ENEE 324 HW #3
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1. Suppose that X is a discrete random variable (rv) with range $S_X \subset N := \{1, 2, 3, ...\}$. Prove
$$E[X] = \sum_{i \in N} P[X \geq i]$$
Starting with the definition of $E[X]$:
$$E[X] = \sum_{x \in S_x} x \cdot p_X(x)$$
To get here, we rewrite $\sum_{i \in N} P[X \geq i]$
$$\sum_{i \in N} P[X \geq i] = \sum_{i \in N} \sum_{x \geq i} p_X(x)$$
From the definition of N,
$$\sum_{i \in N} \sum_{x \geq i} p_X(x) = \sum_{i=1}^{\infty} \sum_{x \geq i} p_X(x)$$
Logically, this means that for each value of $x = i$, $p_X(x)$ will be added i times. For example, $p_X(1)$ will only be included in the first summation. $p_X(2)$ will be added in the first and second. And thus this simplifies to the definition of the expectation.
Mathematically, it needs a few more steps:
$$\sum_{i=1}^{\infty} \sum_{x \geq i} p_X(x) = \sum_{x=1}^{\infty} \sum_{i=1}^{x} p_X(x) = \sum_{x=1}^{\infty} p_X(x) \sum_{i=1}^{x} 1 = \sum_{x=1}^{\infty} p_X(x) \cdot x$$
Based on the definition $S_X \subset N := \{1, 2, 3, ...\}$,
$$\sum_{x=1}^{\infty} p_X(x) \cdot x = \sum_{x \in S_x} x \cdot p_X(x) = E[X]$$

2. Suppose that X and Y are two discrete rvs, and $S_X = S_Y = \{1, -1\}$

   (a) Suppose that $E[X] = E[Y] = 0$. Show that $p_{X,Y}(1, 1) = p_{X,Y}(-1, -1)$ and $p_{X,Y}(1, -1) = p_{X,Y}(-1, 1)$. Since each rv has only two possible values, 1 and -1, each must have equal probabiltiy 0.5 in order for the expectation to be 0. If X and Y were independent, then this would be simple: $p_{X,Y}(1, 1) = p_{X,Y}(-1, -1) = p_{X,Y}(1, -1) = p_{X,Y}(-1, 1) = 0.25$
   However, even if they aren't independent, their conditional probabilities must be symmetric in order to give an expectation of 0. In other words:
   $$p_Y(1) = p_{Y|X}(1|1) + p_{Y|X}(1|-1) = 0.5 \text{ and } p_Y(-1) = p_{Y|X}(-1|1) + p_{Y|X}(-1|-1) = 0.5$$
   Since the expecation is 0, if $p_{Y|X}(1|1) > p_{Y|X}(1|-1)$ (Y is more likely to be 1 if X is 1 than if it is -1), then $p_{Y|X}(-1|1) < p_{Y|X}(-1|-1)$ is necessary to ensure that the total $p_Y(1) = p_Y(-1) = 0.5$. More specifically, if $p_{Y|X}(1|1) \neq p_{Y|X}(1|-1)$, then
   $$p_{Y|X}(1|1) - p_{Y|X}(1|-1) = p_{Y|X}(-1|-1) - p_{Y|X}(-1|1)$$
   Rearranging the earlier equations to solve for $p_{Y|X}(1|1), p_{Y|X}(1|-1), p_{Y|X}(-1|1),$ and $p_{Y|X}(-1|-1)$, then subsituting into the equality above and simplifying yeilds:
   $$p_{Y|X}(1|1) = p_{Y|X}(-1|-1) \text{ and } p_{Y|X}(1|-1) = p_{Y|X}(-1|1)$$
   Each side of the above equations can be multiplied by $p_X(1) = 0.5$ or $p_X(-1) = 0.5$ to turn the conditional probability into the desired joint probability, proving that $p_{X,Y}(1, 1) = p_{X,Y}(-1, -1)$ and $p_{X,Y}(1, -1) = p_{X,Y}(-1, 1)$.

(b) Assume $E[X] = E[Y] = 0$. Let $p = 2p_{X,Y}(1,1)$. Find $Var(X)$ and $Var(Y)$. Write $Cov(X,Y)$ in terms of p.

$$Var(X) = E[X^2] - E[X]^2 = 1 - 0 = 1 \text{ and } Var(Y) = E[Y^2] - E[Y]^2 = 1 - 0 = 1$$

$$Cov(X,Y) = \sigma_{X,Y} = E[(X - E[X])(Y - E[Y])] = E[X \cdot Y]$$

$$E[X \cdot Y] = \sum x \cdot y \cdot p_{X,Y}(x,y) = p_{X,Y}(1,1) - p_{X,Y}(1,-1) - p_{X,Y}(-1,1) + p_{X,Y}(-1,-1)$$

Since we showed that $p_{X,Y}(1,1) = p_{X,Y}(-1,-1)$ and $p_{X,Y}(1,-1) = p_{X,Y}(-1,1)$ above, this can be simplified.

$$p_{X,Y}(1,1) - p_{X,Y}(1,-1) - p_{X,Y}(-1,1) + p_{X,Y}(-1,-1) = 2p_{X,Y}(1,1) - 2p_{X,Y}(1,-1)$$

In addition,

$$p_{X,Y}(1,1) + p_{X,Y}(1,-1) + p_{X,Y}(-1,1) + p_{X,Y}(-1,-1) = 1 = 2p_{X,Y}(1,1) + 2p_{X,Y}(1,-1)$$

$$2p_{X,Y}(1,-1) = 1 - 2p_{X,Y}(1,1)$$

Substituting this into $\sigma_{X,Y}$ and replacing $2p_{X,Y}(1,1)$ with p gives:

$$Cov(X,Y) = \sigma_{X,Y} = 2p_{X,Y}(1,1) - (1 - 2p_{X,Y}(1,1))p - (1-p) = 2p - 1$$

3. Let X and Y be two discrete rv with joint PMF

$$p_{X,Y}(x,y) = \begin{cases} 0.1 & x = 1, 2, \cdots, 10, y = 1, 2, \cdots, 10, \\ 0 & otherwise \end{cases}$$

(a) What is the PMF of $W = min(X,Y)$?
There are 100 possible combinations for $x = 1, 2, \cdots, 10, y = 1, 2, \cdots, 10$. Only one has a minimum value of 10. Nineteen have a minimum value of 1. And so on in between.

$$p_W(w) = \begin{cases} 0.19 & w = 1 \\ 0.17 & w = 2 \\ 0.15 & w = 3 \\ 0.13 & w = 4 \\ 0.11 & w = 5 \\ 0.9 & w = 6 \\ 0.7 & w = 7 \\ 0.5 & w = 8 \\ 0.3 & w = 9 \\ 0.1 & w = 10 \\ 0 & otherwise \end{cases}$$

(b) What is the PMF of $Z = max(X,Y)$?

Same result as W, but with the order reversed.

$$p_Z(z) = \begin{cases} 0.19 & z = 10 \\ 0.17 & z = 9 \\ 0.15 & z = 8 \\ 0.13 & z = 7 \\ 0.11 & z = 6 \\ 0.9 & z = 5 \\ 0.7 & z = 4 \\ 0.5 & z = 3 \\ 0.3 & z = 2 \\ 0.1 & z = 1 \\ 0 & otherwise \end{cases}$$

4. Tom and Mary want to have 2 girls together. Each time they have a baby, it is a girl with a probability of 0.6. They stop having any more babies when they have two girls. Let N_1 be the number of boys till the first girl and N_T the total number of children they have together.

   (a) Let $B = \{\text{third baby is a boy}\}$. What is the conditional joint PMF $p_{N_1,N_T|B}(n_1, n_T)$?
   This limits the available values for N_1 and N_T. Because the third child is a boy, $N_1 \neq 2$ and $N_T > 3$. This sets the minimum for N_1 and reduces the possible permutations for where the first girl was born.

$$p_{N_1,N_T|B}(n_1, n_T) = \begin{cases} (0.316)(0.4)^{n_T-4}(0.6) & n_1 = 0, 1, n_T \geq 4 \text{ (One of the 1st two children is a girl)} \\ (0.4)^{n_T-3}(0.6)^2 & n_1 \geq 3, n_T \geq n_1 + 2 \text{ (Neither of the 1st two children is a girl)} \\ 0 & otherwise \end{cases}$$