwork Question 4.3.4

Find the value of each acute angle (ie, $0 < \theta < \frac{\pi}{2}$).

(a) $\cos\theta = \dfrac{\sqrt{2}}{2}$ (b) $\csc\theta = 2$ (c) $\cot\theta = \sqrt{3}$

Example 4.3.5

Find the exact values of x and r in the right triangle below:



We need to relate each unknown side to the known side and the known angle using the appropriate trig function.

From the marked angle, 8 is the ______________ and x is the ______________.

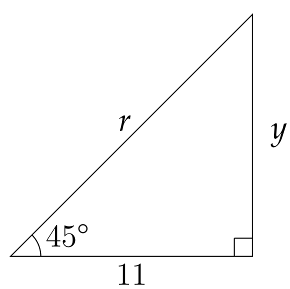
The trig function that relates these two is _____. Plugging in and solving for x:

From the marked angle, 8 is the ______________ and r is the ______________.

The trig function that relates these two is _____. Plugging in and solving for r:

Classwork Question 4.3.5

Find the exact values of y and r in the right triangle below:



4.4 Trigonometric Functions of Any Angle

Definition: Trig Functions of Any Angle

If the point (x, y) lies on the terminal side of the angle θ, we define the quantity r as:

Then the six trig functions of the angle θ associated with (x, y) are:

$\sin\theta =$ $\qquad$ $\tan\theta =$ $\qquad$ $\sec\theta =$

$\cos\theta =$ $\qquad$ $\csc\theta =$ $\qquad$ $\cot\theta =$

Example 4.4.1

The point $(-12, 5)$ lies on the terminal side of angle θ. Evaluate all six trig functions of θ.

We calculate $r =$ ______________ $=$ ______________ $=$ ______________ $=$ _____.

Then the six trig functions are:

$\sin\theta =$ $\qquad$ $\tan\theta =$ $\qquad$ $\sec\theta =$

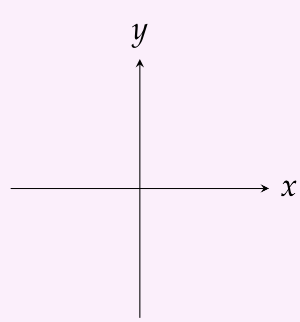
$\cos\theta =$ $\qquad$ $\csc\theta =$ $\qquad$ $\cot\theta =$

Classwork Question 4.4.1

The point $(6, -8)$ lies on the terminal side of angle θ. Evaluate all six trig functions of θ.

Depending on the quadrant where an angle lies, certain trig functions will be positive.

Formula 4.4.1 Where Trig Functions are Positive



Example 4.4.2

Determine the quadrant in which θ lies:

(a) $\sin\theta > 0,\ \cos\theta < 0$ (b) $\cot\theta > 0,\ \sec\theta > 0$

For (a), $\sin\theta > 0$ means we are in Quadrant _____ or _____. $\cos\theta < 0$ means we are in Quadrant _____ or _____. The overlap is the correct answer, Quadrant _____.

For (b), since $\cot\theta =$ _____ and $\sec\theta =$ _____, we know __________ and __________.

Thus, __________ means we are in Quadrant _____ or _____. __________ means we are in Quadrant _____ or _____. The overlap is the correct answer, Quadrant _____.

Classwork Question 4.4.2

Determine the quadrant in which θ lies:

(a) $\tan\theta < 0,\ \cos\theta > 0$ (b) $\csc\theta < 0,\ \sec\theta < 0$

Example 4.4.3

Evaluate all six trig functions using the given information: $\sin\theta = \frac{3}{5},\ \cos\theta < 0$

Since sin is positive and cos is negative, θ lies in Quadrant _____.

Since $\sin\theta = \frac{3}{5}$, we know __________ and __________. Solving for _____:

Since we are in Quadrant _____, we know to take the _____ case. Thus, the six trig functions are:

$\sin\theta =$	$\tan\theta =$	$\sec\theta =$
$\cos\theta =$	$\csc\theta =$	$\cot\theta =$

Classwork Question 4.4.3

Evaluate all six trig functions using the given information: $\tan\theta = \frac{5}{12}$, $\sin\theta < 0$

Example 4.4.4

Find two different solutions to each of the following, with $0 \le \theta < 2\pi$.

(a) $\cos\theta = \dfrac{1}{2}$ (b) $\cot\theta = 1$

For (a), we want _____ $= \frac{1}{2}$ in the unit circle.

This corresponds with the two angles _____ and _____.

For (b), recall that $\cot\theta =$ __________. Thus, we want _______________ in the unit circle.

This corresponds with the two angles _____ and _____.

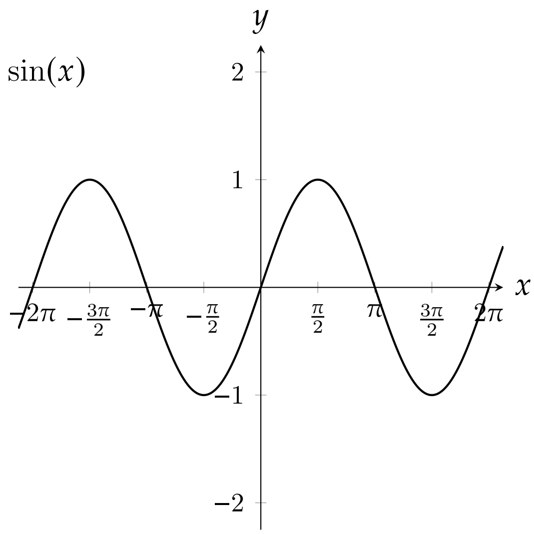
Classwork Question 4.4.4

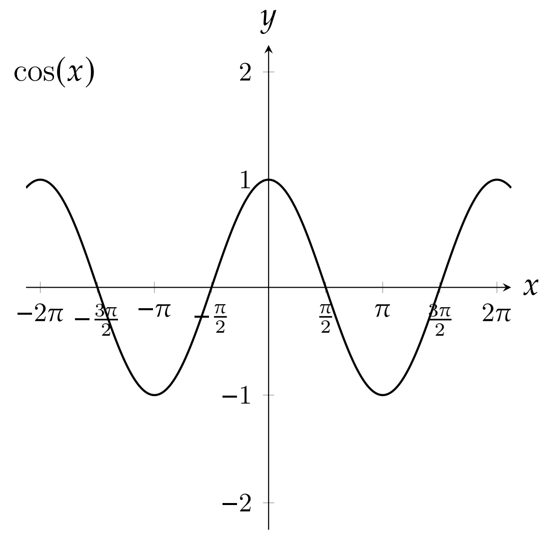
Find two different solutions to each of the following, with $0 \le \theta < 2\pi$.

(a) $\sin\theta = -\dfrac{\sqrt{3}}{2}$ (b) $\sec\theta = \sqrt{2}$

4.5 Graphs of Sine and Cosine

Recall the graphs of $\sin(x)$ and $\cos(x)$:





The **period** of a sine/cosine graph is ______________________________________.

Sine and cosine both have a period of _____.

The **midline** of a sine/cosine graph is ______________________________________.

Sine and cosine both have the midline __________.

The **amplitude** of a sine/cosine graph is ______________________________________.

The amplitude is also ______________________________________.

Sine and cosine both have an amplitude of _____.

Definition: Transformed Sine/Cosine Equation

The equations of the transformed sine/cosine functions are:

The **phase shift** of a transformed sine/cosine function is _________________________.

Formula 4.5.1 Period, Midline, Amplitude, and Phase Shift (From Transformed Equation)

Given the transformed equation $y = d + a\sin(bx - c)$ or $y = d + a\cos(bx - c)$:

Period = _____ Midline = _____ Amplitude = _____ Phase Shift = _____

Formula 4.5.2 Sketching the Graph of a Transformed Sine/Cosine Function

To sketch the graph of $y = d + a\sin(bx - c)$ or $y = d + a\cos(bx - c)$:

1.

2.

3.

Example 4.5.1

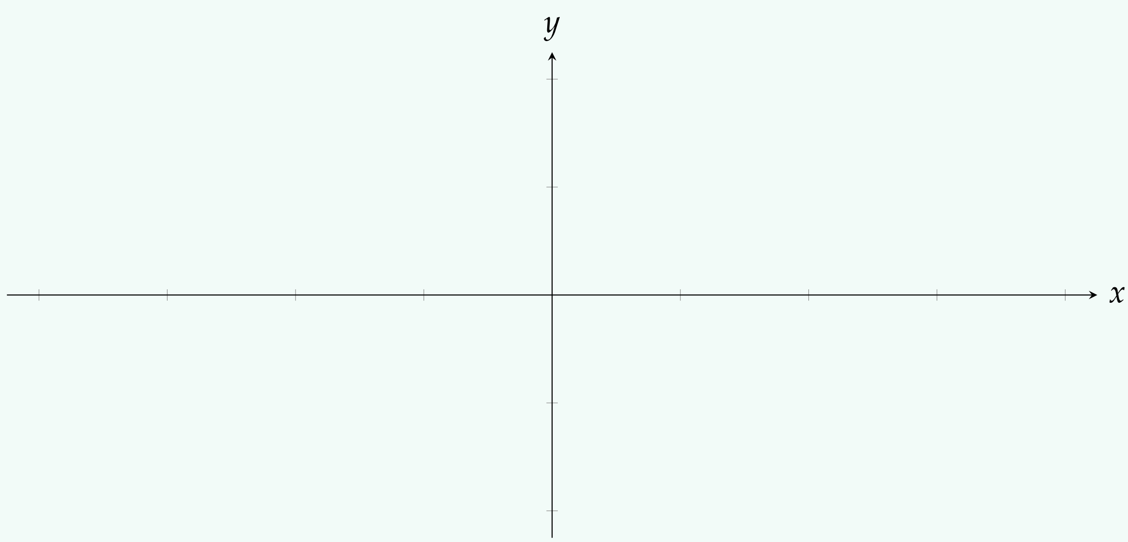
Find the period, midline, amplitude, and phase shift of the transformed sine function, and sketch a graph of it: $f(x) = 1 - \frac{1}{2}\sin(\pi x - \pi)$. Include at least two full periods in your sketch.

Period = ____ Midline = ____ Amplitude = ____ Phase Shift = ____

We can see that $f(x)$ completes one cycle over the interval ______.

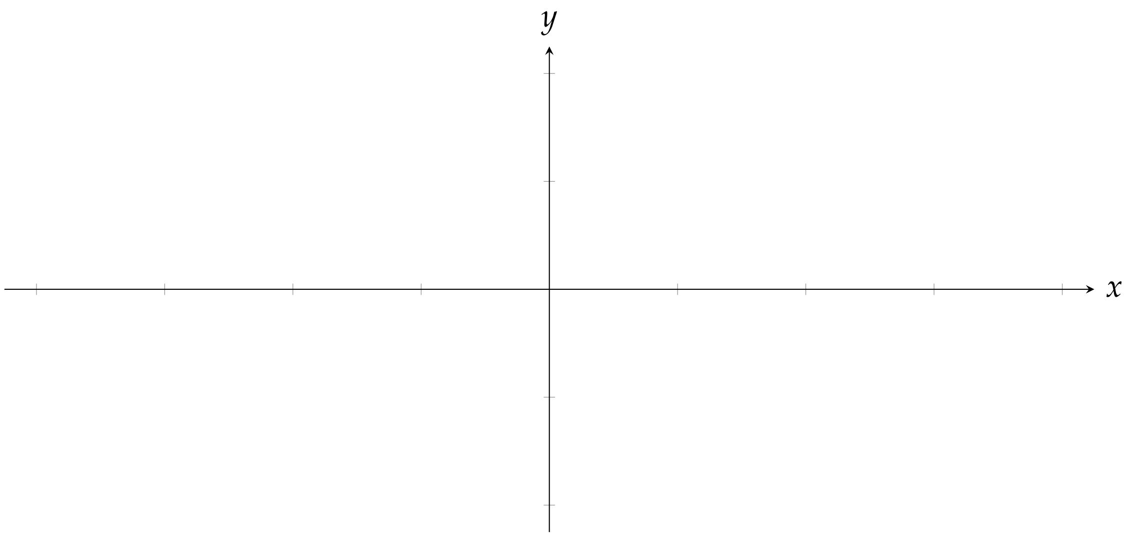
We need to apply the following transformations to the basic shape of $\sin(x)$:

Now we can sketch our graph: Now we sketch the graph:



Classwork Question 4.5.1

Find the period, midline, amplitude, and phase shift of the transformed sine function, and sketch a graph of it: $-\frac{1}{2} + \frac{3}{2}\sin(2\pi x + \frac{\pi}{2})$. Include at least two full periods in your sketch.



Example 4.5.2

Write the equation of a transformed cosine curve with an amplitude of 2, a period of 4π, a left phase shift of $\frac{\pi}{2}$, and shifted down 3 units.

From the given information, we can determine $a =$ _____ and $d =$ _____.

Since the period is 4π, we solve for b using the __________ equation:

Since the phase shift is $\frac{\pi}{2}$, we solve for c using the __________ equation:

Since the phase shift is left, we put a $+$ / $-$ on _____.

Thus, the transformed cosine curve is _______________

Classwork Question 4.5.2

Write the equation of a transformed cosine curve with an amplitude of 3, a period of 2, a right phase shift of 1, and shifted up 7 units.

Formula 4.5.3 Midline and Amplitude (From Graph)

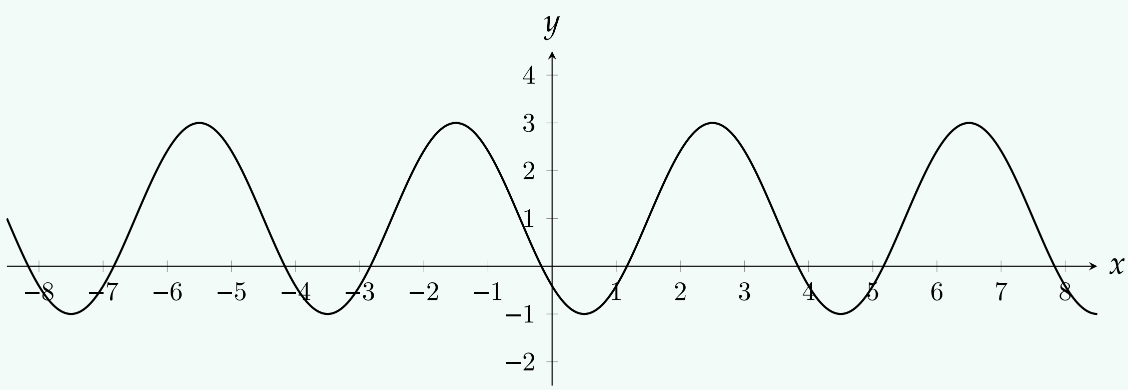
Given the graph of a transformed sine/cosine curve:

Amplitude =

Midline =

Example 4.5.3

The graph below depicts a graph of a function of the form $f(x) = d + a\cos(bx - c)$. Write the equation of the function.



We can see $y_{\max} =$ _____ and $y_{\min} =$ _____. Thus, $|a| =$ _____ and $d =$ _____.

Next, identify an interval where the graph completes one period, such as _____. Using the period equation, we can solve for b:

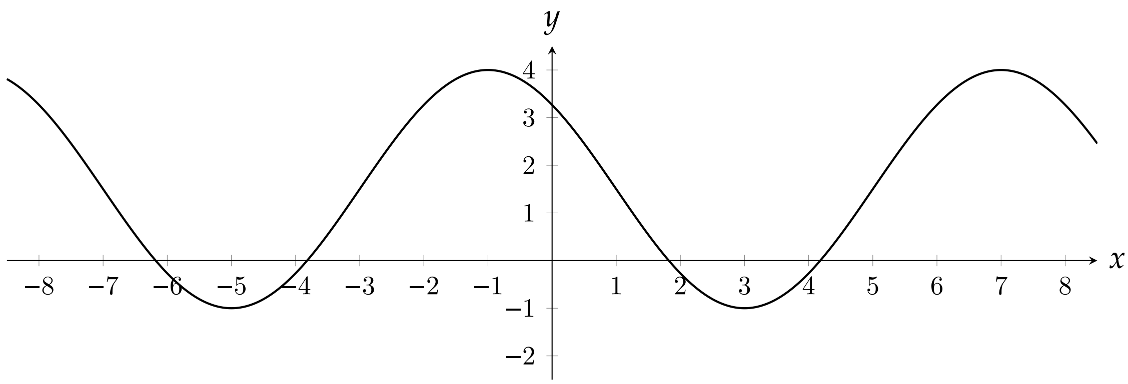
Our chosen interval is shifted __________. Solving for c using the phase shift:

To determine if a is + or −, we plug in any point and solve. Let's plug in _______:

Thus the equation is _________________________.

Classwork Question 4.5.3

The graph below depicts a graph of a function of the form $f(x) = d + a\cos(bx - c)$. Write the equation of the function.



Example 4.5.4

The number of prey fish in a lake follows a transformed cosine function. The maximum number of fish is 10,000, which occurs on day 5. The minimum number of fish is 4,000, which occurs on day 25. Write a function $f(t)$ that models the number of fish in thousands, where t is the number of days.

We can see $y_{\max} =$ _____ and $y_{\min} =$ _____. Thus, $|a| =$ _____ and $d =$ _____.

Between the maximum and minimum, _____ days pass. This is _____ a period, so the period is _____. Using this, we can solve for b:

Since the maximum occurs on day 5, the phase shift is _______________. Solving for c:

To determine if a is positive or negative, we can plug in the point _______ and solve:

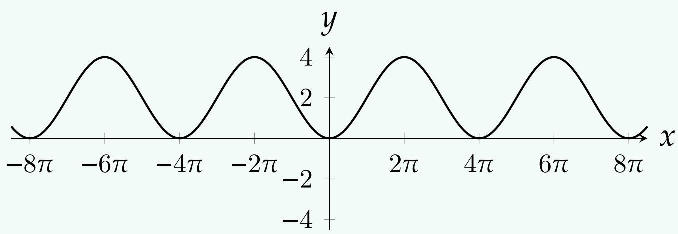
Thus the equation is _________________________.

Classwork Question 4.5.4

The depth of the tides at a beach follows a transformed cosine function. The minimum depth is 2 feet, which occurs after 3 hours. The maximum depth is 18 feet, which occurs after 11 hours. Write a function $f(t)$ that models the depth of the tides, where t is the number of hours.

Example 4.5.5

State the amplitude, period, and vertical shift of the transformed sine function $f(x)$ below. Using the graph, evaluate $f(10\pi)$.

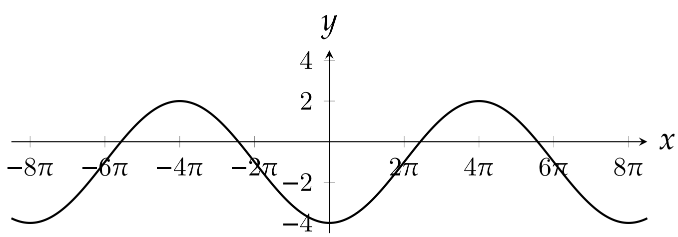


We can see $y_{\text{max}} =$ _____ and $y_{\text{min}} =$ _____, and the graph repeats every _____ units. So:

Since the period is _____, we can deduce that $f(10\pi) = f($_____$) =$ _____.

Classwork Question 4.5.5

State the amplitude, period, and vertical shift of the transformed cosine function $f(x)$ below. Using the graph, evaluate $f(12\pi)$.



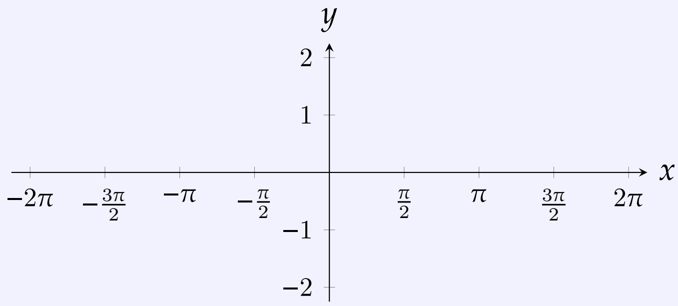
4.6 Graphs of Other Trigonometric Functions

Definition: Tangent Function

The **tangent function** is

Domain: Range: Vertical Asymptotes:

x-intercepts: y-intercept: Even / Odd / Neither



The period of tangent is ____.

Formula 4.6.1 Period and Vertical Asymptotes of a Transformed Tangent Function

Given a transformed tangent function $y = d + a\tan(bx - c)$:

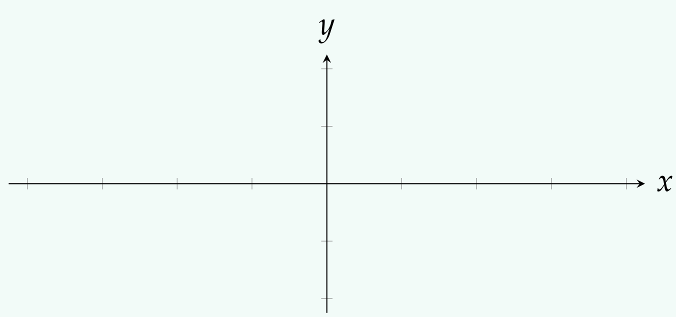
Period = Vertical Asymptotes =

Example 4.6.1

Find the period of $f(x) = \tan(2x)$. Find its vertical asymptotes and sketch a graph.

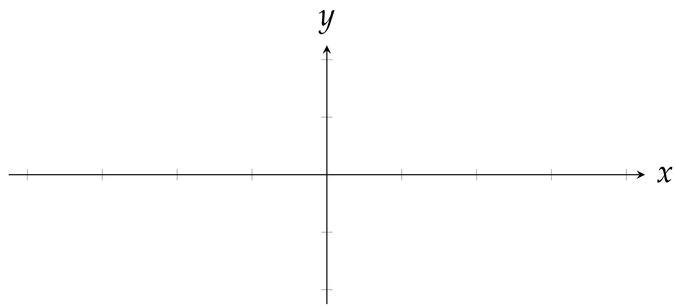
The period is ____. The vertical asymptotes are:

Now we sketch:



Classwork Question 4.6.1

Find the period of $f(x) = \tan(\frac{1}{2}x)$. Find its vertical asymptotes and sketch a graph.

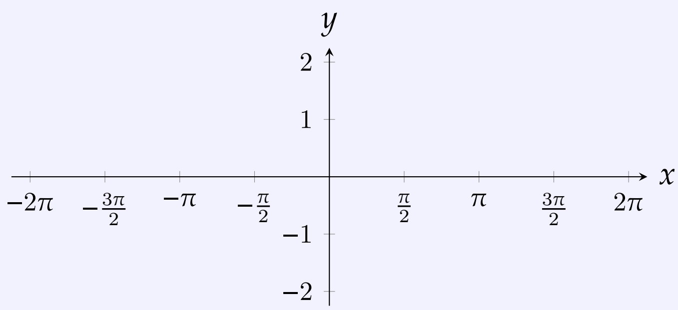


Definition: Secant Function

The **secant function** is

Domain: Range: Vertical Asymptotes:

x-intercepts: y-intercept: Even / Odd / Neither

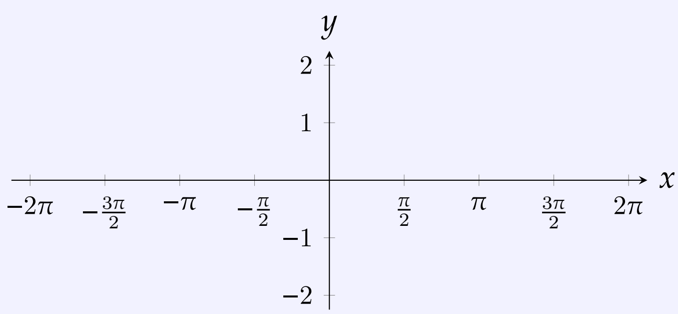


Definition: Cosecant Function

The **cosecant function** is

Domain: Range: Vertical Asymptotes:

x-intercepts: y-intercept: Even / Odd / Neither

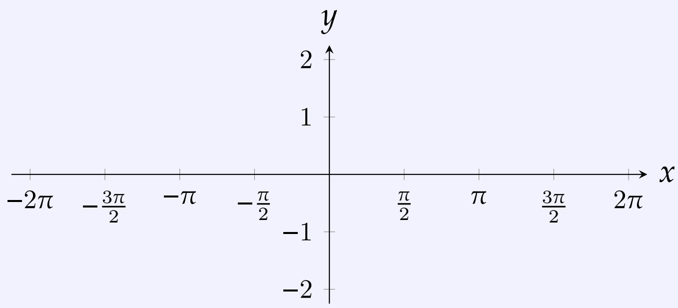


Definition: Cotangent Function

The **cotangent function** is

Domain: Range: Vertical Asymptotes:

x-intercepts: y-intercept: Even / Odd / Neither



The period of secant and cosecant is _____.

The period of cotangent is _____.

Formula 4.6.2 Period and Vertical Asymptotes of a Transformed Secant Function

Given a transformed secant function $y = d + a\sec(bx - c)$:

Period = Vertical Asymptotes =

Formula 4.6.3 Period and Vertical Asymptotes of a Cosecant Tangent Function

Given a transformed cosecant function $y = d + a\csc(bx - c)$:

Period = Vertical Asymptotes =

Formula 4.6.4 Period and Vertical Asymptotes of a Transformed Cotangent Function

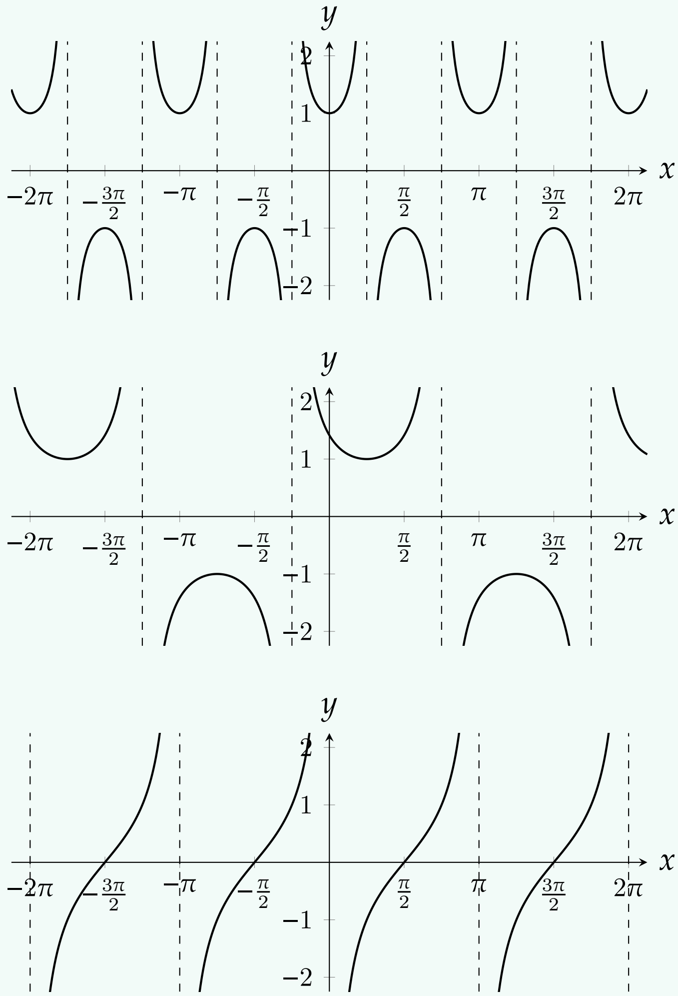
Given a transformed cotangent function $y = d + a\cot(bx - c)$:

Period = Vertical Asymptotes =

Example 4.6.2

Match each function below to its graph.

$f(x) = -\cot(x) \qquad g(x) = \sec(2x) \qquad h(x) = \csc(x + \frac{\pi}{4})$



Instead of trying to plot each function on our own and match the graphs, let's look at the key characteristics of each graph and function, and see if we can match them up that way.

Of the functions sec, csc, and cot, only __________ has x-intercepts.

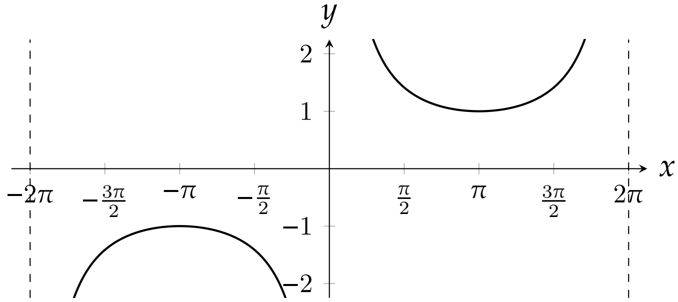
$\sec(2x)$ has a y-intercept at _______.

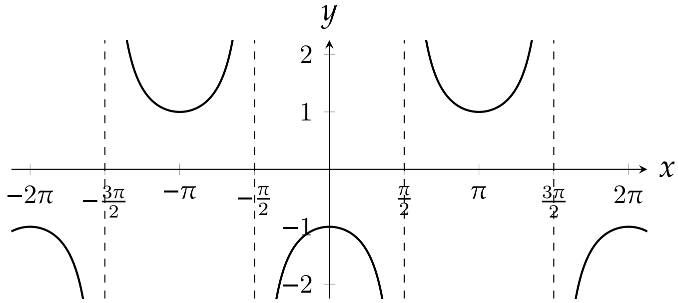
$\csc(x + \frac{\pi}{4})$ has vertical asymptotes at $x =$ __________.

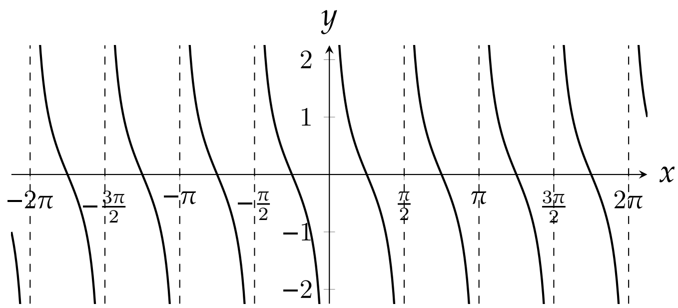
Classwork Question 4.6.2

Match each function below to its graph.

$f(x) = -\sec(x)$ $\quad$ $g(x) = \cot(2x)$ $\quad$ $h(x) = \csc(\frac{1}{2}x)$







Example 4.6.3

Find all solutions to $\tan(x) = \sqrt{3}$ in the interval $[-2\pi, 2\pi]$.

First, let's find the positive solutions. They are:

For each positive solution, we can find a coterminal angle by _______________:

Classwork Question 4.6.3

Find all solutions to $\sec(x) = \sqrt{2}$ in the interval $[-2\pi, 2\pi]$.

Example 4.6.4

A bird is flying west 120 feet above the ground. You are standing on the ground, looking up at the bird as it flies by.

(a) Draw a diagram to represent the situation. Let x represent the angle of elevation and d represent the ground distance between you and the bird.
(b) Write d as a function of x.

Our diagram is:

The trig function that relates the known side and d is ________. Solving for d:

Classwork Question 4.6.4

An airplane is flying west 10 miles above the ground. You are standing on the ground, look up at the plane as it flies by.

(a) Draw a diagram to represent the situation. Let x represent the angle of elevation and d represent the ground distance between you and the plane.
(b) Write d as a function of x.

4.7 Inverse Trigonometric Functions

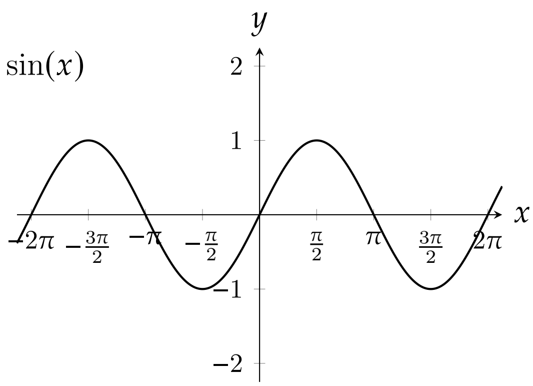
None of the trig functions we have studied have an inverse. Why?

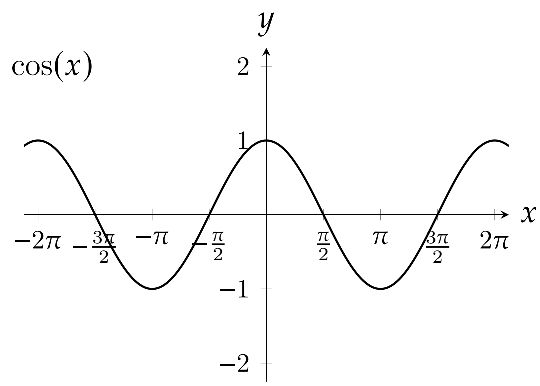
So if trig functions don't have inverses, is there no way to "undo" them? There is a way! We can define a **partial inverse**.

Definition: Partial Inverse

If f is a function that isn't one-to-one, then a **partial inverse** of f is

For sin and cos, we can restrict the domains so that the functions are one-to-one:





Domain restriction for sin:

Domain restriction for cos:

Definition: Inverse Sine Function

The **inverse sine function** is

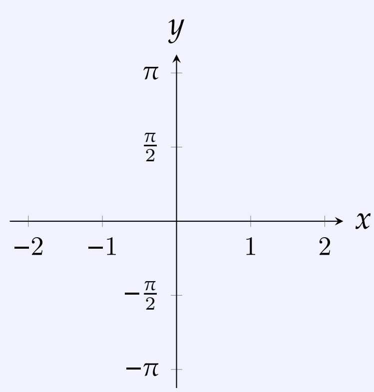
Domain:

Range:

x-intercepts:

y-intercept:

Even / Odd / Neither

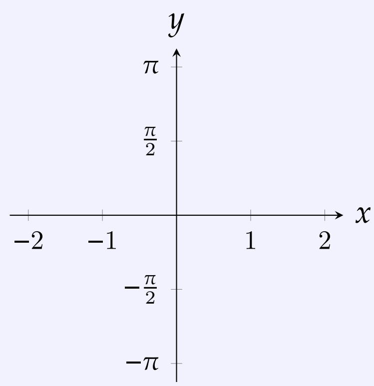


Definition: Inverse Cosine Function

The **inverse cosine function** is

Domain: Range:

x-intercepts: y-intercept: Even / Odd / Neither



Remember that the output of an inverse trig function is an angle.

Example 4.7.1

Evaluate each of the following, if possible:

(a) $\sin^{-1}(\frac{1}{2})$ (b) $\arccos(0)$ (c) $\arcsin(-1)$ (d) $\cos^{-1}(2)$

For (a), what angle θ in ________ gives us ________________? The answer is ___________.

For (b), what angle θ in ________ gives us ________________? The answer is ___________.

For (c), what angle θ in ________ gives us ________________? The answer is ___________.

For (d), what angle θ in ________ gives us ________________? The answer is ___________.

Note:

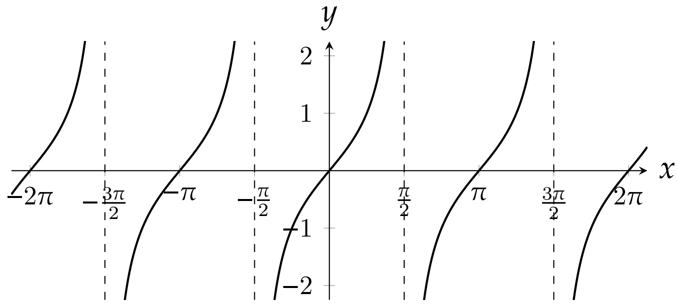
Remember $\sin^{-1}(x) \neq \frac{1}{\sin(x)}$ and $\cos^{-1}(x) \neq \frac{1}{\cos(x)}$! $\frac{1}{\sin(x)}$ is $\csc(x)$ and $\frac{1}{\cos(x)}$ is $\sec(x)$.

Classwork Question 4.7.1

Evaluate each of the following, if possible:

(a) $\arccos(-\frac{\sqrt{3}}{2})$ (b) $\sin^{-1}(\pi)$ (c) $\cos^{-1}(1)$ $\arcsin(0)$

We can also restrict the domain of tan to get a partial inverse:



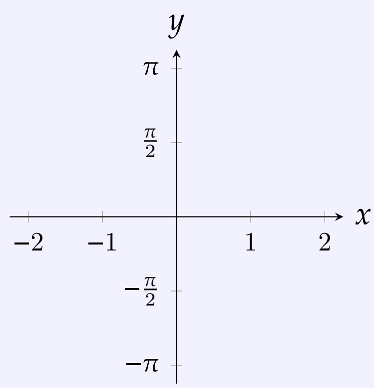
Domain restriction for tan:

Definition: Inverse Tangent Function

The **inverse tangent function** is

Domain: Range: Horizontal Asymptotes:

x-intercepts: y-intercept: Even / Odd / Neither



Example 4.7.2

Evaluate each of the following, if possible:

(a) $\tan^{-1}(1)$ (b) $\arctan(\sqrt{3})$

For (a), what angle θ in ________ gives us ______________? The answer is __________.

For (b), what angle θ in ________ gives us ______________? The answer is __________.

Classwork Question 4.7.2

Evaluate each of the following, if possible:

(a) $\arctan(0)$ (b) $\tan^{-1}(-\frac{\sqrt{3}}{3})$

Partial inverses don't always perfectly cancel, so be careful!

Example 4.7.3

Evaluate each of the following, if possible:

(a) $\arcsin(\sin(\frac{5\pi}{2}))$ (b) $\cos(\arccos(\frac{1}{2}))$ (c) $\sin(\sin^{-1}(2\pi))$

For (a), we first evaluate ______________ = ______________.

Plugging this in to the outer function: ______________ = ______________.

For (b), we first evaluate ______________ = ______________.

Plugging this in to the outer function: ______________ = ______________.

For (c), the inner function is ______________, so the whole expression is ______________.

Classwork Question 4.7.3

Evaluate each of the following, if possible:

(a) $\cos^{-1}(\cos(\frac{9\pi}{4}))$ (b) $\sin(\arcsin(\sqrt{3}))$ (c) $\sin(\sin^{-1}(1))$

If we mix an inverse trig function with a different trig function, we can sketch a right triangle to help us evaluate it. We will use an angle θ in our triangle to represent the inverse trig function.

Example 4.7.4

Evaluate $\tan(\arccos(5/7))$.

First, we sketch a right triangle. We will label an angle θ for reference, and use it to represent the inverse trig function:

Since θ = __________, $\cos\theta$ = _____, so we will label that in our right triangle above. Solving for the missing side using the Pythagorean Theorem:

Thus, $\tan(\arccos(5/7))$ = _____.

Classwork Question 4.7.4

Evaluate $\cos(\sin^{-1}(2/9))$.

Example 4.7.5

Find an expression equal to $\sin(\tan^{-1}(x))$ without any trig or inverse trig functions.

Here, we have an x instead of a given number, but that is okay! We will treat x like a fraction $\frac{x}{1}$.

We sketch a right triangle. We will label an angle θ for reference:

Since θ = __________, $\tan\theta$ = __________, so we will label that in our right triangle above. Solving for the missing side using the Pythagorean Theorem:

Thus, $\sin(\tan^{-1}(x))$ = _______________.

Classwork Question 4.7.5

Find an expression equal to $\tan(\arcsin(x))$ without any trig or inverse trig functions.

Inverse trig functions can be used to solve certain word problems.

Example 4.7.6

You are standing 200 feet away from a rocket ship. The ship blasts off vertically into space. Let s be the height of the spaceship and θ be the angle of elevation.

(a) Draw a diagram to represent this situation. Write θ as a function of s.
(b) What is θ when the rocket is 10,000 feet in the air?
(c) What value does θ approach as $s \to \infty$?

For (a), let's draw our diagram:

We can see that ______________, so $\theta =$ ______________.

For (b), we plug in $s =$ _________, which gives $\theta =$ _________ = _________.

As $s \to \infty$, _________ approaches its ___________________ at _________.

Thus, as $s \to \infty$, θ approaches _______.

Classwork Question 4.7.6

You are standing 3000 feet away from a satellite that is being launched into space. Let h be the height of the spaceship and α be the angle of elevation.

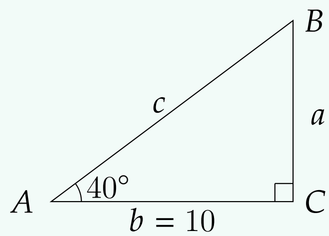
(a) Draw a diagram to represent this situation. Write α as a function of h.
(b) What is α when the satellite is 9000 feet in the air?
(c) What value does α approach as $h \to \infty$?

4.8 Applications and Models

Solving a right triangle refers to finding the measure of each angle and the length of each side.

Example 4.8.1

Solve the right triangle shown below:



Since the angles $A + B =$ _____, we find that $B =$ _____.

If we choose the angle A, then a is ____________ and $b = 10$ is ____________.

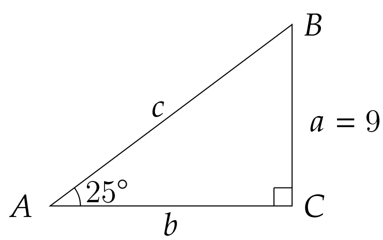
Thus, the trig function we need is _____. Plugging in and solving:

If we choose the angle A, then c is the ____________ and $b = 10$ is ____________.

Thus, the trig function we need is _____. Plugging in and solving:

Classwork Question 4.8.1

Solve the right triangle shown below:



Example 4.8.2

A car drives 3000 feet down a mountain that is 1200 feet tall. Find the angle of elevation from the base to the top of the mountain.

First we sketch a right triangle:

Using trig functions, we see that ____________. Thus, $\theta =$ ____________.

Classwork Question 4.8.2

A 30-foot cable connects the top of a pole to the ground. The point where the cable is tied to the ground is 8 feet from the base of the pole. Find the angle of elevation formed by the cable.

Example 4.8.3

A 10-foot plank of wood is resting against a wall. The angle of elevation is $65°$. Find the height on the wall where the plank is resting.

First we sketch a right triangle:

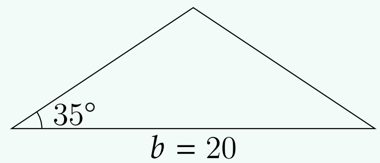
Using the referenced angle, we use the trig function _______ and solve:

Classwork Question 4.8.3

A plank of wood is resting against a wall. The angle of elevation is $40°$, and the bottom of the plank is 8 feet away from the wall. Find the length of the plank.

Example 4.8.4

The **altitude** of a triangle is the height from the horizontal base to the opposite corner. Find the altitude of the isosceles triangle below:



First, we will sketch in the altitude line.

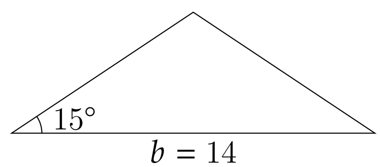
From the diagram, we can see that the base is ____________. Thus, we can sketch a right triangle inside our original diagram.

From the referenced angle, _____ is the ____________ and _____ is the ____________.

So we need the trig function _____. Solving gives:

Classwork Question 4.8.4

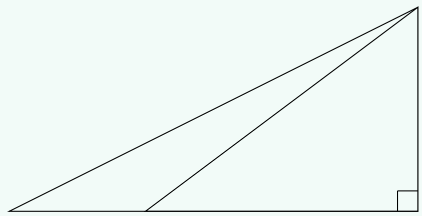
Find the altitude of the isosceles triangle below:



Example 4.8.5

From where you are standing, the angle of elevation to the top of a statue is 10°. After moving 25 feet closer, the angle of elevation is 15°. Find the height of the statue.

These "two triangle" problems can be very challenging. The key is to draw and label a careful diagram:



We can see that _____ is the ____________ side for both triangles.

For the smaller triangle, the ____________ side has length _____.

For the larger triangle, the ____________ side has length __________.

While we could use _____ for both triangles, we will instead use _____. We get:

We can substitute out _____ for _______ in the second equation, since that is the value of the first equation. Now we can solve:

Note:

You can only break up addition in the numerator, not in the denominator!

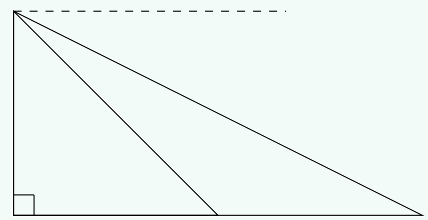
Classwork Question 4.8.5

From where you are standing, the angle of elevation to the top of a building is 35°. After walking 40 feet back, the angle of elevation is 20°. Find the height of the building.

Example 4.8.6

You are floating in a hot air balloon 1500 feet above the ground. Looking down, you see two towns. The angle of depression to Town A is 32° and the angle of depression to Town B is 48°. Find the distance between the two towns.

Again, we will label a diagram:



The angle of depression is the ________________ of the angle in the upper-left of each triangle. Thus, the angles in the triangle are:

We can see that _____ is the ____________ side for both triangles.

For the smaller triangle, the ____________ side has length _____.

For the larger triangle, the ____________ side has length __________.

We can relate these using _____. We get:

We can substitute out _____ for _______ in the second equation, since that is the value of the first equation. Now we can solve:

Classwork Question 4.8.6

From the top of a building, you see two of your friends in the distance. The angle of depression to your first frend is 18° and the angle of depression to your second friend is 34°. If the building is 100 feet tall, how far apart are your friends?

Definition: Simple Harmonic Motion

An object is said to be in **simple harmonic motion** if its displacement from the origin is given by one of two equations:

a is called the **amplitude**, which is the maximum displacement of the object.
ω is called the **angular rate**.

The **period** is the amount of time for an object in simple harmonic motion to complete one full cycle. The **frequency** is the number of cycles completed per second. The equations for period and frequency are:

Period: $T =$ Frequency: $f =$

Example 4.8.7

An object undergoes simple harmonic motion described by the equation $d = 12\cos(40\pi t)$. Find the amplitude and period. How many cycles does the object complete in 30 seconds?

From the equation, we can see that the amplitude is _____.

The angular rate is _______. Using this we can calculate the period:

To find the cycles completed in 30 seconds, we first find the frequency:

The object completes _______ cycles in one second. Therefore, it completes _______ cycles in 30 seconds.

Classwork Question 4.8.7

An object undergoes simple harmonic motion described by the equation $d = 5\cos(0.2\pi t)$. Find the amplitude and period. How many cycles does the object complete in 60 seconds?

Chapter 5

Analytic Trigonometry

5.1 Using Fundamental Identities

An **identity** refers to an equation of two or more functions that is true for every input value. In trigonometry, there are many identites.

Formula 5.1.1 Pythagorean Identities

The following three identities relate various trig functions:

1.

2.

3.

Note:

$\sin^2\theta$ means $(\sin\theta)^2$. The squaring occurs after taking the sine of θ.

Example 5.1.1

Use the Pythagorean Identities to simplify the expressions:
(a) $(1-\cos^2\theta)\csc\theta$ (b) $(1-\csc^2\theta)\cos^2\theta$

For (a), if we rearrange the first Pythagorean identity, we get:

Plugging this in and simplifying:

For (b), if we rearrange the third Pythagorean identity, we get:

Plugging this in and simplifying:

Classwork Question 5.1.1

Use the Pythagorean identities to simplify the expressions:
(a) $\cot\theta(\sec^2\theta-1)$ (b) $(\sin^2\theta-1)\sec^2\theta$

Formula 5.1.2 Reciprocal and Quotient Identities

$\dfrac{1}{\sin\theta} =$

$\dfrac{1}{\csc\theta} =$

$\dfrac{1}{\cos\theta} =$

$\dfrac{1}{\sec\theta} =$

$\dfrac{1}{\tan\theta} =$

$\dfrac{1}{\cot\theta} =$

$\tan\theta =$

$\cot\theta =$

Example 5.1.2

Given $\sin\theta = -\dfrac{1}{3}$ and $\cos\theta = \dfrac{2\sqrt{2}}{3}$, find all six trig functions of θ.

We can use the reciprocal and quotient identities to solve this:

$\sin\theta =$

$\csc\theta =$

$\cos\theta =$

$\sec\theta =$

$\tan\theta =$

$\cot\theta =$

Classwork Question 5.1.2

Given $\sin\theta = \dfrac{2}{5}$ and $\cos\theta = \dfrac{\sqrt{21}}{5}$, find all six trig functions of θ.

Each trig function has an associated "co" trig function:

_______________ and _______________ are cofunctions.

_______________ and _______________ are cofunctions.

_______________ and _______________ are cofunctions.

Each function and its cofunction are related.

Formula 5.1.3 Cofunction Identities

$\sin(\frac{\pi}{2} - \theta) =$

$\csc(\frac{\pi}{2} - \theta) =$

$\cos(\frac{\pi}{2} - \theta) =$

$\sec(\frac{\pi}{2} - \theta) =$

$\tan(\frac{\pi}{2} - \theta) =$

$\cot(\frac{\pi}{2} - \theta) =$

Example 5.1.3

Simplify the expression $\dfrac{\sec^2(\frac{\pi}{2} - \theta)}{\sec\theta}$

From the cofunction identities, we know that $\sec(\frac{\pi}{2} - \theta) =$ __________.

Thus, we can substitute and simplify:

Classwork Question 5.1.3

Simplify the expression $\dfrac{\sin\theta}{\cos^2(\frac{\pi}{2} - \theta)}$

We can factor and multiply trig expressions just like we would with simpler expressions.

Example 5.1.4

Fully factor the expression $\sin^2\theta + 10\sin\theta + 21$ and simplify if possible.

Let's make $y =$ ________. Now our expression becomes ________________. We can factor this and then substitute back out for y:

Classwork Question 5.1.4

Fully factor the expression $\tan^2\theta - 5\tan\theta + 6$ and simplify if possible.

Example 5.1.5

Fully factor the expression $\cos\theta + \tan^2\theta\cos\theta$ and simplify if possible.

First we factor out ________:

We can simplify using the Pythagorean identity _____________________:

Classwork Question 5.1.5

Fully factor the expression $\sin\theta - \sin\theta\cos^2\theta$ and simplify if possible.

Example 5.1.6

Multiply and simplify the expression $(8\sin\theta - 8\cos\theta)^2$.

First, we will FOIL out the expression:

We can use the Pythagorean identity ________________ to simplify the expression:

Classwork Question 5.1.6

Multiply and simplify the expression $(2\csc\theta + 2)(2\csc\theta - 2)$

Example 5.1.7

Simplify the expression $\dfrac{\sin\theta}{\cos\theta} + \dfrac{\cos\theta}{\sin\theta}$.

For this, we need to get a common denominator. To do this, multiply the denominators together.

In this case, it our common denominator is ____________. For each fraction, multiply the "missing" term to the numerator:

We can simplify this using the Pythagorean identity ________________:

Classwork Question 5.1.7

Simplify the expression $\dfrac{1}{1+\sin\theta} + \dfrac{1}{1-\sin\theta}$

Example 5.1.8

Simplify the expression $\tan^2\theta - \sec^2\theta$.

Using the Pythagorean identity ____________________, we can simplify:

Classwork Question 5.1.8

Simplify the expression $\csc^2\theta - \cot^2\theta$

Example 5.1.9

Factor and simplify the expression $\dfrac{6\csc^2\theta - 6}{\csc\theta + 1}$

It may be tempting to use a Pythagorean identity, but we will factor instead. Factoring the numerator and canceling gives:

Classwork Question 5.1.9

Factor and simplify the expression $\dfrac{\sin^2\theta - \cos^2\theta}{\sin\theta - \cos\theta}$

5.3 Solving Trigonometric Equations

In previous sections, we found solutions to problems like $\sin\theta = \frac{1}{2}$ for θ in some interval, like $[0, 2\pi]$ or $[-2\pi, 2\pi]$. If we want to find all of the solutions in $(-\infty, \infty)$, we need to find a **general solution**.

To find a general solution, first find all solutions in an interval corresponding to one period of the trig function in question. We then add a **general term** of n times the period, where n is any integer. That is, a general solution takes the form:

$$x = (\text{solution in interval of one period}) + n \cdot (\text{period})$$

Example 5.3.1

Solve the equation $\sin(x) + \sqrt{2} = -\sin(x)$.

First, we will solve for $\sin(x)$:

The period of sine is _____, so we will first find solutions in _______:

We then add our general term. Since the period is _____, the general term is _______:

Classwork Question 5.3.1

Solve the equation $\cos(x) - 1 = -\cos(x)$

Example 5.3.2

Solve the equation $3\tan^2(x) - 1 = 0$.

First, we will solve for $\tan(x)$:

The period of tangent is _____, so we will first find solutions in _______:

We then add our general term. Since the period is _____, the general term is _______:

Classwork Question 5.3.2

Solve the equation $4\sin^2(x) - 3 = 0$

If multiple trig functions are involved, collect everything on one side, factor, and then solve each factor on its own.

Example 5.3.3

Solve $\cot(x)\cos^2(x) = 2\cot(x)$.

First, we will move everything to the left and factor:

Now we will solve each factor separately. For our first factor:

Our second factor has ______________ because ____________________________. Therefore our solutions are:

Note:
Always solve trig equations by factoring, do not divide! Dividing can miss solutions.

Classwork Question 5.3.3

Solve $\cos(x)\sin(x) = -4\cos(x)$

A trig equation is said to be of the **quadratic type** if it has the form

$$a(\text{trig function})^2 + b(\text{trig function}) + c = 0$$

Example 5.3.4

Solve the equation $2\sin^2(x) - \sin(x) - 1 = 0$.

First we factor the left hand side:

Now we will solve each factor separately. For our first factor:

For our second factor:

Classwork Question 5.3.4

Solve the equation $2\cos^2(x) - 3\cos(x) + 1 = 0$.

If an equation involves single trig functions, we have two strategies. The first is to rewrite the equation using trig identities.

Example 5.3.5

Solve the equation $\sin(x) + \cos(x)\cot(x) = 2$ for all x in $[0, 2\pi)$

We can simplify the equation by finding a common denominator and using the Pythagorean identity:

Since we are only looking in $[0, 2\pi)$, our solutions are:

Classwork Question 5.3.5

Solve the equation $\sin(x)(\tan(x) + \cot(x)) = 2$ for all x in $[0, 2\pi)$.

Another method is to square both sides to produce an equation of the quadratic type. When we do this, however, we must check for **extraneous solutions**.

Example 5.3.6

Solve the equation $\sin(x) + 1 = \cos(x)$ for all x in $[0, 2\pi)$.

We will square both sides, simplify using a Pythagorean identity, and then factor:

Now we solve each factor:

Finally, we have to check each answer:

Classwork Question 5.3.6

Solve the equation $\cos(x) + 1 = \sin(x)$ for all x in $[0, 2\pi)$.

If the x inside of the trig function is multiplied or divided by a constant, we call that a **multiple angle** equation.

Example 5.3.7

Solve the equation $2\cos(3x) - \sqrt{2} = 0$. How many solutions are there in $[0, 2\pi)$?

First we will isolate the trig function:

We will find our general solution as normal, but with $3x$ instead of x:

Finally, we divide through by 3 for each solution:

Normally there are _____ solutions in $[0, 2\pi)$, but now we have _____.

Classwork Question 5.3.7

Solve the equation $2\sin(4x) + \sqrt{3} = 0$. How many solutions are there in $[0, 2\pi)$?

Example 5.3.8

Solve the equation $2\cos(\frac{1}{2}x) - \sqrt{3} = 0$. How many solutions are there in $[0, 2\pi)$?

Solving the multiple angle equation:

Normally there are _____ solutions in $[0, 2\pi)$, but now we have _____.

Classwork Question 5.3.8

Solve the equation $2\sin(\frac{1}{3}x) + \sqrt{3} = 0$. How many solutions are there in $[0, 2\pi)$

Example 5.3.9

Solve the equation $3\tan^3(2x) + \tan(2x) = 0$.

Here, we first factor and then solve each factor:

Classwork Question 5.3.9

Solve the equation $\cos^3(5x) - \cos(5x) = 0$