Sigma Algebra

Jung Park

July 28, 2023

Contents

1 Sigma Algebra

1.1 Definition

Algebra : Let X be a nonempty set. An **Algebra** of set X is a nonempty collection $\mathcal{A}$ of subsets of X that is
1) Closed under finite union: If E_1,...,$E_n \in A$, then $\bigcup_{j=1}^{n} E_j \in A$.
2) Closed under finite Intersection: If E_1,...,$E_n \in A$, then $\bigcap_{j=1}^{n} E_j \in A$.
3) Closed under complement: If $E \in A$, then $E^c \in A$.
4) $\emptyset \in A$.
5) $X \in A$.

Condition 2,4,5 and be derived from condition 1 and 3, using De Morgan's Law.

σ**-Algebra(Field)** : Let X be a nonempty set. An σ**-Algebra(Field)** of set X is a nonempty collection $\mathcal{A}$ of subsets of X that is
1) Closed under Countable union: If E_1,...,$E_n \in A$, then $\bigcup_{j=1} E_j \in A$.
2) Closed under Countable Intersection: If E_1,...,$E_n \in A$, then $\bigcap_{j=1} E_j \in A$.
3) Closed under complement: If $E \in A$, then $E^c \in A$.
4) $\emptyset \in A$.
5) $X \in A$.

The only difference between algebra and σ-algebra is that it's closed under finite or countable union, respectively.

It is worth noting that an algebra A is a σ-algebra provided that it is closed under countable disjoint unions. Suppose $\{E_j\}_1^\infty \subset A$. A set

$$F_k = E_k \setminus \left[\bigcup_{j=1}^{k-1} E_j\right] = E_k \cap \left[\bigcup_{j=1}^{k-1} E_j\right]^c.$$

Then the F_k's belong to A and are disjoint, and $\bigcup_{j=1}^{\infty} E_j = \bigcup_{k=1}^{\infty} F_j$. This technique of replacing a sequence of sets by a disjoint sequence is frequently employed in future proofs.

Intersection of any family of σ-algebra on X is also a σ-algebra, but their union is not necessarily σ-algebra.

1.2 Property