ENEE 381 HW #3
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I recently injured my wrist, which has made my already poor handwriting downright illegible. While it recovers, I'm using LaTeXto write up homework assignments. If that is unacceptable, let me know.
Thanks,
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1) **Problem 9.14:** Calculate the *exact* reflection and transmission coeffecients, *without* assuming that $\mu_1 = \mu_2 = \mu_0$. Confirm that $R + T = 1$.

$$\beta = \frac{\mu_1 v_1}{\mu_2 v_2} \tag{1}$$

$$E_{0R} = \frac{1-\beta}{1+\beta} E_{0I} \text{ and } E_{0T} = \frac{2}{1+\beta} E_{0I} \tag{2}$$

$$I_I = \frac{\epsilon_1 v_1}{2} |E_{0I}|^2 \text{ and } I_R = \frac{\epsilon_1 v_1}{2} |E_{0R}|^2 \text{ and } I_T = \frac{\epsilon_2 v_2}{2} |E_{0T}|^2 \tag{3}$$

Plugging the equations from (2) in for E_{0R} and E_{0T} in (3), we get:

$$R = \frac{I_R}{I_I} = \frac{\frac{\epsilon_1 v_1}{2} (\frac{1-\beta}{1+\beta})^2 E_{0I}^2}{\frac{\epsilon_1 v_1}{2} E_{0I}^2} = (\frac{1-\beta}{1+\beta})^2 = \frac{1 - 2\frac{\mu_1 v_1}{\mu_2 v_2} + \frac{\mu_1^2 v_1^2}{\mu_2^2 v_2^2}}{1 + 2\frac{\mu_1 v_1}{\mu_2 v_2} + \frac{\mu_1^2 v_1^2}{\mu_2^2 v_2^2}} \tag{4}$$

and

$$T = \frac{I_T}{I_I} = \frac{\frac{\epsilon_2 v_2}{2} (\frac{2}{1+\beta})^2 E_{0I}^2}{\frac{\epsilon_1 v_1}{2} E_{0I}^2} = \frac{\epsilon_2 v_2}{\epsilon_1 v_1} (\frac{2}{1+\beta})^2 = \frac{\epsilon_2 v_2}{\epsilon_1 v_1} \frac{4}{1 + 2\frac{\mu_1 v_1}{\mu_2 v_2} + \frac{\mu_1^2 v_1^2}{\mu_2^2 v_2^2}} \tag{5}$$

Show that $R + T = 1$:

$$R + T = \frac{1 - 2\frac{\mu_1 v_1}{\mu_2 v_2} + \frac{\mu_1^2 v_1^2}{\mu_2^2 v_2^2} + 4\frac{\epsilon_2 v_2}{\epsilon_1 v_1}}{1 + 2\frac{\mu_1 v_1}{\mu_2 v_2} + \frac{\mu_1^2 v_1^2}{\mu_2^2 v_2^2}} \tag{6}$$

For that to equal 1, we must show that

$$4\frac{\epsilon_2 v_2}{\epsilon_1 v_1} - 2\frac{\mu_1 v_1}{\mu_2 v_2} = 2\frac{\mu_1 v_1}{\mu_2 v_2} \text{ or that } \frac{\epsilon_2 v_2}{\epsilon_1 v_1} = \frac{\mu_1 v_1}{\mu_2 v_2} \tag{7}$$

For that we need the equations for v_1 and v_2:

$$v_1 = \frac{1}{\sqrt{\mu_1 \epsilon_1}} \text{ and } v_2 = \frac{1}{\sqrt{\mu_2 \epsilon_2}} \tag{8}$$

Then rearrange Equation (7) into:

$$\frac{\mu_2 \epsilon_2}{\mu_1 \epsilon_1} = \frac{v_1^2}{v_2^2} \tag{9}$$

And substitute for v_1 and v_2 to get:

$$\frac{\mu_2\epsilon_2}{\mu_1\epsilon_1} = \frac{\frac{1}{\sqrt{\mu_1\epsilon_1}}^2}{\frac{1}{\sqrt{\mu_2\epsilon_2}}^2} = \frac{\mu_2\epsilon_2}{\mu_1\epsilon_1} \tag{10}$$

This proves the first version of Equation (7), which means we can substitute into Equation (6):

$$R + T = \frac{1 - 2\frac{\mu_1 v_1}{\mu_2 v_2} + \frac{\mu_1^2 v_1^2}{\mu_2^2 v_2^2} + 4\frac{\epsilon_2 v_2}{\epsilon_1 v_1}}{1 + 2\frac{\mu_1 v_1}{\mu_2 v_2} + \frac{\mu_1^2 v_1^2}{\mu_2^2 v_2^2}} = \frac{1 + 2\frac{\mu_1 v_1}{\mu_2 v_2} + \frac{\mu_1^2 v_1^2}{\mu_2^2 v_2^2}}{1 + 2\frac{\mu_1 v_1}{\mu_2 v_2} + \frac{\mu_1^2 v_1^2}{\mu_2^2 v_2^2}} = 1 \tag{11}$$

2) **Problem 9.15:** Prove that for a normal incident wave, the reflected and transmitted waves have the same polarization.

At normal incidence, there is no component of the E or B field that is perpendicular to the boundary. It is all parallel. Therefore, we are only concerned with the parallel boundary conditions. Specifically:

$$(\tilde{E}_{0I} + \tilde{E}_{0R})_{x,y} = (\tilde{E}_{0T})_{x,y} \tag{12}$$

$$\frac{1}{\mu_1}(\tilde{B}_{0I} + \tilde{B}_{0R})_{x,y} = \frac{1}{\mu_2}(\tilde{B}_{0T})_{x,y} \tag{13}$$

Substituting $\tilde{B} = \frac{1}{v}\tilde{E}$ gives

$$\frac{1}{\mu_1 v_1}(\tilde{E}_{0I} + \tilde{E}_{0R})_{x,y} = \frac{1}{\mu_2 v_2}(\tilde{E}_{0T})_{x,y} \tag{14}$$

If the polarization of the incident wave is strictly in the $\hat{x}$ direction, then the x, y components of Equation (12) are:

$$\tilde{E}_{0I} + \tilde{E}_{0R}\cos\theta_R = \tilde{E}_{0T}\cos\theta_T \tag{15}$$

$$\tilde{E}_{0R}\sin\theta_R = \tilde{E}_{0T}\sin\theta_T \tag{16}$$

Likewise, the x, y components of Equation (14) are, respectively:

$$\frac{1}{\mu_1 v_1}\tilde{E}_{0I} + \frac{1}{\mu_1 v_1}\tilde{E}_{0R}\cos\theta_R = \frac{1}{\mu_2 v_2}\tilde{E}_{0T}\cos\theta_T \tag{17}$$

$$\frac{1}{\mu_1 v_1}\tilde{E}_{0R}\sin\theta_R = \frac{1}{\mu_2 v_2}\tilde{E}_{0T}\sin\theta_T \tag{18}$$

We will consider just the y components:

$$\tilde{E}_{0R}\sin\theta_R = \tilde{E}_{0T}\sin\theta_T \tag{19}$$

$$\frac{\tilde{E}_{0R}}{\mu_1 v_1}\sin\theta_R = \frac{\tilde{E}_{0T}}{\mu_2 v_2}\sin\theta_T \tag{20}$$

There are only two circumstances under which both Equation (19) and (20) can be true:
Either $\mu_1 v_1 = \mu_2 v_2$, in which case both sides of the boundary are electromagnetically identical and there *is no* boundary,
or $sin\theta_R = sin\theta_T = 0$ which means that $\theta_R = \theta_T = 0$.

3) **Problem 9.18:** I did this one by hand, don't yet know how to do graphs in LaTeX. It's not pretty.

4) Derive the Fresnel coefficients for an incident wave polarized in the y direction, perpendicular to the incident plane.

This one has a lot of steps. The final answers are Equations (32) and (35) on the next page.

Since Ẽ is in the y direction, then B̃ will be in the incident plane (x, z). Therefore there are three boundary conditions to deal with:

$$(\tilde{E}_{0I} + \tilde{E}_{0R})_y = (\tilde{E}_{0T})_y \tag{21}$$

$$(\tilde{B}_{0I} + \tilde{B}_{0R})_z = (\tilde{B}_{0T})_z \tag{22}$$

$$\frac{1}{\mu_1}(\tilde{B}_{0I} + \tilde{B}_{0R})_x = \frac{1}{\mu_2}(\tilde{B}_{0T})_x \tag{23}$$

Which can be expanded or rearranged into:

$$E_{0I} + E_{0R} = E_{0T} \tag{24}$$

$$(B_{0I} - B_{0R})\cos\theta_I = B_{0T}\cos\theta_t \tag{25}$$

The latter of which can be rewritten as

$$\frac{1}{v_1}(E_{0I} - E_{0R})\cos\theta_I = \frac{1}{v_2}E_{0T}\cos\theta_t \tag{26}$$

Since $n = \frac{c}{v}$ we can rewrite that as

$$n_1(E_{0I} - E_{0R})\cos\theta_I = n_2 E_{0T}\cos\theta_t \tag{27}$$

Using Equation (24), plug in $E_{0I} + E_{0R}$ for E_{0T} to get

$$n_1(E_{0I} - E_{0R})\cos\theta_I = n_2(E_{0I} + E_{0R})\cos\theta_t \tag{28}$$

Which can be rearranged into:

$$\frac{E_{0R}}{E_{0I}} = \frac{n_1\cos\theta_I - n_2\cos\theta_T}{n_1\cos\theta_I + n_2\cos\theta_T} \tag{29}$$

Subbing in $n = \frac{n_2}{n_1}$

$$\frac{E_{0R}}{E_{0I}} = \frac{\cos\theta_I - n\cos\theta_T}{\cos\theta_I + n\cos\theta_T} \tag{30}$$

To get this in terms of θ_I, use the law of reflection, $n_1 \sin\theta_I = n_2 \sin\theta_T$ or $\sin\theta_I = n \sin\theta_T$

$$\cos\theta_T = \sqrt{1 - \sin{\theta_T}^2} = \sqrt{1 - \frac{sin{\theta_I}^2}{n^2}} \tag{31}$$

Substitute this into Equation (30) to get everything in terms of n and θ_I:

$$\frac{E_{0R}}{E_{0I}} = \frac{\cos\theta_I - \sqrt{n^2 - sin{\theta_I}^2}}{\cos\theta_I + \sqrt{n^2 - sin{\theta_I}^2}} \tag{32}$$

To find $\frac{E_{0T}}{E_{0I}}$, we solve Equation (24) for E_{0R} and plug that into Equation (26) to get:

$$\frac{1}{v_1}(E_{0I} - (E_{0T} - E_{0I}))\cos\theta_I = \frac{1}{v_2}E_{0T}\cos\theta_t \tag{33}$$

Once again substituting $n = \frac{1}{v}$ and rearranging terms, we get:

$$\frac{E_{0T}}{E_{0I}} = \frac{2n_1\cos\theta_I}{n_1\cos\theta_I + n_2\cos\theta_T} \tag{34}$$

Repeating the sequence of substitutions used to make Equations (30), (31), and (32) gives:

$$\frac{E_{0T}}{E_{0I}} = \frac{2\cos\theta_I}{\cos\theta_I + \sqrt{n^2 - sin{\theta_I}^2}} \tag{35}$$

Brewster's angle occurs were $\frac{E_{0R}}{E_{0I}} = 0$. This requires the numerator of Equation (32) to be 0.

$$\cos\theta_I - \sqrt{n^2 - sin{\theta_I}^2} = 0 \tag{36}$$

$$\cos\theta_I = \sqrt{n^2 - sin{\theta_I}^2} \tag{37}$$

$$\sqrt{1 - sin{\theta_I}^2} = \sqrt{n^2 - sin{\theta_I}^2} \tag{38}$$

$$1 - sin{\theta_I}^2 = n^2 - sin{\theta_I}^2 \tag{39}$$

$$1 = n^2 \tag{40}$$

$$n = 1 \tag{41}$$

Brewster's Angle can only exist if $n = 1$ or $n_1 = n_2$, in which case there is no reflection because there is no boundary. Otherwise, it is impossible.