AUGUST 2024 MEASURE PRELIM TUTORING PROBLEM SET 1

Notation: m denotes the Lebesgue measure on $\mathbb{R}^n$. $[x_n]$ is used to denote a sequence.

1. (Jan 2023) Prove or disprove three of the following statements.
   (a) If $A \subset \mathbb{R}$ and $m(A) > 0$, then there exist $x, y \in A$ such that $x - y \notin \mathbb{Q}$.
   (b) Let $f : [0,1] \to \mathbb{R}$. If the sets $\{x \in [0,1] : f(x) = c\}$ are Lebesgue measurable for all $c \in \mathbb{R}$, then f is measurable.
   (c) Let $(X, \mathcal{F}, \mu)$ be a finite measure space. Let $[f_n], f : X \to \mathbb{R}$ be measurable functions. If $f_n \to f$ in measure, then $f_n \to f$ almost everywhere.
   (d) If μ and ν are two measures on the same measurable space $(X, \mathcal{F})$ that have exactly the same sets of measure 0, then $L^\infty(\mu) = L^\infty(\nu)$
2. (Aug 2019) Let $g : [0,1] \to \mathbb{R}$ be a nonnegative Lebesgue measurable function.
   (a) Prove that as $n \to \infty$, the numbers $I_n = \int g^n \, dm$ converge to a nonnegative limit that may be infinity.
   (b) If $I_n = C < \infty$ for all n, prove that there exists some Lebesgue measurable set $A \subset [0,1]$ such that $g = \chi_A$ m-almost everywhere.
3. (Aug 2023) Let $1 < p_1 < p_2 < \infty$.
   (a) Let $f_i \in L^{p_i}(m)$, $i = 1, 2$, be nonnegative measurable functions. Find $r = r(p_1, p_2)$ for which $(f_1 f_2)^r \in L^1(m)$
   (b) If $s \neq r(p_1, p_2)$, show there exist nonnegative $f_i \in L^{p_i}(m)$, $i = 1, 2$, for which $(f_1 f_2)^s \notin L^1(m)$
4. (Aug 2022) Let E and F be Borel subsets of $\mathbb{R}^2$ such that
   $$m^1(E_x) = m^1(F_x) \quad \text{for all } x \in \mathbb{R}$$
   where $A_x = \{y \in \mathbb{R} : (x, y) \in A\}$ and m^1 is the 1-dimensional Lebesgue measure. Show that $m^2(E) = m^2(F)$, where m^2 is the 2-dimensional Lebesgue measure.
5. (Aug 2023) Let $(X, \mathcal{A}, \mu)$ be a finite measure space. Prove $f \in L^1(\mu)$ if and only if
   $$\sum_{k=1}^{\infty} 2^k \mu(\{x \in X : |f(x)| \geq 2^k\}) < \infty$$
6. (Aug 2020) Suppose that W is a Lebesgue nonmeasurable subset of $[0,1]$. Prove that there exists some $0 < \varepsilon < 1$ such that for any Lebesgue measurable subset E of $[0,1]$ with $m(E) \geq \varepsilon$, the set $W \cap E$ must be Lebesgue nonmeasurable.

AUGUST 2024 MEASURE PRELIM TUTORING PROBLEM SET 2

Notation: m denotes the Lebesgue measure on $\mathbb{R}^n$. $[x_n]$ is used to denote a sequence.

1. (Aug 2018) Show that there exists a Borel measure ν on $\mathbb{R}$ that satisfies both
   (a) ν and m are mutually singular
   (b) $0 < \nu(B(x,r)) < \infty$ for all $x \in \mathbb{R}$ and $r > 0$
2. (Jan 2016) Let $K \subset \mathbb{R}$ be a compact interval and $f_n : K \to \mathbb{R}$ a sequence of functions.
   (a) Suppose that the sequence $[f_n]$ converges almost everywhere on K with respect to m. Show that $[f_n]$ converges in measure.
   (b) Must the conclusion of part (a) be true if K is not compact? Give a proof or counterexample.
   (c) Suppose that all f_n are differentiable and
      i. There exists $M > 0$ such that $\|f_n'\|_\infty < M$ for all n, and
      ii. For each n there exists x_n such that $f_n(x_n) = 0$

   Prove that there is a subsequence of $[f_n]$ that converges uniformly on K to a continuous limit function f.
3. (Aug 2013) Let $[q_n]$ be an enumeration of the rationals in $[0,1]$. Consider the series
$$s(x) = \sum_{n=1}^{\infty} \frac{1}{n^2\sqrt{|x - q_n|}}$$
   (a) Prove that s converges m-almost everywhere.
   (b) Prove that s is unbounded on any non-empty open subinterval of $[0,1]$.
4. (Aug 2023) Let $A \subset \mathbb{R}$ be Lebesgue measurable with finite Lebesgue measure. Prove
$$\lim_{|x|\to 0} m(A \cap (x + A)) = m(A)$$
   Here $x + A = \{x + y : y \in A\}$
5. (Aug 2020)
   (a) Let (X, Σ, μ) and (X, Σ, ν) be two measure spaces with $\nu(X) < \infty$. Prove that ν is absolutely continuous with respect to μ if and only if for every $\varepsilon > 0$ there exists $\delta > 0$ such that if $A \in \Sigma$ with $\mu(A) < \delta$ then $\nu(A) < \varepsilon$.
   (b) Give an example of a pair of measure spaces (X, Σ, μ) and (X, Σ, ν) such that ν is absolutely continuous with respect to μ, but given $\varepsilon > 0$ there is no $\delta > 0$ such that $\nu(A) < \varepsilon$ for every $A \in \Sigma$ with $\mu(A) < \delta$.
6. (Jan 2017) Let $f, g \in L^1(\mathbb{R}^n, m)$ be nonnegative functions such that
$$\liminf_{k\to\infty} \frac{\int_{B(x,1/k)} f(y)\,dy}{\int_{B(x,1/k)} g(y)\,dy} \le 1$$
   For x-almost everywhere in $\mathbb{R}^n$. Prove that $f \le g$ almost everywhere.

AUGUST 2024 MEASURE PRELIM TUTORING PROBLEM SET 3

Notation: m denotes the Lebesgue measure on $\mathbb{R}^n$. $[x_n]$ is used to denote a sequence.

1. (Aug 2018) Let $T = \{(x,y) \in \mathbb{R}^2 : 0 \le |x| \le y \le 1\}$ and let μ be the restriction of m to T. Let $f \in L^2(T,\mu)$. Prove that
   (a) $f \in L^1(T,\mu)$
   (b) $\liminf_{y\to 0^+} \int_{-y}^{y} |f(x,y)|\,dx = 0$
2. (Jan 2022) Let $f : \mathbb{R} \to \mathbb{R}$ be Lebesgue measurable. Prove there exists a constant $C > 0$ such that
$$\|f\|_1 \le C(\|f\|_2 + \|x^2 f\|_2)$$
3. (Jan 2015) Let $(X, \mathcal{A}, \mu)$ be a σ-finite measure space and $p \in [1,\infty)$. Let $f : X \to \mathbb{R}$ be $\mathcal{A}$-measurable and define
$$R_p(x) = x^{p-1}\mu(\{|f| > x\})$$
   (a) Prove if $f \in L^p(\mu)$, then $\lim_{x\to\infty} xR_p(x) = 0$
   (b) Prove $f \in L^p(\mu)$ if and only if $R_p \in L^1([0,\infty), m)$ (Lebesgue measure)
4. (Aug 2016) Compute the limit
$$\lim_{j\to\infty} \int_{-j}^{j} \frac{\sin(x^j)}{x^{j-2}}\,dx$$
   and provide justification for all steps in your reasoning.
5. (Jan 2022) Prove or disprove the following statements:
   (a) If $S^{n-1} = \{x \in \mathbb{R}^n : |x| = 1\}$, then $m(S^{n-1}) = 0$.
   (b) Every nonnegative continuous $f \in L^1(\mathbb{R})$ satisfies $\limsup_{x\to\infty} f(x) \in [0,\infty)$.
   (c) If $f : (a,b) \to \mathbb{R}$ is differentiable, then f' is Lebesgue measurable.
6. (Aug 2015) Let f and $[f_n]$ be measurable functions on a measure space $(X, \mathcal{A}, \mu)$ and suppose that for any $\varepsilon > 0$ we have
$$\sum_{n=1}^{\infty} \mu(\{x \in X : |f_n(x) - f(x)| > \varepsilon\}) < \infty.$$
   Prove that $f_n \to f$ μ-almost everywhere.

AUGUST 2024 MEASURE PRELIM TUTORING PROBLEM SET 4

Notation: m denotes the Lebesgue measure on $\mathbb{R}^n$. $[x_n]$ is used to denote a sequence.

1. (Aug 2021) Let $(X, \mathcal{A}, \mu)$ be a measure space and $[f_n], f \in L^1(\mu)$ such that $f_n \to f$ in $L^1(\mu)$. Show that if $\sup_n \|f_n\|_{L^4(\mu)} < \infty$, then $f \in L^2(\mu)$ and $f_n \to f$ in $L^2(\mu)$.

2. (Aug 2023) Let
$$f_n(x) = \frac{1}{1 + x^{\frac{\sqrt{n}}{\log(n+2023)}}}, \quad x \geq 0, n \in \mathbb{N}$$
Find $\lim_{n\to\infty} \int_0^\infty f_n \, dm$

3. (Jan 2022) Let $f : [0,1] \to \mathbb{R}$ be a Borel measurable function with $\int_0^1 |f(t)| \, dt < \infty$. Prove that the function
$$h(x) = \int_x^1 t^{-1} f(t) \, dt$$
is integrable on $[0,1]$ and $\int_0^1 f(t) \, dt = \int_0^1 h(x) \, dx$.

4. (Aug 2020) Let $\{E_1, \ldots, E_m\}$ be a finite family of Lebesgue measurable subsets of $\mathbb{R}^n$ and let $k > 0$ be a positive integer. Let $E \subset \mathbb{R}^n$ be a measurable subset with $m(E) > 0$. Suppose that almost every $x \in E$ belongs to at least k of the E_j. Prove there is at least one E_ℓ with $m(E_\ell) \geq \frac{k}{m} m(E)$.

5. (Aug 2017) Prove or disprove three of the following statements:
   (a) If $[f_n]$ is a sequence of measurable functions that converges in $L^1(\mathbb{R})$, then it converges in measure.
   (b) If $[f_n]$ is a sequence of integrable functions that converges almost everywhere in $[0,1]$, then it converges in $L^1([0,1])$.
   (c) If $[f_n]$ is a sequence of measurable functions that converges almost everywhere in $[0,1]$, then it converges in $L^\infty([0,1])$.
   (d) If $[f_n]$ is a sequence of measurable functions that converges in $L^1(\mathbb{R})$, then it converges almost everywhere.

6. (Jan 2017) Let $g \in L^1(\mathbb{R}^n, m)$ such that
$$\int g(x)\phi(x) \, dx = 0$$
for all $\phi \in C_c(\mathbb{R}^n)$, ie the space of all continuous functions with compact support on $\mathbb{R}^n$. Prove that $g = 0$ almost everywhere.

AUGUST 2024 MEASURE PRELIM TUTORING PROBLEM SET 5

Notation: m denotes the Lebesgue measure on $\mathbb{R}^n$. $[x_n]$ is used to denote a sequence.

1. (Aug 2019) Let $0 < a < b < \infty$ and consider the function
$$f(x) = \frac{1}{x^a + x^b} \quad x > 0$$
For which values of p is $f \in L^p(0, \infty)$?

2. (Aug 2018) Let $(X, \mathcal{A}, \mu)$ be a measure space.
   (a) Prove that if $[f_n], [g_n], f, g \in L^1(\mu)$, $|f_n| \leq g_n$ for all n, $f_n \to f$ μ-ae, $g_n \to g$ μ-ae, and $\int g_n \, d\mu \to \int g \, d\mu$, then $\int f_n \, d\mu \to \int f \, d\mu$.
   (b) Let $1 \leq p < \infty$. Suppose $[f_n], f \in L^p(\mu)$ and that $f_n \to f$ μ-ae. Prove that $\int |f_n - f|^p \, d\mu \to 0$ if and only if $\int |f_n|^p \, d\mu \to \int |f|^p \, d\mu$.

3. (Jan 2023) Let $p : \mathbb{R}^n \to \mathbb{R}$ be a (nontrivial) polynomial in n variables, with $n \geq 2$. Prove that the set $p^{-1}(\{0\}) \subset \mathbb{R}^n$ has Lebesgue measure 0.

4. (Jan 2018) If $\mu \ll \nu$ and $\nu \ll \mu$ are finite measures on a measurable space $(X, \mathcal{A})$, show that the Radon-Nikodym derivatives satisfy $\frac{d\mu}{d\nu}\frac{d\nu}{d\mu} = 1$ μ-almost everywhere.

5. (Aug 2021) Let $f : \mathbb{R}^n \to (0, \infty)$ be a Lebesgue measurable function with $\|f\|_{L^1(\mathbb{R}^n)} = 1$. Show that if $E \subset \mathbb{R}^n$ is Lebesgue measurable with $m(E) \in (0, \infty)$, then
$$\int_E \log f \, dm \leq -m(E) \log \big(m(E)\big)$$

6. (Jan 2016) Compute
$$\lim_{n \to \infty} \int_0^n \left(1 + \frac{x}{n}\right)^n e^{-2x} \, dx$$
Provide full justification for all steps in your reasoning.

AUGUST 2024 MEASURE PRELIM TUTORING PROBLEM SET 6

Notation: m denotes the Lebesgue measure on $\mathbb{R}^n$. $[x_n]$ is used to denote a sequence.

1. (Jan 2022) Let $(X, \mathcal{A}, \mu)$ be a measure space and $[f_n], f$ be measurable functions. Show that if $f_n \geq 0$ for all n and $f_n \to f$ in measure, then
$$\int f \, d\mu \leq \liminf_{n\to\infty} \int f_n \, d\mu$$

2. (Jan 2017) Let $f \in L^2(\mathbb{R}, m)$ and set $F(x) = \int_0^x f(t)\, dt$. Prove there exists a constant $C \geq 0$ such that
$$|F(x) - F(y)| \leq C|x - y|^{1/2}$$
for all $x, y \in \mathbb{R}$.

3. (Jan 2017) Prove or disprove three of the following statements:
   (a) If $[f_n]$ is a Cauchy sequence in $L^2(\mathbb{R}^n, m)$, then $[f_n]$ converges almost everywhere.
   (b) If $[f_n]$ is a sequence of measurable functions that converges in $L^\infty(\mathbb{R}^n, m)$, then $[f_n]$ converges almost everywhere.
   (c) If $U \subset \mathbb{R}^n$ is a subset whose boundary has Lebesgue outer measure 0, then U is Lebesgue measurable.
   (d) Let $(X, \mathcal{A}, \nu)$ be a measure space and suppose μ is a signed measure on $(X, \mathcal{A})$ satisfying $\mu \ll \nu$. If $\nu(A) = 0$ then $\mu^+(A) = \mu^-(A) = 0$, where $\mu = \mu^+ - \mu^-$ is the Jordan decomposition of μ.

4. (Aug 2018) Prove that if $f : \mathbb{R} \to \mathbb{R}$ is Lebesgue measurable, then
$$\int_{\mathbb{R}} f^4 \, dm = 4 \int_0^\infty m(\{x \in \mathbb{R} : |f(x)| > t\})\, t^3 \, dt$$

5. (Aug 2021) Let $f : \mathbb{R} \to \mathbb{R}$ be a function. Suppose there exists some $C \geq 0$ such that
$$|f(x) - f(y)| \leq C|x - y|, \quad \text{for all } x, y \in \mathbb{R}$$
   (a) Prove that $m^*(f(A)) \leq Cm^*(A)$ for all $A \subset \mathbb{R}$ (m^* is Lebesgue outer measure)
   (b) Prove that f maps Lebesgue measurable sets to Lebesgue measurable sets.

6. (Aug 2019) Let $A \subset \mathbb{R}^n$ be a Lebesgue measurable subset with $m(A) < \infty$ and let $t \in (0, m(A)/2)$. Prove there exist disjoint Lebesgue measurable subsets $B, C \subset A$ such that $m(B) = m(C) = t$.