MATH 1020 - Class Slides

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Fall 2023

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Welcome!

Welcome to MATH 1020 - Problem Solving!

My name is Jacob Brown. I prefer to be called Jake.

Take a flashcard from up front and write:

- Your name as it appears on the course roster
- Your preferred name and its pronunciation
- Your pronouns
- Your major
- Your dream career after graduation

About These Slides

I will post the class slides on HuskyCT, but they are not enough on their own to learn the material. You should attend class and read the textbook, as well.

I will color-code certain parts of the slides:

Blue Boxes

Blue boxes will contain things like definitions and important formulas. You should copy these down in your notes, as they are likely to appear on homework and exams.

Red Boxes

Red boxes will contain information that I feel is interesting to talk about, but will not be expected for you to know for homework or exams.

Note: Anything NOT in a red box is fair game for homework and quizzes/exams!

During class, I will ask questions to check your understanding of the material. We will use Socrative for this.

Socrative is a free website where you anonymously respond to the question I ask. I receive the responses and if the class is not in consensus about the answer, I will ask you all to turn to one another and discuss your thoughts. Then I will run the question again to see we have reached an agreement.

To access Socrative, search “Socrative student login” and click the link that comes up. Then type “BROWN4663” into the Room Name box. You don’t need to make an account.

Question

What is $2 + 2$?

A. 1
B. 2
C. 3
D. 4
E. I don't know

Question

What is $2 + 2$?

A. 1
B. 2
C. 3
D. 4
E. I don't know

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Logical Reasoning

Solving a problem requires you to think logically about the information you have.

There are two main types logical reasoning: inductive and deductive. Both methods have benefits and drawbacks.

Inductive Reasoning

Definition: Inductive Reasoning

Inductive reasoning is the process of reaching a general conclusion by examining specific examples.

You probably use inductive reasoning without even knowing it!

The conclusion reached by using inductive reasoning is called a **conjecture**.

Examples of Inductive Reasoning

Each of the following is an example of inductive reasoning. The underlined portion of each example is the conjecture.

A sequence of numbers goes $2, 4, 6, 8, 10, \_$. Each number is 2 greater than the previous, <u>so the number that fills in the blank is 12.</u>

The dining hall has been busy at noon for the past three days, <u>so it will be busy at noon today.</u>

A small coffee is \$1, a medium coffee is \$2, a large coffee is \$3, <u>so the extra large coffee is \$4.</u>

Question

Does the following statement use inductive reasoning?

"A ball dropped from a height of 1 meter bounces back to a height of 0.8 meters. The same ball dropped from a height of 2 meters bounces back to a height of 1.6 meters. Therefore, the ball always bounces back to 80% of its original height."

A. Yes
B. No
C. I don't know

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Question

Does the following statement use inductive reasoning?

"A ball dropped from a height of 1 meter bounces back to a height of 0.8 meters. The same ball dropped from a height of 2 meters bounces back to a height of 1.6 meters. Therefore, the ball always bounces back to 80% of its original height."

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Counterexamples

The conjecture formed by inductive reasoning might not be correct!

3 is not divisible by 8, and $3^2 = 9$ is also not divisible by 8. The same is true for 2 and $2^2 = 4$, as well as 5 and $5^2 = 25$. I make the following conjecture: "If n is not divisible by 8, then n^2 is not divisible by 8."

My conjecture is false! 4 is not divisible by 8, but $4^2 = 16$ is divisible by 8. In order for my conjecture to be true, it needs to *always* be true.

Definition: Counterexample

A **counterexample** is an example that disproves a conjecture by showing it is not always true.

Deductive Reasoning

Definition: Deductive Reasoning

Deductive reasoning is the process of reaching a conclusion by applying general assumptions, procedures, or principles.

In order to use deductive reasoning, you need more information than just examples. Instead, you need information about what is *always* true.

Examples of Deductive Reasoning

"Every beehive has a queen bee. There is a beehive outside my window. Therefore, there is a queen bee in the beehive outside my window."

In this example, we are given the general principle: "Every beehive has a queen bee."

"The sum of two odd numbers is an even number. 3^{523} and 5^{937} are odd numbers, so $3^{523} + 5^{937}$ is an even number."

In this example, we are given the general principle: "The sum of two odd numbers is an even number."

Logic Puzzles

One Joker and four aces with different suits (Spade, Club, Heart, Diamond) are placed facedown on a table. Use the hints below to determine the position for each card.

- 1. The club is to the immediate right of the heart.
- 2. Neither the diamond nor the joker is next to the spade.
- 3. Neither the diamond nor the spade is next to the heart.
- 4. The spade is in the spot on the far right.

	Far left	Left	Middle	Right	Far right
Spade					
Club					
Heart					
Diamond					
Joker					

Logic Puzzles

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	Far left	Left	Middle	Right	Far right
Spade	X	X	X	X	O
Club					X
Heart					X
Diamond					X
Joker					X

Logic Puzzles

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	Far left	Left	Middle	Right	Far right
Spade	X	X	X	X	O
Club	X	X	X	O	X
Heart				X	X
Diamond				X	X
Joker				X	X

Logic Puzzles

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Club	X	X	X	O	X
Heart	X	X	O	X	X
Diamond			X	X	X
Joker			X	X	X

Logic Puzzles

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	Far left	Left	Middle	Right	Far right
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Club	X	X	X	O	X
Heart	X	X	O	X	X
Diamond	O	X	X	X	X
Joker	X	O	X	X	X

Question

Consider the two statements below:

1. "The product of two even numbers is even, 4^{100} and 100^4 are even, so $4^{100} \times 100^4$ is even."

2. "4^{100} and 100^4 are even, and $4^{100} \times 100^4$ is even, so the product of two even numbers is even."

Do these two statements use the same type of logical reasoning?

A. Yes, they both use inductive reasoning.
B. Yes, they both use deductive reasoning.
C. No, 1 uses inductive reasoning and 2 uses deductive reasoning.
D. No, 1 uses deductive reasoning and 2 uses inductive reasoning.
E. I don't know

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Question

Consider the two statements below:

1. “The product of two even numbers is even, 4^{100} and 100^4 are even, so $4^{100} \times 100^4$ is even.”

2. “4^{100} and 100^4 are even, and $4^{100} \times 100^4$ is even, so the product of two even numbers is even.”

Do these two statements use the same type of logical reasoning?

A. Yes, they both use inductive reasoning.
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C. No, 1 uses inductive reasoning and 2 uses deductive reasoning.
D. No, 1 uses deductive reasoning and 2 uses inductive reasoning.
E. I don’t know

Definition: Sequence

A **sequence** is an ordered list of numbers. Each number is the list is called a **term**, and the n-th term in the sequence is written as $\boldsymbol{a_n}$.

The little number at the bottom right of a is called a "subscript" or "index."

a_1 represents the first term
a_2 represents the second term
a_3 represents the third term
$\vdots$
a_n represents the n-th term

For example, $1, 3, 29, 100, 2, \ldots$ is a sequence.
$a_1 = 1$, $a_2 = 3$, $a_3 = 29$, $a_4 = 100$, $a_5 = 2$, $\ldots$

Predicting Terms in a Sequence

Consider the sequence below:

$$1, 7, 13, 19, 25, 31, \dots$$

We can predict a_7 by using inductive reasoning.

Notice that each term is 6 greater than the one before it, so our conjecture is that $a_7 = 37$.

Difference Tables

What about this sequence?

$$1, 2, 4, 7, 11, 16, \ldots$$

For sequences in general, we can try to find a pattern by using a difference table.

Technique: Constructing a Difference Table

Sequence:	1		2		4		7		11		16
First Differences:		1		2		3		4		5	
Second Differences:			1		1		1		1		

Technique: Constructing a Difference Table

We can use the difference table to predict the next term:

Sequence:	1		2		4		7		11		16		22
First Differences:		1		2		3		4		5		6	
Second Differences:			1		1		1		1		1		

If needed, you can calculate third differences, fourth differences, etc, until the differences are all the same.

Question

Use a difference table to predict a_6 for the sequence below:

$$3, 5, 9, 15, 23, \ldots$$

A. 24
B. 31
C. 33
D. 37
E. I don't know

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Question

Use a difference table to predict a_6 for the sequence below:

$$3, 5, 9, 15, 23, \ldots$$

A. 24
B. 31
C. 33
D. 37
E. I don't know

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The n-th Term Formula

For some sequences, you can create a formula where you plug in n and you get out a_n, the n-th term in the sequence.

Definition: n-th Term Formula

An ***n*-th Term Formula** is an equation $a_n = \cdots$, where you plug in n and you get out the n-th term in the sequence.

For example, the n-th term formula for the sequence $2, 4, 6, 8, 10, 12, \ldots$ is $a_n = 2n$.

Using an n-th Term Formula

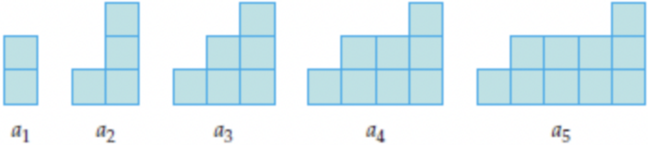
Let's say that a sequence follows the n-th term formula given by $a_n = n^2 - n$. Then the first five terms in the sequence are:

$$a_1 = (1)^2 - 1 = 1 - 1 = 0$$
$$a_2 = (2)^2 - 2 = 4 - 2 = 2$$
$$a_3 = (3)^2 - 3 = 9 - 3 = 6$$
$$a_4 = (4)^2 - 4 = 16 - 4 = 12$$
$$a_5 = (5)^2 - 5 = 25 - 5 = 20$$

So the sequence goes: $0, 2, 6, 12, 20, \ldots$

Creating an n-th Term Formula

Creating an n-th term formula can be tricky, but the general idea is to look for "n" in each step.



In the sequence above, we can see there a_1 has two groups of 1 tile, a_2 has two groups of 2 tiles tiles, a_3 has two groups of 3 tiles, and so on. So for each a_n, there are two groups of n tiles. Thus, $a_n = 2n$.

Recursive Definition of a Sequence

Another way to represent a sequence is to use a recursive definition.

Definition: Recursive Definition of a Sequence

A **recursive definition** of a sequence is one in which each successive term is defined by using preceding terms. You will be given some of the first few terms, $a_1, a_2, a_3, \dots$, called the **initial values**, and a formula $a_n = \cdots$, called the **recursion**.

The recursion will look like $a_n = \cdots$, where the right hand side will contain terms written as a_{n-1}, a_{n-2}, and so on.

Example of Recursion

A sequence has $a_1 = 2$, $a_2 = 5$, and $a_n = 2a_{n-1} - a_{n-2}$. Calculate a_3.

First, we will substitute 3 in place of n in the recursion:
$a_3 = 2a_{3-1} - a_{3-2} = 2a_2 - a_1$

Now we substitute our initial values:
$a_3 = 2(5) - (2) = 10 - 2 = 8$

So $a_3 = 8$.

If we wanted to calculate a_4, we would use a_3 and a_2. In general, to find a specific term, you need to find all the ones that come before it.

Question

A sequence has $a_1 = 1$, $a_2 = 2$, and $a_n = 3a_{n-1} + 2a_{n-2}$. Calculate a_4. (Hint: You'll need to find a_3 first!)

A. 20
B. 22
C. 28
D. 32
E. I don't know.

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Question

A sequence has $a_1 = 1$, $a_2 = 2$, and $a_n = 3a_{n-1} + 2a_{n-2}$. Calculate a_4. (Hint: You'll need to find a_3 first!)

A. 20
B. 22
C. 28
D. 32
E. I don't know.

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The Fibonacci Sequence, named after the Italian mathematician Leonardo Bonacci, is an important sequence of numbers that comes up in many areas of mathematics.

Definition: Fibonacci Sequence

The **Fibonacci Sequence** is the sequence of numbers defined by the recursive definition where $F_1 = 1$, $F_2 = 1$ and $F_n = F_{n-1} + F_{n-2}$. That is, each term is the sum of the previous two.

The Fibonacci Sequence goes: $1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, \ldots$

The Golden Ratio

The Golden Ratio

If you divide consecutive terms in the Fibonacci Sequence, a pattern arises:
$\frac{8}{5} = 1.6$, $\frac{13}{8} = 1.625$, $\frac{21}{13} = 1.61538\ldots$, $\frac{34}{21} = 1.61905\ldots$

The numbers approach a specific number called the Golden Ratio! We use the Greek letter ϕ (phi) to denote the Golden Ratio, and
$\phi = \frac{1+\sqrt{5}}{2} = 1.61828\ldots$

The Golden Ratio is approximately equal to the conversion factor between kilometers and miles, so you can use consecutive Fibonacci Sequence to get a rough conversion. For example, to convert 55 miles to kilometers, look at the next Fibonacci number, which is 89. The actual conversion is 88.5 miles, so it's pretty close!

Question

A sequence has $a_1 = 2$, $a_2 = 0$, and $a_n = 5a_{n-1} + 4a_{n-2}$. Calculate a_3.

A. 6
B. 8
C. 10
D. 12
E. I don't know

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Question

A sequence has $a_1 = 2$, $a_2 = 0$, and $a_n = 5a_{n-1} + 4a_{n-2}$. Calculate a_3.

A. 6
B. 8
C. 10
D. 12
E. I don't know

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Question

A sequence has $a_1 = 2$, $a_2 = 0$, and $a_n = 5a_{n-1} + 4a_{n-2}$. Calculate a_4. (Use a_3 from the previous question)

A. 32
B. 36
C. 40
D. 48
E. I don't know

Room Name: BROWN4663

Question

A sequence has $a_1 = 2$, $a_2 = 0$, and $a_n = 5a_{n-1} + 4a_{n-2}$. Calculate a_4. (Use a_3 from the previous question)

A. 32
B. 36
C. 40
D. 48
E. I don't know

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Polya's Problem-Solving Strategy

George Polya, a Hungarian mathematician who made great contributions to many areas of mathematics, created a general problem-solving strategy that can be used for almost any problem you come across.

Technique: Polya's Problem-Solving Strategy

1. Understand the problem.
2. Devise a plan.
3. Carry out the plan.
4. Review the solution.

Understand the Problem

It is impossible to solve a problem that you don't understand. To help understand a problem, try the following:

- Restate the problem in your own words.
- Break the problem down into pieces.
- Draw a picture.
- Relate the problem to something you have done before.
- Think about the goal of the problem.

Understanding the problem will give you a sense of direction about how to solve the problem.

Devise a Plan

There are many common techniques to choose as a plan:

- List out the known and unknown information.
- Add a grid or units to a diagram.
- Make a table or chart.
- Work backwards.
- Try a different, yet related, problem.
- Look for a pattern.
- Write an equation.

Carry Out the Plan

When working on a plan, be sure to:

- Work neatly and explain each step.
- Keep your plan in mind.
- Modify the plan as needed.

Review the Solution

Reviewing your solution helps you make sure you correctly solved the problem:

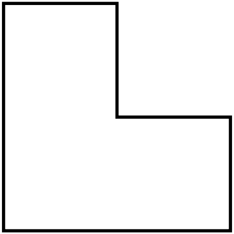
- Ensure you actually answered the question being asked.
- Work through the problem knowing your answer and see if you get the same result.
- Check if your answer is plausible.

Example of Polya's Problem-Solving Process

Alice, Bob, Charley, Diane, and Edgar all want to split a square cake. Edgar greedily eats the entire top-right quarter of the cake. After kicking Edgar out of the friend group, the four friends want to cut up the remaining cake into exactly four pieces that are exactly the same size and shape. How should they cut the cake to do so?

Example: Understand the Problem

Let's draw a picture to help us see what we are dealing with. The remaining cake is a square with the top-right quarter removed, so it looks like:



We want to cut this shape into four pieces that are the same size and shape. We can't simply cut it up into a bunch of squares and distribute them equally, because we have to get exactly four pieces.

Example: Devise a Plan

We are given the shape of the remaining cake and the goal of cutting it into four pieces of equal size and shape.

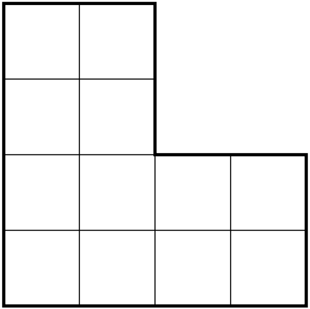
We could try drawing out the pieces, but how will we know they are *exactly* the same size and shape?

We would like to know how big to make each piece, but we aren't given any information about the size of the cake itself. To help with that, we should put a grid on the cake.

Using our grid, we can then decide how big to make each piece. Once we know the size, we can then work out the shape of each piece.

Example: Carry Out the Plan

Let's put our grid on the cake:

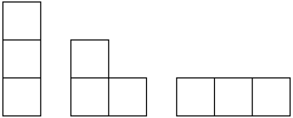


Now we can see that there are 12 squares total. Since there are 4 friends, each one should get 3 squares of cake.

Example: Carry Out the Plan

Each piece should be three squares joined together. Let's think about what shapes we can make with three squares.

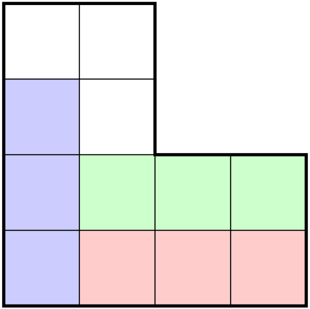
The only possible shapes are:



We can rotate and flip the pieces and they will remain the same size and shape, but we can't mix and match pieces. Let's see if we can fit these shapes into our grid.

Example: Carry Out the Plan

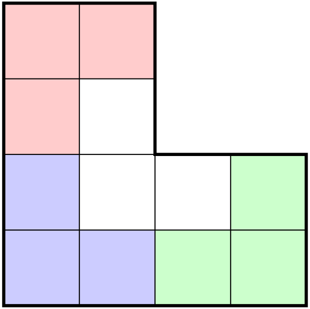
Using the rectangular pieces:



It looks like we can't use the rectangular pieces because the last piece will need to be an L-shaped piece. So let's try the L-shaped pieces.

Example: Carry Out the Plan

Using the L-shaped pieces:



This works!

Example: Review the Solution

Did we correctly solve the problem we were asked to solve?

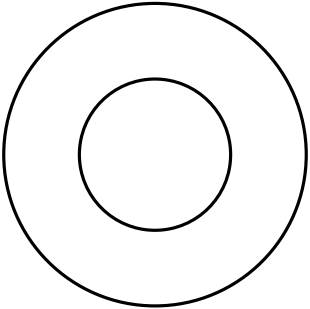
Yes! We need four pieces that are the same size and shape. Each piece is an L-shape that consists of three grid squares, so they are indeed the same size and shape. Finally, we got exactly four pieces. This is exactly what we were asked to do.

Example 2

Alice, Bob, Charley, Diane, and Edgar (they forgave him for the cake incident) want to split a donut. They want to cut the donut into five pieces, but only using two cuts. This time, they don't mind how big or small each piece is, nor do they care about the shape. As long as each of them gets a piece, no matter how small, they are happy. How can they cut the donut into five pieces using two cuts?

Example 2: Understand the Problem

Again, let's draw a picture to help us out:

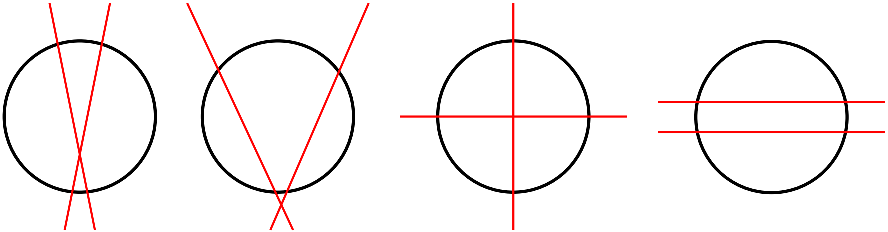


We need to find two cuts that divide the donut into five pieces.

Example 2: Devise a Plan

Let's consider a different, but related, problem to help us come up with some ideas.

Imagine instead of a donut, they were sharing a pizza. Let's do some examples:



It looks like we get 4 pieces if the lines cross inside the circle, and 3 pieces if they cross outside the circle or not at all.

Example 2: Devise a Plan

We should look for cuts that cross at some point on the donut in order to get more pieces.

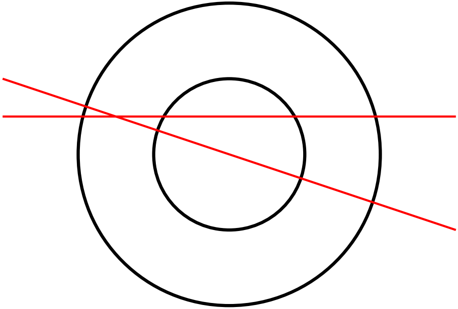
Let's think more about the pizza problem. Is it possible to get five pieces from a pizza while using two cuts?

The answer is no! No matter how hard you try, you can only get at most four pieces when cutting a pizza using two cuts.

The difference between the pizza and the donut is that the donut has a hole. So the hole in the donut must be the key to solving this problem. For our plan, we should focus on cuts that pass through the hole.

Example 2: Carry Out the Plan

We want two cuts that cross at some point on the donut and pass through the hole. If we try making cuts with those conditions:



We get five pieces!

Example 2: Review the Solution

We were asked to get five pieces using two cuts, which is what we did.

Example 3

You have been captured by an evil wizard. In order to escape, you have to win at his game. The rules of the game are as follows:

In between you and the wizard is a large cauldron. The wizard gives you a bag of white marbles. He adds between 10 and 20 black marbles into the cauldron, but you don't know how many he adds. You then need to add between 10 and 20 marbles. The wizard then randomly removes marbles from the cauldron two at a time. If they are the same color, he adds one black marble to the cauldron. If they are different colors, you add one white marble to the cauldron. You win if the last marble in the cauldron is white.

How many marbles should you add to the cauldron to guarantee your freedom?

Example 3: Understand the Problem

The rules are a bit complicated, but our goal is simple: choose your number of marbles to ensure that the last marble in the cauldron is white.

Example 3: Devise a Plan

At first, it looks like there is no way to guarantee a win. You don't know the number of black marbles the wizard adds, and even if you did, the marbles are removed randomly!

Let's try working backwards. We want to end up with one white marble in the cauldron, so let's think about what must happen right before then.

Example 3: Carry out the Plan

To end up with one white marble, the wizard must have removed two marbles that are different colors right before then. So right before, there was one white marble and one black marble.

Let's make a chart to keep track of things:

White	Black
1	0
1	1

We'll read the chart from the bottom up.

Example 3: Carry out the Plan

Let's think about the step that gets us to have one marble of each color in the cauldron.

There are two ways to get there. The first is that a white marble was added. So a white marble and a black marble were removed, but one white marble was added, so in total we lost one black marble. Since we are going backward, we add one black marble:

White	Black
1	0
1	1
1	2

Example 3: Carry out the Plan

The second way is that a black marble was added. This means either two black marbles were removed, or two white marbles were removed. If it was two black marbles, in total we lost one black marble, so we add one:

White	Black
1	0
1	1
1	2

If it was two white marbles, in total we lost two white marbles and gained one black marble, so going back, we add two white marbles and subtract one black marble:

White	Black
1	0
1	1
3	0

Example 3: Carry out the Plan

Let's consider our three possible charts:

White	Black
1	0
1	1
1	2

White	Black
1	0
1	1
1	2

White	Black
1	0
1	1
3	0

Are there any patterns?

Looking at the white marble column, we only have odd numbers. Is that always the case?

Example 3: Carry out the Plan

Let's think about what happens during the removal process.

If we remove a white marble and a black marble, we lose one black marble and the number of white marbles stays the same.

If we remove two black marbles, again we lose one black marble and the number of white marbles stays the same.

If we remove two white marbles, we lose two white marbles and gain one black marble.

So the number of white marbles either stays the same or drops by two in any removal.

Example 3: Carry out the Plan

Lastly, we can notice the fact that the evenness or oddness (called the "parity") of a number doesn't change if you subtract two from it.

Now we have our strategy!

We want to end up with 1 white marble in the cauldron. Since 1 is an odd number, and we can never change the fact that the number of white marbles in the cauldron must be an odd number, we should add an odd number of white marbles to the cauldron.

Example 3: Review the Solution

We could ensure this solution works by running an experiment.

Try cutting out some black and white paper as the marbles, add in a random number of black marbles and an odd number of white marbles, then randomly remove them according to the rules.

Repeat this multiple times. If you keep ending up with a white marble, you know the solution is correct.

Example 4

What is the digit in the units place of 2^{522}?

Example 4: Understand the Problem

We want to find the digit in the units place of 2^{522}.

Remember that the units place is the number furthest to the right in a number with no decimals.

For example, 3 is the digit in the units place of 204930201943.

Example 4: Devise a Plan

The number is too big to put into a calculator and just check the last digit, so we need to find another way to determine the digit in the units place.

Let's try experimenting with some smaller powers of 2 and see if we can notice a pattern.

We can then use inductive reasoning to extend our pattern, and finally use deductive reasoning to show our conjecture is correct.

Example 4: Carry out the Plan

Let's make a table to keep track of things:

Power of 2	Result
0	1
1	2
2	4
3	8
4	16
5	32
6	64
7	128
8	256
9	512
10	1024

We can see a pattern in our table. With the exception of 1 at the beginning, the digits in the units place go $2, 4, 8, 6, 2, 4, 8, 6, 2, 4, \ldots$

Example 4: Carry out the Plan

We can see that our pattern repeats every four terms, so let's look at the remainder of the power of 2 when we divide the power by 4:

Power of 2	Result	Remainder of Power
0	1	0
1	2	1
2	4	2
3	8	3
4	16	0
5	32	1
6	64	2
7	128	3
8	256	0
9	512	1
10	1024	2

Now we can relate the power of 2 to the digit in the units place by looking at the remainder when dividing by 4.

Example 4: Carry out the Plan

It appears that the pattern is the following:

Remainder of Power	Last Digit
0	6
1	2
2	4
3	8

Example 4: Carry out the Plan

So we can look at our original question and use the pattern.

We wanted to find the digit in the units place of 2^{522}, so we will find the remainder of 522 when we divide it by 4.

We get 2, so the digit in the units place of 2^{522} is 4.

Example 4: Review the Solution

Our number is too big to put into a calculator, but we can double check that our answer is plausible.

We know that 2^{522} is an even number because it is divisible by 2. So we know that the last digit cannot be 1, 3, 5, 7, or 9. Furthermore, we know that the last digit cannot be 0 because that would mean 2^{522} is divisible by 5, which it is not.

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Sets

Definition: Sets

A **set** is any group or collection of objects. The objects inside a set are called **elements** or **members**.

For example, the collection of all the students in this room is a set.

Almost everything you can think of can be put into a set!

Writing Sets

We use a capital letter to represent a set. For example, we may say the set of all colors is represented by C. The choice of letter doesn't matter.

We write the contents of a set as an equation:

$$C = \{\text{red}, \text{blue}, \text{green}, \dots\}$$

We use {} (called "curly brackets" or "curly braces") to denote what is in the set.

Definition: Roster Method

The **roster method** of writing a set is done by listing the elements of the set inside curly brackets.

If an element is in a set, we use the symbol $\in$ to denote that. For example, pink $\in C$.

Some Special Sets

Certain sets of numbers are used so often in mathematics that they get their own names:

Special Sets

The set of **natural numbers** or **counting numbers** is $N = \{1, 2, 3, 4, 5, \dots\}$.

The set of **whole numbers** is $W = \{0, 1, 2, 3, 4, 5, \dots\}$.

The set of **integers** is $I = \{\dots, -3, -2, -1, 0, 1, 2, 3, \dots\}$.

The set containing no elements is called the **empty set** and is denoted by $\emptyset$.

Other Special Sets

More Special Sets

Any number than can be written as an infinite decimal is called a **real number**. The set of real numbers is denoted by $\mathbb{R}$.

The set of **rational numbers**, denoted by $\mathbb{Q}$, consists of all the numbers whose decimals either terminate or repeat.

The set of **irrational numbers**, denoted by $\mathbb{R} \setminus \mathbb{Q}$, consists of all the numbers whose decimals never terminate nor repeat.

Many ancient Greek mathematicians believed that every number was rational. According to legend, Hippasus, a student of Pythagoras, proved that $\sqrt{2}$ was irrational, and was thrown off a ship for proving his mentor wrong. The truth of the legend is not well supported, however.

Subsets

Definition: Subset

Given two sets, A and B, we say A is a **subset** of B if every element in A is also an element in B. We write this as $A \subseteq B$.

For example, $\{1, 4, 5\} \subseteq \{1, 2, 3, 4, 5\}$.

For any set A, we have:
$A \subseteq A$
$\emptyset \subseteq A$

Question

Let E be the set of even natural numbers. Which of the following is FALSE?

A. $E = \{2, 4, 6, 8, \dots\}$
B. $E \subseteq N$
C. $17 \in E$
D. $\{2, 44, 198\} \subseteq E$
E. I don't know

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Question

Let E be the set of even natural numbers. Which of the following is FALSE?

A. $E = \{2, 4, 6, 8, \ldots\}$
B. $E \subseteq N$
C. $\boldsymbol{17 \in E}$
D. $\{2, 44, 198\} \subseteq E$
E. I don't know

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Set Operations and Cardinality

Definition: Union and Intersection

The **union** of two sets, A and B, is the set consisting of every element in A, or B, or both. We write the union as $A \cup B$.

The **intersection** of two sets, A and B, is the set consisting of every element in both A and B. We write the intersection as $A \cap B$.

For example, if $A = \{1, 4, 7, 9\}$ and $B = \{2, 3, 4\}$, then:
$A \cup B = \{1, 2, 3, 4, 7, 9\}$
$A \cap B = \{4\}$

Definition: Cardinality

The **cardinality** of a set is the number of elements in the set. The cardinality of a set A is denoted by $n(A)$.

For example, $n(A) = 4$ and $n(B) = 3$.

Question

Let S be the set of all sandwiches, and let C be the set of all meals containing chicken. The set $S \cap C$ represents:

A. The set of all sandwiches or meals with chicken.
B. The set of all sandwiches that also have chicken.
C. The set of all sandwiches that do not contain chicken.
D. The set of all chicken meals that are not sandwiches.
E. I don't know.

Room Name: BROWN4663

Question

Let S be the set of all sandwiches, and let C be the set of all meals containing chicken. The set $S \cap C$ represents:

A. The set of all sandwiches or meals with chicken.
B. The set of all sandwiches that also have chicken.
C. The set of all sandwiches that do not contain chicken.
D. The set of all chicken meals that are not sandwiches.
E. I don't know.

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Surveys

Let's say I ran a survey of 100 people about whether or not they like cheese pizza and/or pepperoni pizza.

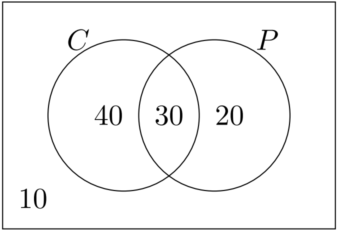
40 respond that they like only cheese, 20 respond that they like only pepperoni, 30 respond that they like both, and 10 respond that they don't like either.

If C is the set of people who like cheese pizza and P is the set of people who like pepperoni pizza, we can determine:

$n(C) = 70$
$n(P) = 50$
$n(C \cup P) = 90$
$n(C \cap P) = 30$

Venn Diagrams

We can represent the survey using a Venn Diagram:



Again, we see:

$n(C) = 70$
$n(P) = 50$
$n(C \cup P) = 90$
$n(C \cap P) = 30$

Inclusion-Exclusion Principle

Formula: Inclusion-Exclusion Principle

Given any two sets, A and B, the following equation holds:

$$n(A \cup B) = n(A) + n(B) - n(A \cap B)$$

The Inclusion-Exclusion Principle turns a Venn Diagram into an equation.

For example, if $n(A) = 10$, $n(B) = 15$, and $n(A \cap B) = 5$, then:

$$n(A \cup B) = 10 + 15 - 5 = 20$$

Further Uses of Inclusion-Exclusion Principle

We can use algebra to rearrange the Inclusion-Exclusion equation as needed:

A survey shows 50 students like jazz music, 20 students like rock music, and 60 students like either jazz or rock. How many like both?

Let's say J is the set of students who like jazz music and R is the set of students who like rock music. Then we know:

$n(J) = 50$
$n(R) = 20$
$n(J \cup R) = 60$

Further Uses of Inclusion-Exclusion Principle

Plugging these into the equation and solving using algebra gives us:

$$\begin{aligned}
60 &= 50 + 20 - n(J \cap R) && \\
60 &= 70 - n(J \cap R) && \text{Add 50 and 20} \\
n(J \cap R) + 60 &= 70 && \text{Add } n(J \cap R) \text{ to both sides} \\
n(J \cap R) &= 10 && \text{Subtract 60 from both sides}
\end{aligned}$$

So 10 students like both jazz music and rock music.

As long as you have three out of the four sets in the Inclusion-Exclusion equation, you can use algebra to solve for the last one.

Question

Netflix surveys its customers about what kind of shows they watch. 500 say they watch comedies, 700 say they watch dramas, and 1100 say they watch either comedies or dramas. How many watch both comedies and dramas?

A. 100
B. 200
C. 400
D. 600
E. I don't know

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Question

Netflix surveys its customers about what kind of shows they watch. 500 say they watch comedies, 700 say they watch dramas, and 1100 say they watch either comedies or dramas. How many watch both comedies and dramas?

A. 100
B. 200
C. 400
D. 600
E. I don't know

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A Bigger Survey

A music teacher surveyed 495 students. The results are:

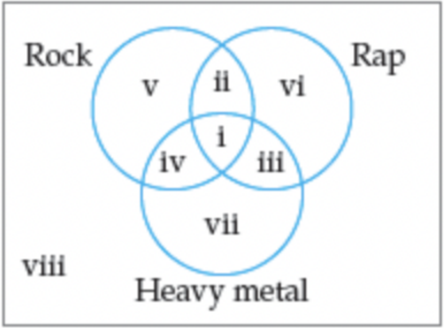
320 students like rap music.
395 students like rock music.
295 students like heavy metal music.
280 students like both rap music and rock music.
190 students like both rap music and heavy metal music.
245 students like both rock music and heavy metal music.
160 students like all three.

How many students:

a. Like exactly one of the music types?
b. Like exactly two of the music types?
c. Do not like rock music?

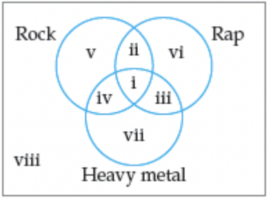
Triple Venn Diagram

We can make a triple Venn Diagram to help us sort out the information:



Our goal is to figure out the number of people in each region of the Venn Diagram.

Filling out the Venn Diagram

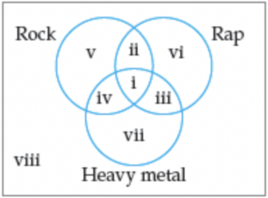


Region i corresponds with liking all three music types, so we write 160 in there.

Region ii corresponds with liking rap and rock, but not heavy metal. The total number of students who like rap and rock is 280. Since 160 of these like heavy metal, the remaining number is $280 - 160 = 120$. So we write 120 in region ii.

We can use the same logic to determine we need to write 30 in region iii and 85 in region iv.

Filling out the Venn Diagram



Region v corresponds with liking only rock music. We know that the total number of students who like rock is 395. We add up regions i, ii, and iv to get $160 + 120 + 85 = 365$, the number of students who like rock and some other type of music. The remaining number is $395 - 365 = 30$, so we write 30 in region v.

We can use the same logic to determine we need to write 10 in region vi and 20 in region vii. Finally, we write 40 in region viii because those are the leftover students.

Answering the Questions

a. Like exactly one of the music types?

These are the students in regions v, vi, and vii, so $30 + 10 + 20 = 60$ students like exactly one music type.

b. Like exactly two of the music types?

These are the students in regions ii, iii, and iv, so $120 + 30 + 85 = 235$ like exactly two music types.

c. Do not like rock music?

These are the students who are NOT in regions i, ii, iv, or v. There are $160 + 120 + 85 + 30 = 395$ who like rock, so $495 - 395 = 100$ students do not like rock music.

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Graph Theory

The word "graph" can have multiple meanings in mathematics. One you're probably familiar with is the graph of a function (think $y = mx + b$). But there is another type of graph that we use in math!

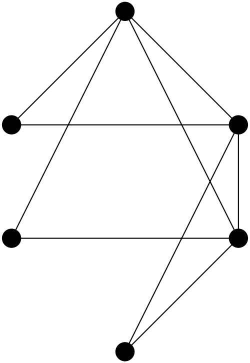
The mathematical field of **graph theory** deals with objects and the connections between them.

For example, you can use a graph to represent the cities in Connecticut and the roads between them.

Definition: Graph

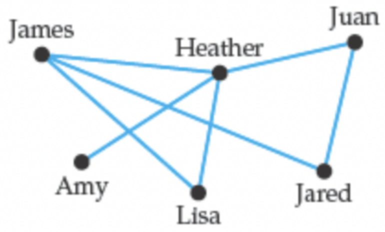
A **graph** is a set of points with line segments and/or curves that connect the points. The points are called **vertices** and the line segments/curves are called **edges**.

Shown below is an example of a graph:



Using a Graph to Represent Something

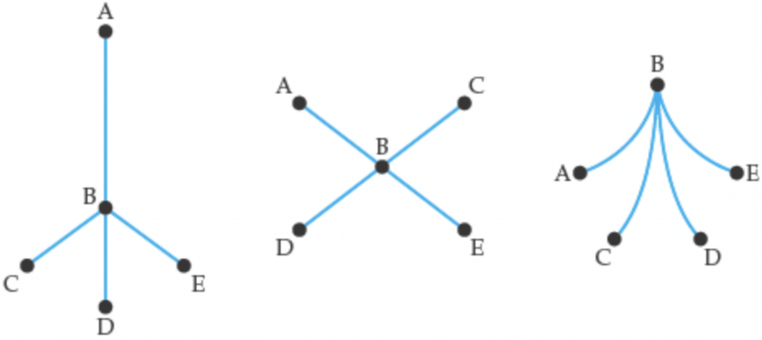
While graphs are interesting on their own, our main purpose is to use them to represent something. The following graph represents the friendships of a group of students. Two students are joined by an edge if they are friends with one another.



Equivalent Graphs

The only thing that matters when drawing a graph is which vertices are connected by edges.

Two graphs are called “equivalent” if the same vertices are connected by edges in both graphs. For example, the following graphs are equivalent:



Creating a Graph

To create a graph to represent something, you need two pieces of information: what the objects are, and what the connections between them are.

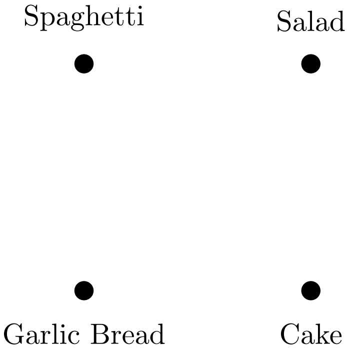
Let's create a graph to represent a meal. The meal consists of spaghetti, salad, garlic bread, and cake.

- The spaghetti requires flour, eggs, tomatoes, and garlic.
- The salad requires lettuce, tomatoes, and onions.
- The garlic bread requires a baguette, garlic, and butter.
- The cake requires flour, sugar, butter, and eggs.

We'll connect two parts of the meal if they share a required ingredient.

Creating a Graph

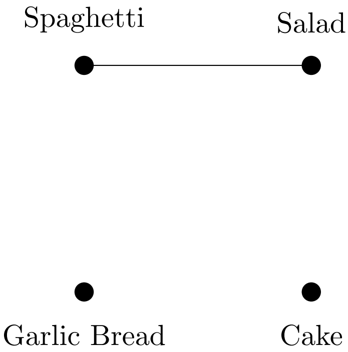
First, draw and label the vertices.



Give yourself lots of room to work!

Creating a Graph

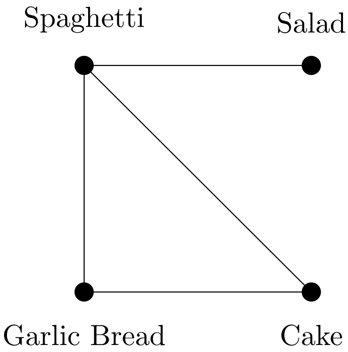
Next, add in the edges. The spaghetti and the salad both require tomatoes, so we'll connect them with an edge:



Repeat this for all other connections.

Creating a Graph

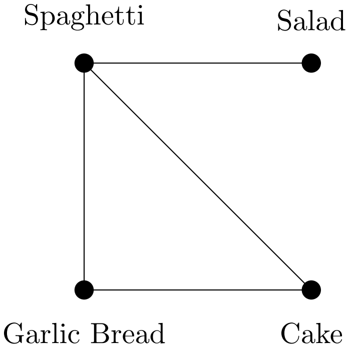
The final graph should look like this:



Multiple Edges and Loops

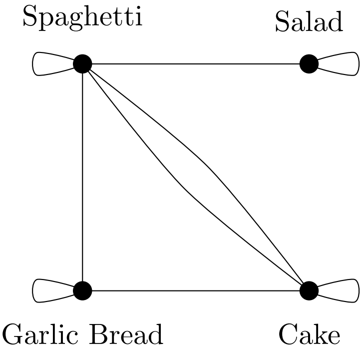
Graphs can have multiple edges between vertices, or it can have edges that connect a vertex to itself (called a loop).

Let's add multiple edges and loops to our graph for the meal:



Multiple Edges and Loops

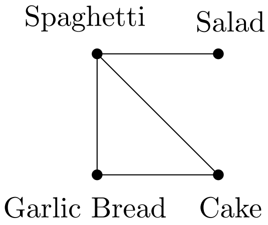
Since the spaghetti and cake both required flour and eggs, we can use a double edge to represent that. Also, every dish obviously shares ingredients with itself, so we can add a loop to each vertex:



Graph Terminology

Two vertices are **adjacent** if there is an edge between them.

The **degree** of a vertex is the number of edges that originate at the vertex.

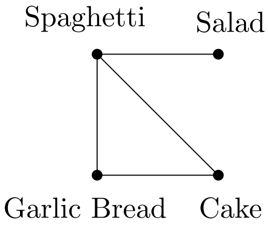


For example, Garlic Bread and Spaghetti are adjacent, Cake and Salad are not adjacent, and Spaghetti has degree 3.

Graph Terminology

A **path** between two vertices connects the vertices by traveling along edges.

A **circuit** is a path that starts and ends at the same vertex.

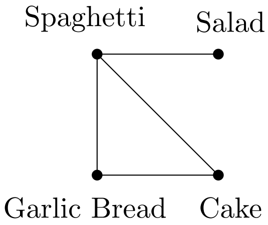


There is a path from Garlic Bread to Salad by going Garlic Bread to Spaghetti to Salad. There is a circuit starting at Cake by going to Spaghetti to Garlic Bread and back to Cake.

Graph Terminology

A graph is **connected** if any two vertices can be joined by a path.

A graph is **complete** if every possible edge is drawn between the vertices.



This graph is connected, but it is not complete, because there is no edge between Cake and Salad, nor is there an edge between Garlic Bread and Salad.

Question

Suppose you want to draw a graph to represent the following information. There are four students with different majors: Math, English, Art History, and Nursing.

- Math requires calculus, statistics, and academic writing.
- English requires academic writing, poetry, and history.
- Art History requires pottery, painting, and history.
- Nursing requires biology, academic writing, and statistics.

Draw an edge if two majors share a required class. Which vertex has degree equal to 1?

A. Math B. English C. Art History
D. Nursing E. I don't know.

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Question

Suppose you want to draw a graph to represent the following information. There are four students with different majors: Math, English, Art History, and Nursing.

- Math requires calculus, statistics, and academic writing.
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A. Math B. English **C. Art History**
D. Nursing E. I don't know.

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Question

Suppose you want to draw a graph to represent the following information. There are four students with different majors: Math, English, Art History, and Nursing.

- Math requires calculus, statistics, and academic writing.
- English requires academic writing, poetry, and history.
- Art History requires pottery, painting, and history.
- Nursing requires biology, academic writing, and statistics.

Draw an edge if two majors share a required class. Is the graph connected? Is the graph complete?

A. Yes, Yes B. Yes, No
C. No, Yes D. No, No E. I don't know.

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Question

Suppose you want to draw a graph to represent the following information. There are four students with different majors: Math, English, Art History, and Nursing.

- Math requires calculus, statistics, and academic writing.
- English requires academic writing, poetry, and history.
- Art History requires pottery, painting, and history.
- Nursing requires biology, academic writing, and statistics.

Draw an edge if two majors share a required class. Is the graph connected? Is the graph complete?

A. Yes, Yes **B. Yes, No**
C. No, Yes D. No, No E. I don't know.

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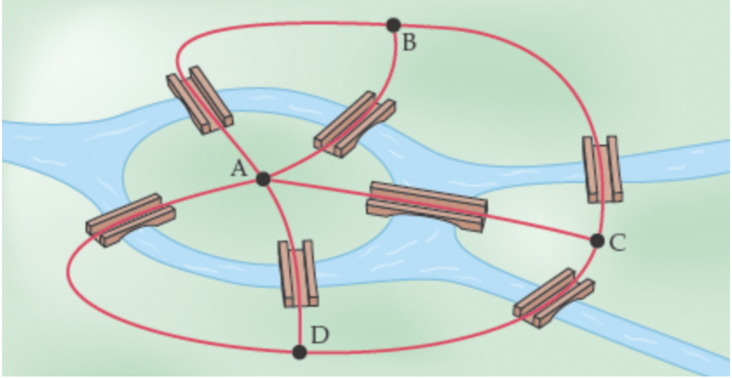
The Seven Bridges of Königsberg

The city of Königsberg in Prussia (now Kaliningrad, Russia) had seven bridges arranged as shown below:



A longstanding puzzle was to plan a walk through the city that crossed every bridge exactly once that ended where it started.

The Seven Bridges of Königsberg

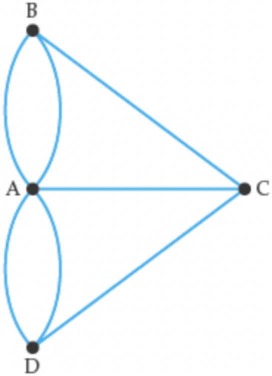
Many people tried to find a path that worked, but nobody was able to succeed. Some started to think it was impossible, but weren't sure how to explain why.

It wasn't until 1735 when Swiss mathematician Leonard Euler proved that it was, in fact, impossible.

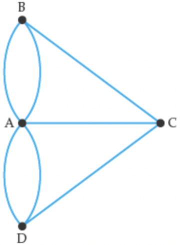
To understand his solution, we need to turn our problem into a graph.

The Seven Bridges of Königsberg

We'll represent each plot of land as a vertex and draw an edge between the vertices if there is a bridge connecting them. Since some plots have multiple bridges connecting them, we will use multiple edges:



Euler Circuit

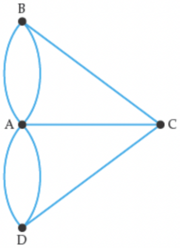


Euler wanted to figure out if it was possible to find a circuit on this graph that crossed each edge exactly once. Such circuits were eventually named after him.

Definition: Euler Circuit

An **Euler circuit** is a circuit on a graph that crosses each edge exactly once. A graph with an Euler circuit is called **Eulerian**.

The Seven Bridges of Königsberg



It turns out that it is impossible to find an Euler circuit on this graph! The reason why is because of the degrees of the vertices.

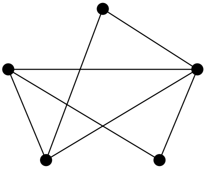
Euler proved that only certain graphs can have an Euler circuit. In fact, he came up with a very simple test that says whether or not a graph is Eulerian.

The Eulerian Graph Theorem

A connected graph is Eulerian if and only if every vertex has even degree.

The phrase “if and only if” means that in any graph where the degree of every vertex is even, the graph has an Euler circuit. It also means that if even one vertex has odd degree, the graph CANNOT be Eulerian.

Graph A



Degrees of Graph B: $\{2, 4, 6, 4, 2, 2, 4, 2\}$.

Graph A is drawn above. Graph B is not shown, but the degrees of each vertex are listed above. Which graph(s), if any, have an Euler circuit?

A. A and B   B. A only   C. B only
D. Neither A nor B   E. I don't know.

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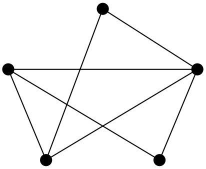
Graph A



Degrees of Graph B: $\{2, 4, 6, 4, 2, 2, 4, 2\}$.

Graph A is drawn above. Graph B is not shown, but the degrees of each vertex are listed above. Which graph(s), if any, have an Euler circuit?

A. A and B
B. A only
C. B only
D. Neither A nor B
E. I don't know.

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Euler Paths & The Euler Path Theorem

What if we don't need a circuit? Does this change things?

Definition: Euler Path

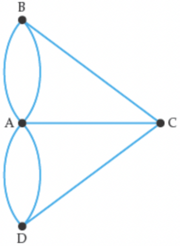
An **Euler path** is a path that crosses each edge exactly once, without the need to start and end at the same vertex. Sometimes an Euler path is called an "Euler walk."

The answer is yes!

The Euler Path Theorem

A connected graph has Euler path if and only if every vertex has even degree, or there are exactly two vertices with odd degree, and the rest are even.

The Seven Bridges of Königsberg

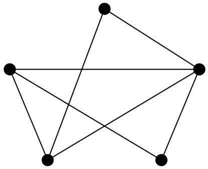


Does the graph of the Königsberg bridge problem have an Euler path?

No, there are four vertices of odd degree.

Question

Graph A



Degrees of Graph B: $\{2, 4, 6, 4, 2, 2, 4, 2\}$.

Graph A is drawn above. Graph B is not shown, but the degrees of each vertex are listed above, and we assume Graph B is connected. Which graph(s), if any, have an Euler path?

A. A and B
B. A only
C. B only
D. Neither A nor B
E. I don't know.

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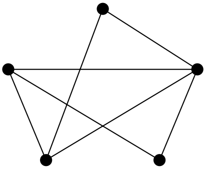
Question

Graph A 

Degrees of Graph B: $\{2, 4, 6, 4, 2, 2, 4, 2\}$.

Graph A is drawn above. Graph B is not shown, but the degrees of each vertex are listed above, and we assume Graph B is connected. Which graph(s), if any, have an Euler path?

A. A and B B. A only C. B only
D. Neither A nor B E. I don't know.

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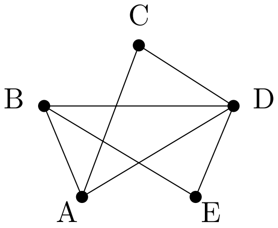
Finding an Euler Circuit/Path

Strategy to Find an Euler Circuit/Path

1. Check if the graph has any vertices of odd degree.
   1a. If looking for circuit, stop.
   1b. If there are more than 2 vertices of odd degree, stop.
2. If there are any vertices of odd degree, start there. Otherwise, pick any vertex you like.
3. Traverse the edges, keeping track of the vertices in the order you visit them.
   3a. This will require trial and error!
4. When done, double check you actually crossed every edge.

There can be many different Euler circuits/paths in a graph, if it has one.

Example of Finding Euler Path



Let's try to find an Euler circuit/path on this graph. Since vertices A and B have degree 3, we can't find an Euler circuit, but we can find an Euler path. Let's start at A.

After some guess and check, we can see that A-D-C-A-B-E-D-B is an Euler path. Another could be A-B-E-D-C-A-D-B.

A-B-E-D-B-A-D-C-A is not an Euler circuit/path, because is crosses the edge between A and B twice.

Hamiltonian Circuits

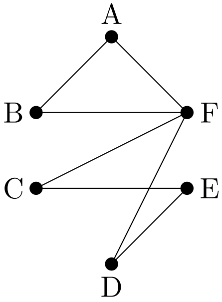
There is another type of circuit we can look at on a graph. Instead of traveling each edge once, we can visit each vertex once.

Definition: Hamiltonian Circuit

A **Hamiltonian circuit** is a circuit on a graph that visits each vertex once (and only once). A graph that has a Hamiltonian circuit is called **Hamiltonian**.

Finding a Hamiltonian circuit in a graph is unfortunately rather difficult. Most of the time, you need to guess-and-check.

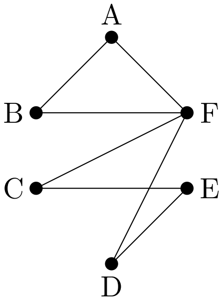
Let's say you make a graph to represent your errand run. The vertices represent locations you need to visit and the edges represent bus routes between the locations:



If you live at vertex A, is there a way to visit all of the locations you need by traveling on the bus routes (without visiting the same place more than once) and return home at the end?

Usefulness of Hamiltonian Circuits

If we can find a Hamiltonian circuit, then you can achieve your goal.



This graph does NOT have a Hamiltonian circuit. If we keep trying, we end up having to cross vertex F more than once every time.

Planar Graphs

Earlier, we mentioned how edges on a graph can intersect without creating vertices, which means they don't really matter.

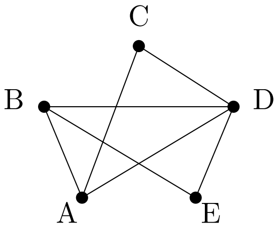
Sometimes, however, we ARE interested in graphs whose edges don't intersect.

Definition: Planar Graph

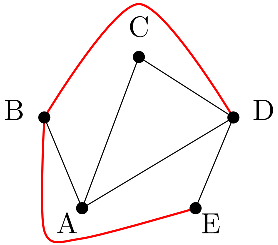
A graph is called **planar** if it is possible to draw it in a way where the edges don't intersect.

Warning: Just because a graph is drawn with intersecting edges, doesn't mean it's not planar! If you can re-draw the graph without edge crossings while keeping the same vertices connected by edges, the graph is planar.

Re-Drawing a Graph to be Planar

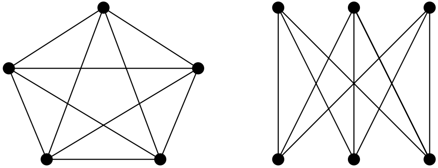


The current drawing has edge crossings, but we can eliminate the edge crossings by re-drawing the graph as follows:



Non-Planar Graphs

Certain graphs cannot be re-drawn to avoid edge crossings. In particular, the following two graphs are not planar:



The graph on the left is the complete graph with 5 vertices. The graph on the right is sometimes called the Utilities Graph.

Showing that a graph is nonplanar is pretty difficult, but it is possible.

Scheduling Meetings

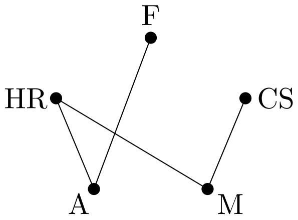
A company has five departments: Finance, Human Resources, Administration, Marketing, and Customer Service. Each department needs to have a meeting, and they want to be as efficient as possible with scheduling. Some employees belong to more than one department, shown in the table below:

	Finance	Human Resources	Administration	Marketing	Customer Service
Finance	—		X		
Human Resources		—	X	X	
Administration	X	X	—		
Marketing		X		—	X
Customer Service				X	—

Scheduling Meetings

	Finance	Human Resources	Administration	Marketing	Customer Service
Finance	—		X		
Human Resources		—	X	X	
Administration	X	X	—		
Marketing		X		—	X
Customer Service				X	—

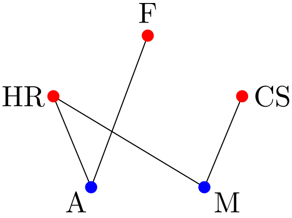
Our graph is:



Scheduling Meetings

Let's use our graph to schedule the meetings. We'll color a vertex red if the meeting will be at 12 PM, and we'll color a vertex blue if the meeting will be at 1 PM.

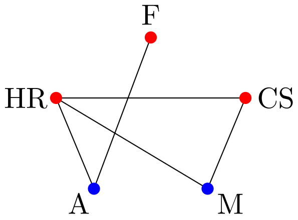
If two departments share a person, we can't schedule their meetings at the same time. So we need to make sure adjacent vertices are not the same color.



Scheduling Meetings

What would happen if someone from Customer Service joined the Human Resources department?

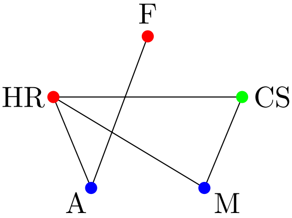
Then we add an edge to our graph:



But now Customer Service and Human Resources have a meeting at the same time.

Scheduling Meetings

To fix that, we add a third meeting time. We'll color a vertex green if the meeting will be at 2 PM.



Now every department can meet.

Graph Coloring

What we just did was called a graph coloring. The basic idea of a graph coloring is to color in each vertex while making sure that vertices connected by edges have different colors.

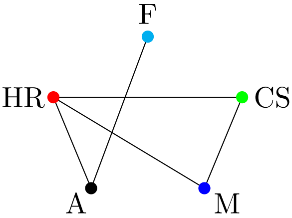
Definition: Graph Coloring

A **graph coloring** (also called a **coloring of a graph**) is a rule that assigns each vertex a color such that adjacent vertices have different colors.

If a graph coloring uses n colors, we say the graph is n-colorable.

Chromatic Number

Returning to our company, we could have colored each vertex its own color:

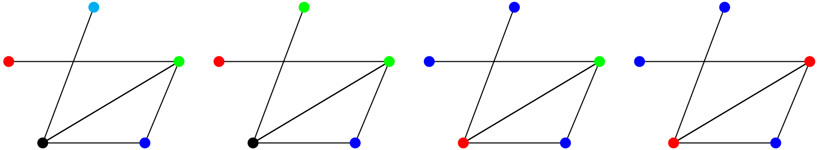


This is a valid coloring, but it's inefficient. We are most interested in a coloring that uses the fewest number of colors possible.

Definition: Chromatic Number

The **chromatic number** of a graph is the minimum number of colors needed to color the graph.

Question



Shown above are four (potential) colorings of the same graph. What is the chromatic number of the graph?

A. 2
B. 3
C. 4
D. 5
E. I don't know.

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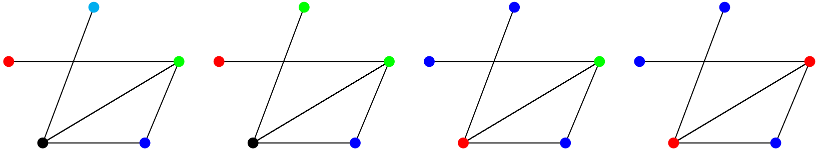
Question



Shown above are four (potential) colorings of the same graph. What is the chromatic number of the graph?

A. 2
B. 3
C. 4
D. 5
E. I don't know.

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Properties of Graph Colorings

Properties of Graph Colorings

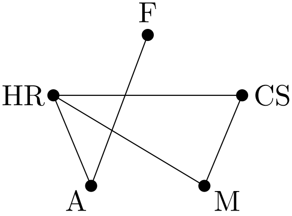
If a graph is n-colorable, then it can also be colored using more than n colors.

If a graph is n-colorable, then the chromatic number of the graph is less than or equal to n.

A graph is 2-colorable if and only if it has no circuits that consist of an odd number of vertices.

2-Colorable Graph Theorem

A graph is 2-colorable if and only if it has no circuits that consist of an odd number of vertices.



This graph has a circuit consisting of an odd number of vertices (CS-HR-M-CS), so it is not 2-colorable.

If you can find just one circuit consisting of an odd number of vertices, you need at least 3 colors to color the graph.

Question

For a certain graph, you know the following information:

- There is a circuit consisting of 7 vertices.
- You can color the graph using 6 colors.

What can you say about the chromatic number of the graph?

A. It is exactly 2.
B. It is exactly 6.
C. It is greater than or equal to 2 and less than or equal to 6.
D. It is greater than 2 and less than or equal to 6.
E. I don't know.

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Question

For a certain graph, you know the following information:

- There is a circuit consisting of 7 vertices.
- You can color the graph using 6 colors.

What can you say about the chromatic number of the graph?

A. It is exactly 2.
B. It is exactly 6.
C. It is greater than or equal to 2 and less than or equal to 6.
D. It is greater than 2 and less than or equal to 6.
E. I don't know.

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The Four Color Theorem

The Four Color Theorem states that every planar graph is 4-colorable. The reason people were interested in it is because every map of land/countries/states can be represented as a planar graph.

This means you can color in any map such that countries who share a border are not the same color, and you need at most four colors.

The theorem went unproved for many years. It wasn't until computers came about that the theorem was proved in 1976 by Haken and Appel.

This was the first time a computer was used in a mathematical proof, which was very controversial at the time because people didn't trust computers to give the right answer. In modern times, computers are used all the time, and are readily accepted.

Finding a Coloring of a Graph

There is no general way to find a coloring of a graph that uses the fewest number of colors.

First, it's good to check if the graph has any circuits that consist of an odd number of vertices. If it doesn't, you know it is 2-colorable. Otherwise, you need 3 or more colors.

From there, it is simply a matter of guess-and-check.

Greedy Algorithm

One strategy is known as a "greedy algorithm." It goes like this:

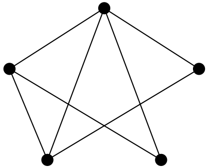
Initial step: Pick any vertex and color it any color. Add the color to a separate list.

Repeat the following:

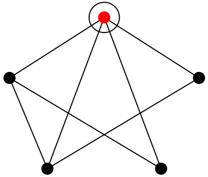
1. Pick any uncolored vertex and check the vertices adjacent to it.
2. If there is a color on your list that is not used by any of the adjacent vertices, color the vertex that color.
3. If all colors on your list are taken, add a new color to your list and color the vertex that color.

Stop when you have colored every vertex.

This process guarantees a coloring, but does NOT guarantee that it uses the fewest colors possible.



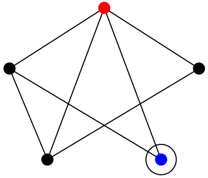
Let's color the above graph using a greedy algorithm. Let's pick this vertex and color it red:



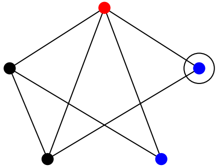
Our list of colors is: red.

Example of Greedy Algorithm

Now let's pick this one:

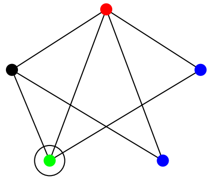


Since it is adjacent to a red vertex, we have to color it blue. Our list of colors is: red, blue. Now let's pick:

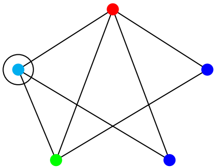


It's adjacent to a red vertex, but we still have blue on our list, so we'll color it blue. Our list is still: red, blue.

Now let's pick this one:



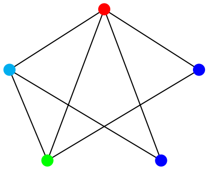
It's adjacent to a red vertex and a blue vertex. We are out of colors, so let's color it green. Our list of colors is: red, blue. green. Finally:



It's adjacent to a red, blue, and green vertex, so we need to color it cyan.

Example of Greedy Algorithm

We got a coloring of our graph using 4 colors:



However, we only need 3:

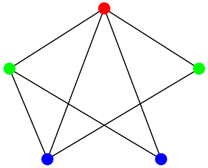


Table of Contents

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Base Ten

What does the number 538 really mean?

It means we have 8 groups of 1, 3 groups of 10, and 5 groups of 100. If we add these up, we get our number:

$$5 \times 100 + 3 \times 10 + 8 \times 1 = 538$$

So the way we write a number encodes information about the groups we have.

In this case we were looking at groups that were powers of 10. That is, groups of $10^0 = 1$, groups of $10^1 = 10$, groups of $10^2 = 100$, etc. And the number of groups for any particular power is never 10 or greater. This is because 10 groups of some power is 1 group of the next power.

Different Groups

But what is so special about 10? What if we wanted to think about 538 using groups that were powers of 5?

The powers of 5 are: $5^0 = 1, 5^1 = 5, 5^2 = 25, 5^3 = 125, 5^4 = 625, \ldots$

Then 538 contains 4 groups of 125, 1 group of 25, 2 groups of 5, and 3 groups of 1:

$$4 \times 125 + 1 \times 25 + 2 \times 5 + 3 \times 1 = 538$$

If we write this number like we do using groups of powers of 10, we could say that $538 = 4123_{\text{five}}$, where we write "five" as a subscript to let us know the number 4123 uses groups that are powers of 5.

Definition: Base System

A **base system** is a method of writing a number using groups that are the powers of a given number.

The specific number b being used for the groups is called the **base**, and any number written using b as its base is said to be written "in base b."

For example 4123_{five} is 538 written in base 5.

The "normal" numbers we use in everyday life are written in base 10. If a number doesn't have a subscript with a specific base, the number is in base 10.

Question

Consider the number $17 = 10001_{\text{two}}$. Which of the following statements correctly uses the terminology for base systems?

A. 2 is 10001_{two} written in base 17.
B. 10001_{two} is 2 written in base 17.
C. 17 is 2 written in base 10001_{two}.
D. 10001_{two} is 17 written in base 2.
E. I don't know.

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Question

Consider the number $17 = 10001_{\text{two}}$. Which of the following statements correctly uses the terminology for base systems?

A. 2 is 10001_{two} written in base 17.
B. 10001_{two} is 2 written in base 17.
C. 17 is 2 written in base 10001_{two}.
D. 10001_{two} is 17 written in base 2.
E. I don't know.

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Question

What does the digit 3 represent in the number 63781_{nine}?

A. 3 groups of 9^1
B. 3 groups of 9^2
C. 3 groups of 9^3
D. 3 groups of 9^4
E. I don't know.

Question

What does the digit 3 represent in the number 63781_{nine}?

A. 3 groups of 9^1
B. 3 groups of 9^2
C. 3 groups of 9^3
D. 3 groups of 9^4
E. I don't know.

Things to Keep in Mind for Different Bases

No digit in your number can be greater than or equal to your base. For example 374_{six} and 452_{five} are both invalid numbers.

When the base is greater than 10, we start using letters:

A=10
B=11
C=12
D=13
E=14
F=15

For example $1C_{\text{sixteen}} = (1 \times 16^1) + (12 \times 16^0) = 28$

Converting from Base b to Base 10

What if I only gave you 4123_{five}, how would you convert this into a number in base 10?

Remember that the position of each digit encodes the group size, and each digit encodes the number of groups.

It's often helpful to write out the group sizes for each position:

$$\ldots, 5^4, 5^3, 5^2, 5^1, 5^0$$

Then multiply each digit with its corresponding group size, and add up the results:

$$(4 \times 5^3) + (1 \times 5^2) + (2 \times 5^1) + (3 \times 5^0)$$

$$= (4 \times 125) + (1 \times 25) + (2 \times 5) + (3 \times 1)$$

$$= 500 + 25 + 10 + 3 = 538$$

Converting from Base b to Base 10

Converting from Base b to Base 10

1. Compute the group sizes for each digit in the base:

$$\ldots, b^4, b^3, b^2, b^1, b^0$$

2. Multiply each digit with its corresponding group size.
3. Add up the results.

Question

Convert 123_{four} into base 10.

A. 27
B. 34
C. 48
D. 57
E. I don't know.

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Question

Convert 123_{four} into base 10.

A. 27
B. 34
C. 48
D. 57
E. I don't know.

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What is 75 written in base 4? The method involves building the number right to left by using the remainders after repeated division by 4.

$75/4 = 18$ with a remainder of 3. Put 3 as the units digit. Divide by 4 again.

$18/4 = 4$ with a remainder of 2. Put 2 on the left of 3. Divide by 4 again.

$4/4 = 1$ with a remainder of 0. Put the 0 on the left of the 2. Since $1 < 4$, we stop and put the 1 on the left of the 0. Our number is 1023_{four}.

Converting from Base 10 to Base b

1. Divide the number by b.
2. Put the remainder as the units digit.
3. If the quotient (the number you get by dividing) is greater than or equal to b, repeat steps 1 and 2, putting each remainder the left of the previous remainder.
4. If the quotient is less than b, put the quotient to the left of the last remainder.

Question

Convert 33 to base 3.

A. 11_{three}
B. 102_{three}
C. 1020_{three}
D. 1120_{three}
E. I don't know.

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Question

Convert 33 to base 3.

A. 11_{three}
B. 102_{three}
C. 1020_{three}
D. 1120_{three}
E. I don't know.

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Addition in Base 10

Everyone is familiar with addition in base 10. You add the digits right to left, carrying as needed:

$$\begin{array}{ccccc} & & 1 & & \\ & 2 & 3 & 6 & 5 \\ + & 3 & 4 & 7 & 1 \\ \hline & 5 & 8 & 3 & 6 \end{array}$$

You carry a 1 whenever the sum of the digits is greater than 9.

It may seem silly to review this, but it's helpful to keep it in mind when adding in other bases.

Addition in Other Bases

Let's add $1242_{\text{five}} + 432_{\text{five}}$:

$$\begin{array}{ccccc} & 1 & 2 & 4 & 2_{\text{five}} \\ + & & 4 & 3 & 2_{\text{five}} \\ \hline \end{array}$$

$2_{\text{five}} + 2_{\text{five}} = 4_{\text{five}}$. Since 4 is less than 5, we don't need to carry:

$$\begin{array}{ccccc} & 1 & 2 & 4 & 2_{\text{five}} \\ + & & 4 & 3 & 2_{\text{five}} \\ \hline & & & & 4_{\text{five}} \end{array}$$

Now we have to be careful! $4_{\text{five}} + 3_{\text{five}} \neq 7_{\text{five}}$, because 7 is not a digit in base 5. Instead, we need to convert the 7 to base 5. Doing this gives $4_{\text{five}} + 3_{\text{five}} = 12_{\text{five}}$. Since this is a two-digit number in base 5, we carry:

$$\begin{array}{ccccc} & & 1 & & \\ & 1 & 2 & 4 & 2_{\text{five}} \\ + & & 4 & 3 & 2_{\text{five}} \\ \hline & & & 2 & 4_{\text{five}} \end{array}$$

Now we add $1_{\text{five}} + 2_{\text{five}} + 4_{\text{five}} = 12_{\text{five}}$ and carry:

$$\begin{array}{ccccc} & 1 & 1 & & \\ & 1 & 2 & 4 & 2_{\text{five}} \\ + & & 4 & 3 & 2_{\text{five}} \\ \hline & & 2 & 2 & 4_{\text{five}} \end{array}$$

Addition in Other Bases

Finally, we add $1_{\text{five}} + 1_{\text{five}} = 2_{\text{five}}$:

$$\begin{array}{ccccc} & 1 & 1 & & \\ & 1 & 2 & 4 & 2_{\text{five}} \\ + & & 4 & 3 & 2_{\text{five}} \\ \hline & 2 & 2 & 2 & 4_{\text{five}} \end{array}$$

So $1242_{\text{five}} + 432_{\text{five}} = 2224_{\text{five}}$.

Addition in Other Bases

To add in base b:

1. Stack the numbers on top of one another, aligning the digits.
2. Add each column of digits right to left using base 10.
3. Convert the result to base b. If it has two digits, carry the first one.
4. Repeat until all digits have been added.

Question

Add $23_{\text{four}} + 13_{\text{four}}$.

A. 36_{four}
B. 32_{four}
C. 102_{four}
D. 113_{four}
E. I don't know.

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Question

Add $23_{\text{four}} + 13_{\text{four}}$.

A. 36_{four}
B. 32_{four}
C. 102_{four}
D. 113_{four}
E. I don't know.

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Subtraction in Base 10

Subtraction in base 10 goes right to left, borrowing if you subtract a larger digit from a smaller one:

$$
\begin{array}{rrrrr}
 & & 6 & & \\
 & 7 & \cancel{7} & {}^{1}6 & 5 \\
- & 3 & 4 & 7 & 1 \\
\hline
 & 4 & 2 & 9 & 4
\end{array}
$$

Again, this may seem silly to review, but the idea of borrowing will be important when subtracting in different bases.

Subtraction in Other Bases

Let's subtract $463_{\text{seven}} - 124_{\text{seven}}$:

$$\begin{array}{rccc} & 4 & 6 & 3_{\text{seven}} \\ - & 1 & 2 & 4_{\text{seven}} \\ \hline \end{array}$$

Since $4_{\text{seven}} > 3_{\text{seven}}$, we need to borrow:

$$\begin{array}{rccc} & & 5 & \\ & 4 & \cancel{6} & {}^{1}3_{\text{seven}} \\ - & 1 & 2 & 4_{\text{seven}} \\ \hline & & & 6_{\text{seven}} \end{array}$$

When subtracting in just a single column, it's often helpful to convert to base 10, subtract, and then convert back. So for $13_{\text{seven}} - 4_{\text{seven}}$, it would be $10 - 4 = 6 = 6_{\text{seven}}$.

Now we continue. We can directly subtract $5_{\text{seven}} - 2_{\text{seven}}$:

$$\begin{array}{ccccc} & & 5 & \\ & 4 & \cancel{6} & {}^{1}3_{\text{seven}} \\ - & 1 & 2 & 4_{\text{seven}} \\ \hline & & 3 & 6_{\text{seven}} \end{array}$$

We can also directly subtract $4_{\text{seven}} - 1_{\text{seven}}.$:

$$\begin{array}{ccccc} & & 5 & \\ & 4 & \cancel{6} & {}^{1}3_{\text{seven}} \\ - & 1 & 2 & 4_{\text{seven}} \\ \hline & 3 & 3 & 6_{\text{seven}} \end{array}$$

So $463_{\text{seven}} - 124_{\text{seven}} = 336_{\text{seven}}$

Subtraction in Other Bases

Subtraction in Other Bases

To subtract in base b:

1. Stack the numbers on top of one another, aligning the digits.
2. Subtract each column of digits right to left.
3. If you are subtracting a larger digit from a smaller digit, borrow from the next digit.
4. When subtracting in a single column, you may find it helpful to convert to base 10, subtract, then convert back.
5. Repeat until all digits have been subtracted.

Question

Compute $337_{\text{eight}} - 66_{\text{eight}}$.

A. 251_{eight}
B. 271_{eight}
C. 351_{eight}
D. 357_{eight}
E. I don't know.

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Question

Compute $337_{\text{eight}} - 66_{\text{eight}}$.

A. 251_{eight}
B. 271_{eight}
C. 351_{eight}
D. 357_{eight}
E. I don't know.

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Multiplication in Base 10

When multiplying in base 10, we work one digit at a time:

$$\begin{array}{rccc} & 1 & & \\ & 1 & 7 & 3 \\ \times & & 3 & \color{red}{2} \\ \hline & \color{red}{3} & \color{red}{4} & \color{red}{6} \end{array}$$

When working with digits in the 10s, 100s, etc, place, we add zeroes to the end, corresponding with the place value:

$$\begin{array}{rcccc} & & 2 & & \\ & & 1 & 7 & 3 \\ \times & & & \color{red}{3} & 2 \\ \hline & & 3 & 4 & 6 \\ + & \color{red}{5} & \color{red}{1} & \color{red}{9} & \color{red}{0} \\ \hline & 5 & 5 & 3 & 6 \end{array}$$

We get the product by adding the numbers together.

Multiplication in Other Bases

Let's multiply $231_{\text{four}} \times 32_{\text{four}}$:

$$\begin{array}{rccc} & 2 & 3 & 1_{\text{four}} \\ \times & & 3 & 2_{\text{four}} \\ \hline \end{array}$$

First, we focus on the 2_{four}:

$$\begin{array}{rccc} & 2 & 3 & 1_{\text{four}} \\ \times & & 3 & \color{red}{2_{\text{four}}} \\ \hline \end{array}$$

What we will do is multiply the digits in base 10 and convert back to base 4.

Multiplication in Other Bases

$2 \times 1 = 2$, which equals 2_{four} because 2 is a digit in base 4. So we just write 2_{four} down:

$$\begin{array}{cccc} & 2 & 3 & 1_{\text{four}} \\ \times & & 3 & \color{red}{2_{\text{four}}} \\ \hline & & & \color{red}{2_{\text{four}}} \end{array}$$

Next we multiply $3 \times 2 = 6$. We need to convert the 6 into base 4, which gives us $6 = 12_{\text{four}}$. Since this is a two-digit number, we have to carry:

$$\begin{array}{cccc} & 1 & & \\ & 2 & 3 & 1_{\text{four}} \\ \times & & 3 & \color{red}{2_{\text{four}}} \\ \hline & & \color{red}{2} & \color{red}{2_{\text{four}}} \end{array}$$

Multiplication in Other Bases

Finally we multiply $2 \times 2 = 4$, then we add the carried 1 to get $4 + 1 = 5$. We need to convert this 5 into base 4, which gives us $5 = 11_{\text{four}}$:

$$\begin{array}{ccccc} & & 1 & & \\ & & 2 & 3 & 1_{\text{four}} \\ \times & & & 3 & 2_{\text{four}} \\ \hline & 1 & 1 & 2 & 2_{\text{four}} \end{array}$$

Now we have to work on the next digit. Like in base 10, we add a 0 on the end:

$$\begin{array}{ccccc} & & 2 & 3 & 1_{\text{four}} \\ \times & & & 3 & 2_{\text{four}} \\ \hline & 1 & 1 & 2 & 2_{\text{four}} \\ & & & & 0_{\text{four}} \end{array}$$

Multiplication in Other Bases

Now we multiply. $1 \times 3 = 3$, which is 3_{four}:

$$\begin{array}{ccccc} & & 2 & 3 & 1_{\text{four}} \\ \times & & & \color{red}{3} & 2_{\text{four}} \\ \hline & 1 & 1 & 2 & 2_{\text{four}} \\ & & & \color{red}{3} & \color{red}{0_{\text{four}}} \end{array}$$

Now we multiply $3 \times 3 = 9$. We need to convert this to base 4, which gives us $9 = 21_{\text{four}}$. We carry the 2:

$$\begin{array}{ccccc} & & 2 & & \\ & & 2 & 3 & 1_{\text{four}} \\ \times & & & \color{red}{3} & 2_{\text{four}} \\ \hline & 1 & 1 & 2 & 2_{\text{four}} \\ & & \color{red}{1} & \color{red}{3} & \color{red}{0_{\text{four}}} \end{array}$$

Multiplication in Other Bases

Now we multiply $2 \times 3 = 6$, and add the carried 2 to get $6 + 2 = 8$. We need to convert this into base 4, $8 = 20_{\text{four}}$:

$$\begin{array}{cccccc}
 & & & 2 & & \\
 & & & 2 & 3 & 1_{\text{four}} \\
\times & & & & \color{red}{3} & 2_{\text{four}} \\
\hline
 & & 1 & 1 & 2 & 2_{\text{four}} \\
 & \color{red}{2} & \color{red}{0} & \color{red}{1} & \color{red}{3} & \color{red}{0_{\text{four}}}
\end{array}$$

Finally, we add the results

$$\begin{array}{cccccc}
 & & & 2 & & \\
 & & & 2 & 3 & 1_{\text{four}} \\
\times & & & & \color{red}{3} & 2_{\text{four}} \\
\hline
 & & 1 & 1 & 2 & 2_{\text{four}} \\
+ & 2 & 0 & 1 & 3 & 0_{\text{four}} \\
\hline
 & 2 & 1 & 3 & 1 & 2_{\text{four}}
\end{array}$$

Multiplication in Other Bases

To multiply numbers in base b:

1. Stack the numbers on top of one another, aligning the digits.
2. Start with the digit furthest to the right on the bottom number.
3. Multiply that digit with each digit in the top number. Do this by multiplying in base 10 then converting the result to base b.
4. Move to the next digit to the left. Add a zero to the end each time you move left. (That is, add one zero for the second digit, two zeroes for the third, three zeroes for the fourth, and so on).
5. Repeat until all digits in the bottom number have been used.
6. Add up the results of each multiplication.

Question

Multiply $45_{\text{six}} \times 14_{\text{six}}$.

A. 312_{six}
B. 401_{six}
C. 450_{six}
D. 1202_{six}
E. I don't know.

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Question

Multiply $45_{\text{six}} \times 14_{\text{six}}$.

A. 312_{six}
B. 401_{six}
C. 450_{six}
D. 1202_{six}
E. I don't know.

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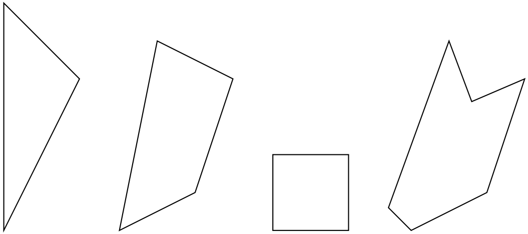
Polygons

The most basic figures we can construct are ones composed of some number of straight line segments:



Definition: Polygon

A **polygon** is a closed figure composed of three or more line segments.

Perimeter

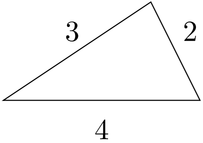
There are many ways to measure a figure, one being the perimeter.

Definition: Perimeter

The **perimeter** of a figure is the distance around the figure.

The perimeter could represent the length of fence around a garden, bias binding around a tablecloth, a walk around town, etc.

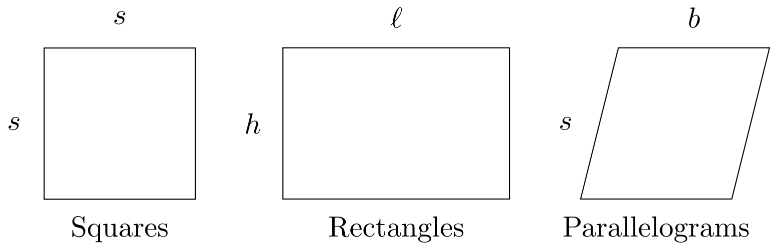
For polygons, we get the perimeter by adding the lengths of each side:



For example, the perimeter of this triangle is $4 + 2 + 3 = 9$. If we are given units, like ft, we would say the perimeter is 9 ft. Sometimes you need to convert units!

Perimeter Formulas for Special Quadrilaterals

If a figure has four sides, it is called a quadrilateral. Some special quadrilaterals are:



These figures have special formulas:

Formulas: Perimeters of Squares, Rectangles, and Quadrilaterals

$P_{\text{square}} = 4s$	s is side length
$P_{\text{rectangle}} = 2\ell + 2h$	ℓ is length and h is height
$P_{\text{parallelogram}} = 2b + 2s$	b is the base and s is the slanted side

Question

You are decorating the border of a large rectangular sign with colorful tape. The sign has a length of 25 feet and a height of 32 feet. The tape is sold in rolls of 20 feet. How many rolls must you buy in order to decorate the entire sign?

A. 5 rolls
B. 6 rolls
C. 7 rolls
D. 8 rolls
E. I don't know.

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Question

You are decorating the border of a large rectangular sign with colorful tape. The sign has a length of 25 feet and a height of 32 feet. The tape is sold in rolls of 20 feet. How many rolls must you buy in order to decorate the entire sign?

A. 5 rolls
B. 6 rolls
C. 7 rolls
D. 8 rolls
E. I don't know.

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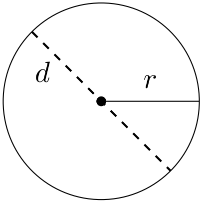
Circles

A special figure that is not a polygon is the circle.



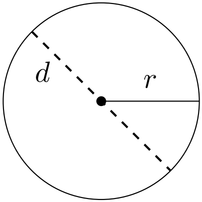
Circle Terminology

The **radius** of a circle is the distance from the center of the circle to a point on the circle, which we denote with r.

The **diameter** of a circle is the distance between two opposite points on a circle, which we denote with d.

The **circumference** of a circle is the perimeter of the circle.

Circumference of a Circle



Formula: Circumference of a Circle

The diameter and radius are related: $d = 2r$ or $r = \dfrac{d}{2}$.

The circumference of a circle is given by: $C = 2\pi r$ or $C = \pi d$.

The number π (pi) is $3.14159265\ldots$ We will often leave our answer in terms of π.

Leaving Your Answer in Terms of π

When a question says to "leave your answer in terms of π," what it means is that we won't plug in the numerical value of π.

Instead, we will combine all the numbers besides π into a single number and leave π alone.

For example, if a question asks you to calculate the circumference of a circle with a radius of 4, here's what we do:

$$C = 2\pi r$$
$$C = 2\pi(4)$$
$$C = 8\pi$$

When we leave our answer in terms of π, we say the circumference is 8π (said aloud as "eight pi").

Question

A plate with a diameter of 1 foot will be decorated around the edge with gold thread. How many feet of gold thread are needed? Leave your answer in terms of π.

A. $\pi/4$ feet
B. $\pi/2$ feet
C. π feet
D. 2π feet
E. I don't know.

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Question

A plate with a diameter of 1 foot will be decorated around the edge with gold thread. How many feet of gold thread are needed? Leave your answer in terms of π.

A. $\pi/4$ feet
B. $\pi/2$ feet
C. π feet
D. 2π feet
E. I don't know.

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Area

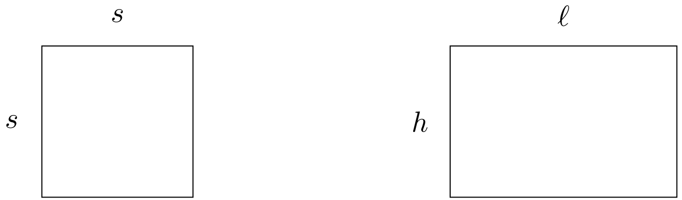
Another way we can measure figures is using area.

Definition: Area

The **area** is the amount of surface in a figure.

For general polygons, we don't have an easy way to calculate the area, so we will only look at the area for special figures.

Area of Squares and Rectangles



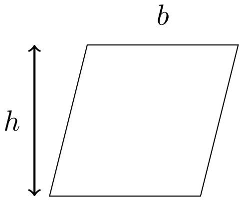
Squares and rectangles have nice area formulas:

Formulas: Areas of Squares and Rectangles

$A_{\text{square}} = s^2$	s is side length
$A_{\text{rectangle}} = \ell h$	ℓ is length and h is height

In both formulas, you multiply the lengths of the sides. In a square the side lengths are the same, so we get $s \times s = s^2$.

Area of a Parallelogram



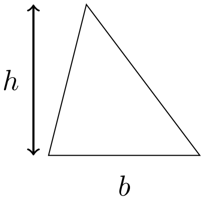
Parallelograms also have a nice formula:

Formula: Area of Parallelogram

$A_{\text{parallelogram}} = bh$ $\qquad$ b is the base and h is the height

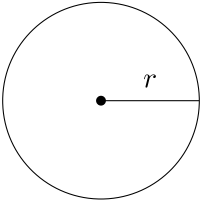
Careful! The height h is NOT the same as s, the length of the slanted side. Pay very close attention to what information the question is giving you when dealing with parallelograms.

Area of a Triangle



Formula: Area of Triangle

$A_{\text{triangle}} = \frac{1}{2}bh$ b is the base and h is the height



Formula: Area of a Circle

$A_{\text{circle}} = \pi r^2$ r is the radius

Be sure to pay attention to whether you are given the radius or the diameter in the question. If you are given the diameter, you need to divide it by 2 to get the radius in order to use that in the area formula.

Question

A triangular pennant has been cut from fabric and has a total area of 100 cm^2. If the height of the triangle is 20 cm, how long is the base?

A. 5 cm
B. 10 cm
C. 20 cm
D. 80 cm
E. I don't know.

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Question

A triangular pennant has been cut from fabric and has a total area of 100 cm^2. If the height of the triangle is 20 cm, how long is the base?

A. 5 cm
B. 10 cm
C. 20 cm
D. 80 cm
E. I don't know.

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Question

A pizza parlor offers a medium pizza that is 12 inches in diameter and a large pizza that is 16 inches in diameter. How much MORE pizza do you get when you order a large compared to a medium? Leave your answer in terms of π.

A. 28π in^2
B. 36π in^2
C. 64π in^2
D. 112π in^2
E. I don't know.

Room Name: BROWN4663

Question

A pizza parlor offers a medium pizza that is 12 inches in diameter and a large pizza that is 16 inches in diameter. How much MORE pizza do you get when you order a large compared to a medium? Leave your answer in terms of π.

A. $\mathbf{28}\pi$ in^2
B. 36π in^2
C. 64π in^2
D. 112π in^2
E. I don't know.

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Geometric Solids

Just like with 2-dimensional figures, we can create formulas for the volumes and surface areas of 3-dimensional geometric solids.

Definition: Geometric Solid

A **geometric solid** is a 3-dimensional region bounded by surfaces.

Some common solids are cubes, rectangular prisms, cylinders, cones, and spheres.

Volume

For 2-dimensional figures, the area was a measurement of how much surface was contained in the figure.

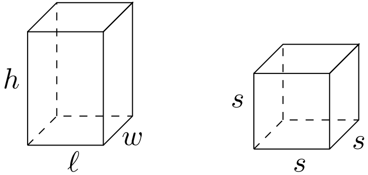
For 3-dimensional geometric solids, we have a similar notion, called the volume.

Definition: Volume

The **volume** of a geometric solid is the amount of space occupied by that solid.

Volume can be used to describe the amount of water in a bottle, trash in a landfill, air inside a basketball, etc.

Volume of a Rectangular Prism

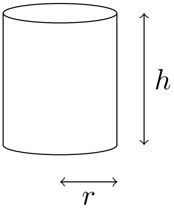


A rectangular prism is a geometric solid where all the faces (sides) are rectangles. If all the faces are squares, it is called a cube.

Formulas: Volumes of Rectangular Prism and Cube

$V_{\text{rectangular prism}} = \ell w h$	ℓ is length, w is width, h is height.
$V_{\text{cube}} = s^3$	s is side length.

Volume of a Cylinder



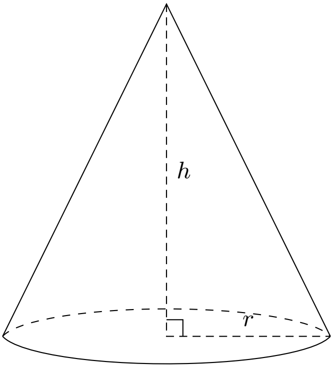
A cylinder is a geometric solid where the base is a circle extended up by some amount of height.

Formula: Volume of a Cylinder

$V_{\text{cylinder}} = \pi r^2 h$ $\qquad$ r is radius of circle at base, h is the height.

If it's helpful, you can think about the volume formula as the area of the circle at the base (πr^2) multiplied by the height h.

Volume of a Cone

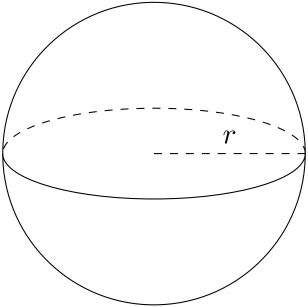


A cone is a geometric solid obtained by shrinking the top circle of a cylinder down to a single point.

Formula: Volume of a Cone

$V_{\text{cone}} = \dfrac{1}{3}\pi r^2 h$ r is radius of circle at base, h is the height.

Volume of a Sphere



A sphere is the 3-dimensional version of a circle. Just like a circle, it has a center, radius, and diameter.

Formula: Volume of a Sphere

$V_{\text{sphere}} = \frac{4}{3}\pi r^3$ r is radius.

Question

A group of kids at the beach build a sand sculpture in the shape of a cone. The circle at the base has a diameter of 2 m, and the height of the sculpture is 4 m. What is the volume of sand in the sculpture? Leave your answer in terms of π.

A. $\frac{4}{3}\pi$ m^3

B. $\frac{16}{3}\pi$ m^3

C. 4π m^3

D. 16π m^3

E. I don't know.

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Question

A group of kids at the beach build a sand sculpture in the shape of a cone. The circle at the base has a diameter of 2 m, and the height of the sculpture is 4 m. What is the volume of sand in the sculpture? Leave your answer in terms of π.

A. $\frac{4}{3}\pi$ m^3

B. $\frac{16}{3}\pi$ m^3

C. 4π m^3

D. 16π m^3

E. I don't know.

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Volumes of Higher-Dimensional Spheres

Volumes of Higher-Dimensional Spheres

We live in a universe with 3 spatial dimensions, but in math, we can imagine what higher dimensions would look like.

In a 4-dimensional universe, spheres would have a volume of $V_{\text{4D sphere}} = \dfrac{\pi^2}{2} r^4$.

In a 5-dimensional universe, it would be $V_{\text{5D sphere}} = \dfrac{8\pi^2}{15} r^5$.

In a 15-dimensional universe, it would be $V_{\text{15D sphere}} = \dfrac{256\pi^7}{2027025} r^{15}$.

These formulas can be obtained by calculating integrals, which come from the mathematical field of calculus.

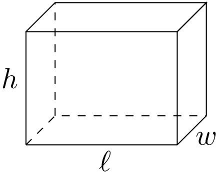
Definition: Surface Area

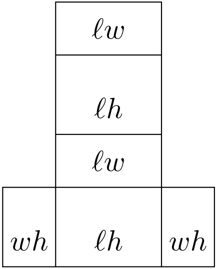
The **surface area** of a geometric solid is the total area of the surfaces that bound the solid.

The surface area can be used to represent the amount of paint needed to paint a room, the amount of metal used in a soda can, the rubber used to make a basketball, etc.

Surface Area of Rectangular Prism

We can imagine "unwrapping" a rectangular prism:





Then the surface area is just the sum of each individual rectangle.

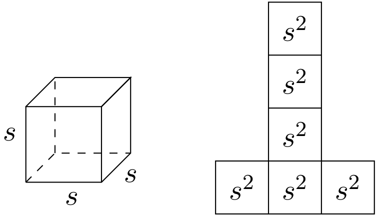
Formula: Surface Area of Rectangular Prism

$S_{\text{rectangular prism}} = 2\ell h + 2\ell w + 2wh$

ℓ is length, w is width, h is height

Surface Area of Cube

We can do the same thing for a cube:

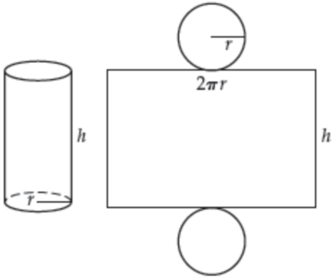


Formula: Surface Area of Cube

$S_{\text{cube}} = 6s^2$ s is side length

Surface Area of Cylinder

We can again do this unwrapping for a cylinder:

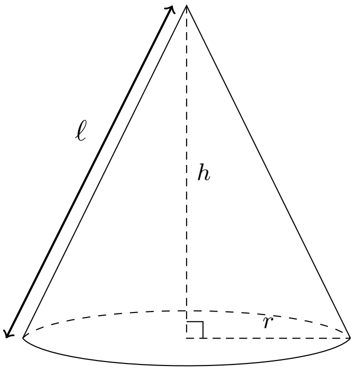


We see we have two circles with radius r and a rectangle with a length of $2\pi r$ and a height of h.

Formula: Surface Area of Cylinder

$S_{\text{cylinder}} = 2\pi r^2 + 2\pi r h$ $\qquad r$ is radius, h is height

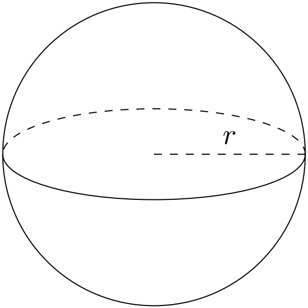
Surface Area of Cone



For a cone, we need the slant height, the length of the diagonal slope of the cone.

Formula: Surface Area of Cone

$S_{\text{cone}} = \pi r^2 + \pi r\ell$ $\qquad$ r is radius, ℓ is slant height



For a sphere, we can't "unwrap" it, so you'll have to memorize the formula.

Formula: Surface Area of Sphere

$S_{\text{sphere}} = 4\pi r^2$ $\quad$ r is the radius.

Question

How much metal is needed to make a cylindrical can with a diameter of 4 cm and a height of 7 cm? Leave your answer in terms of π.

A. 88π cm^2
B. 56π cm^2
C. 44π cm^2
D. 36π cm^2
E. I don't know.

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Question

How much metal is needed to make a cylindrical can with a diameter of 4 cm and a height of 7 cm? Leave your answer in terms of π.

A. 88π cm^2
B. 56π cm^2
C. 44π cm^2
D. 36π cm^2
E. I don't know.

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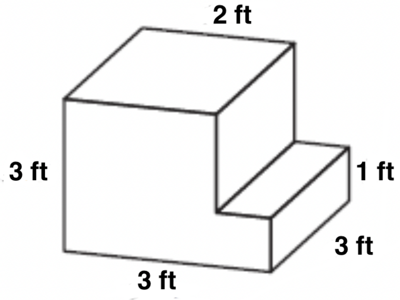
Volumes of Complex Objects

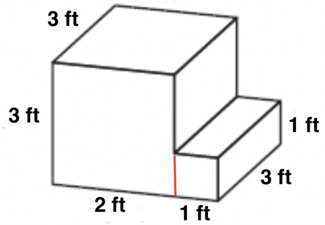
Sometimes, an object is not a simple geometric solid. For example, how could you go about finding the volume of the object below?



There are two main methods we can use.

Method 1: Break it Up

The first method is to break up the object into pieces. For example, we can break up our object like this:



Now our object consists of two rectangular prisms. If we compute their volumes and add them together, we get the volume of the object:

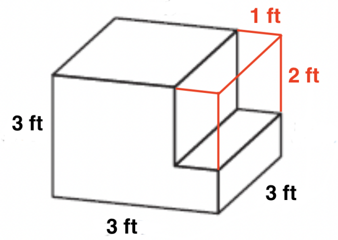
$V_{\text{rectangular prism}} = 3 \times 2 \times 3 = 18$

$V_{\text{rectangular prism}} = 1 \times 3 \times 1 = 3$

So the total volume is 21.

Method 2: Subtract

The second method is to imagine your object as a geometric solid with another geometric solid subtracted from it. For example, we imagine our object like this:



Here, the rectangular prism in red is being subtracted from the whole object. Our whole object is a cube with a side length of 3, so:

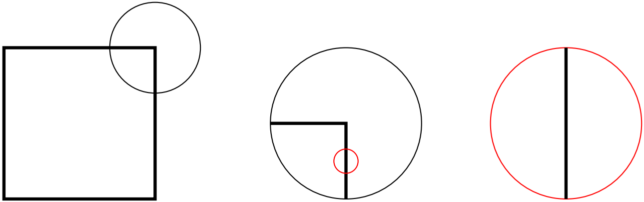
$V_{\text{cube}} = 3 \times 3 \times 3 = 27$

$V_{\text{rectangular prism}} = 1 \times 3 \times 2 = 6$

So the total volume is 21, just like the first method.

Normal Figures

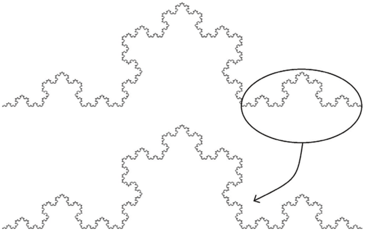
Most figures we we encounter in the real world change the way they look as you zoom in on them:



This means they are not what we call "self-similar." That is, their boundary changes as you zoom in.

Fractals

Some figures, however, have the remarkable property that they look the same no matter how far you zoom in!



Definition: Fractal

A **fractal** is a geometric figure in which a self-similar motif repeats on an ever-dimininishing scale. That is, it looks the same no matter how far you zoom in.

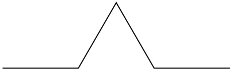
The fractal above is known as the Koch curve.

Generating a Fractal

A fractal can be made by using an iterative process.

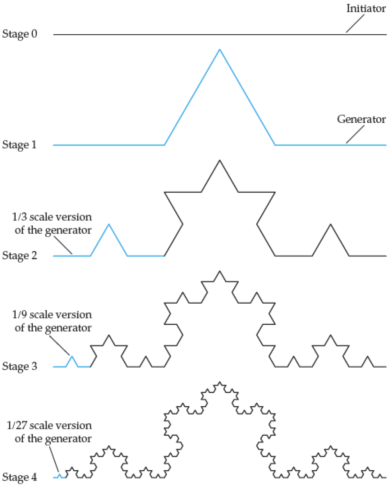
You start with an **initiator** and a **generator**. For example, the initiator of the Koch curve is:

And the generator is:



To generate the fractal, you work in stages, replacing each copy of initiator with a copy of the generator.

Generating the Koch Curve



Question

The Cantor Set is a fractal with initiator:

And generator:

What is the next stage of the Cantor Set?

A.

B.

C.

D.

E. I don't know.

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Question

The Cantor Set is a fractal with initiator:

And generator:

What is the next stage of the Cantor Set?

A.

B.

C.

D.

E. I don't know.

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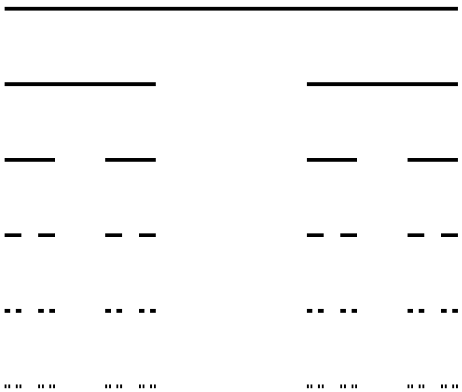
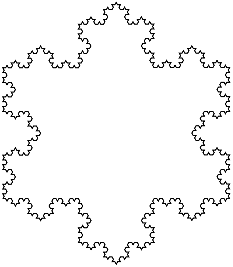
Strange Properties of Fractals



Shown above are more iterations of the Cantor Set. It contains infinitely many points, but if you put all the points together in a line, the total length is zero!

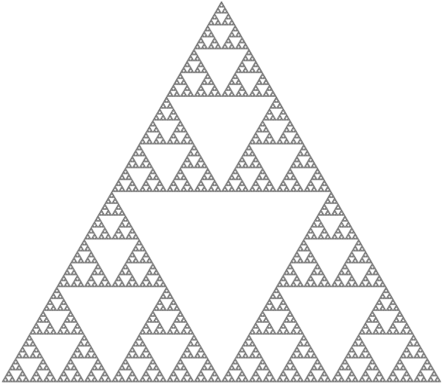
Strange Properties of Fractals



Shown above is the Koch snowflake. It is made by gluing together three copies of the Koch curve. This figure has a perimeter that is infinite in length!

Question: Is the area also infinite?

Strange Properties of Fractals



Shown above is the Sierpinski Gasket. If you add up all the lengths of the line segments, you get infinity, but the area taken up by the "frame" is exactly zero!

Making a Real-Life Fractal

You can make a fractal using just a strip of paper! Cut out a thin strip of printer paper, about 1 inch in width.

Orient the strip vertically in front of you, then fold the top down to the bottom.

Without unfolding, take the top and fold it down to the bottom again. Keep doing this until you can't fold the paper anymore.

Unfold the paper and stand it up on its edge. The result is known as the Dragon Curve! The more folds you do, the more iterations of the Dragon Curve you make.

Fractal Dimension

Dimension of a Fractal

In some sense, fractals live "in between" dimensions. Mathematically, the dimension of a figure has to do with how much its "measure" scales when you scale it up by a factor of 2.

A line segment is 1-dimensional, and when you scale it up by 2, its length scales by a factor of 2. A square is 2-dimensional, and when you scale it up by 2, its area scales by a factor of $4 = 2^2$. A cube is 3-dimensional, and when you scale it up by 2, its volume scales by a factor of $8 = 2^3$.

The dimension of a figure is the exponent on the 2. For fractals, this exponent can be a non-whole number! For example, the Koch curve has a dimension of ≈ 1.2619, and the Sierpinski Gasket has a dimension of ≈ 1.5850.

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First-Degree Equations

One of the goals of mathematics is to model the world around us.

One of the simplest models we can make is known as a first-degree, or linear, equation.

Definition: First-Degree Equation

A **first-degree equation** (also called a linear equation) is an equation of the form $y = mx + b$.

m is the slope, or rate of change.
b is the intercept, or initial value.
x is the independent variable. This is the variable you can change.
y is the dependent variable. This is the variable that changes in response to how you change x.

We will often NOT use x and y, but other letters that more closely represent our situation.

Interpreting a First-Degree Equation

A tank is slowly being filled with water. The amount of water in a tank is given by the equation $W = 2H + 5$, where W is the amount of water in gallons and H is the number of hours the tank has been filling.

Here H is the independent variable and W is the dependent variable.

The slope is 2, which represents the tank is being filled at a rate of 2 gallons per hour.

The intercept is 5, which represents the tank starting with 5 gallons in it at the beginning.

Question

A scientist is heating up a chemical over a bunsen burner. The temperature of the chemical is given by the equation $T = 12M + 40$, where T is the temperature of the chemical in Fahrenheit and M is the number of minutes that have passed. Which of the following statements is correct?

A. The chemical is heating up at a rate of 12° F/min and the initial temperature was 40° F.
B. The chemical is heating up at a rate of 40° F/min and the initial temperature was 12° F.
C. It will take 12 minutes for the chemical to reach a temperature of 40° F.
D. By the time 40 minutes have passed, the chemical will be at a temperature of 12° F.
E. I don't know.

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Question

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D. By the time 40 minutes have passed, the chemical will be at a temperature of 12° F.
E. I don't know.

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Creating a First-Degree Equation

To create a first-degree equation, you first want to define variables (if they aren't already defined for you). That means you'll choose two letters, one to represent the independent variable, and one to represent the dependent variable.

Next, you need to determine the slope and the intercept.

Typically, the slope will deal with rates, like miles per hour, dollars per unit, etc.

The intercept typically deals with initial values or fixed values. This could be things like the initial amount of gas in your car, a one-time payment, etc.

Example of Creating First-Degree Equation

A babysitter charges \$20 per hour for watching the kids, as well as a \$30 flat fee to cover their traveling costs. Let's create a first-degree equation to model this situation.

First, we will define our variables. The letters we choose don't matter, but it's good to pick letters that have some sort of helpful meaning. We will use H to represent the number of hours the babysitter works, and C to represent the total cost.

Next, we want to determine our slope and intercept. The slope will be 20, since that is a rate, and the intercept will be 30, because that is a one-time payment.

So our first-degree equation will be $C = 20H + 30$. We can also write it as $C = 30 + 20H$.

Question

A local band performs at parties for \$5 per song. They also charge a flat \$50 booking fee. Let T represent the total cost of a performance where S songs are performed. Which of the following equations correctly models the situation?

A. $T = 5S + 50$
B. $T = 50S + 5$
C. $S = 5T + 50$
D. $S = 50T + 5$
E. I don't know.

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Question

A local band performs at parties for \$5 per song. They also charge a flat \$50 booking fee. Let T represent the total cost of a performance where S songs are performed. Which of the following equations correctly models the situation?

A. $T = 5S + 50$
B. $T = 50S + 5$
C. $S = 5T + 50$
D. $S = 50T + 5$
E. I don't know.

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Utilizing a First-Degree Equation

There are two main ways we will be utilizing a first-degree equation:

1. Plugging in a given value for the independent variable to obtain a value for the dependent variable.

2. Solving for the value of the independent variable, given a value for the dependent variable.

The key will be to recognize whether you are given the independent variable or the dependent variable in the question.

Plugging in Independent Variable

If you are given the value of the independent variable, you can use it to calculate the value of the dependent variable.

For example, returning to our water tank equation, $W = 2H + 5$, we can calculate the amount of water in the tank after 3 hours of filling:

$$\begin{aligned} W &= 2H + 5 \\ &= 2(3) + 5 \\ &= 6 + 5 \\ &= 11 \end{aligned}$$

So there will be 11 gallons of water in the tanks after 3 hours.

Question

You take your car to the mechanic after it breaks down. It will cost \$700 to get the parts to fix it, and the mechanic charges \$80 per hour for labor. If it takes 5 hours to fix your car, how much do you pay in total?

A. \$400
B. \$700
C. \$1100
D. \$1300
E. I don't know.

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Question

You take your car to the mechanic after it breaks down. It will cost \$700 to get the parts to fix it, and the mechanic charges \$80 per hour for labor. If it takes 5 hours to fix your car, how much do you pay in total?

A. \$400
B. \$700
C. \$1100
D. \$1300
E. I don't know.

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Solving for the Independent Variable

If you are given the value of the dependent variable, you can use it to solve for the value of the independent variable.

For example, again using our water tank equation, $W = 2H + 5$, we can calculate the amount of time it would take for the tank to fill up to 17 gallons:

$$\begin{aligned} W &= 2H + 5 \\ 17 &= 2H + 5 \\ 17 - 5 &= 2H + 5 - 5 \\ 12 &= 2H \\ 12/2 &= 2H/2 \\ 6 &= H \end{aligned}$$

So it will take 6 hours for the tank to fill to 17 gallons.

Question

A museum offers a tour for a high school at a cost of \$5 per student. They also have a documentary screening, which costs \$250 and seats every student. How many students attended the museum if the school sees the documentary and the total cost of the trip was \$1375?

A. 200
B. 225
C. 275
D. 300
E. I don't know.

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Question

A museum offers a tour for a high school at a cost of \$5 per student. They also have a documentary screening, which costs \$250 and seats every student. How many students attended the museum if the school sees the documentary and the total cost of the trip was \$1375?

A. 200
B. 225
C. 275
D. 300
E. I don't know.

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Rates

We use rates all the time in our lives. The speed you drive your car, the amount of money you earn per hour, the electricity consumed by a city each day, etc.

Definition: Rate

A **rate** is a comparison of one quantity in terms of another. We often write rates as a fraction.

For example, if you drive 125 miles and consume 5 gallons of gas, the rate at which you used the gas is $\dfrac{125 \text{ mi}}{5 \text{ gal}}$. We are often more interested when the denominator is 1, so we simplify: $\dfrac{25 \text{ mi}}{1 \text{ gal}}$, or 25 mi/gal.

Definition: Unit Rate

A **unit rate** is a rate with a denominator of 1.

Utilizing Unit Rates

One of the most common ways we use rates is when buying groceries. Suppose there are two bags of apples for sale, a 2 lb bag for \$4, and a 5 lb bag for \$9. Which is the better deal?

The better deal is the one that gives us the lowest unit rate for the cost of apples per pound, so we compute the two unit rates:

$$\text{2 lb bag}: \qquad \frac{\$4}{2\text{ lb}} = \$2/\text{lb}$$

$$\text{5 lb bag}: \qquad \frac{\$9}{5\text{ lb}} = \$1.80/\text{lb}$$

So the 5 lb bag is the better deal.

Question

You have two job offers. For Job 1, you work 40 hours a week and are paid \$1000 every week. For Job 2, you work 30 hours a week and are paid \$720 every week. Which job has the better hourly wage?

A. Job 1
B. Job 2
C. They're the same.
D. Not enough information to tell.
E. I don't know

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Question

You have two job offers. For Job 1, you work 40 hours a week and are paid \$1000 every week. For Job 2, you work 30 hours a week and are paid \$720 every week. Which job has the better hourly wage?

A. Job 1
B. Job 2
C. They're the same.
D. Not enough information to tell.
E. I don't know

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Ratios

We often compare quantities that have the same units. For example, we can think about the amount of hours we spend sleeping every week. Both of these quantities have time units.

Definition: Ratio

A ratio is a comparison of two quantities that have the same units. We can write a ratio in three ways:

1. As a fraction - $\dfrac{a}{b}$
2. As two numbers separated by a colon - $a : b$
3. As two numbers separated by the word “to” - a to b

Note: For a ratio, the order matters. That is, the ratio of a to b is NOT always the same as the ratio of b to a.

Computing Ratios

To compute a ratio a to b, you will divide the quantity a by the quantity b, that is, $\frac{a}{b}$, then put the fraction into simplest form.

For example, let's say the math department has 240 math majors and 25 faculty. The ratio of math majors to faculty is:

$$\frac{240}{25} = \frac{48}{5} = 48 : 5$$

The ratio of math majors to faculty is 48:5. What about the ratio of faculty to members of the department? The total number of people in the department is $240 + 25 = 265$, so the ratio is:

$$\frac{25}{265} = \frac{5}{53} = 5 : 53$$

The ratio of faculty to members of the department is 5:53.

Question

In a fish farm, the population of tilapia of 340 and the population of haddock is 140. What is the ratio of the total population of fish to the population of haddock?

A. 7 : 17
B. 17 : 7
C. 7 : 24
D. 24 : 7
E. I don't know.

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Question

In a fish farm, the population of tilapia of 340 and the population of haddock is 140. What is the ratio of the total population of fish to the population of haddock?

A. 7 : 17
B. 17 : 7
C. 7 : 24
D. 24 : 7
E. I don't know.

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Proportions

There are many times when rates and ratios are equal. For example, the aspect ratio of a 3 by 4 monitor is the same as the aspect ratio of a 6 by 8 monitor. These are called proportions.

Definition: Proportion

A **proportion** is an equation that equates two rates or two ratios. A proportion is always of the form:

$$\frac{a}{b} = \frac{c}{d}$$

Note: None of the values in a proportion will ever be 0.

Solving a Proportion

To solve a proportion, we use cross multiplication. That is we multiply:

$$\frac{a}{b} = \frac{c}{d}$$

To get:

$$ad = bc$$

When solving a proportion, you will be given 3 out of 4 of the variables and will need to find the last one.

Example of Solving a Proportion

Let's solve the proportion $\frac{7}{10} = \frac{x}{15}$.

We cross multiply to get:

$$\begin{aligned} (7)(15) &= 10x \\ 105 &= 10x \end{aligned}$$

Then we divide both sides by 10 to get:

$$\begin{aligned} 105/10 &= 10x/10 \\ 10.5 &= x \end{aligned}$$

Question

Solve the proportion $\dfrac{14}{x} = \dfrac{7}{3}$.

A. $x = 1.5$
B. $x = 3$
C. $x = 6$
D. $x = 7$
E. I don't know.

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Question

Solve the proportion $\dfrac{14}{x} = \dfrac{7}{3}$.

A. $x = 1.5$
B. $x = 3$
C. $x = 6$
D. $x = 7$
E. I don't know.

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Percents

We often use percentages to talk about the portion of some quantity. For example, the unemployment rate is the percent of people in the labor force who don't currently have a job.

Definition: Percent

A **percent** is a ratio with a denominator of 100.

To convert a decimal into a percent, you simply multiply by 100:

$$0.57 \implies 0.57 \times 100 = 57 \implies 0.57 = 57\%$$

To convert a percent to a decimal, you divide by 100:

$$57\% \implies \frac{57}{100} = 0.57 \implies 57\% = 0.57$$

Percents can themselves have decimals, like 23.76%, and can be greater than 100, like 137%.

Writing Percents as a Fraction

We sometimes want to write a percent as a fraction. For example, $25\% = \frac{1}{4}$.

To convert a general percent to a fraction, first remove the % sign, then divide by 100, and finally put the fraction in simplest terms.

For example, for 35%:

$$35\% \implies \frac{35}{100} = \frac{5 \times 7}{5 \times 20} = \frac{7}{20}$$

Calculating Percents

Most of the time, we want to calculate some percent of some quantity. For example, when you calculate the tip on a restaurant bill, you are finding the percent of your subtotal.

Formula: Calculating a Percent

To find $x\%$ of a quantity y, we use the formula:

$$\frac{x}{100} \times y$$

For example, 20% of 50 is:

$$\frac{20}{100} \times 50 = 0.2 \times 50 = 10$$

Another way to think about the formula is to convert the $x\%$ into a decimal, and then multiply that decimal by y.

The Basic Percent Equation

The formula on the previous slide is a nice shortcut for calculating $x\%$ of y. In general, we can use the following formula.

Formula: The Basic Percent Equation

Let P be the percent (written as a decimal), B be the base, and A be the amount. Then we have the following equation:

$$PB = A$$

This equation can help us answer questions like "What percent of 30 is 5?" and "45% of some number is 28, what is the original number?"

The key is to identify which numbers are the percent, base, and amount. Two will be given to you, and you need to find the last one.

Using the Basic Percent Equation

A butcher trims the fat off of a slab of meat. The meat originally had a mass of 500 grams, and after trimming, the meat has a mass of 475 grams. What percent of the meat remained after trimming?

We are looking for the percent P, so we need to determine which number is B and which one is A.

Since the 500 grams is the original mass, it is the base B. Since the 475 grams is the final mass, it is the amount A. So we plug these into the equation and solve for P:

$$\begin{aligned} PB &= A \\ P(500) &= (475) \\ P(500)/500 &= (475)/500 \\ P &= 0.95 \end{aligned}$$

We convert this decimal to a percent to get 95%

Question

40% of some number is 80. What is the original number?

A. 32
B. 40
C. 100
D. 200
E. I don't know

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Question

40% of some number is 80. What is the original number?

A. 32
B. 40
C. 100
D. 200
E. I don't know

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Other Meanings of Percents

Sometimes when we use percents, we are referring to how much a quantity changes in relation to its original amount.

For example, if a store is having a 10% off sale, the 10% does NOT mean every item is only 10% of its original price. Instead, it means that the price has gone down by 10%.

Likewise, a 10% markup means that the price has gone up by 10%.

We need to work with these percents differently.

Percent Increase

Definition: Percent Increase

A **percent increase** is used to show the amount that a quantity increased compared to the original quantity. The formula is:

$$A = B + BP = B(1 + P)$$

A store buys their bags of flour from a wholesale vendor at a price of \$2 per bag. In order to turn a profit, they markup the price by 15%. What is the price they sell the flour for?

First we write the percent as a decimal: $15\% = 0.15$. Our base B is 2. Then we use the formula:

$$A = (2)(1 + 0.15) = 2(1.15) = 2.30$$

So the store sells the flour for \$2.30.

Definition: Percent Decrease

A **percent decrease** is used to show the amount that a quantity decreased compared to the original quantity. The formula is:

$$A = B - PB = B(1 - P)$$

You go to the store and pick out a pair of shoes that cost \$30. You have a coupon for 20% off. How much do you pay?

First we write the percent as a decimal: $20\% = 0.20$. Our base B is 30. Then we use the formula:

$$A = (30)(1 - 0.20) = 30(0.80) = 24$$

So the you pay \$24.

Question

The UConn bookstore buys a math textbook from the publisher for \$160. They markup the price by 25%. How much do you pay for the book?

A. \$25
B. \$40
C. \$185
D. \$200
E. I don't know.

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Question

The UConn bookstore buys a math textbook from the publisher for \$160. They markup the price by 25%. How much do you pay for the book?

A. \$25
B. \$40
C. \$185
D. \$200
E. I don't know.

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Calculating Percent Increase/Decrease

You can use the percent increase formula to calculate a percent increase/decrease when given the base and the amount. To do this, substitute B and A into the formula and solve for P.

For example, if there are 250 students in a class in January, and 300 students in March, the percent increase from January to March is:

$$\begin{aligned} A &= B(1+P) \\ (300) &= (250)(1+P) \\ (300)/250 &= (250)(1+P)/250 \\ 1.2 &= 1+P \\ 1.2-1 &= 1+P-1 \\ 0.2 &= P \end{aligned}$$

So the percent increase was 20%.

Calculating Percent Increase/Decrease

If you get a negative value for P when you finish, that means it is a percent decrease.

For example, if there are 250 students in a class in January, and 225 students in March, the percent "increase" from January to March is:

$$\begin{aligned} A &= B(1+P) \\ (225) &= (250)(1+P) \\ (225)/250 &= (250)(1+P)/250 \\ 0.9 &= 1+P \\ 0.9-1 &= 1+P-1 \\ -0.1 &= P \end{aligned}$$

So the percent "increase" was -10%, which means there was a percent decrease by 10%.

Question

A lemonade stand sells 55 cups of lemonade on Monday, and 44 cups of lemonade on Tuesday. What is the percent increase/decrease in sales from Monday to Tuesday?

A. 25% decrease
B. 25% increase
C. 20% decrease
D. 20% increase
E. I don't know

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Question

A lemonade stand sells 55 cups of lemonade on Monday, and 44 cups of lemonade on Tuesday. What is the percent increase/decrease in sales from Monday to Tuesday?

A. 25% decrease
B. 25% increase
C. 20% decrease
D. 20% increase
E. I don't know

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Interest

When you deposit money into a bank, the teller doesn't put your money into a little box with your name on it. Instead, they use that money to make more money by investing it.

Since they are making money using your money, you deserve a portion of those earnings. They pay you by giving you interest on your money.

Interest is a sum of money paid for the privilege of being able to use someone else's money.

Banks incentivize you to deposit more money into their bank by giving you more interest when you deposit more money into their bank.

Interest works both ways, it also accrues on loans that you take out.

Interest Vocabulary

In finance, there are some common terms used to refer to money and the interest earned by that money.

Definitions: Finance Vocabulary

The amount of money deposited into an account or taken out in a loan is called the **principal**.

The percent used to determine the annual interest payment is called the **interest rate**. Interest rates are always annual, unless specifically stated otherwise.

Interest paid based <u>solely on the original principal</u> is called **simple interest**.

For example, if you deposit \$500 into a bank account that pays 5% annual interest, the \$500 is the principal and the 5% is the interest rate. (If only real interest rates were this good!)

Calculating Simple Interest

Formula: Simple Interest Formula

The simple interest formula is

$$I = Prt$$

I is the interest paid, P is the principal, r is the interest rate (written as a decimal), and t is the time period (in years).

For example, if you deposit \$500 into a bank account with an interest rate of 5%, then the amount of interest you earn after 3 years is:

$$\begin{aligned} I &= Prt \\ I &= (500)(0.05)(3) \\ I &= 75 \end{aligned}$$

You earn \$75 in interest after 3 years

Adjusting the Time Period

As stated earlier, interest rates are always reported annually. If the interest is paid out at a different rate, then we have to adjust the time period in our formula.

For example, if you deposit \$500 into a bank account with an interest rate of 5%, then the amount of interest you earn after 3 months is:

$$I = Prt$$

$$I = (500)(0.05)\left(\frac{3}{12}\right)$$

$$I = 6.25$$

You earn \$6.25 in interest after 3 months. The $\frac{3}{12}$ came from the fact that we are looking at the interest earned after 3 out of the 12 months of the year have passed.

Question

You deposit \$800 into a bank account with a simple interest rate of 3%. What is the amount of interest you earn after 6 months?

A. \$6
B. \$12
C. \$24
D. \$48
E. I don't know.

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Question

You deposit \$800 into a bank account with a simple interest rate of 3%. What is the amount of interest you earn after 6 months?

A. \$6
B. \$12
C. \$24
D. \$48
E. I don't know.

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Solving for the Interest Rate

Much like other problems we have worked on, we can use algebra to solve for other parts of the formula.

Let's say you deposited \$1000 into an account, and after 2 years, you've earned \$140 in interest. What is the interest rate?

$$\begin{aligned} I &= Prt \\ (140) &= (1000)r(2) \\ 140 &= 2000r \\ 140/\textcolor{red}{2000} &= 2000r/\textcolor{red}{2000} \\ 0.07 &= r \end{aligned}$$

The interest rate was 7%.

Future Value and Maturity Value - Simple Interest

The total amount of money you have or owe after the interest is added is called the future value (term used for investments like bank accounts) or the maturity value (term used for loans). They are calculated the same way.

Formula: Future/Maturity Value - Simple Interest

The future/maturity value for simple interest is:

$$A = P + I$$

Where A is the future/maturity value, P is the principal, and I is the interest accrued. If we use our earlier formula for I, we can write the formula as:

$$A = P + Prt \qquad \text{or} \qquad A = P(1 + rt)$$

Any of these formulas will give you the right answer. Choose the one that makes the most sense to you.

Using the Future/Maturity Value Formula

You deposit \$2000 into a bank account with a 4% simple interest rate. What is the future value of the account after 18 months?

$$A = P + Prt$$

$$A = (2000) + (2000)(0.04)\left(\frac{18}{12}\right)$$

$$A = 2000 + 120$$

$$A = 2120$$

Using the other formula:

$$A = P(1 + rt)$$

$$A = (2000)\left(1 + (0.04)\left(\frac{18}{12}\right)\right)$$

$$A = 2000(1.06)$$

$$A = 2120$$

Both formulas give the correct answer, \$2120.

Question

You take out a loan in the amount of \$500. The simple interest rate of the loan is 12%. What is the maturity value of the loan after 9 months?

A. \$45
B. \$500
C. \$545
D. \$560
E. I don't know.

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Question

You take out a loan in the amount of \$500. The simple interest rate of the loan is 12%. What is the maturity value of the loan after 9 months?

A. \$45
B. \$500
C. \$545
D. \$560
E. I don't know.

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Simple vs Compound Interest

In general, simple interest is only used for loans that last 1 year or less.

Most of the time, interest is compound, meaning each interest calculation is based on the new amount after the previous interest payment.

Definition: Compound Interest

Interest that is based on both the original principal and previously earned interest is called **compound interest**.

Compound Interest

Let's say you deposit \$1000 into a bank account with a compound interest rate of 5%. After the first year, you have:

$$A = P + Prt$$
$$A = 1000 + 1000(0.05)(1)$$
$$A = 1000 + 50$$
$$A = 1050$$

During the second year, the compound interest uses the \$1050 as the new principal:

$$A = P + Prt$$
$$A = 1050 + 1050(0.05)(1)$$
$$A = 1050 + 52.50$$
$$A = 1102.50$$

So your balance after 2 years is \$1102.50. Notice that this is \$2.50 more than what you get with simple interest!

Compounding Period

In the real world, you don't get an interest payment every year, you get one every month. The timing of the interest payment will be important.

Definition: Compounding Period

The frequency with which interest is compounded is called the **compounding period**. Some common compounding periods are:

Annual: 1 time per year
Semiannual: 2 times per year
Quarterly: 4 times per year
Monthly: 12 times per year
Daily: 365 times per year

The compounding period will be an important factor in our formula for compound interest.

Compound Interest Formula

Formula: Compound Interest

The compound interest formula is:

$$A = P\left(1 + \frac{r}{n}\right)^{nt}$$

Where A is the amount of money in the account, P is the original deposit, r is the interest rate, n is the number of times per year the interest is compounded, and t is the time period (in years).

There are lots of variables in this formula, so it's important to keep track of everything.

The formula looks somewhat similar to the simple interest formula, so let's see what makes it different.

Reasoning for the Formula

Let's say a bank offers 6% interest, compounded monthly.

That 6% interest is NOT applied every time the interest is compounded. Instead, the bank "breaks up" that 6% into 12 pieces of 0.5%.

Each month, your bank takes your current balance and adds 0.5% of interest. The next month, they do the same thing with your balance at the end of the next month. They do this 12 times, and $12 \times 0.5 = 6$, giving you 6% "total" of interest.

So the $\frac{r}{n}$ comes from the fact that the interest is "broken up," and the nt comes from the fact that the interest is applied multiple times per year. And this nt is in the exponent to account for the fact that your new balance is used in each calculation.

Calculator Tips for Compound Interest

When using your calculator, you have to make sure you are entering things in correctly.

One way to do this is to use parentheses. On a graphing calculator, it would look like this:

$$P * (1 + (r/n)) \wedge (n * t)$$

Where you replace each variable with its value in the question.

Another way is to calculate things piece by piece. That means you first calculate nt, write it down. Then you calculate $1 + \frac{r}{n}$, write it down. Next you calculate $(1 + \frac{r}{n})^{nt}$, then you multiply this by P. If you use this method, do NOT round until the last step.

Compound Interest Example

You deposit \$500 into a bank account that earns 3% interest, compounded monthly. How much money is in your account after 2 years, to the nearest cent?

Since the interest is compounded monthly, we use $n = 12$. Plugging everything into the formula:

$$A = P\left(1 + \frac{r}{n}\right)^{nt}$$

$$A = (500)\left(1 + \frac{(0.03)}{(12)}\right)^{(12)(2)}$$

$$A = 500(1.0025)^{24}$$

$$A = 500(1.061757044262\ldots)$$

$$A = 530.8785\ldots$$

So the amount in the account after 2 years is \$530.88.

Question

You deposit \$1000 into a bank account that earns 2% interest, compounded quarterly. How much money is in your account after 5 years, to the nearest cent?

A. \$1104.90
B. \$1105.08
C. \$1207.29
D. \$1485.95
E. I don't know.

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Question

You deposit \$1000 into a bank account that earns 2% interest, compounded quarterly. How much money is in your account after 5 years, to the nearest cent?

A. \$1104.90
B. \$1105.08
C. \$1207.29
D. \$1485.95
E. I don't know.

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Effect of the Compounding Period

One interesting question is: How does the compounding period affect the interest you earn? Is better to compound more often, less often, or is it somewhere in the middle?

It turns out that you get more interest the more often you compound!

If Person A and Person B both deposit \$1000 into a bank that has an annual interest rate of 5%, but Person A's is compounded monthly and Person B's is compounded quarterly, then Person A will have more money in their account after 1 year.

Continuously Compounded Interest

Since compounding more often gets you more money, it would be better to compound daily than monthly. But even better than daily would be hourly, but even even better would be minute-ly, but even even *even* better would be second-ly...

It would be best if you compounded every possible moment. This type of compounding is called **continuous**.

It has its own formula, $A = Pe^{rt}$, where $e \approx 2.71828182846\ldots$ is a special number called **Euler's number**.

The way you get Euler's number is to look at what you get from $\left(1 + \frac{1}{n}\right)^n$ when you plug in bigger and bigger values for n. In math, we call this a **limit**, which comes from the field of calculus.

Present Value

When investing money, people are often interested in making a certain amount of money after a certain amount of time.

For example, you may want to have \$40,000 saved up 10 years from now. If your bank offers 4% interest compounded monthly, how much must you deposit in order to save that amount?

Definition: Present Value

The **present value** of an investment is the amount you need to originally deposit in order for your investment to have a specific value at a future date.

For example, the present value for the situation above is \$26,830.64

Present Value Formula

Formula: Present Value

The present value formula is:

$$p = \frac{A}{\left(1 + \frac{r}{n}\right)^{nt}}$$

Where p is the present value, A is the desired amount of money, r is the interest rate, n is the number of times per year the interest is compounded, and t is the time period (in years).

Calculating Present Value

A family wants to save money for their child's college fund. They choose a bank that offers 5% interest compounded semiannually. How much do they need to deposit when their child is born to ensure they have \$60,000 saved by the time their child turns 18?

Since the interest is compounded semiannually, $n = 2$. The desired amount of money is $A = 60000$, the interest rate is $r = 0.05$, and $t = 18$. We plug these values into the present value formula:

$$
\begin{aligned}
p &= \frac{A}{\left(1 + \frac{r}{n}\right)^{nt}} \\
p &= \frac{60000}{\left(1 + \frac{0.05}{2}\right)^{(2)(18)}} = \frac{60000}{(1.025)^{36}} \\
p &= 24665.6233981
\end{aligned}
$$

They need to deposit \$24,665.62.

Question

How much do you need to deposit into a bank account that offers 6% interest, compounded quarterly, so that you have \$50,000 saved 20 years from now?

A. \$37123.52
B. \$27563.12
C. \$15327.84
D. \$15194.51
E. I don't know.

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Question

How much do you need to deposit into a bank account that offers 6% interest, compounded quarterly, so that you have \$50,000 saved 20 years from now?

A. \$37123.52
B. \$27563.12
C. \$15327.84
D. \$15194.51
E. I don't know.

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Effective Interest Rate

Let's say you can choose from two banks. Bank 1 offers 4% interest compounded monthly, and Bank 2 offers 5% interest compounded quarterly. Which bank is better?

Bank 2 has a higher interest rate, but as we saw earlier, compounding more often gives you more money. How can we compare these banks?

The way we do this is to turn each of these compound interests into a simple interest, and then compare the two simple interests.

Definition: Effective Interest Rate

The **effective interest rate** is the simple interest rate that would give you the same amount of money after 1 year of compound interest. The original compound interest rate is called the **nominal rate**.

Effective Interest Rate Formula

Formula: Effective Interest Rate

The effective interest rate formula is:

$$E = \left(1 + \frac{r}{n}\right)^n - 1$$

Where E is the effective interest rate, r is the nominal interest rate, and n is the number of times per year the interest is compounded.

Calculating Effective Interest Rate

In our previous example, the effective interest rate for Bank 1 is 4.07%:

$$E = \left(1 + \frac{r}{n}\right)^n - 1$$
$$E = \left(1 + \frac{0.04}{12}\right)^{12} - 1$$
$$E = 0.0407415\ldots$$

And the effective interest rate for Bank 2 is 5.09%:

$$E = \left(1 + \frac{r}{n}\right)^n - 1$$
$$E = \left(1 + \frac{0.05}{4}\right)^{4} - 1$$
$$E = 0.0509453\ldots$$

So Bank 2 is the better option.

Question

What is the effective interest rate of an account that offer 3% interest compounded daily, to the nearest hundredth?

A. 3.04%
B. 3.05%
C. 3.07%
D. 3.09%
E. I don't know.

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Question

What is the effective interest rate of an account that offer 3% interest compounded daily, to the nearest hundredth?

A. 3.04%
B. 3.05%
C. 3.07%
D. 3.09%
E. I don't know.

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Credit Cards

Your credit score is one of the most important things in your life. It is a number that reflects your responsibility in paying back your debts on time.

Building up a good credit score takes time, so it's best to get started early by obtaining a credit card.

Using a credit card, you essentially take out a loan every time you make a purchase. You pay back that loan at the end of each billing cycle.

Credit Card Vocabulary

Definitions: Credit Card Vocabulary

A **finance charge** is an amount paid for the privilege of using credit. This can take the form of interest, or an annual fee. NOTE: Interest on credit cards is applied on a monthly, simple interest basis.

The **billing cycle** is the amount of time during which your purchases are added up. Usually, a billing cycle ends and a new one begins on the same day every month (ie, the 22nd of each month).

After a billing cycle ends, you receive a bill. The **due date** is the date that bill must be paid before interest kicks in (usually one month later).

The **annual percentage rate (APR)** is the effective annual interest rate you pay if you carry a balance on your credit card.

Billing Cycle

Paying attention to your billing cycle is very important for knowing when each payment will be due.

Let's say it's March and your billing cycle runs based on 15th of each month. You make a \$200 purchase on March 18, a \$50 purchase on March 25, a \$70 purchase on April 12, and a \$130 purchase on April 16.

You will receive a bill for \$320 for the March billing cycle. Even though your balance is \$450, the bill is calculated using only your purchases during that billing cycle.

Typically, the due date is a month later. So your \$320 payment needs to be made on or before May 15 in order to avoid paying interest.

Paying on time not only helps you avoid paying more in interest, it builds your credit score

Finance Charge on Outstanding Balance

If you do not pay your balance by the due date, interest kicks in. In fact, it kicks in by double:

- You accrue interest on your outstanding balance.
- You accrue interest immediately on any purchases made after the due date.

For example, let's say your credit card charges 15% interest. If you don't pay the \$320 on on May 15, then you starting owing interest on the \$450 (this includes the purchase made on April 16).

On top of that, let's say you make a \$100 purchase on May 23. You start owing interest on that \$100 as soon as you use your card to make that purchase. That is, you don't get the "grace period" for that purchase.

Calculating a Finance Charge

The most common way to calculate the finance charge is to use the **average daily balance method**:

• At the end of every day, your credit card company checks the balance you currently owe.
• They add that amount you owe to a running total.
• After that month, they divide by the number of days in the billing cycle. This is the average daily balance.
• The average daily balance is used as the principal in the monthly simple interest calculation.

Making more purchases increases your average daily balance, meaning you accrue more interest. Making payments toward the balance (even small ones) reduces your average daily balance, meaning you accrue less interest.

Calculating Average Daily Balance

To calculate the average daily balance, it's best to make a table. Let's say you have an outstanding balance of \$450 on May 15. On May 23 you make a purchase of \$100. On June 3, you make a payment of \$300. The table looks like this:

Dates	**Purchase/ Payment**	**Balance**	**Days until Balance Changes**	**Balance × Number of Days**
May 15 - May 22		\$450	8	\$3,600
May 23 - June 2	\$100	\$550	11	\$6,050
June 3 - June 14	−\$300	\$250	12	\$3,000

Payments are listed as negatives.

Calculating Average Daily Balance

Next, we add up the two columns on the right

Dates	**Purchase/ Payment**	**Balance**	**Days until Balance Changes**	**Balance $\times$ Number of Days**
May 15 - May 22		\$450	8	\$3,600
May 23 - June 2	\$100	\$550	11	\$6,050
June 3 - June 14	–\$300	\$250	12	\$3,000
			31	\$12,650

To get the average daily balance, we divide: $\$12{,}650/31 \approx \$408.06$.

Calculating the Finance Charge

Finally, we use this \$408.06 in the simple interest formula with an interest rate of 15% to calculate the finance charge:

$$
\begin{aligned}
A &= Prt \\
A &= (408.06)(0.15)(1) \\
A &= 61.209
\end{aligned}
$$

So the finance charge is \$61.21.

If you had waited to make that payment of \$300, your average daily balance would have been \$524.19 and your finance charge would have been \$78.63. So it's always better to make a payment as early as possible, even if it is small.

Loans

There are many times in life when you need to take out a loan.

It's important to know how exactly these loans will play out over time. You don't want to make an uninformed decision and be stuck paying back a bad loan.

Comparing loans can be confusing at first, especially when they have different lengths and interest rates. We will learn how to choose the best loan for your needs.

Short-term Loans

Remember that short-term loans (less than one year) typically use simple interest.

Federal law (Truth in Lending Act) requires that you only be charged interest on what you owe, not what you originally borrowed.

For example, let's say you take out a \$2,400 loan with a 10% interest rate that is to be repaid over 6 months, and your monthly payment is \$420.

Each payment reduces the amount you owe, and therefore reduces the interest you need to pay each month.

Annual Percentage Rate

Just like with credit cards, there is an annual percentage rate for loans with simple interest. The APR allows you to compare loans that have different interest rates and different lengths.

The calculation of the actual APR is complicated for loans, but we have a nice formula to approximate it.

Formula: APR Approximation Formula

The APR on a simple interest loans is approximated by the following formula:

$$APR \approx \frac{2nr}{n+1}$$

Where APR is the annual percentage rate, r is the simple interest rate, and n is the number of payments to be made (ie, the length of the loan in months)

Approximating APR

Going back to our example of the \$2,400 loan with a 10% interest rate that is to be repaid over 6 months:

$$APR \approx \frac{2nr}{n+1}$$

$$APR \approx \frac{2(6)(0.10)}{(6)+1}$$

$$APR \approx 0.171428571429\ldots$$

Its APR is about 17.1%, which is higher than the 10% interest rate they advertised!

Question

Which of the following loans has the lowest APR?

A. A \$1000 3-month loan with 10% interest.
B. A \$7000 6-month loan with 10% interest.
C. A \$5000 5-month loan with 9.7% interest.
D. A \$9000 12-month loan with 9.5% interest.
E. I don't know.

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Question

Which of the following loans has the lowest APR?

A. A $1000 3-month loan with 10% interest.
B. A $7000 6-month loan with 10% interest.
C. A $5000 5-month loan with 9.7% interest.
D. A $9000 12-month loan with 9.5% interest.
E. I don't know.

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Calculating Monthly Payments

Whenever you take out a loan, the lender is required by law to tell you the APR on the loan.

Based on this APR, they calculate your payment amount based on the following formula:

Formula: Payment Formula

The payment on a loan based on APR is:

$$PMT = A\left(\frac{\frac{r}{n}}{1-\left(1+\frac{r}{n}\right)^{-nt}}\right)$$

Where PMT is the payment amount, A is the loan amount, r is the annual interest rate (APR), n is the number of payments per year, and t is the length of the loan in years.

Pay attention to the minus sign on the exponent: $-nt$!

Calculator Tips

The formula is very complicated, so you have to work very neatly to make sure you get the correct answer.

One trick is to store certain values. Here's how you do that on a TI 83/84:

• Calculate the value you want to store.
• Press the STO→ button on the bottom left.
• Press the ALPHA button on the top left, and choose a letter in which to store your value.
• Press ENTER.

The letter now holds the value you stored.

Calculating Monthly Payment

Let's say you take out a \$7000 5-year loan with an interest rate of 10%, that is to be paid monthly. What is your monthly payment?

First, we'll calculate $\frac{r}{n}$ and store that result as the letter C. On your calculator, type 0.1/12, ENTER, STO→, ALPHA, PRGM, ENTER.

To make sure you did it correctly, type, ALPHA, PRGM, ENTER, you should see a C on the left of your screen, and the number 0.0083333... on the right.

Now we can calculate the payment. We have $nt = (12)(5) = 60$. Type the following into your calculator, paying very close attention to the parentheses:

$$7000 * (C/(1 - (1 + C) \wedge (-60)))$$

You should get 148.729312979 (or something that matches to the first four digits after the decimal)

Using the Finance Function on TI 83/84 Calculator

A TI 83/84 has a lot of interesting apps! One in particular a finance tool that you can use to calculate monthly payments.

Press APPS, ENTER, ENTER. You will bring up a screen with a bunch of variables. Fill them in according to the table below. N is nt, I% is the actual interest percent, not the decimal, PV is the initial loan amount, FV is always 0, P/Y is n, and C/Y is n:

N	60
I%	10
PV	7000
PMT	
FV	0
P/Y	12
C/Y	12

Using the Finance Function on TI 83/84 Calculator

Scroll up to PMT, then press ALPHA, ENTER. Your table will now look like this:

N	60
I%	10
PV	7000
PMT	–148.7293
FV	0
P/Y	12
C/Y	12

This tells you the monthly payment is \$148.73. The calculator gives a negative number because it subtracts from the amount you owe.

Choosing the Best Loan

For the \$7000 5-year loan with an interest rate of 10%, you pay \$148.73 per month. Since there are 60 total payments, this comes out to a total repayment of \$8923.80.

What if the loan was only 3 years, with the same interest rate? Then your monthly payment is \$225.87, more than \$75 greater per month than the 5-year loan.

But there are now only 36 total payments, which comes out to a total repayment of \$8131.32. You save nearly \$800 in interest payments!

In general, the best loan for you to choose is the one with the shortest length whose monthly payments you can reasonably afford. This minimizes the amount you pay in interest.

Student Loans

Student loans allow people to attend college without having to pay the full tuition up front. Loans offered by the Department of Education are called Stafford loans, and generally have lower interest rates than private student loans.

There are two types of Stafford loans:

- Subsidized: The loan does not accrue interest while the student is in school, only after graduation.
- Non-subsidized: The loan accrues interest even while the student is in school.

Monthly Payment for Student Loans: Subsidized

We will use the same formula to calculate monthly payments for student loans. The key is to pay attention to if the loan is subsidized or non-subsidized.

Let's say you obtain a 15-year subsidized student loan of \$30,000 at an annual interest rate of 6%. What is your monthly payment after you graduate in 4 years?

For this question, the "graduate in 4 years" is irrelevant because the loan is subsidized. No interest accrues until you graduate. So your amount A in the payment formula is 30,000. Plugging everything in gives a monthly payment of \$253.16.

Monthly Payment for Student Loans: Non-Subsidized

Let's say instead, you obtain a 15-year non-subsidized student loan of \$30,000 at an annual interest rate of 6%. What is your monthly payment after you graduate in 4 years?

For this question, you can't ignore the "graduate in 4 years" because the loan is non-subsidized. Interest accrues while you are in school.

We use the simple interest formula to calculate the amount:

$$\begin{aligned} A &= P(1+rt) \\ &= (30,000)(1+(0.06)(4)) \\ &= (30,000)(1.24) \\ &= 37,200 \end{aligned}$$

Your monthly payment using this amount is \$313.92, about \$60 more per month than the subsidized loan.

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Counting

Counting can sometimes be very hard in mathematics! It's quite easy to count the number of people in a room, products in a store, houses in a town, etc.

But there are other things that we may be interested in counting where it isn't obvious how to count them.

For example, how many different ways can everyone in this room arrange themselves in the chairs? Some chairs are empty, some rows may have more students than others, there are many factors we need to consider.

The branch of mathematics that deals with counting things that aren't obvious how to count is called **combinatorics**.

Sample Space

The way mathematicians count is to look at every possible outcome of an event. Our current arrangement in the room is one of the possible ways we could arrange ourselves. If someone changes seats, we get a new arrangement.

Definition: Sample Space

The **sample space** is a set that contains all the possible outcomes of an event.

If you roll a die, the sample space is $\{1, 2, 3, 4, 5, 6\}$. If you pick a finger on your hand, the sample space is $\{\text{Thumb}, \text{Index}, \text{Middle}, \text{Ring}, \text{Pinky}\}$.

The cardinality of the sample space is the number we are counting. Remember that the cardinality of a set is the number of elements in the set.

Counting by Making a List

We can count something by finding its sample space. To find the sample space, we can make a list.

For example, I can choose to add cream and/or sugar to my coffee. How many different ways can I make my coffee?

I can count this by listing out every possibility for what I add to my coffee: $\{\text{Nothing}, \text{Cream}, \text{Sugar}, \text{Cream and Sugar}\}$.

So I have 4 ways to make my coffee.

Sample Space for Sequence of Events

Sometimes we want to look at things that happen in a sequence. For example, you can toss a coin and then roll a die.

The sample space for this is NOT $\{\mathrm{H}, \mathrm{T}, 1, 2, 3, 4, 5, 6\}$. Remember, the sample space is all possible outcomes. We need to account for the cases where you get a tails and a 3, a head and a 1, a head and a 5, etc.

For these cases, it's often helpful to make a table:

H1	H2	H3	H4	H5	H6
T1	T2	T3	T4	T5	T6

The sample space is $\{\mathrm{H1}, \mathrm{H2}, \mathrm{H3}, \mathrm{H4}, \mathrm{H5}, \mathrm{H6}, \mathrm{T1}, \mathrm{T2}, \mathrm{T3}, \mathrm{T4}, \mathrm{T5}, \mathrm{T6}\}$.

Question

What is the sample space for choosing a letter from the word "CAR," and then choosing a digit from the number 381?

A. {C, A, R, 3, 8, 1}
B. {C3, C8, C1, A3, A8, A1, R3, R8, R1}
C. {C3, A8, R1}
D. {3C, 8C, 1C, 3A, 8A, 1A, 3R, 8R, 1R}
E. I don't know.

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Question

What is the sample space for choosing a letter from the word “CAR,” and then choosing a digit from the number 381?

A. {C, A, R, 3, 8, 1}
B. {C3, C8, C1, A3, A8, A1, R3, R8, R1}
C. {C3, A8, R1}
D. {3C, 8C, 1C, 3A, 8A, 1A, 3R, 8R, 1R}
E. I don’t know.

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Counting with a Tree Diagram

One visual way to count the possible outcomes of a sequence of events is to use a tree diagram.

Definition: Tree Diagram

A **tree diagram** consists of points and edges, organized into layers. Each layer represents an event, and each point represents a possible outcome for an event in the layer.

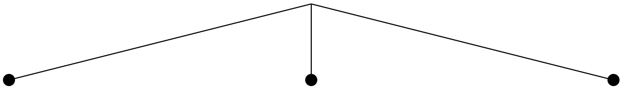
To get the next layer, we first count how many possible outcomes there are for the next layer. If there are n possible outcomes, we connect each point in the previous layer to n nodes in the next layer.

The number of possible outcomes for the sequence of events is the number of points in the final layer.

Tree Diagram Example

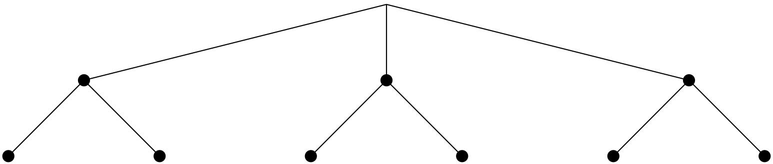
Let's say an ice cream stand has three flavors of ice cream, two flavors of sauce, and two types of sprinkles. How many different sundaes can be made?

We'll use a tree diagram to count this. For the first layer, we have three choices, so we draw three points:



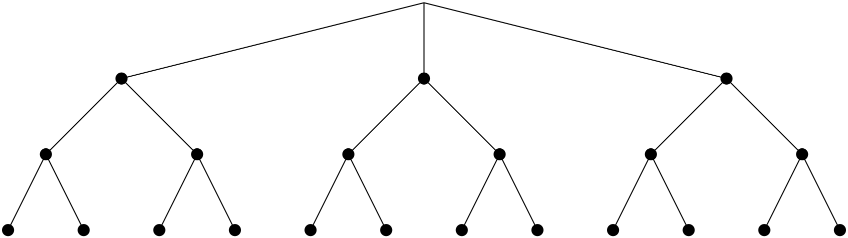
Tree Diagram Example

For the second layer, we have two choices of sauce. So to each point in the first layer we connect two points in the next layer.



Tree Diagram Example

For the last layer, we have two choices of sprinkles. So to each point in the second layer we connect two points in the last layer.



We can see there are 12 points in the last layer, so there are 12 possible sundaes.

Question

A florist arranges bouquets where there are three choices of flowers, two choices of wrapping paper, and three choices for notecards. Use a tree diagram to count the number of possible bouquets.

A. 8
B. 12
C. 15
D. 18
E. I don't know.

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Question

A florist arranges bouquets where there are three choices of flowers, two choices of wrapping paper, and three choices for notecards. Use a tree diagram to count the number of possible bouquets.

A. 8
B. 12
C. 15
D. 18
E. I don't know.

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A Giant Counting Problem

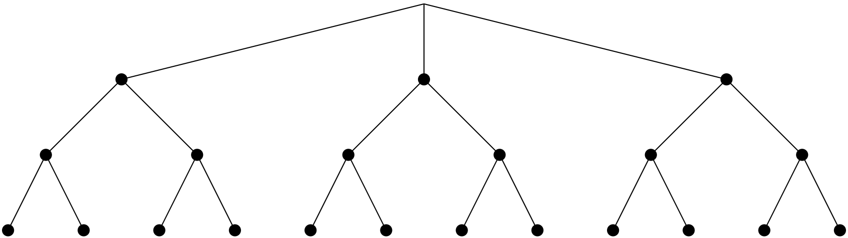
Let's say a family wants to redecorate their bedroom. There are 321 colors they can paint the walls, 34 dressers, 12 bed frames, 21 lamps, and 38 curtains. How many different ways can they redecorate their room?

We could, in theory, count this using a tree diagram. Our first layer would have 321 points, and our second layer would add 34 points to each of those 321 points, and so on.

This would take forever and kill many trees, so let's try another approach.

Our approach will be to look at how the tree diagram counts our total possibilities.

Revisiting the Tree Diagram



In our first layer, we have 3 points. In the second layer, we have $2+2+2=6$ points. We can write this addition as a multiplication: $6=3\times 2$.

In our last layer, we have $2+2+2+2+2+2=12$ points. We can again write this as a multiplication: $12=6\times 2$. If we substitute our multiplication above, we get $12=3\times 2\times 2$.

So it seems that the total number of possible outcomes is the product of the number of possible outcomes for each event. That is, we get the total by multiplying each step together.

This turns out that this is true for all types of counting with sequences of events! It's called the Counting Principle.

Definition: The Counting Principle

The number of possible outcomes for a sequence of k events where the events have $n_1, n_2, \ldots, n_k$ outcomes is $n_1 \times n_2 \times \cdots \times n_k$

Returning to our sundae example, there were 3 events (ice cream, sauce, sprinkles), so $k = 3$. Since we have 3 choices for ice cream, $n_1 = 3$. Likewise, we have $n_2 = 2$ and $n_3 = 2$. The total number of sundaes is:

$$n_1 \times n_2 \times n_3 = 3 \times 2 \times 2 = 12$$

For our bedroom redecoration example, we have 5 events, so $k = 5$. Then we have $n_1 = 321, n_2 = 34, n_3 = 12, n_4 = 21, n_5 = 38$. The total number of bedrooms is:

$$n_1 \times n_2 \times n_3 \times n_4 \times n_5 = 321 \times 34 \times 12 \times 21 \times 38 = 104,512,464$$

Question

How many outfits can you make if you have 9 shirts, 7 pairs of pants, and 3 pairs of shoes?

A. 19
B. 63
C. 189
D. 203
E. I don't know.

Room Name: BROWN4663

Question

How many outfits can you make if you have 9 shirts, 7 pairs of pants, and 3 pairs of shoes?

A. 19
B. 63
C. 189
D. 203
E. I don't know.

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Determining the Sequence of Events

Some questions will describe a process for a sequence of events without explicitly stating what the number of possible outcomes for each event is.

For example, how many four digit numbers can you create using the digits 1 through 9, if each digit may be used repeatedly? What about if no digit can be used more than once?

For questions like these, we need to determine the number of events, as well as the number of possible outcomes for each event.

Counting with Replacement

Let's count how many four digit numbers you can create using the digits 1 through 9, if each digit may be used repeatedly. This is an example of counting **with replacement**.

We are creating a four digit number, so we have to pick four digits, meaning the number of events is 4.

For our first digit, we have 9 choices, so $n_1 = 9$. For our second digit, we are allowed to repeat so again we have 9 choices, meaning $n_2 = 9$. The same is true for $n_3 = 9$ and $n_4 = 9$. Therefore, we can use the Counting Principle to calculate the number of four digit numbers we can make:

$$n_1 \times n_2 \times n_3 \times n_4 = 9 \times 9 \times 9 \times 9 = 9^4 = 6561$$

Counting without Replacement

Now let's count how many four digit numbers you can create using the digits 1 through 9, if no digit can be used more than once. This is an example of counting **without replacement**.

We are still creating a four digit number, so again we have to pick four digits, meaning the number of events is sill 4.

For our first digit, we have 9 choices, so $n_1 = 9$. For our second digit, we cannot repeat the first digit, meaning we only have 8 choices, so $n_2 = 8$. We can use the same logic to conclude that $n_3 = 7$ and $n_4 = 6$. Therefore, we can use the Counting Principle to calculate the number of four digit numbers we can make:

$$n_1 \times n_2 \times n_3 \times n_4 = 9 \times 8 \times 7 \times 6 = 3024$$

Question

How many strings of 3 letters can you make using the letters: A, B, C, D, E, F, G, if no letter can be repeated? (Note that a string of letters is just some number of letters in any order. For example, ABC, GAB, BEG are all strings of letters)

A. 210
B. 343
C. 451
D. 2187
E. I don't know.

Room Name: BROWN4663

Question

How many strings of 3 letters can you make using the letters: A, B, C, D, E, F, G, if no letter can be repeated? (Note that a string of letters is just some number of letters in any order. For example, ABC, GAB, BEG are all strings of letters)

A. 210
B. 343
C. 451
D. 2187
E. I don't know.

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Arranging Objects

Let's say I have four distinct objects. How many ways can I arrange them in a line?

In the first spot, I have 4 choices, so $n_1 = 4$. In the second spot, I only have 3 choices, so $n_2 = 3$. Continuing this logic, $n_3 = 2$ and $n_4 = 1$. So the number of arrangements is:

$$n_1 \times n_2 \times n_3 \times n_4 = 4 \times 3 \times 2 \times 1 = 24$$

If I had 5 objects, the same steps would give me $5 \times 4 \times 3 \times 2 \times 1 = 120$ ways. We call this kind of multiplication a factorial.

Definition: Factorial

n **factorial** is the product of all the natural numbers 1 through n. We write it as $n!$ and it is calculated as:

$$n! = n \times (n-1) \times (n-2) \times \cdots \times 3 \times 2 \times 1$$

Properties of Factorial

Properties of Factorial

We can "pull apart" a factorial. That is, $n! = n \times (n-1)!$.

This is helpful for allowing us to cancel things. For example:

$$\frac{8!}{5!} = \frac{8 \times 7 \times 6 \times 5!}{5!} = \frac{8 \times 7 \times 6 \times \cancel{5!}}{\cancel{5!}} = 8 \times 7 \times 6 = 336$$

There are times when we need to use $0!$. The rule is that $0! = 1$, even though $0!$ doesn't make sense according to the multiplication formula.

Permutations

Definition: Permutation

A **permutation** is an arrangement of distinct objects in a definite order.

We use permutations quite frequently. For example, when you use the shuffle feature on a playlist, it gives you a permutations of the songs.

If you are given n objects, the number of permutations is $n!$

Permutations of Subsets of Objects

Let's say you have 10 songs in a playlist, but you can only listen to the first three songs before you have to go to class. How many different arrangements of songs can you listen to?

We can use the Counting Principle to calculate this: $10 \times 9 \times 8 = 720$. In general, we can use the following formula.

Formula: Permutation Formula

The number of permutations of k objects drawn from a pool of n objects is:

$$P(n,k) = \frac{n!}{(n-k)!}$$

We can see this gives us the same answer:

$$P(10,3) = \frac{10!}{(10-3)!} = \frac{10!}{7!} = \frac{10 \times 9 \times 8 \times 7!}{7!} = 10 \times 9 \times 8 = 720$$

Question

In the Olympics, 206 countries compete for the gold, silver, and bronze medals. How many different ways can the winners be chosen for a specific event?

A. $206!$

B. $\dfrac{206!}{3}$

C. $\dfrac{206!}{3!}$

D. $206 \times 205 \times 204$

E. I don't know.

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Question

In the Olympics, 206 countries compete for the gold, silver, and bronze medals. How many different ways can the winners be chosen for a specific event?

A. 206!

B. $\dfrac{206!}{3}$

C. $\dfrac{206!}{3!}$

D. $206 \times 205 \times 204$

E. I don't know.

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Restricted Arrangements

Sometimes, there are restrictions on how we can arrange things.

For example, let's say a history class is doing presentations, and there are 10 groups in total. If we have no restrictions on the arrangement, there are $10! = 3,628,800$ possible orders they can present their projects.

Now let's say 4 of the groups did presentations on ancient history, and 6 of the groups did presentations on modern history. In how many orders can they present their projects if all of the groups that did the same topic have to present in a row?

Applying Multiple Counting Techniques

To answer the second question, we need to combine some of our counting techniques.

First, let's calculate the number of arrangements within each topic. Since 4 groups did ancient history, there are 4! ways to arrange them, and since 6 groups did modern history, there are 6! ways to arrange them.

Now, we have two options on the order they go in. Either ancient history can go first or modern history can go first. Using the Counting Principle, we get that the number of orders they can go in is:

$$2 \times 4! \times 6! = 34,560$$

Question

How many ways can a store arrange 8 brands of cereal on a shelf if 3 brands are organic, 5 are non-organic, and all of the organic/non-organic cereals are in a row?

A. $8!$
B. $3! \times 5!$
C. $2 \times 3! \times 5!$
D. $5! + 3!$
E. I don't know.

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Question

How many ways can a store arrange 8 brands of cereal on a shelf if 3 brands are organic, 5 are non-organic, and all of the organic/non-organic cereals are in a row?

A. $8!$
B. $3! \times 5!$
C. $2 \times 3! \times 5!$
D. $5! + 3!$
E. I don't know.

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Combinations

Everything we have talked about up to this point has involved collections of objects where the order matters.

But there are many times where the order doesn't matter. For example, if a store wants to hire 3 cashiers from a pool of 12 applicants, the order in which they choose them doesn't make a difference. Situations like these are called combinations.

Definition: Combination

A **combination** is a collection of objects where the order does not matter.

Deciding Between Permutation and Combination

When doing problems involving collections of objects, the key is to determine if the question is asking about a permutation or a combination.

To determine which one a question is asking about, ask yourself these questions:

- Is the order an important factor?
- Are there any distinctions being made for each choice?
- Would the situation be different if the choices were the same, but rearranged?

For permutations, the answer to these questions is generally "yes," and for combinations, the answer is generally "no."

Counting Combinations

To count combinations, we first count the permutations, and then undo the rearrangements by dividing.

For example, consider picking 3 letters from the set $\{a, b, c, d, e, f\}$. The number of permutations is:

$$P(n,k) = \frac{n!}{(n-k)!} = \frac{6!}{3!} = 6 \times 5 \times 4 = 120$$

But we want to not consider the order. Let's say our 3 letters were a, b, c. We only want to count this once, but the permutations are counted six times:

$$a, b, c \qquad a, c, b \qquad b, a, c \qquad b, c, a \qquad c, a, b \qquad c, b, a$$

To fix that, we divide by the number of rearrangements, which is $3! = 6$. Therefore, the number of combinations is $120/6 = 20$.

Combination Formula

Formula: Combinations

The number of combinations k objects drawn from a pool of n objects is:

$$C(n,k) = \frac{n!}{k! \times (n-k)!}$$

Another way to think about it is the number of permutations divided by $k!$:

$$C(n,k) = \frac{P(n,k)}{k!}$$

For example, how many ways can you choose 3 toppings for a pizza if there are 12 toppings to choose from? Since the order doesn't matter, we use a combination:

$$C(12,3) = \frac{12!}{3! \times (12-3)!} = \frac{12!}{3! \times 9!} = \frac{12 \times 11 \times 10 \times 9!}{3! \times 9!} = \frac{1320}{6} = 220$$

Question

An award show is giving out 5 identical trophies to a group of 20 nominees. How many ways can they distribute the awards?

A. $\dfrac{20!}{15!}$

B. $\dfrac{20!}{5! \times 15!}$

C. $\dfrac{5!}{20!}$

D. $\dfrac{5!}{15! \times 20!}$

E. I don't know.

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Question

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A. $\dfrac{20!}{15!}$

B. $\dfrac{20!}{5! \times 15!}$

C. $\dfrac{5!}{20!}$

D. $\dfrac{5!}{15! \times 20!}$

E. I don't know.

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Drawing from Multiple Pools

Sometimes, you need to count something drawn multiple different pools.

For example, suppose a 6-person committee is going to be made from a group of 5 economists and 7 marketers. If there are no restrictions, we proceed as normal:

$$C(12,6) = \frac{12!}{6! \times (12-6)!} = \frac{12!}{6! \times 6!} = 924$$

But what if the committee needs and equal number of economists and marketers? Now we have to combine our techniques again. There are $C(5,3) = 10$ ways to choose the economists and $C(7,3) = 35$ ways to choose the marketers.

Using the Counting Principle, we get there are $10 \times 35 = 350$ ways to make the committee.

Random Events

There are many times in life when the outcome of something is unknown, but you have an idea for how likely each outcome is to occur.

For example, when you flip a coin, you don't know if it will land heads or tails. But you do know that, in theory, each outcome will occur half the time.

This is an example of a **random event**, an event whose definite outcome is unknown, but the likelihood each outcome occurs is known.

Probability

For a random event, the likelihood of each outcome is known. We can denote this mathematically by using a number between 0 and 1 (inclusive).

Definition: Probability

The **probability** of an event is a number between 0 and 1 (inclusive) that describes how likely the event is to occur.

A probability closer to 1 means the event is more likely to occur, and a probability closer to 0 means the event is less likely to occur.

If the probability of an event is 1, the event is guaranteed to occur. If the probability of an event is 0, the event can never occur.

Determining Probability

To find the probability of an event, you need to find the sample space.

Next, you count the number of elements in the sample space that meet the conditions you are looking for.

Finally, you divide the number of elements that meet your conditions by the number of elements in the sample space.

Formula: Probability

The probability of an event E is given by

$$P(E) = \frac{n(E)}{n(S)}$$

Where $P(E)$ is the probability, $n(S)$ is the number of elements in the sample space, and $n(E)$ is the number of elements that meet the conditions.

Example of Probability

Let's say you roll a dodecahedral (12-sided) die. First, the sample space is:

$$S = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\}$$

What is the probability you roll a 4?

There is only one element in the sample space that meets the condition, so the probability is $\frac{1}{12}$.

What is the probability of rolling an even number?

There are 6 elements that meet this condition, so the probability is $\frac{6}{12} = \frac{1}{2}$.

Question

You roll a dodecahedral (12-sided) die. What is the probability you get a 9 or higher?

A. 1/12
B. 1/4
C. 1/3
D. 1/2
E. I don't know.

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Question

You roll a dodecahedral (12-sided) die. What is the probability you get a 9 or higher?

A. 1/12
B. 1/4
C. 1/3
D. 1/2
E. I don't know.

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Complements

If you are given the probability of an event E occurring, you can use that to calculate the probability that the event does not occur. This is called the **complement** of the event, and we write it as E^C.

Formula: Complement Probability

If an event E occurs with probability $P(E)$, the probability that E will not occur is

$$P(E^C) = 1 - P(E)$$

For example, if E is the event you roll a 9 or higher on a 12-sided die, then E^C means you do not roll a 9 or higher, ie, you roll an 8 or lower. The probability is $P(E^C) = 1 - P(E) = 1 - 1/3 = 2/3$.

You can also use this formula to calculate the probability of event using the probability of its complement. Using some algebra gives us $P(E) = 1 - P(E^C)$.

Using Complements

Using complements is helpful for questions that involve “at least” some number of events occurring.

For example, you flip a fair coin 6 times. What is the probability of getting at least 2 tails?

Instead of thinking about this in terms of at least 2 tails, let’s think about the complement.

If we don’t get at least 2 tails, it means we either got 0 heads or 1 tails. So we can instead calculate those possibilities.

Using Complements

Using the Counting Principle, there are $2^6 = 64$ possible outcomes (2 possibilities at each step, with 6 steps total).

There is only 1 way to get 0 tails, HHHHHH. There are 6 ways to get 1 tail. We can get it on the first throw, the second, the third, etc.

So the probability of getting 0 tails or 1 tail is $7/64$.

If E is the event of getting at least 2 tails, then $P(E^C) = 7/64$. Therefore, $P(E) = 57/64$.

Empirical Probability

Up to now, we've dealt with **theoretical** probabilities. We assumed that the coin was fair, the die was properly balanced, etc.

But there are many times when can only calculate the probability of something using collected data.

For example, the probability that someone is born left-handed is about 0.12, or 12%. The way this probability was obtained was by looking at the total population of people, and seeing how many of them were born left-handed.

This is an example of an **empirical probability**, a probability calculated based on data.

Calculating Empirical Probabilities

Formula: Empirical Probability

The empirical probability of an event E is given by

$$P(E) = \frac{\text{Number of data points meeting conditions}}{\text{Total number of data points}}$$

For example, if there are 213 math majors and 25323 total students at a university, the probability that a randomly-chosen student is a math major is $\dfrac{213}{25323}$.

Calculating Empirical Probabilities Using a Table

Sometimes the data are presented in a table:

	Math	English	Art	Total
Freshman	50	60	40	150
Sophomore	40	50	50	140
Junior	25	50	30	105
Senior	15	60	30	105
Total	130	220	150	500

The probability that a student is a freshman is $\dfrac{150}{500} = \dfrac{3}{10}$

The probability that a student is a junior math major is $\dfrac{25}{500} = \dfrac{1}{20}$

Question

	Math	English	Art	Total
Freshman	50	60	40	150
Sophomore	40	50	50	140
Junior	25	50	30	105
Senior	15	60	30	105
Total	130	220	150	500

What is the probability that a student is a senior art major?

A. $3/50$ B. $1/5$ C. $3/10$
D. $1/20$ E. I don't know.

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Question

	Math	English	Art	Total
Freshman	50	60	40	150
Sophomore	40	50	50	140
Junior	25	50	30	105
Senior	15	60	30	105
Total	130	220	150	500

What is the probability that a student is a senior art major?

A. 3/50 B. 1/5 C. 3/10
D. 1/20 E. I don't know.

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Punnet Squares

One example of empirical probability is a Punnet Square. Let's say the height of a certain plant is determined by a dominant allele T corresponding to tall plants, and a recessive allele t corresponding to short (or dwarf) plants. A plant is tall if it has at least one T allele.

If both parent plants have genotype Tt, then we can compute the probability that the offspring plants will be tall by making a Punnet Square:

	T	t
T	TT	Tt
t	Tt	tt

Three of the four possible genotypes have at least one T allele, so the probability the offspring will be tall is 3/4.

The Birthday Paradox

The Birthday Paradox

How many people do you need to gather in a room so that there is a probability of 0.5 of two people sharing a birthday?

It turns out you only need 23 people! The calculation of the probability is quite tedious, but it's doable. The reason why it is so low is because we are looking at pairs of students and seeing if they share a birthday. There are $C(23, 2) = 253$ pairs, which is much greater than just the number of people.

For a class of 32 students, the probability of two people sharing a birthday is about 0.75. A group of 57 brings the probability up to 0.99.

For a group of 365 people, the probability they all have different birthdays is 1.455×10^{-157}. This is about the same probability of flipping a coin and getting tails 521 times in a row.

Odds

If you have ever watched horse racing, you may have heard someone say "The odds against that horse winning are 1 to 5." What does that mean?

It means that for every 1 win that horse has, they will have 5 losses. In other words, the horse will win 1 out of every 6 races.

This way of communicating the likelihood of some event happening is known as **odds**.

Calculating Odds

To find the odds of an event, we need to find the favorable outcomes and the unfavorable outcomes.

The favorable outcomes are the ones that satisfy some condition. Think of winning races, making baskets, correct gambles, etc.

The unfavorable outcomes are the ones that fail to meet the condition. Think of losing races, missing baskets, incorrect gambles, etc.

Tally up each of these separately.

Calculating Odds

Formulas: Odds

Suppose the event E has a favorable outcomes and b unfavorable outcomes.

The odds **in favor of** E are given by $\dfrac{a}{b}$.

The odds **against** E are given by $\dfrac{b}{a}$.

It's important to pay very close attention to whether the odds presented are in favor of the event, or are against the event.

Example of Odds

Suppose that a baseball player gets a hit 3 times during a game, and strikes out 2 times. What are the odds in favor of the player getting a hit?

The favorable outcomes are the hits, and the unfavorable outcomes are the strikeouts. Therefore the odds in favor of the player getting a hit are $\frac{3}{2}$.

On the other hand, the odds against the player getting a hit are $\frac{2}{3}$.

Question

During a hockey player scores 2 shots during a game and misses 4 times. What are the odds against the player scoring a shot?

A. $\dfrac{2}{4}$

B. $\dfrac{4}{2}$

C. $\dfrac{2}{6}$

D. $\dfrac{6}{2}$

E. I don't know.

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Question

During a hockey player scores 2 shots during a game and misses 4 times. What are the odds against the player scoring a shot?

A. $\frac{2}{4}$

B. $\frac{4}{2}$

C. $\frac{2}{6}$

D. $\frac{6}{2}$

E. I don't know.

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Converting Between Probability and Odds

If you know the probability of an event, you can calculate the odds of that event. Likewise, if you know the odds, you can calculate the probability of the event.

Formulas: Converting Between Probability and Odds

If the odds **in favor** of an event E are $\frac{a}{b}$, the probability of event E occurring is:

$$P(E) = \frac{a}{a+b}$$

If the probability of an event E is $P(E)$, the odds **in favor** of E are

$$\frac{P(E)}{1 - P(E)}$$

Example: Odds to Probability

Let's say the odds in favor of a team winning their next game are $\dfrac{5}{6}$. What is the probability they will win their next game?

We can use the formula to convert the odds to a probability:

$$
\begin{aligned}
P(E) &= \frac{a}{a+b} \\
&= \frac{5}{5+6} \\
&= \frac{5}{11}
\end{aligned}
$$

So the probability they will win is $P(E) = \dfrac{5}{11}$.

Example: Probability to Odds

Let's say the probability of a team winning their next game is $P(E) = 0.2$. What are the odds in favor of them winning?

We can use the other formula to convert the probability to odds:

$$\begin{aligned} Odds &= \frac{P(E)}{1 - P(E)} \\ &= \frac{0.2}{1 - 0.2} \\ &= \frac{0.2}{0.8} \\ &= \frac{1}{4} \end{aligned}$$

So the odds in favor of them winning are $\frac{1}{4}$.

Question

The odds in favor of a horse winning a race are $\frac{5}{3}$. What is the probability the horse will win?

A. 3/8
B. 3/5
C. 5/8
D. 5/3
E. I don't know.

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Question

The odds in favor of a horse winning a race are $\frac{5}{3}$. What is the probability the horse will win?

A. 3/8
B. 3/5
C. 5/8
D. 5/3
E. I don't know.

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Playing the Lottery

Scratch-off lottery tickets often have various prizes, where each prize has its own probability of occurring.

For example, a ticket may give \$0 with a probability of 0.8, \$10 with a probability of 0.15, \$50 with a probability of 0.04, and \$200 with a probability of 0.01.

How much can you "expect" to win from this ticket? If the ticket costs \$10, is it a worthwhile investment?

90% of the time, you lose or break even. But maybe the small chance of winning big cancels this out.

We could find out the "expected" value by buying many lottery tickets and taking the average of what we win.

Expectation

Definition: Expectation

For an event X with multiple outcomes, the average of those outcomes in the long run is called the **expectation** of X. We write the expectation as $E[X]$.

We can calculate the expected value of an event using the following formula.

Formula: Expectation

If an event X has the possible outcomes $S_1, S_2, \ldots S_k$, each with probability $P(S_1), P(S_2), \ldots, P(S_k)$, the expectation of X is:

$$E[X] = S_1 \cdot P(S_1) + S_2 \cdot P(S_2) + \cdots + S_k \cdot P(S_k)$$

That is, you multiply each outcome with the probability that it will occur, and add them all up.

Calculating Expectation

Let's calculate the expectation for the ticket that gives \$0 with a probability of 0.8, \$10 with a probability of 0.15, \$50 with a probability of 0.04, and \$200 with a probability of 0.01.

$$E[X] = 0 \cdot 0.8 + 10 \cdot 0.15 + 50 \cdot 0.04 + 200 \cdot 0.01 = 0 + 1.50 + 2.00 + 2.00 = 5.50$$

So the expectation is 5.50.

This means if we bought many tickets, our average winning on each ticket would be \$5.50.

If we factor in that each ticket costs \$10, that means, on average, we lose \$4.50 per ticket. Another way to express this is that, when we factor in the cost, the expectation of the ticket is –\$4.50.

Question

A company estimated that a new advertising campaign will pay out \$10,000 with a probability of 0.5, \$20,000 with a probability of 0.3, and \$40,000 with a probability of 0.2. If the campaign will cost \$15,000 to run, what is the company's expectation?

A. –\$10,000
B. \$4,000
C. \$15,000
D. \$19,000
E. I don't know.

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Expectation in Life Insurance

Life insurance companies use expectation to determine whether or not a policy is profitable for them.

They use (very complicated) formulas and mortality tables to determine the probability that a person will die during the term of the policy, and use that in the expectation formula to calculate how much they will gain/lose by selling the policy.

For example, the 2020 mortality tables state that the probability a 65 year old woman will die within the next year 0.007926. How much does the company expect to gain/lose by selling her a one-year \$15,000 policy for \$35?

Expectation in Life Insurance

Let S_1 be the event the woman dies within the next year. Then $P(S_1) = 0.007926$. Then the company pays out the policy and loses \$14,965 (they still collect her \$35 premium).

Let S_2 be the event the woman does not die within the next year. Then $P(S_2) = 1 - P(S_1) = 0.992074$. Then the company gains \$35 from her premium.

Now we calculate the expectation. Note that S_1 is negative because the company loses money.

$$E[X] = -14,965 \cdot 0.007926 + 35 \cdot 0.992074 = -83.89$$

So the company would expect lose \$83.89 for each of these policies. Thus, they should make the premium high enough so that the expectation is positive.

The St. Petersburg Paradox

Suppose a gambler offers for you to play the following game:

He will flip a fair coin until it comes up tails. If it takes n throws, he will pay you 2^n dollars. For example, if the coin shows tails on the first flip, $n = 1$ and you get \$2. If he gets 9 heads and then a tail, $n = 10$ and you get \$1024. What is the maximum you should pay to play this game in order to have a positive expectation?

$$\begin{aligned}
E[X] &= 2^1 \cdot P(T) + 2^2 \cdot P(HT) + 2^3 \cdot P(HHT) + 2^4 \cdot P(HHHT) + \dots \\
&= 2 \cdot \frac{1}{2} + 4 \cdot \frac{1}{4} + 8 \cdot \frac{1}{8} + 16 \cdot \frac{1}{16} + \dots \\
&= 1 + 1 + 1 + 1 + \dots
\end{aligned}$$

The expectation is infinite! That means you "should" pay any amount of money to play the game.