ENEE 324 HW #4
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1. Suppose that a random variable (RV) Y has the following cumulative distribution function (CDF):

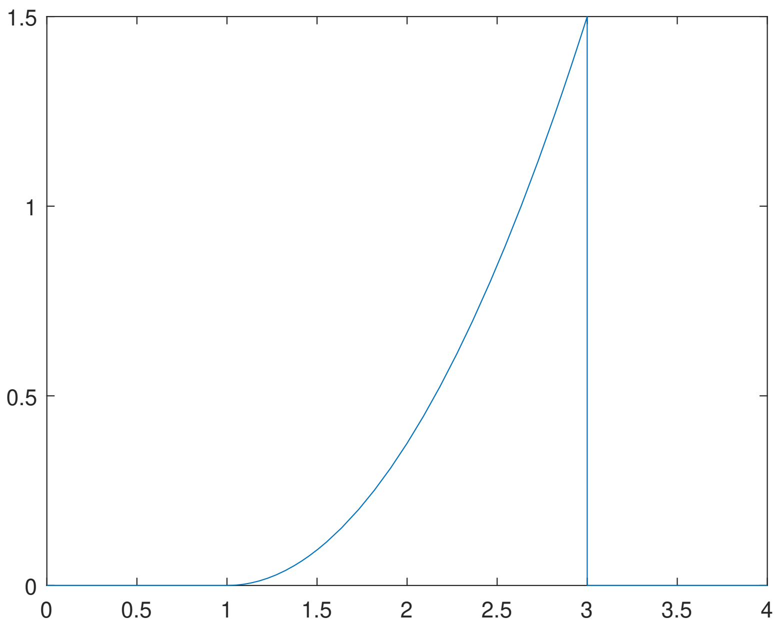
$$F_Y(y) = \begin{cases} 0 & ify < 1 \\ \frac{1}{8}(y-1)^3 & if 1 \leq y < 3 \\ 1 & ify \geq 3 \end{cases}$$

(a) Find and sketch the probability density function (PDF) of RV Y.
The PDF is the derivative of the CDF.

$$PDF = f_Y(y) = \begin{cases} 0 & ify < 1 \\ \frac{3}{8}(y-1)^2 & if 1 \leq y < 3 \\ 0 & ify \geq 3 \end{cases}$$

I could sketch this by hand, but it looks better this way. Using Matlab:

```
syms y(x);
y(x) = piecewise(x<1, 0, 1<x<3, (3/8)*(x-1)^2, x>3, 0);
fplot(y, [0,4])
```



(b) Compute $P[0.5 < Y \leq 2.5]$

$$P[0.5 < Y \leq 2.5] = \int_{0.5}^{2.5} f_Y(y)dy = \int_{1}^{2.5} 38(y-1)^2 dy = \frac{1}{8}(y-1)^3 \text{ at } y = 2.5 = 0.421875$$

(c) Compute $E[Y]$ and $Var(Y)$.

$$E[Y] = \int_{-\infty}^{infty} y \cdot f_Y(y)dy = \int_1^3 \frac{3y}{8}(y-1)^2 dy = \int_1^3 \frac{3}{8}(y^3 - 2y^2 + y)dy = [\frac{3}{32}y^4 - \frac{1}{4}y^3 + \frac{3}{16}y^2]_1^3$$

$$= [\frac{243}{32} - \frac{27}{4} + \frac{27}{16}] - [\frac{3}{32} - \frac{1}{4} + \frac{3}{16}] = [\frac{243}{32} - \frac{216}{32} + \frac{54}{32}] - [\frac{3}{32} - \frac{8}{32} + \frac{6}{32}] = \frac{81}{32} - \frac{1}{32} = \frac{80}{32} = 2.5 = E[Y]$$

To find $Var(Y)$, need $E[Y^2]$.

$$E[Y^2] = \int_{-\infty}^{infty} y^2 \cdot f_Y(y)dy = \int_1^3 \frac{3y^2}{8}(y-1)^2 dy = \int_1^3 \frac{3}{8}(y^4 - 2y^3 + y^2)dy = [\frac{3}{40}y^5 - \frac{3}{16}y^4 + \frac{1}{8}y^3]_1^3$$

$$= [\frac{729}{40} - \frac{243}{16} + \frac{27}{8}] - [\frac{3}{40} - \frac{3}{16} + \frac{1}{8}] = [\frac{1458}{80} - \frac{1215}{80} + \frac{270}{80}] - [\frac{6}{80} - \frac{15}{80} + \frac{10}{80}] = \frac{513}{80} - \frac{1}{80} = \frac{512}{80} = 6.4 = E[Y^2]$$

$$Var[Y] = E[Y^2] - E[Y]^2 = 6.4 - 6.25 = 0.15$$

2. Rachel drives to the BWI airport to see off a friend. She can park her car in one of two hourly parking garages. The parking garages charge \$1 per hour. However, the first parking garage charges for full hour for any fraction, even if the car is parked only for 1 second, while the second parking garage charges only for the amount of time the car is parked there. Rachel will park for Y number of hours, where Y is exponentially distributed with parameter 0.3. Let Z be the amount Rachel pays for parking.

   (a) Suppose that Rachel parks her car at the first parking garage. Compute the PMF of RV Z and $E[Z]$.
   Z is geometric, starts at a value of 1, and has a parameter of $p = 0.3$.

   $$p_Z(z) = \begin{cases} (0.7)^{z-1}(0.3) & for z = 1, 2, 3... \\ 0 & otherwise \end{cases}$$

   The expectation for a geometric RV is $\frac{1}{p}$, so $E[Z] = 0.3^{-1} = 3.33$

   (b) Suppose that Rachel goes to the second parking garage instead. What is $E[Z]$?
   In this case, $Z = \$1 \times Y$, so Z is an exponential distribution with parameter $\lambda = 0.3$ and expectation $E[Z] = \lambda^{-1} = 3.33$. I'm not sure I did this right, because it seems like Z should be higher for the first garage, yet the expecatation is the same.

3. Pam takes the shuttle bus back to her place after school. Suppose that the waiting time for the shuttle bus in minutes is an exponential RV Z with parameter $\lambda = 0.1$.

   (a) Compute the probability $P[A|B]$, where $A = \{Z > 5minutes\}$ and $B = \{Z > 2minutes\}$.
   Exponential RVs are memoryless. Therefore, $P[\{Z > 5minutes\}|\{Z > 2minutes\}] = P[\{Z > 3minutes\}]$

   $$P[\{Z > 3minutes\}] = \lambda e^{-3\lambda} = 0.1e^{-0.3} = 0.0741$$

   (b) Suppose that Pam has waited for the shuttle bus for 5 minutes, but the shuttle bus did not show up. Given this, what is the conditional expected value of Z?
   Again, exponential functions are memoryless, therefore the time waiting has no impact on how soon the bus will come. The normal expected value of Z is $E[Z] = 0.1^{-1} = 10$. If it hasn't come for 5 minutes, that is added on and the new expected value is 15 minutes.

4. Consider an optical fiber transmission line that uses unipolar nonreturn-to-zero signaling (also called on-off signaling). When bit 1 is sent, the signal we send is $m > 0$ and when bit is 0, we send 0. When a maximum likelihood detector is used, the probability of bit error (called bit error rate) is $Q\frac{\alpha \cdot m}{2\sigma}$, where α is the signal attenuation and σ^2 is the noise variance. Suppose that $\alpha = 10^4$. [For this problem you will need the table of or Q function posted along with the homework assignment.]

   (a) Assume $\sigma^2 = 10^{-6}$. What is the minimum required value of m if we want to ensure that the bit error rate does not exceed 10^{-4}?

   (b) If $m = 1$, hat is the maximum noise variance we can tolerate subject to the same constraint that the bit error rate does not exceed 10^4?

5. In the Internet, messages (e.g., emails) are broken into smaller pieces, called packets, before being transmitted. Packets are lost at a router (switch) due to limited buffer (physical memory) at the router. Let $T_i, i = 1, 2, \cdots$, denote the time at which i-th packet loss occurs with $T_0 = 0$ and $X_i = T_i T_{i1}$ be the inter-loss time between the i-th and $(i1)$-th packet losses (as shown in Figure 1). Suppose that the inter-loss times $X_i, i = 1, 2, \cdots$, are independent and exponentially distributed with parameter , i.e., X_i exponential() for all $i = 1, 2, \cdots$.

   (a) Let N be the number of packet losses that occur during the interval [0, 2]. Find the PMF of RV N and compute $E[N]$.

   (b) Compute the probability $P[T_4 \leq 4]$.

6. Suppose a RV $XN(\mu, \sigma^2)$. We define Y $= g(X) = e^X$. Compute the PDF function of the RV Y .