ENEE 324 HW #3
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1. Suppose that X is a discrete random variable (rv) with range $S_X \subset N := \{1, 2, 3, ...\}$. Prove

$$E[X] = \sum_{i \in N} P[X \geq i]$$

Starting with the definition of $E[X]$:

$$E[X] = \sum_{x \in S_x} x \cdot p_X(x)$$

To get here, we rewrite $\sum_{i \in N} P[X \geq i]$

$$\sum_{i \in N} P[X \geq i] = \sum_{i \in N} \sum_{x \geq i} p_X(x)$$

From the definition of N,

$$\sum_{i \in N} \sum_{x \geq i} p_X(x) = \sum_{i=1}^{\infty} \sum_{x \geq i} p_X(x)$$

Logically, this means that for each value of $x = i$, $p_X(x)$ will be added i times. For example, $p_X(1)$ will only be included in the first summation. $p_X(2)$ will be added in the first and second. And thus this simplifies to the definition of the expectation.
Mathematically, it needs a few more steps:

$$\sum_{i=1}^{\infty} \sum_{x \geq i} p_X(x) = \sum_{x=1}^{\infty} \sum_{i=1}^{x} p_X(x) = \sum_{x=1}^{\infty} p_X(x) \sum_{i=1}^{x} 1 = \sum_{x=1}^{\infty} p_X(x) \cdot x$$

Based on the definition $S_X \subset N := \{1, 2, 3, ...\}$,

$$\sum_{x=1}^{\infty} p_X(x) \cdot x = \sum_{x \in S_x} x \cdot p_X(x) = E[X]$$

2. Suppose that X and Y are two discrete rvs, and $S_X = S_Y = \{1, -1\}$

(a) Suppose that $E[X] = E[Y] = 0$. Show that $p_{X,Y}(1, 1) = p_{X,Y}(-1, -1)$ and $p_{X,Y}(1, -1) = p_{X,Y}(-1, 1)$.
Since each rv has only two possible values, 1 and -1, each must have equal probabiltiy 0.5 in order for the expectation to be 0. If X and Y were independent, then this would be simple: $p_{X,Y}(1, 1) = p_{X,Y}(-1, -1) = p_{X,Y}(1, -1) = p_{X,Y}(-1, 1) = 0.25$
However, even if they aren't independent, their conditional probabilities must be symmetric in order to give an expectation of 0. In other words:

$$p_Y(1) = p_{Y|X}(1|1) + p_{Y|X}(1|-1) = 0.5 \text{ and } p_Y(-1) = p_{Y|X}(-1|1) + p_{Y|X}(-1|-1) = 0.5$$

Since the expecation is 0, if $p_{Y|X}(1|1) > p_{Y|X}(1|-1)$ (Y is more likely to be 1 if X is 1 than if it is -1), then $p_{Y|X}(-1|1) < p_{Y|X}(-1|-1)$ is necessary to ensure that the total $p_Y(1) = p_Y(-1) = 0.5$. More specifically, if $p_{Y|X}(1|1) \neq p_{Y|X}(1|-1)$, then

$$p_{Y|X}(1|1) - p_{Y|X}(1|-1) = p_{Y|X}(-1|-1) - p_{Y|X}(-1|1)$$

Rearranging the earlier equations to solve for $p_{Y|X}(1|1), p_{Y|X}(1|-1), p_{Y|X}(-1|1)$, and $p_{Y|X}(-1|-1)$, then subsituting into the equality above and simplifying yeilds:

$$p_{Y|X}(1|1) = p_{Y|X}(-1|-1) \text{ and } p_{Y|X}(1|-1) = p_{Y|X}(-1|1)$$

Each side of the above equations can be multiplied by $p_X(1) = 0.5$ or $p_X(-1) = 0.5$ to turn the conditional probability into the desired joint probability, proving that $p_{X,Y}(1, 1) = p_{X,Y}(-1, -1)$ and $p_{X,Y}(1, -1) = p_{X,Y}(-1, 1)$.

(b) Assume $E[X] = E[Y] = 0$. Let $p = 2p_{X,Y}(1,1)$. Find $Var(X)$ and $Var(Y)$. Write $Cov(X,Y)$ in terms of p.

$$Var(X) = E[X^2] - E[X]^2 = 1 - 0 = 1 \text{ and } Var(Y) = E[Y^2] - E[Y]^2 = 1 - 0 = 1$$
$$Cov(X,Y) = \sigma_{X,Y} = E[(X - E[X])(Y - E[Y])] = E[X \cdot Y]$$
$$E[X \cdot Y] = \sum x \cdot y \cdot p_{X,Y}(x,y) = p_{X,Y}(1,1) - p_{X,Y}(1,-1) - p_{X,Y}(-1,1) + p_{X,Y}(-1,-1)$$

Since we showed that $p_{X,Y}(1,1) = p_{X,Y}(-1,-1)$ and $p_{X,Y}(1,-1) = p_{X,Y}(-1,1)$ above, this can be simplified.

$$p_{X,Y}(1,1) - p_{X,Y}(1,-1) - p_{X,Y}(-1,1) + p_{X,Y}(-1,-1) = 2p_{X,Y}(1,1) - 2p_{X,Y}(1,-1)$$

In addition,

$$p_{X,Y}(1,1) + p_{X,Y}(1,-1) + p_{X,Y}(-1,1) + p_{X,Y}(-1,-1) = 1 = 2p_{X,Y}(1,1) + 2p_{X,Y}(1,-1)$$
$$2p_{X,Y}(1,-1) = 1 - 2p_{X,Y}(1,1)$$

Substituting this into $\sigma_{X,Y}$ and replacing $2p_{X,Y}(1,1)$ with p gives:

$$Cov(X,Y) = \sigma_{X,Y} = 2p_{X,Y}(1,1) - (1 - 2p_{X,Y}(1,1))p - (1-p) = 2p - 1$$

3. Let X and Y be two discrete rv with joint PMF

$$p_{X,Y}(x,y) = \begin{cases} 0.1 & x = 1,2,\cdots,10, y = 1,2,\cdots,10, \\ 0 & otherwise \end{cases}$$

(a) What is the PMF of $W = min(X,Y)$?
There are 100 possible combinations for $x = 1,2,\cdots,10, y = 1,2,\cdots,10$. Only one has a minimum value of 10. Nineteen have a minimum value of 1. And so on in between.

$$p_W(w) = \begin{cases} 0.19 & w = 1 \\ 0.17 & w = 2 \\ 0.15 & w = 3 \\ 0.13 & w = 4 \\ 0.11 & w = 5 \\ 0.9 & w = 6 \\ 0.7 & w = 7 \\ 0.5 & w = 8 \\ 0.3 & w = 9 \\ 0.1 & w = 10 \\ 0 & otherwise \end{cases}$$

(b) What is the PMF of $Z = max(X,Y)$?
Same result as W, but with the order reversed.

$$p_Z(z) = \begin{cases} 0.19 & z = 10 \\ 0.17 & z = 9 \\ 0.15 & z = 8 \\ 0.13 & z = 7 \\ 0.11 & z = 6 \\ 0.9 & z = 5 \\ 0.7 & z = 4 \\ 0.5 & z = 3 \\ 0.3 & z = 2 \\ 0.1 & z = 1 \\ 0 & otherwise \end{cases}$$

4. Tom and Mary want to have 2 girls together. Each time they have a baby, it is a girl with a probability of 0.6. They stop having any more babies when they have two girls. Let N_1 be the number of boys till the first girl and N_T the total number of children they have together.

   (a) Let $B = \{\text{third baby is a boy}\}$. What is the conditional joint PMF $p_{N_1,N_T|B}(n_1, n_T)$?
   This limits the available values for N_1 and N_T. Because the third child is a boy, $N_1 \neq 2$ and $N_T > 3$. This sets the minimum for N_1 and reduces the possible permutations for where the first girl was born.

$$p_{N_1,N_T|B}(n_1, n_T) = \begin{cases} (0.316)(0.4)^{n_T-4}(0.6) & n_1 = 0, 1, n_T \geq 4 \text{ (One of the 1st two children is a girl)} \\ (0.4)^{n_T-3}(0.6)^2 & n_1 \geq 3, n_T \geq n_1 + 2 \text{ (Neither of the 1st two children is a girl)} \\ 0 & otherwise \end{cases}$$

   (b) Compute $E[N_1|B]$ and $E[N_T|B]$.
   N_1 is related to a geometric rv, the birth of the first girl. Ignoring B, $E[N_1 + 1] = \frac{1}{p} = \frac{5}{3}$. However, the expectation is calculated as a sum, $E[X] = \sum_{x \in S_x} x \cdot p_X(x)$. The effect of B is to remove one of the possible values of N_1. Specifically, $N_1 \neq 2$ or the first girl can't be the third child. Since the original expectation is a sum of all possible values, the conditional expectation can be obtained by subtracting the disallowed one.

$$E[N_1 + 1] = \frac{\frac{5}{3} - 3p_{N_1+1}(3)}{1 - p_{N_1+1}(3)} = \frac{\frac{5}{3} - 3(0.4^2 \cdot 0.6)}{1 - (0.4^2 \cdot 0.6)} = \frac{\frac{5}{3} - 0.288}{1 - 0.096} = 1.525$$

   This is the expected value for the birth of the first girl given B. $E[N_1|B]$ will therefore be $1.525 - 1 = 0.525$. N_T is a pascal rv, with an expected value of $\frac{2}{0.6} = 3\frac{1}{3}$. The condition B removes the possibility of $N_T = 2, 3$, so these values should be removed from the expectation and the probability of the others bumped up.

$$E[N_T|B] = \frac{3\frac{1}{3} - (2p_{N_T}(2) + 3p_{N_T}(3))}{1 - (p_{N_T}(2) + p_{N_T}(3))} = \frac{3\frac{1}{3} - (2(0.6)^2 + 3(0.4)(0.6)^2)}{1 - ((0.6)^2 + (0.4)(0.6)^2)} = \frac{3\frac{1}{3} - (0.72 + 0.432)}{1 - (0.36 + 0.144)} = \frac{2.181}{0.496} = 4.40$$

   (c) Find the conditional PMFs $p_{N_1|N_T}(n_1|5)$ and $p_{N_T|N_!}(n_T|2)$.
   If $N_T = 5$, then N_1 can be 0, 1, 2 or 3. Each of these is equally likely.

$$p_{N_1|N_T}(n_1|5) = \begin{cases} 0.25 & n_1 = 0, 1, 2, 3 \\ 0 & otherwise \end{cases}$$

   From a given value of N_1, N_T is a geometric rv with parameter 0.6.

$$p_{N_T|N_!}(n_T|2) = \begin{cases} (0.4)^{n_T-4}(0.6) & n_T = 4, 5, 6, ... \\ 0 & otherwise \end{cases}$$

5. The number of dry days following a rainy day is geometrically distributed with a parameter that depends on the amount of rain we have. Suppose that it is raining today. The amount of rain Y we will get today is 0.2, 0.5, and 0.8 inches with probability 0.3, 0.3, and 0.4, respectively, e.g., $P(\text{we get 0.2 inches of rain today}) = 0.3$. Suppose that the parameter of the geometric rv X, which is the number of dry days we will have after today before another rainy day, is equal to the amount of rain we have today.

   (a) Find the joint PMF of X and Y.
   Working from $p_{X,Y}(x, y) = p_{X|Y}(x|y)p_Y(y)$,

$$p_{X|Y}(x|y) = \begin{cases} (1 - y)^{x-1}y & x = 0, 1, ... \\ 0 & otherwise \end{cases}$$

   Substituting in the values for y and multipling by their respective probabilities:

$$p_{X,Y}(x, y) = \begin{cases} (0.3)(0.8)^{x-1}(0.2) & y = 0.2, x = 0, 1, 2, ... \\ (0.3)(0.5)^x & y = 0.5, x = 0, 1, 2, ... \\ (0.4)(0.2)^{x-1}(0.8) & y = 0.8, x = 0, 1, 2, ... \\ 0 & otherwise \end{cases}$$

(b) Compute E[X].
The expectation of a geometric series is $\frac{1}{p}$. Since the parameter for X depends on Y:

$$E[X] = \sum_{Y \in S_Y} p_Y(y)\frac{1}{y} = (0.3)\frac{1}{0.2} + (0.3)\frac{1}{0.5} + (0.4)\frac{1}{0.8} = 1.5 + 0.6 + 0.48 = 2.58$$

(c) Compute the correlation coefficient of X and Y.
Correlation Coefficient $\rho_{X,Y} = \frac{Cov(X,Y)}{Var(X) \cdot Var(Y)}$. For this, we need the Expectation and Variance of both X and Y, as well as the Covariance. $E[X]$ was calculated in part (b).

$$E[Y] = (0.3)(0.2) + (0.3)(0.5) + (0.4)(0.8) = 0.06 + 0.15 + 0.32 = 0.53$$

$$E[Y^2] = (0.3)(0.2)^2 + (0.3)(0.5)^2 + (0.4)(0.8)^2 = 0.012 + 0.075 + 0.256 = 0.343 \text{ and } (E[Y])^2 = 0.281$$

$$\text{Therefore } Var[Y] = 0.343 - 0.281 = 0.062$$

Finding $Var[X]$ is worse. $(E[X])^2 = 6.66$. That's the easy part. To find $E[X^2]$, break it into a sum:

$$E[X^2] = \sum_{Y \in S_Y} E[X^2|Y = y]p_Y(y)$$

This breaks it into the respective geometric terms. Then use $E[(X^2)] = E(X) + [E(X(X1))]$. Then find the solution on stack exchange.

For a geometric rv X, $E[X(X1)] = \dfrac{2(1-p)}{p^2}$ and $E[(X^2)] = E(X) + [E(X(X1)] = \dfrac{1}{p} + \dfrac{2(1-p)}{p^2} = \dfrac{2-p}{p^2}$

In this case, $p = y$, so we sum the results for each value of y multiplied by their respective probability:

$$E[X^2] = \sum_{Y \in S_Y} E[X^2|Y = y]p_Y(y) = (0.3)\frac{2-0.2}{0.2^2} + (0.3)\frac{2-0.5}{0.5^2} + (0.4)\frac{2-0.8}{0.8^2} = 13.5 + 1.8 + .75 = 16.05$$

$$\text{Finally, } Var[X] = E[X^2] - E[X]^2 = 16.05 - 6.66 = 9.39$$

Now we just need the covariance, $\sigma_{X,Y} = E[(X - E[X])(Y - E[Y])]$. And I don't even know where to begin with this. But the Correlation Coefficient will be the Covariance devided by the product of the variances.

6. Suppose that next years revenue R of a company depends on its rating X this year. If the rating X is 3, its revenue has a binomial distribution with parameter $(n, p) = (100, 0.5)$. If the rating is 2, the revenue is geometrically distributed with parameter 1/30. If its rating is 1, the revenue is uniformly distributed on {0, 1,..., 50}.
Because the probabitities for the values of X are not specified, I will be using $p_X(x)$ for $x \in [1, 2, 3]$ in my equations.

(a) Find the joint PMF of R and X.
Like 5(a) but with R changing type.

$$p_{X,R}(x,r) = \begin{cases} (p_X(3))\begin{pmatrix}100\\ r\end{pmatrix}(0.5)^{100} & x = 3, r = \{0, 1, 2, ..., 100\} \\ (p_X(2))(1-\frac{1}{30})^{r-1}(\frac{1}{30}) & x = 2, r = 0, 1, 2, ... \\ (p_X(1))(\frac{1}{51}) & x = 1, r = \{0, 1, 2, ..., 50\} \\ 0 & otherwise \end{cases}$$

(b) Compute E[R].
Like 5(b), treat this as a sum of $E[R|X]$ multiplied by the probabilities of X:

$$E[R] = \sum_{X \in S_X} E[R|X]p_X(x) = E[R|X = 1]p_X(1) + E[R|X = 2]p_X(2) + E[R|X = 3]p_X(3)$$

$$E[R] = 25p_X(1) + 30p_X(2) + 50p_X(3)$$

(c) Find the conditional PMF $p_{X|R}(x|100)$.

For this, we need the Bayes Rule: $p_{X|R}(x|r) = p_{R|X}(r|x)\frac{p_X(x)}{p_R(r)}$. Thus,$p_{X|R}(1|100) = p_{R|X}(100|1)\frac{p_X(1)}{p_R(100)}$, $p_{X|R}(2|100) = p_{R|X}(100|2)\frac{p_X(2)}{p_R(100)}$ and $p_{X|R}(3|100) = p_{R|X}(100|3)\frac{p_X(3)}{p_R(100)}$. The first of these is clearly 0 because R cannot be 100 if X is 1. Start by calculating the conditional probabilities for R:

$$p_{R|X}(100|3) = 0.5^{100} = 7.89 \cdot 10^{-31} \text{ This is essentially 0}$$

$$p_{R|X}(100|2) = (\frac{29}{30})^{99}\frac{1/30}{=}0.00116$$

$p_{R|X}(100|2$ is small, but greater than 0 and more than 10^{27} times larger than $p_{X|R}(3|100)$. Therefore, we can ignore the rest of the Bayes Rule calculation and conclude that if $R = 100$, then $X = 2$.

$$p_{X|R}(x|100) = \begin{cases} 1 & x = 2 \\ 0 & Otherwise \end{cases}$$