AUGUST 2024 MEASURE PRELIM TUTORING PROBLEM SET 1 SOLUTIONS

1. (Jan 2023) Prove or disprove three of the following statements.

(a) If $A \subset \mathbb{R}$ and $m(A) > 0$, then there exist $x, y \in A$ such that $x - y \notin \mathbb{Q}$.

What to spot: $\mathbb{Q}$ has Lebesgue measure 0, but $m(A) > 0$.

This is true. Suppose $m(A) > 0$ and that for every $x, y \in A$, we have $x - y \in \mathbb{Q}$. Select a point $x \in A$ and consider the translation $A - x$. Then $0 \in A - x$, and by the stated property, $y - 0 = y \in \mathbb{Q}$ for every $y \in A - x$. Thus $A - x \subset \mathbb{Q}$, and by monotonicity and translation invariance of the Lebesgue measure,

$$0 = m(\mathbb{Q}) \geq m(A - x) = m(A)$$

A contradiction of $m(A) > 0$.

(b) Let $f : [0, 1] \to \mathbb{R}$. If the sets $\{x \in [0, 1] : f(x) = c\}$ are Lebesgue measurable for all $c \in \mathbb{R}$, then f is measurable.

What to spot: If this were true, then any injective function would be measurable.

This is false. Vitali's Theorem states that there is a Lebesgue non-measurable set $W \subset [0, 1]$ since $m([0, 1]) > 0$. Let $W^c = [0, 1] \setminus W$ and define the function $f(x) = x\chi_W - x\chi_{W^c}$. f is injective on $[0, 1]$, so the preimage of any point is either empty or a singleton, both of which are measurable. Thus $\{x \in [0, 1] : f(x) = c\}$ is measurable for all c. But $f^{-1}([0, 1]) = W$, meaning f is not measurable (the preimage of a measurable set is non-measurable under f).

(c) Let $(X, \mathcal{F}, \mu)$ be a finite measure space. Let $[f_n], f : X \to \mathbb{R}$ be measurable functions. If $f_n \to f$ in measure, then $f_n \to f$ almost everywhere.

What to spot: Any time you are asked a true/false question about modes of convergence, recall which modes imply which other, and try out some common counterexamples: typewriter sequence, $f_n = n^2\chi_{[0,1/n]}$, $g_n = \chi_{[n,n+1]}$, etc.

This is false. Let $X = [0, 1]$, $\mathcal{F} = \mathcal{B}[0, 1]$ and $\mu = m$. Let $[f_n]$ be the typewriter sequence defined by

$$f_n(x) = \begin{cases} 1 & x \in [\frac{n-2^k}{2^k}, \frac{n-2^k+1}{2^k}] \text{ for } k \geq 0 \text{ and } 2^k \leq n < 2^{k+1} \\ 0 & \text{otherwise} \end{cases}$$

Let $f = 0$. Then $f_n \to f$ in measure because for any $\varepsilon > 0$ fixed, take k large enough so $2^{-k} < \varepsilon$, then $m(\{x : |f_n(x)| > \varepsilon\}) \leq 2^{-k} < \varepsilon$ when $n \geq 2^k$. But $\lim_n f_n(x)$ fails to exist for each x since $\limsup_n f_n(x) = 1$ and $\liminf_n f_n(x) = 0$ for each x. Thus f_n fails to converge to f (or to any function) almost everywhere.

(d) If μ and ν are two measures on the same measurable space $(X, \mathcal{F})$ that have exactly the same sets of measure 0, then $L^\infty(\mu) = L^\infty(\nu)$.

$\boxed{\text{What to spot: } L^\infty \text{ means bounded outside a set of measure 0, and } \mu \text{ and } \nu \text{ have the same measure 0 sets.}}$

This is true. Let $f \in L^\infty(\mu)$, then there is a set $A \in \mathcal{F}$ and a constant $M > 0$ such that $\mu(A) = 0$ and $|f| < M$ on $X \setminus A$. Since μ and ν have the same sets of measure 0, it follows that $\nu(A) = 0$, so $f \in L^\infty(\nu)$. By symmetry, $f \in L^\infty(\nu) \implies f \in L^\infty(\mu)$, so $L^\infty(\mu) = L^\infty(\nu)$.

2. (Aug 2019) Let $g : [0,1] \to \mathbb{R}$ be a nonnegative Lebesgue measurable function.

(a) Prove that as $n \to \infty$, the numbers $I_n = \int g^n \, dm$ converge to a nonnegative limit that may be infinity.

What to spot: We have a sequence of integrals, so we will need to apply MCT and/or DCT as appropriate.

First, we observe that g^n doesn't necessarily behave the same everywhere as $n \to \infty$. If $0 \le g(x) < 1$, then $g^n(x) \downarrow 0$, so the sequence is not increasing and MCT fails here. If $g(x) > 1$, then $g^n(x) \uparrow \infty$, which cannot be dominated by an integrable function so DCT fails here. If $g(x) = 1$, then $g^n(x) = 1$ for all n (which doesn't really affect much here). Since neither MCT nor DCT works on its own, we need to find a way to use both.

Define the sets $E_0 = \{x \in [0,1] : 0 \le g(x) < 1\}$, $E_1 = \{x \in [0,1] : g(x) = 1\}$, and $E_2 = \{x \in [0,1] : g(x) > 1\}$. Observe they are all measurable. We then have $I_n = \int_{E_0} g^n \, dm + \int_{E_1} g^n \, dm + \int_{E_2} g^n \, dm$ and we can treat each integral separately.

Since $g^n = 1$ on E_1 for all n, we have $\int_{E_1} g^n \, dm = \int_{E_1} 1 \, dm = m(E_1)$. On E_0, we observe that $g^n \le 1$ for all n, so we can apply DCT with the dominating function 1 to see that $\lim_n \int_{E_0} g^n \, dm = \int_{E_0} \lim_n g^n \, dm = \int_{E_0} 0 \, dm = 0$. On E_2, since $g^n \uparrow \infty$, we can apply the MCT to see that $\lim_n \int_{E_2} g^n \, dm = \int_{E_2} \lim_n g^n \, dm = \int_{E_2} \infty \, dm = \infty \cdot m(E_2)$. Here, we adopt the convention that $\infty \cdot 0 = 0$. Thus, $I_n \to m(E_1) + \infty \cdot m(E_2)$, which is finite when $m(E_2) = 0$ and infinite when $m(E_2) > 0$.

(b) If $I_n = C < \infty$ for all n, prove there exists some Lebesgue measurable set $A \subset [0,1]$ such that $g = \chi_A$ m-ae.

What to spot: Since $I_1 = I_2 = I_3 = \dots$, we ought to expect $g^1 = g^2 = g^3 = \dots$ almost everywhere, which would imply $g(x) \in \{0,1\}$ almost everywhere.

Let E_1 and E_2 be defined as in part (a). Define $E_0' = \{x \in [0,1] : g(x) = 0\}$ and $E_0'' = \{x \in [0,1] : 0 < g(x) < 1\}$.

We will first show $m(E_2) = 0$. Indeed, if $m(E_2) > 0$, then $I_n \to \infty$, as we saw in part (a). The only way for a constant sequence to converge to ∞ is for $I_n = C = \infty$ for all n, but we specify $C < \infty$. Thus $m(E_2) = 0$.

We will now show $m(E_0'') = 0$. If $m(E_0'') > 0$, then $\int_{E_0''} g \, dm = M > 0$, so $I_1 = \int_{E_1} g \, dm + \int_{E_0'} g \, dm + \int_{E_0''} dm = m(E_1) + M$. Thus $I_n = m(E_1) + M$ for all n, so $I_n \to m(E_1) + M$, but we saw in part (a) that $I_n \to m(E_1) + \infty \cdot m(E_2) = m(E_1)$, a contradiction. So $m(E_0'') = 0$

Thus $m(E_1 \cup E_0') = 1$, so we have that $g(x) \in \{0,1\}$ m-almost everywhere, and in fact, $g = \chi_{E_1}$ almost everywhere.

3. (Aug 2023) Let $1 < p_1 < p_2 < \infty$.

(a) Let $f_i \in L^{p_i}(m)$, $i = 1, 2$, be nonnegative measurable functions. Find $r = r(p_1, p_2)$ for which $(f_1 f_2)^r \in L^1(m)$

What to spot: To show a function belongs to one L^p space given functions in other L^p spaces, Hölder's inequality is generally helpful.

For Hölder's inequality, we need to choose s and t (I'm using s and t to avoid confusion with p and p_1 and p_2) so that $\frac{1}{s} + \frac{1}{t} = 1$, while also ensuring that we can cancel out the exponents on f_1 and f_2 so that we end up with $(\int f_1^{p_1}\, dm)^{\text{something}}$ and $(\int f_2^{p_2}\, dm)^{\text{something}}$. Therefore, we need $rs = p_1$, $rt = p_2$, and $\frac{1}{s} + \frac{1}{t} = 1$ simultaneously. A little algebra gives $\frac{1}{s} = \frac{r}{p_1}$ and $\frac{1}{t} = \frac{r}{p_2}$, so,

$$\frac{r}{p_1} + \frac{r}{p_2} = 1 \implies r = \frac{1}{\frac{1}{p_1} + \frac{1}{p_2}} = \frac{p_1 p_2}{p_1 + p_2}$$

To verify our work, let $s = \frac{p_1}{r}$ and $t = \frac{p_2}{r}$. Then $\frac{1}{s} + \frac{1}{t} = 1$ as shown above, so by Hölder's inequality,

$$\int (f_1 f_2)^r\, dm \leq \left(\int f_1^{rs}\, dm\right)^{1/s} \left(\int f_2^{rt}\, dm\right)^{1/t} = \left(\int f_1^{p_1}\, dm\right)^{1/s} \left(\int f_2^{p_2}\, dm\right)^{1/t} < \infty$$

Thus $(f_1 f_2)^r \in L^1$.

(Side note: A more general version of Hölder's inequality states that if $\frac{1}{p} + \frac{1}{q} = \frac{1}{r}$, then $\|fg\|_r \leq \|f\|_p \|g\|_q$. Try proving this as an exercise!)

(b) If $s \neq r(p_1, p_2)$, show there exist nonnegative $f_i \in L^{p_i}(m)$, $i = 1, 2$, for which $(f_1 f_2)^s \notin L^1(m)$

What to spot: The class functions $\frac{1}{x^q}$ is helpful for these kinds of problems because $\int_0^1 \frac{1}{x^q}$ converges for $q < 1$ and diverges for $q \geq 1$, and $\int_1^\infty \frac{1}{x^q}$ converges for $q > 1$ and diverges for $q \leq 1$.

First, suppose $s > r(p_1, p_2)$. Let $f_1 = 1/x^{1/p_1 - \delta} \cdot \chi_{[0,1]}$ and $f_2 = 1/x^{1/p_2 - \delta} \cdot \chi_{[0,1]}$, where $\delta > 0$ is a quantity that we will specify later on. Then $f_1 \in L^{p_1}$ and $f_2 \in L^{p_2}$ because $1 - p_1\delta < 1$ and $1 - p_2\delta < 1$:

$$\int f_1^{p_1}\, dm = \int_0^1 \frac{1}{x^{p_1(1/p_1 - \delta)}}\, dm = \int_0^1 \frac{1}{x^{1 - p_1\delta}}\, dm < \infty$$

$$\int f_2^{p_2}\, dm = \int_0^1 \frac{1}{x^{p_2(1/p_2 - \delta)}}\, dm = \int_0^1 \frac{1}{x^{1 - p_2\delta}}\, dm < \infty$$

We compute:

$$(f_1 f_2)^s = \left(\frac{1}{x^{1/p_1 + 1/p_2 - 2\delta}}\right)^s \chi_{[0,1]} = \frac{1}{x^{s/p_1 + s/p_2 - 2\delta s}} \chi_{[0,1]}$$

Our goal is to choose δ so that $s/p_1 + s/p_2 - 2\delta s \geq 1$. Recall from our work above that $\frac{r}{p_1} + \frac{r}{p_2} = 1$. Since $s > r$, we have $\frac{s}{p_1} + \frac{s}{p_2} = \alpha > 1$. Thus, we need ensure we can choose δ so that $\alpha - 2\delta s > 1$. Indeed, solving for δ results in the condition $(\alpha - 1)/2s > \delta$, which we can satisfy because $(\alpha - 1)/2s > 0$. Choose, say, $\delta = (\alpha - 1)/4s$ and we are done.

The case for $s < r(p_1, p_2)$ proceeds similarly. Let $f_1 = 1/x^{1/p_1+\delta} \cdot \chi_{[1,\infty)}$ and $f_2 = 1/x^{1/p_2+\delta} \cdot \chi_{[1,\infty)}$, where $\delta > 0$ is a quantity that we will specify later on. Then $f_1 \in L^{p_1}$ and $f_2 \in L^{p_2}$ because $1 + p_1\delta > 1$ and $1 + p_2\delta > 1$:

$$\int f_1^{p_1}\, dm = \int_1^\infty \frac{1}{x^{p_1(1/p_1+\delta)}}\, dm = \int_1^\infty \frac{1}{x^{1+p_1\delta}}\, dm < \infty$$

$$\int f_2^{p_2}\, dm = \int_1^\infty \frac{1}{x^{p_2(1/p_2+\delta)}}\, dm = \int_1^\infty \frac{1}{x^{1+p_2\delta}}\, dm < \infty$$

We compute:

$$(f_1 f_2)^s = \left(\frac{1}{x^{1/p_1+1/p_2+2\delta}}\right)^s \chi_{[1,\infty)} = \frac{1}{x^{s/p_1+s/p_2+2\delta s}} \chi_{[1,\infty)}$$

Our goal is to choose δ so that $s/p_1 + s/p_2 + 2\delta s < 1$. Recall from our work above that $\frac{r}{p_1} + \frac{r}{p_2} = 1$. Since $s < r$, we have $\frac{s}{p_1} + \frac{s}{p_2} = \alpha < 1$. Thus, we need ensure we can choose δ so that $\alpha + 2\delta s < 1$. Indeed, solving for δ results in the condition $(1 - \alpha)/2s > \delta$, which we can satisfy because $(1 - \alpha)/2s > 0$. Choose, say, $\delta = (1 - \alpha)/4s$ and we are done.

4. (Aug 2022) Let E and F be Borel subsets of $\mathbb{R}^2$ such that

$$m^1(E_x) = m^1(F_x) \quad \text{for all } x \in \mathbb{R}$$

where $A_x = \{y \in \mathbb{R} : (x,y) \in A\}$ and m^1 is the 1-dimensional Lebesgue measure. Show that $m^2(E) = m^2(F)$, where m^2 is the 2-dimensional Lebesgue measure.

What to spot: Recall that $m^2 = m^1 \times m^1$ for Borel sets. Also, problems involving x-sections typically call for the Fubini-Tonelli Theorem.

Recall that

$$m^n\big|_{\mathcal{B}(\mathbb{R}^n)} = m^1\big|_{\mathcal{B}(\mathbb{R})} \times \cdots \times m^1\big|_{\mathcal{B}(\mathbb{R})}$$

where m^n is the n-dimensional Lebesgue measure and $\mathcal{B}(\mathbb{R}^n)$ is the Borel σ-algebra on $\mathbb{R}^n$. Since the Borel σ-algebra is σ finite, we may use Tonelli's Theorem to compute $m^2(E)$ since $\chi_E \geq 0$,

$$m^2(E) = \int_{\mathbb{R}^2} \chi_E \, dm^2 = \int_{\mathbb{R}} \int_{\mathbb{R}} \chi_E(x,y) \, dy \, dx = \int_{\mathbb{R}} m^1(E_x) \, dx$$

Same reasoning applies for F:

$$m^2(F) = \int_{\mathbb{R}^2} \chi_F \, dm^2 = \int_{\mathbb{R}} \int_{\mathbb{R}} \chi_F(x,y) \, dy \, dx = \int_{\mathbb{R}} m^1(F_x) \, dx$$

Thus

$$m^2(E) = \int_{\mathbb{R}} m^1(E_x) \, dx = \int_{\mathbb{R}} m^1(F_x) \, dx = m^2(F)$$

5. (Aug 2023) Let $(X, \mathcal{A}, \mu)$ be a finite measure space. Prove $f \in L^1(\mu)$ if and only if

$$\sum_{k=1}^{\infty} 2^k \mu(\{x \in X : |f(x)| \geq 2^k\}) < \infty$$

What to spot: For problems involving things like $\mu(\{x : \|f(x)\| \geq \lambda\})$, it's often helpful to replace μ by an integral of an indicator function.

Suppose $f \in L^1(\mu)$. Define the sets $A_k = \{x \in X : |f(x)| \geq 2^k\}$ $E_k = A_k \setminus A_{k+1} = \{x \in X : 2^{k+1} > |f(x)| \geq 2^k\}$ for $k \geq 1$, and $E_0 = \{x \in X : 2 > |f(x)| \geq 0\}$, and notice all are measurable. Moreover, the E_ks form a partition of X. Define the function $g = \sum_{k=1}^{\infty} 2^k \chi_{A_k}$ and notice that, by Beppo-Levi's Theorem,

$$\int g\, d\mu = \int \sum_{k=1}^{\infty} 2^k \chi_{A_k}\, d\mu = \sum_{k=1}^{\infty} \int 2^k \chi_{A_k}\, d\mu = \sum_{k=1}^{\infty} 2^k \mu(\{x \in X : |f(x)| \geq 2^k\})$$

So it suffices to show $\int g\, d\mu < \infty$. For any fixed $x \in E_k$ for $k \geq 1$, we have $x \in A_i$ for $1 \leq i \leq k$, but $x \notin E_{k+1}$. Therefore, $g(x) = \sum_{i=1}^{k} 2^i \leq 2^{k+1}$. Since $|f(x)| < 2^{k+1}$, we can see that $g(x) \leq 2|f(x)|$ for every $x \in E_k$ with $k \geq 1$. If $x \in E_0$, then $g(x) = 0$, so we in fact have $g(x) \leq 2|f(x)|$ for all x. Thus

$$\int g\, d\mu \leq \int 2|f|\, d\mu = 2\|f\|_1 < \infty$$

Now suppose $\sum_k 2^k \mu(\{|f(x)| \geq 2^k\}) < \infty$. By the Rising Sun Lemma, we can write

$$\int_X |f|\, d\mu = \int_0^{\infty} \mu(\{x \in X : |f(x)| \geq t\})\, dt$$

We can rewrite the left-hand side as

$$\int_0^{\infty} \mu(\{x \in X : |f(x)| \geq t\})\, dt = \int_0^{2} \mu(\{x \in X : |f(x)| \geq t\})\, dt + \sum_{k=1}^{\infty} \int_{2^k}^{2^{k+1}} \mu(\{x \in X : |f(x)| \geq t\})\, dt$$

For the first integral, note that $\mu(\{x \in X : |f(x)| \geq t\}) \leq \mu(X)$ for all t, so $\int_0^2 \mu(\{x \in X : |f(x)| \geq t\})\, dt \leq \int_0^2 \mu(X)\, dt = 2\mu(X) < \infty$ since we assumed X to be a finite measure space.

For the second integral, if $t \in [2^k, 2^{k+1})$, then $\mu(\{x \in X : |f(x)| \geq t\}) \leq \mu(\{x \in X : |f(x)| \geq 2^k\})$. Therefore,

$$\begin{aligned}
\sum_{k=1}^{\infty} \int_{2^k}^{2^{k+1}} \mu(\{x \in X : |f(x)| \geq t\})\, dt &\leq \sum_{k=1}^{\infty} \int_{2^k}^{2^{k+1}} \mu(\{x \in X : |f(x)| \geq 2^k\})\, dt \\
&= \sum_{k=1}^{\infty} (2^{k+1} - 2^k) \mu(\{x \in X : |f(x)| \geq 2^k\}) \\
&\leq \sum_{k=1}^{\infty} 2^{k+1} \mu(\{x \in X : |f(x)| \geq 2^k\})
\end{aligned}$$

Putting everything together, we have

$$\begin{aligned}\int |f|\,d\mu &\le 2\mu(X) + \sum_{k=1}^{\infty} 2^{k+1}\mu(\{x \in X : |f(x)| \ge 2^k\}) \\ &= 2\mu(X) + 2\sum_{k=1}^{\infty} 2^{k}\mu(\{x \in X : |f(x)| \ge 2^k\}) < \infty\end{aligned}$$

So $f \in L^1$.

6. (Aug 2020) Suppose that W is a Lebesgue nonmeasurable subset of $[0,1]$. Prove that there exists some $0 < \varepsilon < 1$ such that for any Lebesgue measurable subset E of $[0,1]$ with $m(E) \geq \varepsilon$, the set $W \cap E$ must be Lebesgue nonmeasurable.

> What to spot: If no such ε exists, we can find sets E_n that "fill up" $[0,1]$ while $W \cap E_n$ remains measurable for each n.

Suppose to the contrary that for each $0 < \varepsilon < 1$, there is a Lebesgue measurable subset E with $m(E) \geq \varepsilon$, but $W \cap E$ is Lebesgue measurable. Thus we can define a sequence of measurable sets $[E_n]$ so that $m(E_n) \geq 1 - \frac{1}{n}$ and $W \cap E_n$ is measurable for all n. Define $E = \bigcup_n E_n$. Let $F_k = \bigcup_{n=1}^k E_n$. Then F_k is measurable for all k, $F_k \subset F_{k+1} \subset \ldots$, $m(F_k) \geq m(E_k) = 1 - \frac{1}{k}$ and $\bigcup_k F_k = \bigcup_n E_n = E$ Then by continuity from below, $m(E) = m(\bigcup_n E_n) = m(\bigcup_k F_k) = \lim_k m(F_k) \geq \lim_k 1 - \frac{1}{k} = 1$. Since $E \subset [0,1]$, $m(E) \leq 1$, so $m(E) = 1$.

We claim these conditions prove W is measurable. Indeed, we have

$$W = (W \cap E) \cup (W \cap E^C) = \left(W \cap \bigcup_n E_n \right) \cup (W \cap E^C) = \bigcup_n (W \cap E_n) \cup (W \cap E^C)$$

Since $m(E^C) = 1 - m(E) = 0$, it follows that $W \cap E^C$ is Lebesgue measurable as a subset of a Lebesgue null set (recall the Lebesgue measure is complete over the Lebesgue measurable sets). By construction, $W \cap E_n$ is measurable for each n, so $\bigcup_n (W \cap E_n)$ is measurable as a countable union of measurable sets. But then W itself is a union of measurable sets and therefore measurable, a contradiction.

AUGUST 2024 MEASURE PRELIM TUTORING PROBLEM SET 2 SOLUTIONS

1. (Aug 2018) Show that there exists a Borel measure ν on $\mathbb{R}$ that satisfies both

    1. ν and m are mutually singular
    2. $0 < \nu(B(x,r)) < \infty$ for all $x \in \mathbb{R}$ and $r > 0$

What to spot: The Lebesgue measure is 0 on any subset of the rationals, so try to construct a measure that is positive only on the rationals.

Let $[q_i]$ be an enumeration of the rationals, and define the set function ν as

$$\nu(E) = \sum_{\{i:q_i \in E\}} 2^{-i}$$

To justify that ν is a Borel measure, first note that $\nu(\emptyset) = 0$ since the sum defining ν ranges over an empty set. If $[E_n]$ is a sequence of disjoint Borel sets, then

$$\nu\left(\bigcup_n E_n\right) = \sum_{\{i:q_i \in \bigcup_n E_n\}} 2^{-i} = \sum_n \sum_{\{i:q_i \in E_n\}} 2^{-i} = \sum_n \nu(E_n)$$

So ν is a Borel measure.

We can see that ν and μ are mutually singular, since $A = \mathbb{Q}$ and $B = \mathbb{R} \setminus \mathbb{Q}$ are two measurable sets such that $A \cup B = \mathbb{R}$, $A \cap B = \emptyset$, $m(A) = 0$, and $\nu(B) = 0$

To show $0 < \nu(B(x,r)) < \infty$ for all $x \in \mathbb{R}$ and $r > 0$, note that $\nu(\mathbb{R}) = \sum_{\{i:q_i \in \mathbb{R}\}} 2^{-i} = \sum_{i=1}^{\infty} 2^{-i} = 1$, so $\nu(B(x,r)) < \infty$ by monotonicity. Furthermore, since the rationals are dense in $\mathbb{R}$, any ball $B(x,r)$ contains at least one rational number. Thus the set $\{i : q_i \in B(x,r)\}$ is nonempty, which means $\nu(B(x,r)) = \sum_{\{i:q_i \in B(x,r)\}} 2^{-i} > 0$.

2. (Jan 2016) Let K be a compact interval in $\mathbb{R}$ and let $f_n : K \to \mathbb{R}$ be a sequence of functions.

(a) Suppose that the sequence $[f_n]$ converges almost everywhere on K with respect to m. Show that $[f_n]$ converges in measure.

What to spot: Since K is compact, it has finite measure. In a finite measure space, convergence almost everywhere implies almost uniform convergence.

Let f be the almost-everywhere limit function and fix $\varepsilon, \delta > 0$. We wish to show that $m(\{x \in K : |f(x) - f_n(x)| > \delta\}) < \varepsilon$ for sufficiently large n. Because K is a compact set and m is a Radon measure, $m(K) < \infty$. Since $f_n \to f$ almost everywhere and K is a finite measure space, Egorov's Theorem states there exists a measurable set E with $m(E) < \varepsilon$ such that $f_n \to f$ uniformly on $K \setminus E$. Thus, there is some $N \in \mathbb{N}$ such that $\sup_{x \in K \setminus E} |f(x) - f_n(x)| < \delta$ for all $n > N$. We therefore have that $\{x \in K : |f(x) - f_n(x)| > \delta\} \subseteq E$ whenever $n > N$. Thus by monotonicity, $m(\{x \in K : |f(x) - f_n(x)| > \delta\}) \leq m(E) < \varepsilon$, for all $n > N$.

(b) Must the conclusion of part (a) be true if K is not compact? Give a proof or counterexample.

What to spot: Recall that convergence almost everywhere does not imply convergence in measure.

No, the conclusion may be false if K is not compact. Let $K = [0, \infty)$ and define $f_n = \chi_{[n,n+1]}$ and $f \equiv 0$. Then $f_n \to f$ pointwise everywhere since $f_n(x) = 0$ whenever $n > \lfloor x \rfloor + 1$. However, $m(\{x \in K : |f_n(x) - f(x)| > 0.1\}) = 1$ for all n, so $f_n \not\to f$ in measure. Since the limit functions for convergences almost everywhere and convergence in measure must coincide if they exist, we conclude that $[f_n]$ cannot converge in measure to any function.

(c) Suppose that all f_n are differentiable and

1. There exists $M > 0$ such that $\|f_n'\|_\infty < M$ for all n, and (NOTE: I believe they intended this to mean $|f_n'(x)| < M$ for all $x \in K$. We don't need this per se, but not having it does make the arguments more convoluted. I've written an alternative solution in red.)

2. For each n there exists x_n such that $f_n(x_n) = 0$

Prove that there is a subsequence of $[f_n]$ that converges uniformly on K to a continuous limit function f.

What to spot: The statement to prove is the exact conclusion of the Arzelà-Ascoli Theorem.

We want to invoke the Arzelà-Ascoli Theorem, so we need to show that the sequence $[f_n]$ is uniformly bounded and uniformly equicontinuous. In this case, uniform equicontinuity follows rather quickly. Since $\|f_n'\|_\infty < M$ for all n, each f_n is Lipschitz with a common Lipschitz constant M. That is, for each n and $x, y \in K$, $|f_n(x) - f_n(y)| < M|x - y|$. Thus, for any $\varepsilon > 0$, if we take $\delta < \varepsilon/M$, then $|x - y| < \delta$ implies $|f_n(x) - f_n(y)| < M|x - y| < \varepsilon$, and this holds regardless of the value of n. So $[f_n]$ is uniformly equicontinuous.

To show uniform boundedness, we claim that $|f_n(x)| \leq M \cdot m(K)$ for all $n \in \mathbb{N}$ and $x \in K$. First, since each f_n is Lipschitz, we conclude the each f_n is absolutely continuous on the compact interval K. Thus, we may invoke the Fundamental Theorem of Calculus for Lebesgue Integrals. Using the condition that there is some x_n with $f_n(x_n) = 0$, for any $n \in \mathbb{N}$ and $x \in K$, the Fundamental Theorem of Calculus gives

$$f_n(x) = f_n(x) - f_n(x_n) = \int_{x_n}^{x} f_n'(t)\, dt$$

Taking absolute values and simplifying,

$$|f_n(x)| = \left| \int_{x_n}^{x} f_n'(t)\, dt \right| \leq \int_{x_n}^{x} |f_n'(t)|\, dt \leq \int_{x_n}^{x} M\, dt = M|x_n - x| \leq M \cdot m(K)$$

Thus the f_n are uniformly bounded. Since K is a compact interval, and $[f_n]$ is uniformly bounded and uniformly equicontinuous, the Arzelà-Ascoli Theorem implies there exists a subsequence $[f_{n_k}]$ and a continuous function f on K such that $f_{n_k} \to f$ uniformly.

If they really meant $\|f_n'\|_\infty < M$ as in $|f_n'(x)| < M$ for almost every x, then things are a bit more complicated. We need to invoke the following theorem (from Rudin's Real and Complex Analysis, Theorem 7.21):

If $f : [a, b] \to \mathbb{R}$ is differentiable at each point in $[a, b]$ and $f' \in L^1[a, b]$, then

$$f(x) - f(a) = \int_{a}^{x} f'(t)\, dt$$

We are given f_n' is differentiable for each n and $f_n' \in L^1$ for all n since $\|f_n'\|_\infty < M$ for each n, so $\int_K |f_n'|\, dm \leq \int_K \|f_n'\|_\infty \leq M \cdot m(K)$.

Thus each f_n satisfies f_n differentiable almost everywhere, $f_n' \in L^1[a, b]$, and $f(x) - f(a) = \int_a^x f'(t)\, dt$, so each f_n is absolutely continuous. On a compact interval, absolute continuity along with the bound on the derivative implies Lipschitz continuity with constant M since

$$|f_n(x) - f_n(y)| = \left| \int_{x}^{y} f_n'(t)\, dt \right| \leq \int_{x}^{y} |f_n'(t)|\, dt \leq \int_{x}^{y} \|f_n'\|_\infty \leq M|x - y|$$

Then we proceed as above.

3. (Aug 2013) Let $q_1, q_2, \ldots$ be an enumeration of the rationals in $[0,1]$. Consider the infinite series

$$s(x) = \sum_{n=1}^{\infty} \frac{1}{n^2 \sqrt{|x - q_n|}}$$

(a) Prove that s converges m-almost everywhere.

What to spot: To prove a series of positive terms converges almost everywhere, we need to show it is finite almost everywhere. L^1 functions are finite almost everywhere.

We will prove that $s \in L^1([0,1])$, which will mean $s(x) < \infty$ almost everywhere, ie, the series converges almost everywhere. By Beppo-Levi's Theorem (or MCT),

$$\int_{[0,1]} s\, dm = \int_{[0,1]} \sum_{n=1}^{\infty} \frac{1}{n^2 \sqrt{|x - q_n|}}\, dm = \sum_{n=1}^{\infty} \frac{1}{n^2} \int_{[0,1]} \frac{1}{\sqrt{|x - q_n|}}\, dm$$

We want to find a uniform bound on $\int_{[0,1]} \frac{1}{\sqrt{|x-q_n|}}\, dm$. For each n, we can break the integral into two, one where $x \leq q_n$ and the other where $x \geq q_n$ (the overlap doesn't affect the integrals' values since singletons have measure 0).

$$\begin{aligned} \int_{[0,1]} \frac{1}{\sqrt{|x - q_n|}}\, dm &= \int_{[0,q_n]} \frac{1}{\sqrt{q_n - x}}\, dm + \int_{[q_n,1]} \frac{1}{\sqrt{x - q_n}}\, dm \\ &= -2\sqrt{q_n - x}\Big|_0^{q_n} + 2\sqrt{x - q_n}\Big|_{q_n}^{1} \\ &= 2\sqrt{q_n} + 2\sqrt{1 - q_n} \leq 4 \end{aligned}$$

This bound holds for all n, so

$$\int_{[0,1]} s\, dm = \sum_{n=1}^{\infty} \frac{1}{n^2} \int_{[0,1]} \frac{1}{\sqrt{|x - q_n|}}\, dm \leq \sum_{n=1}^{\infty} \frac{4}{n^2} = \frac{2\pi^2}{3}$$

Thus $s \in L^1([0,1])$, as desired. NOTE: If this function were defined on $\mathbb{R}$ instead of $[0,1]$, first prove that $s \in L^1[-n,n]$ for any n using the strategy above. Define $A_n = \{x \in [-n,n] : s(x) = \infty\}$, then $m(A_n) = 0$ and $\{x \in \mathbb{R} : s(x) = \infty\} = \bigcup_n A_n$ has measure 0.

(b) Prove that s is unbounded on any non-empty open subinterval of $[0,1]$.

What to spot: The rationals are dense, so we can pick a sequence that causes one of the terms of s to blow up.

Let I be a nonempty open subinterval of $[0,1]$. Because the rationals are dense in $[0,1]$, there is some rational number $q_N \in I$. Let $[x_j]$ be a sequence of irrational numbers in I such that $x_j \to q_N$. Then $N^{-2}|x_j - q_N|^{-1/2} \to \infty$, and since all other terms in the series are positive, we have $s(x_j) \geq N^{-2}|x_j - q_N|^{-1/2}$ for all j. Thus, $s(x_j) \to \infty$, meaning s is unbounded on I.

4. Let $A \subset \mathbb{R}$ be Lebesgue measurable with finite Lebesgue measure. Prove

$$\lim_{|x| \to 0} m(A \cap (x + A)) = m(A)$$

Here $x + A = \{x + y : y \in A\}$

What to spot: If you're not given much, approximation via nice sets is usually the way to go.

We will first prove the claim for an open set U with $m(U) < \infty$. Let $x_n \to 0$, then

$$\lim_{n\to\infty} m(U \cap (x_n + U)) = \lim_{n\to\infty} \int \chi_{U \cap x_n + U} \, dm$$

Because $\chi_{U \cap x_n + U} \leq \chi_U$ for all n and $\int \chi_U = m(U) < \infty$, we may invoke the Dominated Convergence Theorem so long as we can show $\chi_{U \cap x_n + U} \to \chi_U$. Since U is open, it follows that for any $y \in U$, there is some $\delta > 0$ such that $(y - \delta, y + \delta) \subset U$. Since $|x_n| < \delta$ for sufficiently large n, it follows that $y + x_n \in (y - \delta, y + \delta)$ for sufficiently large n, meaning $\chi_{U \cap x_n + U}(y) \to \chi_U(y)$. Thus $\chi_{U \cap x_n + U} \to \chi_U$ pointwise and by the DCT,

$$\lim_{n\to\infty} m(U \cap (x_n + U)) = \lim_{n\to\infty} \int \chi_{U \cap x_n + U} \, dm = \int \lim_{n\to\infty} \chi_{U \cap x_n + U} \, dm = \int \chi_U \, dm = m(U)$$

Now suppose A is an arbitrary Lebesgue measurable set and let $\varepsilon > 0$. Then there is an open set $U \supset A$ such that $m(U \setminus A) < \varepsilon$. We can decompose $U \cap (x_n + U)$ as

$$U \cap (x_n + U) = [A \cap (A + x_n)] \;\cup\; [(U \setminus A) \cap (A + x_n)] \;\cup\; [U \cap ((U \setminus A) + x_n)]$$

These sets are disjoint and measurable, so we have

$$m(U \cap (x_n + U)) = m(A \cap (A + x_n)) + m((U \setminus A) \cap (A + x_n)) + m(U \cap ((U \setminus A) + x_n))$$

Since $m(U \setminus A) < \varepsilon$, the latter two terms on the right hand side of the equation above are both $< \varepsilon$. Thus

$$m(U \cap (x_n + U)) < m(A \cap (A + x_n)) + 2\varepsilon \implies m(A \cap (A + x_n)) > m(U \cap (x_n + U)) - 2\varepsilon$$

Since $m(U \cap (x_n + U)) \to m(U)$, for n sufficiently large we have $|m(U \cap (x_n + U)) - m(U)| < \varepsilon$, and since $m(U \cap (x_n + U)) \leq m(U)$, we have $m(U) \geq m(U \cap (x_n + U)) - \varepsilon$. Finally, since $U \supset A$, we have $m(U) \geq m(A)$. Thus

$$m(A \cap (A + x_n)) > m(U) - 3\varepsilon \geq m(A) - 3\varepsilon$$

Clearly $m(A) \geq m(A \cap (A + x_n))$, so $m(A) \geq m(A \cap (A + x_n)) > m(A) - 3\varepsilon$. Since ε was arbitrary, the limit holds.

5. (Aug 2020) (a) Let (X, Σ, μ) and (X, Σ, ν) be two measure spaces with $\nu(X) < \infty$. Prove that ν is absolutely continuous with respect to μ if and only if for every $\varepsilon > 0$ there exists $\delta > 0$ such that if $A \in \Sigma$ with $\mu(A) < \delta$ then $\nu(A) < \varepsilon$.

What to spot: If and only if problems with ε-δ conditions can usually be solved by using contradiction both ways.

Suppose the stated ε-δ condition holds. Let $A \in \Sigma$ be a measurable set with $\mu(A) = 0$. If $\nu(A) > 0$, select $\varepsilon = \nu(A)/2$. Then no matter our choice of $\delta > 0$, we have $\mu(A) < \delta$ but $\nu(A) \geq \varepsilon$, so we violate the ε-δ condition, which is a contradiction. Thus $\nu(A) = 0$ and therefore $\nu \ll \mu$.

Now suppose that $\nu \ll \mu$, but suppose that there exists $\varepsilon > 0$ such that for every $\delta > 0$, there exists some $A_\delta \in \Sigma$ such that $\mu(A_\delta) < \delta$ but $\nu(A_\delta) \geq \varepsilon$. Let $[A_n]$ be a sequence of sets such that $A_n \in \Sigma$, $\mu(A_n) < 2^{-n}$, and $\nu(A_n) \geq \varepsilon$ for all n. Let $A = \limsup_n A_n = \bigcap_{n=1}^{\infty} \bigcup_{k \geq n} A_k$. Since $\sum_n \mu(A_n) < \infty$, the Borel-Cantelli Lemma implies $\mu(A) = 0$. On the other hand, since $\nu(X) < \infty$ and the sets $B_n = \bigcup_{k \geq n} A_k$ form a descending chain of sets (ie, $B_1 \supset B_2 \supset \dots$), we may apply continuity from above to derive a contradiction:

$$\nu(A) = \nu\left(\bigcap_{n=1}^{\infty} \bigcup_{k \geq n} A_k\right) = \lim_{n \to \infty} \nu\left(\bigcup_{k \geq n} A_k\right) \geq \lim_{n \to \infty} \nu(A_n) \geq \varepsilon$$

Here, we contradicted $\nu \ll \mu$.

(b) Give an example of a pair of measure spaces (X, Σ, μ) and (X, Σ, ν) such that ν is absolutely continuous with respect to μ, but given $\varepsilon > 0$ there is no $\delta > 0$ such that $\nu(A) < \varepsilon$ for every $A \in \Sigma$ with $\mu(A) < \delta$.

What to spot: Since $\nu(X) < \infty$ was assumed in part (a), we ought to look for a measure with $\nu(X) = \infty$. Moreover, such a measure should blow up on "small" sets.

Let $X = [0, 1]$, $\Sigma = \mathcal{L}$ be the Lebesgue σ-algebra on $[0, 1]$, and $\mu = m$. Let ν be defined on $\mathcal{L}$ by $\nu(A) = \int_A \frac{1}{x}\, dm$. No matter how we choose $\varepsilon > 0$, the set $A = [0, \delta/2]$ has $\mu(A) < \delta$ but $\nu(A) = \infty > \varepsilon$.

6. (Jan 2017) Let $f, g \in L^1(\mathbb{R}^n, m)$ be nonnegative functions such that

$$\liminf_{k\to\infty} \frac{\int_{B(x,1/k)} f(y)\,dy}{\int_{B(x,1/k)} g(y)\,dy} \leq 1$$

For x-almost everywhere in $\mathbb{R}^n$. Prove that $f \leq g$ almost everywhere.

What to spot: The integrals in the problem are reminiscent of the Lebesgue Differentiation Theorem, so we ought to try that.

Suppose to the contrary that $f > g$ on some measurable set A with $m(A) > 0$. By the Lebesgue Differentiation Theorem, there exists a null set N_1 such that

$$\lim_{r\downarrow 0} \frac{1}{m(B(x,r))} \int_{B(x,r)} f(y)\,dy = f(x) \quad \text{for all } x \in \mathbb{R}^n \setminus N_1.$$

Likewise, there is another null set N_2 such that

$$\lim_{r\downarrow 0} \frac{1}{m(B(x,r))} \int_{B(x,r)} g(y)\,dy = g(x) \quad \text{for all } x \in \mathbb{R}^n \setminus N_2.$$

Let N_3 be the set of points where the inequality in the problem statement fails to hold. Then $m(A \setminus (N_1 \cup N_2 \cup N_3)) > 0$, so we can select a point $x \in A \setminus (N_1 \cup N_2 \cup N_3)$ that satisfies $f(x) > g(x)$, the inequality in the problem statement, and the conclusion of the Lebesgue Differentiation Theorem for both f and g. However, these properties cannot all hold simultaneously since as $\frac{1}{k} \downarrow 0$, assuming $g(x) \neq 0$,

$$\liminf_{k\to\infty} \frac{\int_{B(x,1/k)} f(y)\,dy}{\int_{B(x,1/k)} g(y)\,dy} \geq \frac{\liminf_k \frac{1}{m(B(x,1/k))} \int_{B(x,1/k)} f(y)\,dy}{\limsup_k \frac{1}{m(B(x,1/k))} \int_{B(x,1/k)} g(y)\,dy} = \frac{f(x)}{g(x)} > 1$$

If $g(x) = 0$, then because $f(x) > g(x) = 0$, the $\liminf$ is ∞, which still is a contradiction.

AUGUST 2024 MEASURE PRELIM TUTORING PROBLEM SET 3 SOLUTIONS

1. (Aug 2018) Let $T = \{(x, y) \in \mathbb{R}^2 : 0 \leq |x| \leq y \leq 1\}$ and let μ be the restriction of m to T. Let $f \in L^2(T, \mu)$. Prove that

(a) $f \in L^1(T, \mu)$

What to spot: In a finite measure space, $f \in L^p$ implies $f \in L^q$ for all $1 \leq q \leq p$.

By the Cauchy-Schwartz inequality (Hölder with $p = q = 2$), since $\mu(T) = 1 < \infty$:

$$\|f\|_1 = \|f\chi_T\|_1 \leq \|f\|_2\|\chi_T\|_2 = \|f\|_2\mu(T)^{1/2} < \infty$$

(b) $\liminf_{y\to0^+} \int_{-y}^{y} |f(x, y)|\, dx = 0$

What to spot: This one is tricky. If you have little to work with, try to contradict the strongest piece of information you are given (that being $f \in L^2$ in this case).

Suppose $\liminf_{y\to0^+} \int_{-y}^{y} |f(x, y)|\, dx = c > 0$. Let $\varepsilon = c/2$ if c is finite; otherwise let $\varepsilon = 1$ if c is infinite. By the definition of $\liminf$, there exists some $\delta > 0$ such that if $0 < y < \delta$, then $\int_{-y}^{y} |f(x, y)|\, dx > \varepsilon$. Applying the Cauchy-Schwartz inequality to this one-dimensional integral,

$$\int_{-y}^{y} |f(x, y)|\, dx \leq \left(\int_{-y}^{y} |f(x, y)|^2\, dx\right)^{1/2} \left(\int \chi^2_{[-y,y]}\, dx\right)^{1/2} = \left(\int_{-y}^{y} |f(x, y)|^2\, dx\right)^{1/2} \sqrt{2y}$$

Rearranging and squaring both sides, we obtain

$$\int_{-y}^{y} |f(x, y)|^2\, dx \geq \frac{1}{2y} \left(\int_{-y}^{y} |f(x, y)|\, dx\right)^2$$

If $0 < y < \delta$, we then have

$$\int |f(x, y)|^2\, dx \geq \frac{\varepsilon^2}{2y}$$

Since $\mathbb{R}$ with the Lebesgue measure is σ-finite, we can compute the L^2-norm of f on T as an iterated integral using Tonelli's theorem, which allows us to derive a contradiction:

$$\begin{aligned}\|f\|_2^2 = \int_0^1 \int_{-y}^{y} |f(x, y)|^2\, dx\, dy &\geq \int_0^1 \frac{1}{2y} \left(\int_{-y}^{y} |f(x, y)|\, dx\right)^2 dy \\ &\geq \int_0^\delta \frac{1}{2y} \left(\int_{-y}^{y} |f(x, y)|\, dx\right)^2 dy \\ &\geq \int_0^\delta \frac{\varepsilon^2}{2y}\, dy = \infty\end{aligned}$$

Here, we contradicted $\|f\|_2 < \infty$. Thus $\liminf_{y\to0^+} \int_{-y}^{y} |f(x, y)|\, dx = 0$

2. (Jan 2022) Let $f : \mathbb{R} \to \mathbb{R}$ be Lebesgue measurable. Prove there exists a constant $C > 0$ such that

$$\|f\|_1 \leq C(\|f\|_2 + \|x^2 f\|_2)$$

$\boxed{\text{What to spot: } |x^2 f| \geq |f| \text{ whenever } |x| > 1\text{, so break up into the cases of } |x| > 1 \text{ and } |x| \leq 1.}$

Consider the set $[-1,1]$, and break up $\|f\|_1$ as

$$\int |f|\, dm = \int_{[-1,1]} |f|\, dm + \int_{[-1,1]^c} |f|\, dm$$

On $[-1,1]$, we can apply the Cauchy-Schwartz inequality (special case of Hölder's inequality with $p = q = 2$):

$$\int_{[-1,1]} |f|\, dm = \int |f|\chi_{[-1,1]}\, dm \leq \left(\int |f|^2\, dm\right)^{1/2} \left(\int \chi_{[-1,1]}^2\, dm\right)^{1/2} = \sqrt{2}\, \|f\|_2$$

On $[-1,1]^c$, we can rewrite f as $f = \frac{1}{x^2} \cdot x^2 f$, and then apply Cauchy-Schwartz again

$$\int_{[-1,1]^c} |f|\, dm = \int \left|\frac{1}{x^2} \cdot x^2 f\right| \chi_{[-1,1]^c}\, dm \leq \left(\int |x^2 f|^2\, dm\right)^{1/2} \left(\int \left(\frac{1}{x^2}\chi_{[-1,1]^c}\right)^2 dm\right)^{1/2}$$

$$= \|x^2 f\|_2 \cdot \left(\int_{[-1,1]^c} \frac{1}{x^4}\, dm\right)^{1/2} = \frac{\sqrt{2}}{\sqrt{3}} \|x^2 f\|_2$$

Thus, taking $C = \sqrt{2}$, we have

$$\|f\|_1 \leq C(\|f\|_2 + \|x^2 f\|_2)$$

3. (Jan 2015) Let $(X, \mathcal{A}, \mu)$ be a σ-finite measure space and $p \in [1, \infty)$. Let $f : X \to \mathbb{R}$ be $\mathcal{A}$-measurable and define

$$R_p(x) = x^{p-1}\mu(\{|f| > x\})$$

(a) Prove if $f \in L^p(\mu)$, then $\lim_{x\to\infty} xR_p(x) = 0$.

What to spot: We can rewrite $\mu(\{|f| > x\})$ as an integral of an indicator function.

We can rewrite $R_p(x)$ as an integral of an indicator function, namely,

$$R_p(x) = x^{p-1} \int \chi_{\{|f|>x\}}\, d\mu$$

Since $|f| > x$ on the indicated set, we obtain the following inequality

$$xR_p(x) = x^p \int \chi_{\{|f|>x\}}\, d\mu = \int x^p \chi_{\{|f|>x\}}\, d\mu \leq \int |f|^p \chi_{\{|f|>x\}}$$

Select an arbitrary sequence $x_n \to \infty$ and define the sequence of functions $f_n = |f|^p \chi_{\{|f|>x_n\}}$. We have $|f_n| \leq |f|^p$ everywhere for all n and $|f|^p$ is integrable on X since $f \in L^p(\mu)$. Finally, $\lim_n f_n = 0$ almost everywhere since $\lim_n |f|^p \chi_{\{|f|>x_n\}} = |f|^p \chi_{\{|f|=\infty\}} = 0$ (an integrable function can only $= \infty$ on a set of measure 0). Thus, by the Dominated Convergence Theorem

$$\lim_{n\to\infty} x_n R_p(x_n) \leq \lim_{n\to\infty} \int f_n\, d\mu = \int \lim_{n\to\infty} f_n\, d\mu = 0$$

Since the sequence was arbitrary, we conclude the limit $\lim_{x\to\infty} xR_p(x) = 0$ holds.

(b) Prove $f \in L^p(\mu)$ if and only if $R_p \in L^1([0, \infty), m)$ (Lebesgue measure)

What to spot: We have a double integral on a product of σ-finite spaces, so we will use the Fubini-Tonelli Theorem.

Using our representation of $R_p(x)$ as an integral from part (a), we have

$$\int_0^\infty R_p(x)\, dm(x) = \int_0^\infty \int_X x^{p-1}\, \chi_{\{|f|>x\}}\, d\mu\, dm(x)$$

Since the integrand is non-negative, and X and $[0, \infty)$ are σ-finite measure spaces, we can use Tonelli's Theorem to swap the integrals. Our new bounds of integration for the $dm(x)$ integral are 0 and $|f|$ (for each fixed x, it is included in the integral so long as it is $< |f|$). Thus,

$$\int_0^\infty R_p(x)\, dm(x) = \int_X \int_0^{|f|} x^{p-1}\, dm(x)\, d\mu = \int_X \frac{1}{p} x^p \Big|_0^{|f|} d\mu = \frac{1}{p} \int_X |f|^p\, d\mu$$

Thus, $\int_0^\infty R_p(x)\, dm$ is finite if and only if $\int |f|^p\, d\mu$ is finite, proving the desired statement.

4. (Aug 2016) Compute the limit

$$\lim_{j\to\infty}\int_{-j}^{j}\frac{\sin(x^j)}{x^{j-2}}\,dx$$

and provide justification for all steps in your reasoning.

What to spot: We need to use either MCT or DCT since we have a limit of integrals. Since the integrand is not nonnegative, we will need to use DCT.

We can rewrite the integrand as $\frac{\sin(x^j)}{x^{j-2}} = \frac{\sin(x^j)}{x^j}x^2$. Define the sequence of functions $f_j = \frac{\sin(x^j)}{x^j}x^2\chi_{[-j,j]}$ and let g be defined as

$$g(x) = \begin{cases} 1 & x\in[-1,1] \\ \frac{1}{x^2} & x\in[-1,1]^c \end{cases}$$

We observe that g is measurable because it is continuous. We claim that $|f_j|\le g$ for $j\ge 4$. Indeed, if $x\in[-1,1]$, then because $|\sin(x)|\le|x|$ for all x,

$$\left|\frac{\sin(x^j)}{x^j}x^2\right| \le \frac{|x|^j}{|x|^j}|x|^2 = |x|^2 \le 1$$

If $x\in[-1,1]^c$ and $j\ge 4$, then because $|\sin(x)|\le 1$ for all x

$$\left|\frac{\sin(x^j)}{x^j}x^2\right| \le \frac{1}{|x|^j}|x|^2 = \frac{1}{|x|^{j-2}} \le \frac{1}{x^2}$$

We now want to show $g\in L^1$. Define the sequence $g_n = g\chi_{[-n,n]}$. Then each g_n is continuous almost everywhere (therefore measurable), and bounded on the compact set $[-n,n]$, so each g_n is Riemann integrable and the Riemann integral coincides with the Lebesgue integral. Moreover, $g_n\uparrow g$. Therefore, we may compute $\int g\,dm$ as a limit of Riemann integrals using the Monotone Convergence Theorem.

$$\int g\,dm = \int \lim_{n\to\infty} g_n\,dm = \lim_{n\to\infty}\int g_n\,dm = \lim_{n\to\infty}\int_{-n}^{n} g_n(x)\,dx$$

$$= \lim_{n\to\infty} 2\int_0^1 1\,dx + 2\int_1^n \frac{1}{x^2}\,dx = 2 + \lim_{n\to\infty}\left.\frac{-2}{x}\right|_1^n = 2+2 = 4<\infty$$

We now have satisfied all of the hypotheses of the DCT. Since each f_j is continuous almost everywhere, we can also compute the sequence of Riemann integrals $\lim_j\int f_j$ as a sequence of Lebesgue integrals. Computing the limit $\lim_j f_j$, if $|x|<1$, then $x^j\to 0$ as $j\to\infty$. Thus $\frac{\sin(x^j)}{x^j}\to 1$, so $\lim_j f_j = x^2$ for $|x|<1$. If $|x|=1$, then $f_j(x)=\sin(1)$ for all j. If $|x|>1$, then because $x^{j-2}\to\infty$, $\frac{\sin(x^j)}{x^{j-2}}\to 0$. Thus $\lim_j f_j(x) = \begin{cases} x^2 & x\in(-1,1) \\ \sin(1) & x\in\{-1,1\} \\ 0 & x\in[-1,1]^c \end{cases}$.

Since $m(\{-1,1\})=0$, we can ignore that set in the integral and compute,

$$\lim_{j\to\infty}\int_{-j}^{j}\frac{\sin(x^j)}{x^{j-2}}\,dx = \lim_{j\to\infty}\int f_j\,dm = \int\lim_{j\to\infty} f_j\,dm = \int_{-1}^{1} x^2\,dx = \frac{2}{3}$$

5. (Jan 2022) Prove or disprove the following statements:

(a) If $S^{n-1} = \{x \in \mathbb{R}^n : |x| = 1\}$, then $m(S^{n-1}) = 0$.

What to spot: A sphere is "thin" in $\mathbb{R}^n$, so its measure ought to be zero.

This is true. Recall the scaling property of the Lebesgue measure on $\mathbb{R}^n$: If $E \subset \mathbb{R}^n$ is a measurable set and r is a real number, then $m(rE) = |r|^n m(E)$, where $rE = \{rx : x \in E\}$ is the dilation of E by the factor r.

The closed unit ball $B = \{x \in \mathbb{R}^n : |x| \leq 1\}$ is compact, so $m(B) < \infty$ since m is a Radon measure. Let $B_0 = \{x \in \mathbb{R}^n : |x| < 1\}$ denote the open unit ball. We can write $S^{n-1} = B \setminus B_0$, so $m(S^{n-1}) = m(B) - m(B_0)$, and it thus suffices to prove $m(B) = m(B_0)$.

Let $[r_k]$ be a sequence of real numbers satisfying $0 < r_k < 1$ for all k and $r_k \uparrow 1$. We have $r_k B \subset B_0$ for each k, and $\bigcup_k r_k B = B_0$. The sequence of sets $[r_k B]$ form an ascending sequence (ie, $r_1 B \subset r_2 B \subset \dots$). Thus by continuity from below,

$$m(B_0) = m\left(\bigcup_k r_k B\right) = \lim_{k\to\infty} m(r_k B) = \lim_{k\to\infty} |r_k|^n m(B) = m(B)$$

(b) Every nonnegative continuous $f \in L^1(\mathbb{R})$ satisfies $\limsup_{x\to\infty} f(x) \in [0, \infty)$.

What to spot: An L^1 function can be very large on small sets.

This is false. Let the function f be constructed as follows: around each natural number n construct a triangle of height n and base $1/n^3$ and 0 everywhere else. To be precise, let

$$f_n(x) = \begin{cases} 2n^4(x-n) + n & x \in [n - 1/2n^3, n] \\ -2n^4(x-n) + n & x \in [n, n + 1/2n^3] \\ 0 & \text{elsewhere} \end{cases}$$

Let $f = \sum_n f_n$. Then f is nonnegative, continuous and $\int |f| = \sum_n \frac{1}{2} n \frac{1}{n^3} = \sum_n \frac{1}{2n^2} < \infty$, so $f \in L^1(\mathbb{R})$. But $\limsup f(x) = \infty$ since $f(n) = n \to \infty$ as $n \to \infty$.

(c) If $f : (a,b) \to \mathbb{R}$ is differentiable, then f' is Lebesgue measurable.

What to spot: We can define the derivative as a limit of measurable functions.

This is true. Define the sequence $[f_n]$ as

$$f_n(x) = \begin{cases} \frac{f(x+1/n) - f(x)}{1/n} & \text{if } x + 1/n \in (a,b) \\ 0 & \text{otherwise} \end{cases}$$

Each f_n is Lebesgue measurable as the sum/composition of Lebesgue measurable functions (f is Lebesgue measurable by continuity). $f_n(x) \to f'(x)$ for each $x \in (a,b)$ by definition of the derivative, so f' is Lebesgue measurable as the pointwise limit of Lebesgue measurable functions.

6. (Aug 2015) Let f and $[f_n]$ be measurable functions on a measure space $(X, \mathcal{A}, \mu)$ and suppose that for any $\varepsilon > 0$ we have

$$\sum_{n=1}^{\infty} \mu(\{x \in X : |f_n(x) - f(x)| > \varepsilon\}) < \infty.$$

Prove that $f_n \to f$ μ-almost everywhere.

What to spot: Since $\sum_n \mu(A_n) < \infty$ is part of the Borel-Cantelli Lemma, we ought to try to use it here.

Define the sets $A_{n,k} = \{x \in X : |f_n(x) - f(x)| > 1/k\}$ for $n, k \geq 1$ and $B_k = \limsup_n A_{n,k} = \{x \in X : |f_n(x) - f(x)| > 1/k \text{ for } \infty\text{-ly many } n\}$. If x is a point such that $f_n(x) \not\to f(x)$, then there exists some k such that $|f_n(x) - f(x)| > 1/k$ for infinitely many values of n. That is, $x \in B_k$ for some value of k. If we let $B = \bigcup_{k=1}^{\infty} B_k$, then we conclude that $f_n(x) \to f(x)$ if and only if $x \notin B$. Our goal is to show B is measurable and $\mu(B) = 0$.

First we prove the measurability of B. Each $A_{n,k}$ is measurable since $A_{n,k}$ is the preimage of the set $(1/k, \infty)$ under the function $|f_n - f|$, which is measurable since f_n and f are measurable. Since $B_k = \bigcap_{n=1}^{\infty} \bigcup_{j \geq n} A_{j,k}$, B_k is measurable as the countable union/intersection of measurable sets. Finally, B is the countable union of measurable sets, so it is measurable.

To show $\mu(B) = 0$, we will show $\mu(B_k) = 0$ for all k. Indeed, the condition stated in the problem implies that $\sum_{n=1}^{\infty} \mu(\{x \in X : |f_n(x) - f(x)| > 1/k\}) < \infty$ for each $k \geq 1$. Thus $\sum_{n=1}^{\infty} \mu(A_{n,k}) < \infty$, so by the Borel-Cantelli Lemma, $\mu(\limsup_n A_{n,k}) = \mu(B_k) = 0$. Thus, by subadditivy of measure, $\mu(B) = \mu(\bigcup_k B_k) \leq \sum_k \mu(B_k) = \sum_k 0 = 0$.

Putting everything together, we have shown that there exists a measurable set B with $\mu(B) = 0$ such that $f_n(x) \to f(x)$ for all $x \in B^c$. This is precisely the definition of $f_n \to f$ μ-almost everywhere.

AUGUST 2024 MEASURE PRELIM TUTORING PROBLEM SET 4 SOLUTIONS

1. (Aug 2021) Let $(X, \mathcal{A}, \mu)$ be a measure space and $[f_n], f \in L^1(\mu)$ such that $f_n \to f$ in $L^1(\mu)$. Show that if $\sup_n \|f\|_{L^4(\mu)} < \infty$, then $f \in L^2(\mu)$ and $f_n \to f$ in $L^2(\mu)$.

What to spot: Recall that if $1 \le p < q < r \le \infty$, then $f \in L^p \cap L^r$ implies $f \in L^q$

We will begin by showing $[f_n]$ is a Cauchy sequence in $L^2(\mu)$. We can break up $|f_n - f_m|^2 = |f_n - f_m|^{2/3}|f_n - f_m|^{4/3}$ and then apply Hölder's inequality with $p = 3/2$ and $q = 3$,

$$\|f_n - f_m\|_2^2 = \int |f_n - f_m|^2 \, d\mu = \int |f_n - f_m|^{2/3}|f_n - f_m|^{4/3} \, d\mu$$

$$\le \left(\int (|f_n - f_m|^{2/3})^{3/2}\right)^{2/3} \left(\int (|f_n - f_m|^{4/3})^3\right)^{1/3} = \left(\int |f_n - f_m|\right)^{2/3} \left(\int |f_n - f_m|^4\right)^{1/3}$$

From this we conclude that $\|f_n - f_m\|_2 \le \|f_n - f_m\|_{L^1}^{1/3}\|f_n - f_m\|_{L^4}^{2/3}$. Let $\sup_n \|f_n\|_{L^4} = M < \infty$, then we have by the triangle inequality/Minkowski's inequality,

$$\|f_n - f_m\|_2 \le \|f_n - f_m\|_{L^1}^{1/3}(\|f_n\|_{L^4} + \|f_m\|_{L^4})^{2/3} \le \|f_n - f_m\|_{L^1}^{1/3}(2M)^{2/3}$$

Since $[f_n]$ converges in L^1, it is Cauchy in L^1, so we conclude that $\|f_n - f_m\|_{L^2} \to 0$ as $n, m \to \infty$, so $[f_n]$ is Cauchy in L^2. Since L^2 is a Banach space, there exists a function $\widetilde{f} \in L^2$ such that $f_n \to \widetilde{f}$ in the L^2 norm. But since $f_n \to f$ in L^1, and this limit is unique up to redefinition on a null set, we conclude that $f = \widetilde{f}$ almost everywhere and thus $f \in L^2$. This also takes care of $f_n \to f$ in L^2, so we are done.

(To see why $f = \widetilde{f}$ ae, first take a subsequence $[f_{n_k}]$ such that $f_{n_k} \to f$ ae. Then $f_{n_k} \to \widetilde{f}$ in L^2, so a further subsequence $f_{n_{k_\ell}} \to \widetilde{f}$ ae. But $f_{n_{k_\ell}} \to f$ ae as a subsequence of f_{n_k}. So $f = \widetilde{f}$ ae.)

In general, if $1 \le p < q < r \le \infty$, we have $\|f\|_q \le \|f\|_p^\lambda \|f\|_r^{1-\lambda}$, where $\lambda = \dfrac{q^{-1} - r^{-1}}{p^{-1} - r^{-1}}$ (we take $\infty^{-1} = 0$). The proof of this can be found in Folland.

2. (Aug 2023) Let

$$f_n(x) = \frac{1}{1 + x^{\frac{\sqrt{n}}{\log(n+2023)}}}, \quad x \geq 0, n \in \mathbb{N}$$

Find $\lim_{n\to\infty} \int_0^\infty f_n \, dm$

What to spot: We want to use MCT or DCT. Since $\frac{\sqrt{n}}{\log(n+2023)} \to \infty$, we do not have an increasing sequence on $[1, \infty)$, so we need to use DCT.

We want to invoke the Dominated Convergence Theorem, so we need to find a dominating function g. We claim that $g = \chi_{[0,1)} + \frac{1}{1+x^2}\chi_{[1,\infty)}$ is such a function. Indeed, we can clearly see that $|f_n(x)| \leq 1$ whenever $x \in [0, 1]$. On $[1, \infty)$, we first show that $\frac{\sqrt{n}}{\log(n+2023)} \to \infty$, which follows from L'Hôpital's rule,

$$\lim_{n\to\infty} \frac{\sqrt{n}}{\log(n + 2023)} = \lim_{n\to\infty} \frac{(1/2)n^{-1/2}}{(n + 2023)^{-1}} = \lim_{n\to\infty} \frac{n + 2023}{2\sqrt{n}} = \infty$$

Thus, for some $N \in \mathbb{N}$, we have $\frac{\sqrt{n}}{\log(n+2023)} > 2$ for all $n > N$. Thus, for all $n > N$, we have $\frac{1}{1+x^{\frac{\sqrt{n}}{\log(n+2023)}}} \leq \frac{1}{1+x^2}$ for $x \in [1, \infty)$. Thus $|f_n| \leq g$ almost everywhere (in fact, everywhere).

To show $g \in L^1$, first define the sequence $g_n = g\chi_{[0,n]}$. Each g_n is continuous almost everywhere (thus measurable) and bounded on the compact set $[0, n]$, so each g_n is Riemann integrable on $[0, n]$ and the Riemann integral coincides with the Lebesgue integral. Moreover, $g_n \uparrow g$, so we may evaluate $\int g \, dm$ as a limit of Riemann integrals using the Monotone Convergence Theorem,

$$\int g \, dm = \int \lim_n g_n \, dm = \lim_n \int g_n \, dm = \lim_n \int_0^n g_n(x) \, dx = \lim_n \left[\int_0^1 1 \, dx + \int_1^n \frac{1}{1 + x^2} \, dx\right]$$

$$= \lim_n [1 + \arctan(x)\Big|_1^n] = 1 + \frac{\pi}{2} - \frac{\pi}{4} = 1 + \frac{\pi}{4}$$

Thus $g \in L^1$. We have therefore satisfied all conditions of the Dominated Convergence Theorem (for $n > N$, but that's no issue since we only need it to hold for sufficiently large n). We can see that $\lim_n f_n = \chi_{[0,1)} + \frac{1}{2}\chi_{\{1\}}$ because $x^{\frac{\sqrt{n}}{\log(n+2023)}} \to 0$ as $n \to \infty$ on $[0, 1)$, $f_n(1) = \frac{1}{2}$ for all n and $x^{\frac{\sqrt{n}}{\log(n+2023)}} \to \infty$ as $n \to \infty$. Thus

$$\lim_n \int f_n \, dm = \int \lim_n f_n \, dm = \int \chi_{[0,1)} + \frac{1}{2}\chi_{\{1\}} = m([0, 1)) + \frac{1}{2}m(\{1\}) = 1 + 0 = 1$$

3. (Jan 2022) Let $f : [0,1] \to \mathbb{R}$ be a Borel measurable function with $\int_0^1 |f(t)|\,dt < \infty$. Prove that the function

$$h(x) = \int_x^1 t^{-1} f(t)\,dt$$

is integrable on $[0,1]$ and $\int_0^1 f(t)\,dt = \int_0^1 h(x)\,dx$.

What to spot: We have double integrals, so we will use Fubini-Tonelli.

To show h is integrable, we integrate $|h(x)|$,

$$\int_0^1 |h(x)|\,dx = \int_0^1 \left| \int_x^1 t^{-1} f(t)\,dt \right| dx \leq \int_0^1 \int_x^1 |t^{-1} f(t)|\,dt\,dx$$

Since $[0,1]$ with the Lebesgue measure is a σ-finite measure space, we may invoke Tonelli's Theorem to swap the order of integration since the integrand is nonnegative. Take note this also affects our bounds of integration. Carrying out the computation,

$$\int_0^1 \int_x^1 |t^{-1} f(t)|\,dt\,dx = \int_0^1 \int_0^t |t^{-1} f(t)|\,dx\,dt = \int_0^1 t|t^{-1} f(t)|\,dt = \int_0^1 |f(t)|\,dt < \infty$$

Thus h is integrable on $[0,1]$. With that, we can now compute $\int_0^1 h(x)\,dx$ using Fubini's Theorem, since the function $t^{-1} f(t) \chi_{[x,1]}(t)$ is integrable on $[0,1] \times [0,1]$,

$$\int_0^1 h(x)\,dx = \int_0^1 \int_x^1 t^{-1} f(t)\,dt\,dx = \int_0^1 \int_0^t t^{-1} f(t)\,dx\,dt = \int_0^1 t \cdot t^{-1} f(t)\,dt = \int_0^1 f(t)\,dt$$

As desired.

4. (Aug 2020) Let $\{E_1, \ldots, E_m\}$ be a finite family of Lebesgue measurable subsets of $\mathbb{R}^n$ and let $k > 0$ be a positive integer. Let $E \subset \mathbb{R}^n$ be a measurable subset with $m(E) > 0$. Suppose that almost every $x \in E$ belongs to at least k of the E_j. Prove there is at least one E_ℓ with $m(E_\ell) \geq \frac{k}{m} m(E)$.

What to spot: Try rewriting the measures of sets as integrals of indicators.

Define $f = \sum_{j=1}^m \chi_{E_j}$. Because almost every $x \in E$ belongs to at least k of the E_j, we conclude that $f \geq k$ almost everywhere. Thus $\int_E f \, dm \geq km(E)$. On the other hand, direct computation gives $\int_E f \, dm = \sum_{j=1}^m \int_E \chi_{E_j} = \sum_{j=1}^m m(E \cap E_j)$.

Suppose $m(E_j) < \frac{k}{m} m(E)$ for all $1 \leq j \leq m$. Then $\sum_{j=1}^m m(E \cap E_j) < \sum_{j=1}^m \frac{k}{m} m(E) = \frac{k}{m} m(E) \cdot m = km(E)$. This contradicts $\int_E f \, dm \geq km(E)$, so we conclude $m(E_\ell) \geq \frac{k}{m} m(E)$ for at least one value of $\ell \in \{1, \ldots, m\}$.

5. (Aug 2017) Prove or disprove three of the following statements:

(a) If $[f_n]$ is a sequence of measurable functions that converges in $L^1(\mathbb{R})$, then it converges in measure.

What to spot: Chebyshev's inequality.

This is true. Recall Chebyshev's inequality: $\mu(\{x : |f(x)| > \lambda\}) \leq \frac{1}{\lambda^p} \int |f|^p \, d\mu$ for $p \in [1, \infty)$. Let $f_n \to f$ in L^1, then for each $\varepsilon > 0$

$$\lim_{n\to\infty} m(\{x : |f_n(x) - f(x)| > \varepsilon\}) \leq \lim_{n\to\infty} \frac{1}{\varepsilon} \int |f_n - f| \, dm = 0$$

So $f_n \to f$ in measure.

(b) If $[f_n]$ is a sequence of integrable functions that converges almost everywhere in $[0, 1]$, then it converges in $L^1([0, 1])$.

What to spot: Convergence ae doesn't imply convergence in L^1.

This is false. Consider the sequence defined as $f_n = n^2 \chi_{[0,1/n]}$. Then $f_n \to 0$ almost everywhere because for each fixed x, $f_n(x) = 0$ for all $n > \lceil 1/x \rceil + 1$. Moreover, each f_n is integrable with $\|f_n\|_1 = n$. But $[f_n]$ cannot converge in L^1 to any function because $\|f_n\|_1 \to \infty$, and an unbounded sequence cannot converge.

(c) If $[f_n]$ is a sequence of measurable functions that converges almost everywhere in $[0, 1]$, then it converges in $L^\infty([0, 1])$.

What to spot: Convergence in L^∞ means "uniform convergence outside null set"

This is false. Consider the sequence defined as $f_n = n^2 \chi_{[0,1/n]}$. Then $f_n \to 0$ almost everywhere because for each fixed x, $f_n(x) = 0$ for all $n > \lceil 1/x \rceil + 1$. But $[f_n]$ doesn't converge in L^∞ to any function because $\|f_n\|_\infty = n^2 \to \infty$, and an unbounded sequence cannot converge.

(d) If $[f_n]$ is a sequence of measurable functions that converges in $L^1(\mathbb{R})$, then it converges almost everywhere.

What to spot: Convergence in L^1 doesn't imply convergence ae (typewriter sequence).

This is false. Let $[f_n]$ be the typewriter sequence defined by

$$f_n(x) = \begin{cases} 1 & x \in [\frac{n-2^k}{2^k}, \frac{n-2^k+1}{2^k}] \text{ for } k \geq 0 \text{ and } 2^k \leq n < 2^{k+1} \\ 0 & \text{otherwise} \end{cases}$$

Then $f_n \to 0$ in L^1 since $\|f_n\| \leq 2^{-k}$ whenever $n \geq 2^k$. Then take $k \to \infty$. But $\lim_n f_n(x)$ fails to exist for each x since $\limsup_n f_n(x) = 1$ and $\liminf_n f_n(x) = 0$ for each x. Thus f_n fails to converge to f (or to any function) almost everywhere.

6. (Jan 2017) Let $g \in L^1(\mathbb{R}^n, m)$ such that

$$\int g(x)\phi(x)\, dx = 0$$

for all $\phi \in C_c(\mathbb{R}^n)$, ie the space of all continuous functions with compact support on $\mathbb{R}^n$. Prove that $g = 0$ almost everywhere.

$\boxed{\text{What to spot: Recall that } C_c(\mathbb{R}^n) \text{ is dense in } L^1.}$

Let $A = \{x \in \mathbb{R}^n : g(x) \neq 0\}$. Suppose that $m(A) > 0$ and without loss of generality that $A^+ = \{x \in \mathbb{R}^n : g(x) > 0\}$ has $m(A^+) > 0$. Since g is positive of a set of full measure, there exists some $\varepsilon > 0$ such that $g > \varepsilon$ on some measurable set A^+_ε and $m(A^+_\varepsilon) > 0$. We may suppose that A^+_ε is bounded since we can write $A^+_\varepsilon = \bigcup_{r=1}^\infty A^+_\varepsilon \cap B(0,r)$, where $B(0,r) = \{x \in \mathbb{R}^n : |x| \leq r\}$. Continuity from below shows that $m(A^+_\varepsilon \cap B(0,r)) \to m(A^+_\varepsilon)$, and since the limit is positive, at least one of $A^+_\varepsilon \cap B(0,r)$ must eventually have nonzero measure. Then $\int_{A^+_\varepsilon} g > \int_{A^+_\varepsilon} \varepsilon > \varepsilon m(A^+_\varepsilon) > 0$.

We now wish to approximate $g\chi_{A^+_\varepsilon}$ by elements of $C_c(\mathbb{R}^n)$. Recall that $C_c(\mathbb{R}^n)$ is a dense subspace of $L^1(\mathbb{R}^n, m)$, so there is a sequence $[\phi_n] \subset C_c(\mathbb{R}^n)$ such that $\phi_n \to \chi_{A^+_\varepsilon}$ in the L^1 norm. By passing through a subsequence and re-indexing, we may further assume that the convergence also occurs almost everywhere.

Next, we will normalize each ϕ_n so that $|\phi_n(x)| \leq 1$ for all $x \in \mathbb{R}^n$. Define

$$\widetilde{\phi}(x) = \begin{cases} \frac{\phi(x)}{|\phi(x)|} & \text{if } |\phi(x)| > 1 \\ \phi(x) & \text{if } |\phi(x)| \leq 1 \end{cases}$$

Note that $\widetilde{\phi}$ is also an element of $C_c(\mathbb{R}^n)$. Indeed, its support is identical to ϕ and thus also compact. It is continuous since $\widetilde{\phi} = \psi \circ \phi$ where ψ is the following continuous function:

$$\psi(x) = \begin{cases} -1 & x < -1 \\ x & -1 \leq x \leq 1 \\ 1 & x > 1 \end{cases}$$

Moreover, we have $\widetilde{\phi}_n \to \chi_{A^+_\varepsilon}$ almost everywhere. Let x be a point such that $\phi_n(x) \to \chi_{A^+_\varepsilon}(x)$, then by continuity of ψ, $\lim_n \widetilde{\phi}_n(x) = \lim_n \psi(\phi_n(x)) = \psi(\lim_n \phi_n(x)) = \psi(\chi_{A^+_\varepsilon}(x)) = \chi_{A^+_\varepsilon}(x)$ (last equality follows because $\chi_{A^+_\varepsilon}(x) \in \{0, 1\}$, where $\psi(x) = x$ for both values)

Finally, we have that $g\widetilde{\phi}_n \to g\chi_{A^+_\varepsilon}$ pointwise almost everywhere, and $|g\widetilde{\phi}_n| \leq |g|$ for all n. Since $g \in L^1$, we may apply the Dominated Convergence Theorem to get

$$0 = \lim_{n\to\infty} \int g\widetilde{\phi}_n\, dx = \int \lim_{n\to\infty} g\widetilde{\phi}_n\, dx = \int g\chi_{A^+_\varepsilon} > 0$$

Which is a contradiction.

AUGUST 2024 MEASURE PRELIM TUTORING PROBLEM SET 5 SOLUTIONS

1. (Aug 2019) Let $0 < a < b < \infty$ and consider the function

$$f(x) = \frac{1}{x^a + x^b} \quad x > 0$$

For which values of p is $f \in L^p(0, \infty)$?

What to spot: Recall that $\int_0^1 \frac{1}{x^q} < \infty$ if and only if $q < 1$, and that $\int_1^\infty \frac{1}{x^q} < \infty$ if and only if $q > 1$.

We can break up the integral as

$$\int_0^\infty \left(\frac{1}{x^a + x^b}\right)^p dx = \int_0^1 \left(\frac{1}{x^a + x^b}\right)^p dx + \int_1^\infty \left(\frac{1}{x^a + x^b}\right)^p dx$$

For the first integral, we have $\frac{1}{x^a+x^b} \leq \frac{1}{x^a}$, so

$$\int_0^1 \left(\frac{1}{x^a + x^b}\right)^p dx \leq \int_0^1 \left(\frac{1}{x^a}\right)^p dx = \int_0^1 \frac{1}{x^{ap}} dx$$

The integral $\int_0^1 \frac{1}{x^{ap}} dx$ converges if and only if $ap < 1$, so it is necessary that $p < 1/a$. On the other hand, if $p \geq 1/a$, since $x \in [0, 1]$ and $a < b$, we have $x^a \geq x^b$. Thus $\frac{1}{x^a+x^b} \geq \frac{1}{2x^a}$ on $[0, 1]$. We then have

$$\int_0^1 \left(\frac{1}{x^a + x^b}\right)^p dx \geq \int_0^1 \left(\frac{1}{2x^a}\right)^p dx = \frac{1}{2^p} \int_0^1 \frac{1}{x^{ap}} dx \geq \frac{1}{2^p} \int_0^1 \frac{1}{x} dx = \infty$$

For the second integral, we have $\frac{1}{x^a+x^b} \leq \frac{1}{x^b}$, so

$$\int_1^\infty \left(\frac{1}{x^a + x^b}\right)^p dx \leq \int_1^\infty \left(\frac{1}{x^b}\right)^p dx = \int_1^\infty \frac{1}{x^{bp}} dx$$

The integral $\int_1^\infty \frac{1}{x^{bp}} dx$ converges if and only if $bp > 1$, so it is necessary that $p > 1/b$. On the other hand, if $p \leq 1/b$, then since $x \in [1, \infty)$ and $a < b$, we have $x^b \geq x^a$. Thus $\frac{1}{x^a+x^b} \geq \frac{1}{2x^b}$ on $[1, \infty)$. We then have

$$\int_1^\infty \left(\frac{1}{x^a + x^b}\right)^p dx \geq \int_1^\infty \left(\frac{1}{2x^b}\right)^p dx = \frac{1}{2^p} \int_1^\infty \frac{1}{x^{bp}} dx \geq \frac{1}{2^p} \int_1^\infty \frac{1}{x} dx = \infty$$

Putting everything together, we obtain that $f \in L^p(0, \infty)$ if and only if $1/b < p < 1/a$.

2. (Aug 2018) Let $(X, \mathcal{A}, \mu)$ be a measure space.

(a) Prove that if $[f_n], [g_n], f, g \in L^1(\mu)$, $|f_n| \le g_n$ for all n, $f_n \to f$ μ-ae, $g_n \to g$ μ-ae, and $\int g_n \, d\mu \to \int g \, d\mu$, then $\int f_n \, d\mu \to \int f \, d\mu$.

What to spot: This is the generalized DCT, so modify the proof of DCT.

First we will suppose $f_n \to f$ and $g_n \to g$ everywhere. Since $|f_n| \le g_n$ for all n, it follows from the triangle inequality that $g + g_n \ge |f| + |f_n| \ge |f - f_n|$ for each n. Thus, by Fatou's Lemma,

$$\begin{aligned} 0 \le \int g + g_n \, d\mu - \int \lim_{n\to\infty} |f - f_n| \, d\mu &= \int g + g_n - \lim_{n\to\infty} |f - f_n| \, d\mu \\ &= \int \lim_{n\to\infty} (2g_n - |f - f_n|) \, d\mu \\ &\le \liminf_{n\to\infty} \int 2g_n - |f - f_n| \, d\mu \end{aligned}$$

Since $\int g_n \to \int g$, we can distribute the $\liminf$, ie

$$\begin{aligned} 0 \le \liminf_{n\to\infty} \int 2g_n - |f - f_n| \, d\mu &= 2\int g \, d\mu + \liminf_{n\to\infty} \left[- \int |f - f_n| \, d\mu \right] \\ &= 2 \int g \, d\mu - \limsup_{n\to\infty} \int |f - f_n| \, d\mu \end{aligned}$$

We therefore have $\int g + g_n \, d\mu - \int \lim |f - f_n| \, d\mu \le 2 \int g \, d\mu - \limsup \int |f - f_n| \, d\mu$, so taking the limit for g_n and subtracting from both sides gives $\int - \lim |f - f_n| \, d\mu \le - \limsup \int |f - f_n|$. Putting everything together, we have

$$0 = \int 0 \, d\mu = \int - \lim_{n\to\infty} |f - f_n| \, d\mu \le - \limsup_{n\to\infty} \int |f - f_n| \, d\mu \implies \limsup_{n\to\infty} \int |f - f_n| \, d\mu \le 0$$

Since $|f - f_n| \ge 0$ for all n, it follows that $\liminf \int |f - f_n| \ge 0$. We therefore have

$$0 \le \liminf_{n\to\infty} \int |f - f_n| \, d\mu \le \limsup_{n\to\infty} \int |f - f_n| \, d\mu \le 0$$

Thus $\lim \int |f - f_n| = 0$, so $f_n \to f$ in L^1 and $\int f_n \to \int f$.

If we only have almost-everywhere convergence, let $N = \{x : f_n(x) \not\to f(x) \text{ or } g_n(x) \not\to g(x)\}$. Then $\mu(N) = 0$, and $f_n \to f$ and $g_n \to g$ everywhere on $X \setminus N$, so by our work above,

$$\begin{aligned} \lim_{n\to\infty} \int_X |f - f_n| \, d\mu &= \lim_{n\to\infty} \left[\int_{X \setminus N} |f - f_n| \, d\mu + \int_N |f - f_n| \, d\mu \right] \\ &= \lim_{n\to\infty} \int_{X\setminus N} |f - f_n| \, d\mu + 0 \\ &= 0 \end{aligned}$$

(b) Let $1 \leq p < \infty$. Suppose $[f_n], f \in L^p(\mu)$ and that $f_n \to f$ μ-ae. Prove that $\int |f_n - f|^p \, d\mu \to 0$ if and only if $\int |f_n|^p \, d\mu \to \int |f|^p \, d\mu$.

What to spot: Use part (a).

If $\int |f_n - f|^p \, d\mu \to 0$, then $\|f - f_n\|_p \to 0$ and we can use the reverse triangle inequality to obtain

$$\Big| \|f_n\|_p - \|f\|_p \Big| \leq \|f_n - f\|_p \to 0 \implies \|f_n\|_p^p \to \|f\|_p^p \implies \int |f_n|^p \, d\mu \to \int |f|^p \, d\mu$$

Now suppose $\int |f_n| \, d\mu \to \int |f| \, d\mu$. Define $h_n = |\frac{1}{2} f_n - \frac{1}{2} f|^p$, $g_n = \frac{1}{2}|f_n|^p + \frac{1}{2}|f|^p$, and $g = |f|^p$. Then $h \to 0$ and $g_n \to g$ almost everywhere, and $h_n, h, g_n, g \in L^1$ for all n. By assumption, we have that $\int g_n \to \int g$. Thus, we want to show that $|h_n| \leq g_n$ for all n.

For any $p \geq 1$, the mapping $x \mapsto |x|^p$ is convex, so we have

$$|h_n| = |\tfrac{1}{2} f_n - \tfrac{1}{2} f|^p \leq \tfrac{1}{2}|f_n|^p + \tfrac{1}{2}|f|^p = g_n$$

Thus, by the generalized DCT from part (a), $\|h_n\|_1 \to 0$, ie,

$$\|h_n\|_1 = \int |\tfrac{1}{2} f_n - \tfrac{1}{2} f|^p \, d\mu = \frac{1}{2^p} \int |f_n - f|^p \, d\mu \to 0$$

So $\int |f_n - f|^p \, d\mu \to 0$.

3. (Jan 2023) Let $p : \mathbb{R}^n \to \mathbb{R}$ be a (nontrivial) polynomial in n variables, with $n \geq 2$. Prove that the set $p^{-1}(\{0\}) \subset \mathbb{R}^n$ has Lebesgue measure 0.

$\boxed{\text{What to spot: Recall that the measure of a multi-dimensional set can be computed using iterated integrals.}}$

The proof is by induction on n. First the base case: Let $p : \mathbb{R}^2 \to \mathbb{R}$ be a polynomial in 2 variables, say x, y. For each fixed y, define the polynomial $p_y(x) = p(x, y)$. p_y is a polynomial in one variable, so it possesses finitely many real roots. That is, $p_y^{-1}(\{0\})$ has finite cardinality for each y, so $m(p_y^{-1}(\{0\})) = 0$ for each y. Since $\mathbb{R}$ is σ-finite with respect to m and $\chi_{p^{-1}(\{0\})} \geq 0$ almost everywhere, we can apply Tonelli's theorem to obtain

$$m(p^{-1}(\{0\})) = \int_{\mathbb{R}^2} \chi_{p^{-1}(\{0\})} = \int_{\mathbb{R}} \int_{\mathbb{R}} \chi_{p_y^{-1}(\{0\})}(x)\, dx\, dy = \int_{\mathbb{R}} 0\, dy = 0$$

Now suppose by induction that if $p : \mathbb{R}^n \to \mathbb{R}$ is a polynomial in n variables, then $p^{-1}(\{0\}) = 0$. Let $q : \mathbb{R}^{n+1} \to \mathbb{R}$ be a polynomial in $n+1$ variables, say $x_1, \ldots, x_{n+1}$. For each fixed x_{n+1}, define the polynomial $q_{x_{n+1}}(x_1, \ldots, x_n) = q(x_1, \ldots, x_n, x_{n+1})$. $q_{x_{n+1}}$ is therefore a polynomial $\mathbb{R}^n \to \mathbb{R}$ in n variables, so $m(q_{x_{n+1}}^{-1}(\{0\})) = 0$ for each $x_{n+1} \in \mathbb{R}$. As above, we can apply Tonelli's theorem to obtain

$$m(q^{-1}(\{0\})) = \int_{\mathbb{R}^{n+1}} \chi_{q^{-1}(\{0\})} = \int_{\mathbb{R}} \int_{\mathbb{R}^n} \chi_{q_{x_{n+1}}^{-1}(\{0\})}(x_1, \ldots, x_n)\, dm\, dx_{n+1} = \int_{\mathbb{R}} 0\, dx_{n+1} = 0$$

This completes the induction.

4. (Jan 2018) If $\mu \ll \nu$ and $\nu \ll \mu$ are finite measures on a measurable space $(X, \mathcal{A})$, show that the Radon-Nikodym derivatives satisfy $\frac{d\mu}{d\nu}\frac{d\nu}{d\mu} = 1$ μ-almost everywhere.

What to spot: Writing out the Radon-Nikodym derivatives gives $\mu(E) = \int_E \frac{d\mu}{d\nu}\frac{d\nu}{d\mu}\, d\mu$ for every $E \in \mathcal{A}$, so $\frac{d\mu}{d\nu}\frac{d\nu}{d\mu} = 1$ almost everywhere.

By the Radon-Nikodym Theorem, we have $\mu(E) = \int_E \frac{d\mu}{d\nu}\, d\nu$ and $\nu(E) = \int_E \frac{d\nu}{d\mu}\, d\mu$ for every $E \in \mathcal{A}$, and both $\frac{d\mu}{d\nu}, \frac{d\nu}{d\mu}$ are nonnegative and integrable. Thus, we can substitute $d\nu$ in the integral for μ, resulting in $\mu(E) = \int_E \frac{d\mu}{d\nu}\frac{d\nu}{d\mu}\, d\mu$ for every $E \in \mathcal{A}$.

For convenience, define $f = \frac{d\mu}{d\nu}\frac{d\nu}{d\mu}$. Let $F = \{x \in X : f(x) > 1\}$ and suppose $\mu(F) > 0$. Define $F_k = \{x \in X : f(x) > 1 + \frac{1}{k}\}$, and note each F_k is measurable as the preimage $F_k = f^{-1}((1 + \frac{1}{k}, \infty))$. Then $F = \bigcup_k F_k$, and $F_1 \subset F_2 \subset \dots$, so we may apply continuity from below to see that $\mu(F) = \mu(\bigcup_k F_k) = \lim_k \mu(F_k)$. Since $\mu(F) > 0$, there must be some sufficiently large K such that $\mu(F_k) > 0$ for all $k \geq K$. Then $\int_{F_K} f\, d\mu > \int_{F_K} 1 + \frac{1}{K}\, d\mu = (1 + \frac{1}{K})\mu(F_K) > \mu(F_K) = \int_{F_K} f\, d\mu$, a contradiction.

Now let $F = \{x \in X : f(x) < 1\}$ and suppose $\mu(F) > 0$. Define $F_k = \{x \in X : f(x) < 1 - \frac{1}{k}\}$, and note each F_k is measurable as the preimage $F_k = f^{-1}((-\infty, 1 - \frac{1}{k}))$. Then $F = \bigcup_k F_k$, and $F_1 \subset F_2 \subset \dots$, so we may apply continuity from below to see that $\mu(F) = \mu(\bigcup_k F_k) = \lim_k \mu(F_k)$. Since $\mu(F) > 0$, there must be some sufficiently large K such that $\mu(F_k) > 0$ for all $k \geq K$. Then $\int_{F_K} f\, d\mu > \int_{F_K} 1 - \frac{1}{K}\, d\mu = (1 - \frac{1}{K})\mu(F_K) < \mu(F_K)$, a contradiction. Thus $f = 1$ almost everywhere.

5. (Aug 2021) Let $f : \mathbb{R}^n \to (0, \infty)$ be a Lebesgue measurable function with $\|f\|_{L^1(\mathbb{R}^n)} = 1$. Show that if $E \subset \mathbb{R}^n$ is Lebesgue measurable with $m(E) \in (0, \infty)$, then

$$\int_E \log f \, dm \le -m(E) \log \big(m(E)\big)$$

What to spot: We have a concave function and want to prove an inequality with integrals. These are (most of) the conditions for Jensen's inequality, so find a way to use it.

Define the measure $\widetilde{m}(A) = m(A)/m(E)$ for $A \subset E$ measurable. Then $\widetilde{m}(E) = 1$, so $\widetilde{m}$ is a probability measure on E. Recall Jensen's inequality: if ϕ is a convex function on a probability space (X, Ω, μ), then $\int_X \phi \circ f \, d\mu \ge \phi(\int_X f \, d\mu)$. If ϕ is concave, then the inequality is reversed. Since log is concave, it follows that $\int_E \log f \, d\widetilde{m} \le \log(\int_E f \, d\widetilde{m})$. The inequality in the problem statement follows from a few calculations,

$$\begin{aligned}
\int_E \log f \, dm = m(E) \int_E \log f \, d\widetilde{m} &\le m(E) \log \left(\int_E f \, d\widetilde{m} \right) \\
&= m(E) \log \left(\frac{1}{m(E)} \int_E f \, dm \right) \\
&= m(E) \log \left(\frac{1}{m(E)} \right) \\
&= -m(E) \log \big(m(E)\big)
\end{aligned}$$

6. (Jan 2016) Compute

$$\lim_{n\to\infty}\int_0^n \left(1+\frac{x}{n}\right)^n e^{-2x}\,dx$$

Provide full justification for all steps in your reasoning.

What to spot: We are asked to compute a limit of integrals. The integrand is nonnegative, so let's aim to use MCT.

Define the sequence $[f_n]$ as $f_n = \left(1+\frac{x}{n}\right)^n e^{-2x}\chi_{[0,n]}$ for all n. Since each f_n is continuous almost everywhere, bounded, and supported on a compact set, it follows that the Riemann integral $\int_0^n f_n(x)\,dx$ coincides with the Lebesgue integral $\int_{[0,\infty)} f_n\,dm$ for each n. Thus, we may consider the sequence of Riemann integrals as a sequence of Lebesgue integrals.

Since $f_n \geq 0$ for all n, we are only one step away from being able to apply the MCT. All we need to do is show $f_n \uparrow f$ for some f. Since $\left(1+\frac{x}{n}\right)^n \to e^x$, the limit function will be $e^x e^{-2x} = e^{-x}$, so all we need to do is prove that $\left(1+\frac{x}{n}\right)^n$ is an increasing sequence for (almost) every x.

One way to do this is to set $y(n) = \left(1+\frac{x}{n}\right)^n$ and use logarithmic differentiation to show $\frac{dy}{dn} \geq 0$ for all n. However, there is a much more elegant method using the arithmetic-mean-geometric-mean inequality, which states for a finite sequence $[x_i]_{i=1}^n$ of real numbers, $(\prod_{i=1}^n x_i)^{1/n} \leq \frac{1}{n}\sum_{i=1}^n x_i$. Let $x_1 = 1$ and $x_2 = x_3 = \cdots = x_{n+1} = 1+\frac{x}{n}$. Then

$$\left(1+\frac{x}{n}\right)^{n/n+1} = \left(\prod_{i=1}^{n+1} x_i\right)^{1/n+1} \leq \frac{1}{n+1}\sum_{i=1}^{n+1} x_i = \frac{1}{n+1}\left(1+n\left(1+\frac{x}{n}\right)\right) = 1+\frac{x}{n+1}$$

Raising both sides to the power $n+1$ gives

$$\left(1+\frac{x}{n}\right)^n \leq \left(1+\frac{x}{n+1}\right)^{n+1}$$

Thus, the sequence $(1+\frac{x}{n})^n \uparrow e^x$. We may now apply the MCT,

$$\lim_{n\to\infty}\int_0^n \left(1+\frac{x}{n}\right)^n e^{-2x}\,dx = \lim_{n\to\infty}\int_{[0,\infty)} f_n\,dm = \int_{[0,\infty)} \lim_{n\to\infty} f_n\,dm = \int_{[0,\infty)} e^{-x}\,dm = 1$$

To prove $f_n \uparrow f$ using differentiation, let $y(n) = \left(1+\frac{x}{n}\right)^n$ and compute $\frac{y'}{y} = \log(1+\frac{x}{n}) - \frac{x}{n+x}$ using logarithmic differentiation. Thus we need to verify $\log(1+\frac{x}{n}) \geq \frac{x}{n+x}$ for all n and x. Indeed, $\log\left(1+\frac{x}{n}\right) \geq \frac{x}{n+x} \iff 1+\frac{x}{n} \geq \exp(\frac{x}{n+x})$, and

$$\begin{aligned}
\exp(\tfrac{x}{n+x}) &= 1+\tfrac{x}{n+x}+\tfrac{1}{2}(\tfrac{x}{n+x})^2+\tfrac{1}{6}(\tfrac{x}{n+x})^3+\dots \\
&\leq 1+\tfrac{x}{n+x}+(\tfrac{x}{n+x})^2+(\tfrac{x}{n+x})^3+\dots \\
&= \frac{1}{1-\frac{x}{n+x}} = 1+\tfrac{x}{n} \qquad \text{Note } |\tfrac{x}{n+x}|<1
\end{aligned}$$

Thus $\frac{y'}{y} \geq 0$ for all n, x, so $y' \geq 0$ since $y \geq 0$.

AUGUST 2024 MEASURE PRELIM TUTORING PROBLEM SET 5 SOLUTIONS

1. (Jan 2022) Let $(X, \mathcal{A}, \mu)$ be a measure space and $[f_n], f$ be measurable functions. Show that if $f_n \geq 0$ for all n and $f_n \to f$ in measure, then

$$\int f \, d\mu \leq \liminf_{n \to \infty} \int f_n \, d\mu$$

What to spot: Recall that $f_n \to f$ in measure implies a subsequence $f_{n_k} \to f$ almost everywhere.

Let $[f_{n_k}]$ be a subsequence such that $\lim_k \int f_{n_k} = \liminf_n \int f_n$. Then $f_{n_k} \to f$ in measure, so there exists a subsequence such that $f_{n_{k_\ell}} \to f$ almost everywhere. Thus, by Fatou's lemma,

$$\int f \, d\mu = \int \liminf_{\ell} f_{n_{k_\ell}} \, d\mu \leq \liminf_{\ell} \int f_{n_{k_\ell}} \, d\mu = \lim_{k} \int f_{n_k} \, d\mu = \liminf_{n} \int f_n \, d\mu$$

2. (Jan 2017) Let $f \in L^2(\mathbb{R}, m)$ and set $F(x) = \int_0^x f(t)\,dt$. Prove there exists a constant $C \geq 0$ such that

$$|F(x) - F(y)| \leq C|x-y|^{1/2}$$

for all $x, y \in \mathbb{R}$.

What to spot: Write out the $\lvert F(x) - F(y)\rvert$, and then notice that $\lvert x-y\rvert^{1/2} = \lVert\chi_{[y,x]}\rVert_2$

First, we compute $|F(x) - F(y)|$,

$$|F(x) - F(y)| = \left|\int_0^x f(t)\,dt - \int_0^y f(t)\,dt\right| = \left|\int_y^x f(t)\,dt\right| \leq \int_y^x |f(t)|\,dt$$

Then by the Cauchy-Schwartz inequality,

$$\begin{aligned} \int_y^x |f(t)|\,dt &\leq \|f\|_2 \|\chi_{[y,x]}\|_2 \\ &= \|f\|_2 |x-y|^{1/2} \end{aligned}$$

Thus, $|F(x) - F(y)| \leq C|x-y|^{1/2}$ for $C = \|f\|_2$

3. (Jan 2017) Prove or disprove three of the following statements:

(a) If $[f_n]$ is a Cauchy sequence in $L^2(\mathbb{R}^n, m)$, then $[f_n]$ converges almost everywhere.

What to spot: The typewriter sequence is a good counterexample for almost-everywhere convergence.

This is false. Let $[f_n]$ be the typewriter sequence defined by

$$f_n(x) = \begin{cases} 1 & x \in [\frac{n-2^k}{2^k}, \frac{n-2^k+1}{2^k}] \text{ for } k \geq 0 \text{ and } 2^k \leq n < 2^{k+1} \\ 0 & \text{otherwise} \end{cases}$$

Then $f_n \to 0$ in L^2 since $\|f_n\|_2 \leq 2^{-k/2}$ whenever $n \geq 2^k$. Then take $k \to \infty$. Since $[f_n]$ converges in L^2, it is Cauchy in L^2. But $\lim_n f_n(x)$ fails to exist for each x since $\limsup_n f_n(x) = 1$ and $\liminf_n f_n(x) = 0$ for each x. Thus f_n fails to converge to f (or to any function) almost everywhere.

(b) If $[f_n]$ is a sequence of measurable functions that converges in $L^\infty(\mathbb{R}^n, m)$, then $[f_n]$ converges almost everywhere.

What to spot: Convergence in L^∞ means "uniform convergence outside null set"

This is true. Let $f_n \to f$ in L^∞. Then there exists a set N with $m(N) = 0$ such that $\sup_{x \in \mathbb{R}^n \setminus N} |f_n(x) - f(x)| \to 0$. Thus, for each $x \in \mathbb{R}^n \setminus N$, $|f_n(x) - f(x)| \to 0$ as $n \to \infty$. This is the definition of convergence almost everywhere.

(c) If $U \subset \mathbb{R}^n$ is a subset whose boundary has Lebesgue outer measure 0, then U is Lebesgue measurable.

What to spot: Recall that $f_n \to f$ in measure implies a subsequence $f_{n_k} \to f$ almost everywhere.

This is true. We can rewrite $U = \text{int}(U) \cup B$, where $B \subset \partial U$. Since $\partial U = \overline{U} \setminus \text{int}(U)$ is the difference of a closed set and open set, ∂U is a Borel set and thus Lebesgue measurable. Since the Lebesgue measure is complete and we assume $m(\partial U) = 0$, we conclude that B is Lebesgue measurable. Since $\text{int}(U)$ is open, it is also Lebesgue measurable, so U is Lebesgue measurable as the union of two Lebesgue measurable sets.

(d) Let $(X, \mathcal{A}, \nu)$ be a measure space and suppose μ is a signed measure on $(X, \mathcal{A})$ satisfying $\mu \ll \nu$. If $\nu(A) = 0$ then $\mu^+(A) = \mu^-(A) = 0$, where $\mu = \mu^+ - \mu^-$ is the Jordan decomposition of μ.

What to spot: Recall the Hahn Decomposition Theorem and how the Jordan decomposition is defined.

This is true. Recall the Hahn Decomposition Theorem: If μ is a signed measure on $(X, \mathcal{A})$, then there exist $X^+, X^- \in \mathcal{A}$ such that

- $X^+ \cup X^- = X$
- $X^+ \cap X^- = \emptyset$
- X^+ is a positive set
- X^- is a negative set

A set E is called positive if $\mu(A) \geq 0$ for all $A \subset E$ with $A \in \mathcal{A}$. A set E is called negative if $\mu(A) \leq 0$ for all $A \subset E$ with $A \in \mathcal{A}$.

The Jordan decomposition is defined as follows:

$$\mu^+(A) = \mu(A \cap X^+) \qquad \mu^-(A) = \mu(A \cap X^-)$$

If $\nu(A) = 0$, then by monotonicity, $\nu(A \cap X^+) = \nu(A \cap X^-) = 0$. Since $\mu \ll \nu$, we have

$$0 = \mu(A \cap X^+) = \mu^+(A) \qquad 0 = \mu(A \cap X^-) = \mu^-(A)$$

4. (Aug 2018) Prove that if $f : \mathbb{R} \to \mathbb{R}$ is Lebesgue measurable, then

$$\int_{\mathbb{R}} f^4 \, dm = 4 \int_0^\infty m(\{x \in \mathbb{R} : |f(x)| > t\}) \, t^3 \, dt$$

What to spot: Rewrite the measure on the RHS as an integral of an indicator.

First, replace the measure on the right hand side of the equation with the integral of an indicator,

$$4 \int_0^\infty m(\{x \in \mathbb{R} : |f(x)| > t\}) \, t^3 \, dt = 4 \int_0^\infty \int_{\mathbb{R}} \chi_{\{|f(x)|>t\}} \, t^3 \, dm(x) \, dt$$

Since $\mathbb{R}$ with the Lebesgue measure is σ-finite and $\chi_{\{|f(x)|>t\}} \, t^3 \geq 0$, we can use Tonelli's Theorem to swap the order of integration. Since $\chi_{\{|f(x)|>t\}} = 1$ for $0 \leq t \leq |f(x)|$, we have

$$\begin{aligned} 4 \int_0^\infty \int_{\mathbb{R}} \chi_{\{|f(x)|>t\}} \, t^3 \, dm(x) \, dt &= 4 \int_{\mathbb{R}} \int_0^{|f(x)|} t^3 \, dt \, dm(x) \\ &= 4 \int_{\mathbb{R}} \frac{1}{4} t^4 \Big|_0^{|f(x)|} dm(x) \\ &= \int_{\mathbb{R}} f^4 \, dm \end{aligned}$$

5. (Aug 2021) Let $f : \mathbb{R} \to \mathbb{R}$ be a function. Suppose there exists some $C \geq 0$ such that

$$|f(x) - f(y)| \leq C|x - y|, \quad \text{for all } x, y \in \mathbb{R}$$

(a) Prove that $m^*(f(A)) \leq Cm^*(A)$ for all $A \subset \mathbb{R}$ (m^* is Lebesgue outer measure)

What to spot: Use the definition of outer measure.

Let $\mathcal{R}$ be the collection of all covers of A by countably many open intervals and $\mathcal{R}^*$ be the collection of all covers of $f(A)$ by countably many open intervals. That is,

$$\mathcal{R} = \left\{ [I_n]_{n=1}^{\infty} : I_n = (a_n, b_n), a_n < b_n, A \subset \bigcup_{n=1}^{\infty} I_n \right\}$$

$$\mathcal{R}^* = \left\{ [J_n]_{n=1}^{\infty} : J_n = (a_n, b_n), a_n < b_n, f(A) \subset \bigcup_{n=1}^{\infty} J_n \right\}$$

By definition, $m^*(A) = \inf_{[I_n] \in \mathcal{R}} \sum_{n=1}^{\infty} \ell(I_n)$ and $m^*(f(A)) = \inf_{[J_n] \in \mathcal{R}} \sum_{n=1}^{\infty} \ell(J_n)$, where ℓ is the length of the interval. We claim that for each $[I_n] \in \mathcal{R}$, there is a cover of $f(A) \setminus N$, where N is a set of outer measure 0, by countably many open intervals $[J_n]$ such that $\sum_{n=1}^{\infty} \ell(J_n) \leq C \sum_{n=1}^{\infty} \ell(I_n)$.

Fix an arbitrary $[I_n] \in \mathcal{R}$. Since $A \subset \bigcup_n I_n$, it follows that $f(A) \subset \bigcup_n f(I_n)$. Since f is Lipschitz, it is continuous, so it must map connected sets to connected sets. Since each I_n is an interval, it follows that $f(I_n)$ is either an interval or a singleton for each n. Define a countable collection of open intervals by $J_n = \text{int}(I_n)$, and define the set $N = \bigcup_n \partial f(I_n)$ as the collection of all the endpoints of $f(I_n)$ if it is an interval, or the point itself if it is a singleton. N is therefore at most a countable set.

We therefore have that $f(A) \setminus N \subset \bigcup_n J_n$. Let $\mathcal{R}'$ be the collection of all covers of $f(A)$ (minus a set of outer measure 0) by countably many open intervals formed in this manner. Then

$$\inf_{[J_n] \in \mathcal{R}^*} \sum_{n=1}^{\infty} \ell(J_n) = m^*(A) = m^*(f(A) \setminus N) \leq \inf_{[J_n] \in \mathcal{R}'} \sum_{n=1}^{\infty} \ell(J_n)$$

The inequality comes from the fact that (a priori), not every countable cover of $f(A) \setminus N$ must come from some $[I_n]$. For each $[J_n] \in \mathcal{R}'$, we have that $\ell(J_n) \leq C \cdot \ell(I_n)$. Indeed, if $J_n = (a, b)$, define sequences $a_k \downarrow a$ and $b_k \uparrow b$. Then for each k there exist $c_k, d_k \in I_n$ such that $f(c_k) = a_k$ and $f(d_k) = b_k$ for all k. Then by the Lipschitz condition,

$$|b_k - a_k| = |f(d_k) - f(c_k)| \leq C|d_k - c_k| \leq C\ell(I_n)$$

Taking the limit $k \to \infty$ gives $\ell(J_n) = |b - a| \leq C\ell(I_n)$ for all n. Thus, $\sum_{n=1}^{\infty} \ell(J_n) \leq C \sum_{n=1}^{\infty} \ell(I_n)$, and this holds for every $[J_n] \in \mathcal{R}'$. In other words, for each element of $\mathcal{R}$, there is an element $[J_n] \in \mathcal{R}'$ such that $\sum_{n=1}^{\infty} \ell(J_n) \leq C \sum_{n=1}^{\infty} \ell(I_n)$. Therefore, the infimum of the quantity on the left over $\mathcal{R}'$ must be less than or equal to the infimum of the quantity on the right over $\mathcal{R}$. That is,

$$m^*(f(A)) \leq \inf_{[J_n]\in\mathcal{R}'} \sum_{n=1}^{\infty} \ell(J_n) \leq \inf_{[I_n]\in\mathcal{R}} C \sum_{n=1}^{\infty} \ell(I_n) = Cm^*(A)$$

(b) Prove that f maps Lebesgue measurable sets to Lebesgue measurable sets.

What to spot: Recall that if A is Lebesgue measurable, then $A = F \cup N$, where F is an F_σ set and N is a null set.

Let A be Lebesgue measurable. Then we can write $A = F \cup N$, where F is an F_σ set (countable union of closed sets), and N is a null set. We then have

$$f(A) = f(F \cup N) = f(F) \cup f(N)$$

Since $0 = m(N) = m^*(N)$, we have by part (a) that $m^*(f(N)) \leq Cm^*(N) = 0$. Thus, $f(N)$ is measurable as a set with outer measure 0.

To show $f(F)$ is measurable, we will first show that we can express F as a countable union of compact sets. Indeed, we have $F = \bigcup_{n=1}^{\infty} K_n$, where each K_n is closed. We can write each K_n as a countable union of compact sets by writing $K_n = \bigcup_{m=1}^{\infty} K_n \cap [-m, m]$. Thus $F = \bigcup_{n=1}^{\infty} (\bigcup_{m=1}^{\infty} K_n \cap [-m, m])$ is the countable union of compact sets. Then

$$f(F) = f\left(\bigcup_{n=1}^{\infty}\left(\bigcup_{m=1}^{\infty} K_n \cap [-m, m]\right)\right) = \bigcup_{n=1}^{\infty}\bigcup_{m=1}^{\infty} f(K_n \cap [-m, m])$$

Since $K_n \cap [-m, m]$ is compact for each n, m and f is continuous, each $f(K_n \cap [-m, m])$ is compact and therefore Lebesgue measurable. Thus $f(F)$ is Lebesgue measurable as the countable union of Lebesgue measurable sets. Putting everything together, we obtain $f(A)$ is Lebesgue measurable, as desired.

6. (Aug 2019) Let $A \subset \mathbb{R}^n$ be a Lebesgue measurable subset with $m(A) < \infty$ and let $t \in (0, m(A)/2)$. Prove there exist disjoint Lebesgue measurable subsets $B, C \subset A$ such that $m(B) = m(C) = t$.

What to spot: The measure of a set grows "continuously" as you consider a growing ball from the origin.

Consider the function $f : [0, \infty) \to [0, \infty)$ given by $f(r) = m(A \cap B(0, r))$, where $B(0, r) = \{x \in \mathbb{R}^n : |x| < r\}$ is the open ball of radius r centered at the origin. We claim that f is continuous. Fix $r \in [0, \infty)$ and let $r_n \uparrow r$. Then the sequence of sets $[A \cap B(0, r_n)]$ is ascending (ie, $A \cap B(0, r_1) \subset A \cap B(0, r_2) \subset \dots$) and $\bigcup_{n=1}^{\infty} A \cap B(0, r_n) = A \cap B(0, r)$. Thus, by continuity from below, we have

$$\lim_{n\to\infty} f(r_n) = \lim_{n\to\infty} m(A \cap B(0, r_n)) = m\left(\bigcup_{n=1}^{\infty} A \cap B(0, r_n)\right) = m(A \cap B(0, r)) = f(r)$$

Now let $r_n \downarrow r$. Then $m(A \cap B(0, r_1)) < \infty$, the sequence of sets $[A \cap B(0, r_n)]$ is descending (ie, $A \cap B(0, r_1) \supset A \cap B(0, r_2) \supset \dots$), and $\bigcap_{n=1}^{\infty} A \cap B(0, r_n) = A \cap B(0, r)$. Thus, by continuity from above, we have

$$\lim_{n\to\infty} f(r_n) = \lim_{n\to\infty} m(A \cap B(0, r_n)) = m\left(\bigcap_{n=1}^{\infty} A \cap B(0, r_n)\right) = m(A \cap B(0, r)) = f(r)$$

Thus, f is continuous at r^{***}, and note furthermore that it is non-decreasing. For $t \in (0, m(A)/2)$, let $\eta = m(A)/2 - t$. Since $m(A) < \infty$, for each $\varepsilon > 0$, there is some $R > 0$ such that $m(A \cap B(0, R)) \geq m(A) - \varepsilon$. Therefore, f can assume any value in $[0, m(A))$ because of the Intermediate Value Theorem.

Using the Intermediate Value Theorem, let r_1 be a value such that $f(r_1) = t$ (since $t \in [0, m(A))$). Define the set $B = A \cap B(0, r_1)$.

Using the Intermediate Value Theorem again, let r_2 be a value such that $f(r_2) = m(A)/2 + \eta = m(A) - t$ (since $m(A) - t \in [0, m(A))$). Define the set $C = A \setminus B(0, r_2)$. Since $f(r_2) > f(r_1)$ and f is non-decreasing, it follows that $r_2 > r_1$, so $B \cap C = \emptyset$. We can rewrite C as $C = A \setminus (A \cap B(0, r_2))$ since $A \setminus (A \cap B(0, r_2)) = A \cap (A^C \cup B(0, r_2)^C) = A \setminus B(0, r_2)$. Therefore, since $m(A) < \infty$,

$$\begin{aligned} m(C) = m(A \setminus (A \cap B(0, r_2)) &= m(A) - m(A \setminus B(0, r_2)) \\ &= m(A) - f(r_2) \\ &= m(A) - (m(A)/2 + \eta) \\ &= m(A) - (m(A)/2 + m(A)/2 - t) = t \end{aligned}$$

It suffices to only check the cases $r_n \uparrow r$ and $r_n \downarrow r$. Recall generally that a sequence $a_n \to a$ if and only if for each subsequence $[a_{n_k}]$, a further subsequence $a_{n_{k_\ell}} \to a$. We WTS that if $r_n \to r$, then $f(r_n) \to f(r)$. Pick an arbitrary subsequence r_{n_k}, then a further subsequence $r_{n_{k_\ell}}$ is monotone (either $r_{n_{k_\ell}} \uparrow r$ or $r_{n_{k_\ell}} \downarrow r$). Thus if $[f(r_{n_k})]$ is a subsequence, a further subsequence $f(r_{n_{k_\ell}}) \to f(r)$, so $f(r_n) \to f(r)$.