

Linear Time Backbone Determination in a Wireless Sensor Network

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Abstract

A report on implementing algorithms to partition a random geometric graph into bipartite subgraphs. Three different graph geometries are explored: unit square, unit disk, and unit sphere. Nodes are uniformly distributed in the geometry. Then the edges are determined and the verticies are colored using smallest-last vertex ordering and greedy graph coloring. Once coloring has been used to determine the independent color sets, the combinations of the largest are processed to find the largest backbone. All algorithms used in this report are implemented to run in linear time.

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1 Executive Summary

1.1 Introduction

Random geometric graphs (RGGs) are useful for simulating wireless sensor networks placed in different topologies. This project examines three different geometries: Square, Disk, and Sphere. The user supplies parameters for how many nodes they want in the network and how many connections they want for each node. Then, the simulation finds the average radius needed for that number of connections, determines the edges in the graph, colors the graph to find independent sets, pairs the four largest independent sets to find the largest bipartite subgraphs, and cleans these bipartites to find the major component, or backbone, of each bipartite. The cleaning ensures that there are no singular points of failure that could cause the network to become disconnected. In other words, each backbone exists so that there are multiple paths between any two nodes in the backbone.

This creates network backbones from the random geometric graphs that are highly reliable and allow the largest number of wireless sensors to connect to it in only one hop. Additionally, the linear time implementation of this simulation ensures efficient running time regardless of the input size. The organization of the code base also makes it easy to implement new topologies by subclassing the main Topology class that implements all of the algorithms needed to determine the backbone.

All of the code used for this project, including the graphical display of the generated graphs at each stage in the backbone determination process, can be found here:

<https://github.com/jakecarlson1/sensor-network>

1.2 Environment Description

The data structures and topologies for this simulation are implemented in Python2.7. The graphics are generated using Processing.py [3]. All development and benchmarking has been done on a 2014 MacBook Pro with a 3 GHz Intel Core i7 processor and 16 GB of DDR3 RAM running macOS High Sierra 10.13.3.

Processing offers an easy to use API for drawing and rendering shapes two- and three-dimensions. The Processing.py implementation allows the entire use of the Python programming languages and libraries.

A separate data generation script was used to generate the summary tables (Tables 1, 2, 3). Because these benchmarks were run in a separate script, the timing does not measure the time required to draw the graphs using Processing. The figures were generated using the matplotlib library [4]. This library, and a variety of others, could not be imported into Processing.py because the jython interpreter used by Processing only accepts libraries written in raw Python.

The different geometries were implemented in a stand alone Python file and imported into the Processing.py script or the data generation script, depending on what was being run. These classes can then be used directly by Processing or the data generation script. Because there is no intermediary file to hold the generated nodes and edges, there is no additional disk space needed to run the simulation. Everything can be done in system memory managed by Processing.

Benchmark	Order	A	Topology	r	Size	Realized A	Max Deg	Min Deg	Run Time (s)
1	1000	32	Square	0.101	14342	28	44	8	0.120
2	8000	64	Square	0.050	244994	61	88	19	1.321
3	16000	32	Square	0.025	250958	31	54	6	1.803
4	64000	64	Square	0.018	2017915	63	96	17	12.463
5	64000	128	Square	0.025	4003184	125	173	35	20.456
6	128000	64	Square	0.013	4052999	63	98	16	23.839
7	128000	128	Square	0.018	8073498	126	181	29	47.876
8	8000	64	Disk	0.045	245220	61	91	19	1.496
9	64000	64	Disk	0.016	2020620	63	95	16	11.525
10	64000	128	Disk	0.022	4015904	125	174	40	22.158
11	16000	64	Sphere	0.126	512821	64	98	35	25.165
12	32000	128	Sphere	0.126	2045394	127	168	87	103.663
13	64000	128	Sphere	0.089	4095217	127	170	85	178.100

Table 1: Benchmarks for generating RGGs. A: input average degree, r: node connection radius

Benchmark	Max Deg Deleted	Color Sets	Largest Color Set	Terminal Clique Size
1	19	19	78	18
2	38	35	325	33
3	25	25	1151	22
4	41	41	2548	40
5	74	66	1378	50
6	41	40	5047	38
7	75	66	2744	59
8	38	36	319	35
9	40	41	2542	34
10	73	69	1367	60
11	41	38	625	35
12	88	66	676	54
13	85	67	1354	58

Table 2: Benchmarks for coloring RGGs

Benchmark	B1 Colors	B1 Order	B1 Size	B1 Domination	B1 Faces	B2 Colors	B2 Order	B2 Size	B2 Domination	B2 Faces
1	1 & 0	116	292	0.896		1 & 2	101	260	0.803	
2	0 & 3	559	1476	0.982		1 & 3	560	1462	0.980	
3	0 & 2	1832	4586	0.930		1 & 0	1824	4546	0.931	
4	0 & 3	4533	12126	0.977		1 & 0	4561	12070	0.982	
5	0 & 3	2603	7270	0.995		0 & 2	2577	7172	0.989	
6	1 & 0	9105	24234	0.982		0 & 2	9148	24228	0.983	
7	0 & 3	5131	14334	0.992		0 & 2	5135	14282	0.995	
8	1 & 2	539	1446	0.951		0 & 2	555	1446	0.980	
9	1 & 0	4565	12164	0.982		0 & 2	4519	11994	0.979	
10	1 & 3	2595	7216	0.995		1 & 0	2568	7138	0.993	
11	1 & 3	1163	3086	0.992	1925	3 & 2	1133	2988	0.987	1857
12	1 & 3	1291	3662	0.998	2373	0 & 3	1294	3604	0.999	2312
13	3 & 2	2613	7360	0.997	4749	0 & 2	2604	7284	0.998	4682

Table 3: Benchmarks for backbone determination

2 Reduction to Practice

2.1 Data Structure Design

The primary data structure used for this project is an adjacency list. However, to allow for constant time lookup of edges of a node, a Python dictionary is used where the keys are nodes and the values are a list of indicies of adjacent nodes in the original list of nodes. The space needed by the adjacency list is $\Theta(|V| + 2|E|)$. Two entries are used for each edge because they are undirected. This is superior to the adjacency matrix data structure which would require $\Theta(|E|^2)$ space.

In order to make this project maintainable as it is developed along the semester, the object-oriented capabilities of Python are used to design the different geometries. First, a Topology class is defined that creates the interface Processing uses to draw the graphs. This base class implements all of the methods needed for node placement and edge detection in 2D graphs. Then, three subclasses are created: Square, Disk, and Sphere.

The Square and Disk topologies simply need to override the methods for generating nodes and calculating the node radius needed for the desired average degree. The Sphere subclass needs to override a few additional functions because it exists in a 3D space. Other than the methods for generating nodes and calculating the node radius, it also needs to override the function used to draw the graph so that Processing will render the graph properly in 3D.

2.2 Algorithm Descriptions

2.2.1 Node Placement

A different node placement algorithm is required for each of the geometries. For the Square, the coordinates for each node are generated as two random numbers taken from a uniform distribution on the range $[0, 1]$. All of these points are guaranteed to be in the unit square.

For the Disk, a similar method is used. The coordinates for nodes are randomly sampled from a uniform distribution; however, if a node has a distance from the center of the Disk greater than the radius of 1, the coordinates for that node are resampled.

For the Sphere a different method must be used so that all of the nodes are placed on the surface of the Sphere and the volume is vacant. For this geometry, the following equations are used:

$$x = \sqrt{1 - u^2} \cos \theta \quad (1)$$

$$y = \sqrt{1 - u^2} \sin \theta \quad (2)$$

$$z = u \quad (3)$$

where $\theta \in [0, 2\pi]$ and $u \in [-1, 1]$. This is guaranteed to uniformly distribute nodes on the surface area of the sphere [5].

All of these algorithms can be solved in $\Theta(|V|)$ where because each node only needs to be assigned a position once.

2.2.2 Edge Determination

To calculate the node connection radius needed to achieve the desired average connection, the ratio of node coverage to the total area can be used. This ratio must equal the ratio of the total number of nodes to the average degree, or:

$$\frac{A_{geometry}}{A_{node}} = \frac{\text{Num Nodes}}{\text{Avg Deg}} \quad (4)$$

Applying this to each geometry only requires filling in the equation for the area of the geometry and the connection area. This is straight forward for the square and disk. The geometry areas are given by $R^2 = 1$ and $\pi R^2 = \pi$ respectively since these are the unit square and circle. The sphere is slightly more complicated. Since nodes should only be able to connect over the surface of the sphere (following arcs), the connection area is to be taken as the surface area of the spherical cap such that the arc of the cap is twice the length of the connection distance. In other words, a node placed on the surface of the sphere in the center of a spherical cap can connect to any other node that falls in that spherical cap. The equation for the area of the spherical cap is given by

$$S_{cap} = \pi(a^2 + h^2) \quad (5)$$

where a is the distance from the midpoint of the base of the cap to the edge of the base, and h is the distance from the midpoint of the base to the top of the cap (where the node would be) [6]. If we connect these points with a third variable, x , such that x is the actual distance from the node to the edge of its connection area, the Pythagorean theorem can be used to substitute in x^2 for $a^2 + h^2$. The equation for the node connection radius of the unit sphere then looks identical to that of the unit circle. The final list of equations used to calculate node connection radius for a desired average degree are given in Table 4.

Geometry	Geometry Area	Node Area	r
Square	1	πr^2	$r = \sqrt{\frac{\text{Average Deg}}{\pi \times \text{Num Nodes}}}$
Disk	π	πr^2	$r = \sqrt{\frac{\text{Average Deg}}{\text{Num Nodes}}}$
Sphere	4π	πr^2	$r = 2 \times \sqrt{\frac{\text{Average Deg}}{\text{Num Nodes}}}$

Table 4: Equations for node connection radius

There are several methods for finding the edges in the graph. The brute force method checks every node, and for each node checks all other nodes to see if they are close enough to form an edge. The brute force method is $\Theta(|V|^2)$.

The second method to find the edges is the sweep method. This method first sorts the nodes along the x-axis. Then, for any node, we only need to search left and right until the distance along the x-axis is greater than the connection radius for the nodes. This dramatically reduces the search space. The sweep method is $O(nlg(n) + 2rn^2)$ where $n = |V|$ and r is the connection radius. The $nlg(n)$ portion is for the sorting and the $2rn^2$ portion is for measuring the distance between nodes in a sweep step.

The final method to find edges is the cell method. This method places the nodes into cells of area $r \times r$ based on their position in the topology. When the edge detection runs, each node needs to be visited once, but only the cell the node populates and the neighboring cells need to be searched for connections.

The only method that needs to be adjusted for the Sphere is the cell method. Instead of using a two dimensional grid of cells, a three dimensional mesh is needed to divide the topology. The cells then have volume $r \times r \times r$. Only the current cell and the neighboring cells need to be searched.

2.2.3 Graph Coloring

Two algorithms are used for coloring the graphs. The first is smallest-last vertex ordering, which sorts the vertices based on the number of degrees they have. The second is the greedy graph coloring algorithm.

Smallest-last vertex ordering is used to order the nodes for coloring. The steps to this algorithm are as follows [1]:

1. Initialize a representation of your target graph
2. Find the vertex v_j of minimum degree in your representation
3. Update your representation to simulate deleting v_j
4. If there are still vertices in the representation, return to step 1, otherwise terminate with the sequence of vertices removed

This algorithm is linear if each of the above steps is linear. Step 1 is linear if we can build a representation of the graph in linear time. For this, we can use an array of buckets, where each bucket holds the vertices that have the same number of edges as the position of the bucket in the array of buckets. To build this data structure, each node only needs to be visited once, making this linear in both space and time. Next, finding the vertex of minimum degree simply requires finding the lowest index bucket that has a node. This is bounded by the number of buckets, which is bounded by the number of nodes, making Step 2 linear. Next, we have to update the representation of the graph. To do this, we have to look at each node that shares an edge with v_j and move it to the bucket for nodes with one fewer degree. This requires traversing the list of edges for v_j which means Step 3 is linear. Since this is repeated for each node, the runtime of this program is $\Theta(|E| + |V|)$ and the space needed is $\Theta(|V|)$.

After this, a single traversal of the smallest-last vertex ordering is needed to color the graph. As we traverse this list, we check to see if the nodes before it (that are already colored) share an edge with the current node. The node can then be colored with any color it does not share an edge with or, if it shares an edge with all currently used colors, it is assigned a new color. This algorithm is also linear. Each node needs to be visited once and when a node is visited, all previous nodes are checked to see if they are in the edge list of the current node. Because we used smallest last vertex ordering, as we have to check more and more nodes, we get to check fewer and fewer edges. This makes the greedy coloring algorithm $O(|V| + |E|)$.

2.2.4 Backbone Determination

Several algorithms are needed for determining the most suitable backbones for the wireless sensor network. First, the four largest independent sets are paired with each other to generate the largest bipartite subgraphs for the random geometric graph. These bipartites are bound to have minor components that are not connected to the major component, and blocks that are only connected by bridges. These nodes need to be removed in order for the backbone to be considered reliable. Once all of these nodes have been removed from the bipartite, the backbone has been determined. Then, the two backbones with the largest size are selected and their domination (ratio of nodes connected to the backbone) and number of faces (for the sphere topology) are calculated.

The largest independent sets are the largest color sets given by smallest-last vertex coloring. These will be the first four color sets when greedy coloring is used on a sequence of nodes sorted in smallest-last

order. The combination of these four independent sets must be taken to find the six largest bipartite subgraphs.

The bipartite subgraphs need to be cleaned up in order to measure the size and coverage area of the backbone. This can be done by first removing all of the tails in the graph, which are sequences of nodes coming off of a component where the end node has degree one, and all nodes in between have degree two. Then, the major component needs to be determined, which is the component with the largest order. Once the largest component is determined, the minor blocks and the bridges connecting them to the major component need to be removed. A bridge is similar to a tail; it is a chain of edges that, if removed from the graph would increase the number of connected components. These features need to be removed because they do not provide reliability to the wireless sensor network. If a single one of these node were to fail, a portion of the graph would become disconnected from the remaining backbone. This creates a single point of failure that should not occur in a network backbone.

Each of these algorithms can be implemented in linear time. Taking the combinations of the four largest independent color sets can be done by building a bipartite subgraph for each combination where the nodes are copied from the two color sets that make up the bipartite. Each bipartite will then be built in $\Theta(2|V|)$ time and $\Theta(2|V|)$ space where $|V|$ is the number of nodes in each color set. Since there are six ways to choose two items from a set of 4, this runs six times, resulting in $\Theta(12|V|)$ space and time usage for building all of the bipartites.

The tails then need to be removed. This can be done by repeatedly removing all nodes with a degree of one. This will repeatedly remove the last node in the tail until the only remaining node is the node that connected the tail to its component. This will also remove any minor components that consist of a thin chain of nodes with no cycles. This is similar to smallest-last vertex ordering, except the deletion of nodes from the graph stops when there are no more nodes in the bucket for degree one. Since this algorithm is based off of smallest-last vertex ordering, and it ran in $\Theta(|E| + |V|)$, this is bounded above by smallest-last vertex ordering, $O(|E| + |V|)$. However, since the bipartite could have no tails in it, the lower bound of the runtime is $\Omega(|V|)$ which is the amount of time needed to place nodes in their respective buckets based on how many edges they have in the bipartite. Regardless, this will require $\Theta(|V|)$ space to create a representation of the bipartite that can be deleted from.

Next, the major component needs to be determined. This can be done with breadth-first search. BFS will traverse the entire graph, counting the number of nodes that can be reached from some start node. If an entire component has been explored from some start node, and there are still unvisited nodes in the graph, BFS will pick a new start node and begin searching from there. By counting the number of nodes connected to each start node, the size of each component can be determined. The major component can be determined by taking the max of these sizes. BFS works with a queue of nodes to search. At the start of an iteration, the current node is removed from the front of the queue, and all of its neighbors are added to the queue, if they have not already been visited. Since each node is only visited once, the runtime for BFS is $\Theta(|V| + |E|)$. BFS operates in-place on the graph, but a parallel array to the array of nodes is needed to remember if a node has been visited or not. This requires $\Theta(|V|)$ space and time to initialize. All together, this algorithm runs $\Theta(2|V| + |E|)$ time.

Next, the bridges need to be removed from the major components. This can be done by modifying depth-first search to check for back-edges to nodes. If some node and its edges are being searched, it is a bridge if and only if none of the descendants of the nodes connected to the current node have a back-edge to the current node or any of its ancestors. Back-edges can be checked by maintaining a list of visit times given by the DFS algorithm (tin), and a list of the minimum entry time of any ancestor (fup). If the current node's neighbors have descendants with an earlier entry time, then they must have a back-edge to that node. If they have a back-edge with the current node, the minimum entry time of the ancestors would be the current time. If the minimum entry time of the neighbor's ancestors is greater than the current time, it must be a bridge. This is codified in the following formula [8]:

$$fup[v] = \min \begin{cases} tin[v] \\ tin[p] \text{ for all } p \text{ for which } (v, p) \text{ is a back edge} \\ fup[to] \text{ for all } to \text{ for which } (v, to) \text{ is a tree edge} \end{cases} \quad (6)$$

Given this formula, the current edge (v, to) is a bridge if and only if $fup[to] > tin[v]$ in the DFS tree. DFS runs in $\Theta(|V| + |E|)$ and the book-keeping data structures add a total space requirement of $\Theta(2|V|)$.

Once the bridges have been found, the graph needs to be simulated to have them removed, and the resulting connected components need to be searched again for the major component. BFS can be used

again, where if an edge is encountered that is in the set of bridge edges, the neighbors to the current node are not pushed into the queue. Using BFS again has a time and space requirements $\Theta(2|V| + |E|)$ time and $\Theta(|V|)$ space.

With the bridges removed, the major component in each graph has been determined and all single points of failure that could result in the disconnection of backbone nodes have been removed. It is then time to determine the two largest backbones for further evaluation. The size of the backbones (the number of edges) can be determined in linear time by traversing all of the nodes in the backbone and counting the edges that are shared with other nodes in the backbone. This runs in-place on the backbone representation in $\Theta(|V| + |E|)$ time for each backbone that needs to have its size calculated.

The domination of the two largest backbones needs to be calculated. Finding the number of nodes connected directly to the backbone is equivalent to finding the number of nodes that are not connected to the backbone. This can be done by traversing all nodes that are not part of the backbone and, for each of their edges, seeing if the adjacent node is a backbone node. This algorithm requires $\Theta(|V|)$ space and $\Theta(|V| + |E|)$ time to run where $|V|$ is the number of nodes not in the backbone.

Finally, if the topology is a sphere, the number of faces can be determined by using Euler's Polyhedral Formula [7], which is given by:

$$2 = V + F - E \quad (7)$$

$$F = 2 - V + E \quad (8)$$

Where V is the number of vertices, E is the number of edges, and F is the number of faces.

2.3 Algorithm Engineering

2.3.1 Node Placement

It is easy to implement the algorithms for placing nodes in the different geometries using Python's math library. This library offers functions for sampling points on a uniform distribution. For the Square, sampling on a range $[0, 1]$ is sufficient for all of the nodes. Since each node only needs to be placed once, this runs at $\Theta(|V|)$ where.

For the Disk, the node needs to be resampled if it is too far from the center. To do this, the distance function is used to find the distance between the node and the center. If the node is further than 1 from the center, node generation falls into a while loop which iterates until the node is within the unit circle. Since nodes are taken from a uniform distribution, the number of nodes that will need to be resampled is approximately equal to the ratio of the area of the square that circumscribes the unit circle which falls outside of the unit circle to the total area of the square. This is given by:

$$\frac{(2r)^2 - \pi r^2}{(2r)^2} = \frac{4 - \pi}{4} = 0.2146 \quad (9)$$

Since the placement algorithm for each node of the Disk will iterate until the node falls within the unit circle, the total number of iterations N can be found as the sum of the geometric series:

$$N = \sum_{k=0}^{\infty} n(0.2146)^k = \frac{n}{1 - 0.2146} = 1.273n \quad (10)$$

where $n = |V|$. This shows this implementation is $\Theta(n)$.

For the node placement algorithm of the Sphere, again the math library in Python makes this easy. Each node needs two random values pulled from a uniform distribution, two square root operations, one sine operation, and one cosine operation. Each node only needs to be placed once so the runtime of this algorithm is $\Theta(n)$ where $n = |V|$.

2.3.2 Edge Determination

Each method implemented for finding edges has a different time complexity. The brute force method uses an outer loop and an inner loop, which each iterate over every node in the graph. An edge is saved to the adjacency list if the nodes are not the same and the distance between them is less than or equal to the calculated node radius. This is guaranteed to run in $\Theta(n^2)$ where $n = |V|$. The number of times the distance needs to be calculated is $n \times (n - 1)$ because it will not be calculated when the nodes are the

same (distance would be zero, but no edge is drawn here). No additional space is needed for the brute force method so the space complexity is $O(1)$.

The implementation of sweep starts by sorting the nodes along the x-axis. Python lists have a built-in sort function that has $O(n \lg(n))$ time complexity [9]. After this stage, it iterates over every node building a search space which will be scanned for edges. For each node, the list of nodes is searched right $r \times n$ nodes to find those within one radius length of the current node. With the search space built, the search space is iterated over once to find nodes that have a distance less than or equal the node radius. Then, the indices of the nodes are added to the adjacency list entry for each other. My implementation of this runs in $O(n \lg(n) + 2rn)$ where $n = |V|$ and r is the node connection radius. Because the list sort method sorts inplace, the only additional space needed is for the search space. This saves $O(rn)$ nodes and is reset after every iteration.

The cell method implementation works in linear time. In the first step of the method, the cells are initialized as a list of empty lists. There are $(1/r + 1)^2$ cells. The nodes are then iterated over and assigned a cell by dividing their x and y coordinates by the node radius. At this point, the cells are iterated over and, for each node in the cell, the nodes in the current cell and the four forward adjacent cells and the are checked to see if they fall within the node radius of the current node. All together, this implementation runs at $O(n + n + 5nr^2) = O((2 + 5r^2)n)$ where $n = |V|$. The amount of additional space needed is equal to the number of nodes because they are copied into their respective cells. This places the space complexity at $\Theta(n)$.

2.3.3 Graph Coloring

Implementing the smallest-last coloring algorithm involves implementing the smallest-last vertex ordering algorithm and the greedy graph coloring algorithm. For smallest-last vertex ordering, the first thing to do is build the data structure used to represent the graph with deleted nodes. This can be done with a list of sets, where each the index in the list represents the degree of the nodes in that set. The number of sets needed is equal to the maximum degree of the nodes. The index of each node is placed in the set corresponding to the number of edges it has then the RGG. Simultaneously, a dictionary is created that maps each node to the number of degrees it has in the graph with deletions. Each value starts at the number of edges the corresponding node has in the RGG. At this point, we have iterated over all of the nodes once and allocated space for twice the number of nodes by copying them into the sets and using them as the keys for the degrees dictionary.

Because Python dictionaries resize at specific numbers of entries, we can determine the number of additional insertions caused by rehashing while the degrees dictionary is built. Python dictionaries start out with space for 8 entries and quadruple in size until the number of entries is above 50,000, at which point it begins to double in size. Clearly the dictionary grows at a logarithmic rate, but the total number of insertions I for an input size of n is given by:

$$I = \begin{cases} n + 8 \sum_{k=1}^{\log_4 \lceil n/8 \rceil} 4^k & n \leq 50,000 \\ n + 8 \sum_{k=1}^6 4^k + 32768 \sum_{k=1}^{\log_2 \lceil n/32768 \rceil} 2^k & n > 50,000 \end{cases} \quad (11)$$

Fortunately, because the entire dictionary is built before it is used by the smallest-last vertex ordering algorithm, it will never again be resized once the algorithm starts. Unfortunately, the sets resize at a similar rate and it is more difficult to predict how large the sets will need to be when performing smallest-last vertex ordering. The degree dictionary will also be used to index into the sets, so we gain a speed up here by not having to iterate over all of the edges for a node and determining if the node it shares an edge with are in the remaining graph each time we want to sift nodes down to lower set.

After setting up the graph representation, the smallest-last vertex ordering algorithm runs until every node has been removed from the representation. To delete a node, the first non-empty set is selected. This set must contain the next node to remove because it contains all nodes with smallest degree. Before deleting the node from the graph, and moving all adjacent nodes down a set, the current set is checked to see if it has all remaining nodes. If this is the case, the terminal clique has been found, and the size of the terminal clique must be saved. After this check, a node is popped from the end of the current set, and appended to the smallest-last ordering result. Then, all nodes adjacent to the popped node in the original graph are checked to see if they are in the set with its current degree. If it is, the number of degrees for that node can be decremented and the node can be placed into the correct set for its new degree.

The last step is to reverse the order of the smallest-last ordering result because it was built in the opposite order (smallest-first). All together, excluding the initialization of accessory data structures, this

implementation runs in $\Theta(2|V| + 2|E|)$ time and $\Theta(2|V|)$ space since nodes are removed from the buckets and added to the result.

After this the graph needs to be colored. For this, initially each node is assigned a color of -1 in a node color array that is parallel to the original list of nodes. Then, all of the nodes in the smallest-last vertex ordering are iterated over. At each node, a set of colors that is already used by the neighbors of that node is created by iterating over all of its edge nodes and grabbing their color from the node color array. Then, color just has to be incremented from 0 until it does not exist in the search space set and the color has been determined to assign to the node.

Since the smallest-last ordering is used, each time the edges need to be traversed to see if a node is adjacent to the current node, nodes with fewer and fewer edges are being searched. This means that the nodes with the most neighbors are searched first, when the number of other nodes to check is lowest, and the nodes with the fewest neighbors are searched last, when we have the most nodes to check if they share an edge with the current node. All together, this implementation runs in $\Theta(|V| + 2|E|)$ time and $\Theta(|V|)$ space because we need a new array for the colors assigned to each of the nodes.

A step-by-step walkthrough of the smallest-last coloring algorithm is provided to further visualize this algorithm. For this walkthrough, a unit square topology is used with 20 nodes and a node connection radius of 0.4 . The smallest-last vertex ordering deletion process is shown in Figure 1. The coloring phase is shown in Figure 2. In the deletion process, the minimum degree node is removed at each step. If there are multiple nodes with the same minimum degree, one is chosen randomly. Once all nodes have been removed, the smallest-last vertex ordering has been determined. In the coloring phase, the node that was removed last is assigned a color first. As the smallest-last vertex ordering is traversed, each node's neighbors are checked to see if they have been assigned a color. The first color that has not been used by a neighbor is assigned to the node. To complete this walkthrough, the distribution of the color set sizes and the degrees of nodes when deleted is given in Figure 3.

2.3.4 Backbone Determination

Implementing backbone determination requires implementing all of the algorithms needed to create the bipartite subgraphs, remove unwanted nodes, and find the major components. Pairing the independent color sets is the most straightforward algorithm to implement. First, a list of four sets is created to hold the four largest independent color sets. Since the largest color sets will be the first four colors used in the greedy graph coloring implementation, all of the nodes are iterated over and each one is checked to see if its color is less than four. If that is the case, it is added to the independent set at that index in the initial list. Then, the list of independent sets is iterated over and each set is unioned with each remaining set in the list to get all of the combinations of the independent sets. The Python set union operation iterates over all of the items in each set and adds them to a result set. Since this is called three times on each independent set, and because the nodes needed to be iterated over once to place the nodes in their color sets, the total runtime for this implementation is $O(4|V|)$. The total space used by this algorithm is $O(4|V|)$, because four copies are made of each independent set. However, one of each of these copies is removed when the function returns the combinations.

Next, the independent color set pairings need to be cleaned. This is a multi-step process that starts with the removal of tails from the bipartites. Like stated earlier, the algorithm to remove tails is similar to the smallest-last vertex ordering algorithm with an early stopping condition for when the bucket for degree 1 is empty. First, some accessory data structures are initialized to save information about the representation of the graph while nodes are deleted. The buckets are initialized as empty sets. The total number of buckets needed is equal to the degree of the node with the max degree. A map is needed to relate each node to its bucket, which is created by iterating over all of the nodes in the bipartite, and counting the number of edges it shares with other nodes in the bipartite. Then, the nodes are iterated over again and placed in their buckets. At this point, the total space used is $\Theta(2|V|)$ and the time used is $\Theta(2|V| + 2|E|)$.

At this point, the smallest-last vertex algorithm is run until the sets for degree zero and one are empty. Each iteration of the algorithm, all of the nodes in the degree zero and degree one sets are put in a list of nodes to remove. Both sets are checked so that any nodes in the graph that are not connected to a component are removed. These nodes are then iterated over and each edge it shares with a node in the bipartite is checked to see if the neighbor needs to be moved down a bucket. Once all neighbors have been moved down, the node is removed from the bipartite subgraph. This runs in similar time as smallest-last vertex ordering, $\Theta(2|V| + 2|E|)$. The only additional space needed by the algorithm is the space needed to hold the list of nodes in the first two buckets, however, once the nodes have been copied into the list, the buckets they were in are cleared. Regardless, this can use $O(|V|)$ in the worst case. All

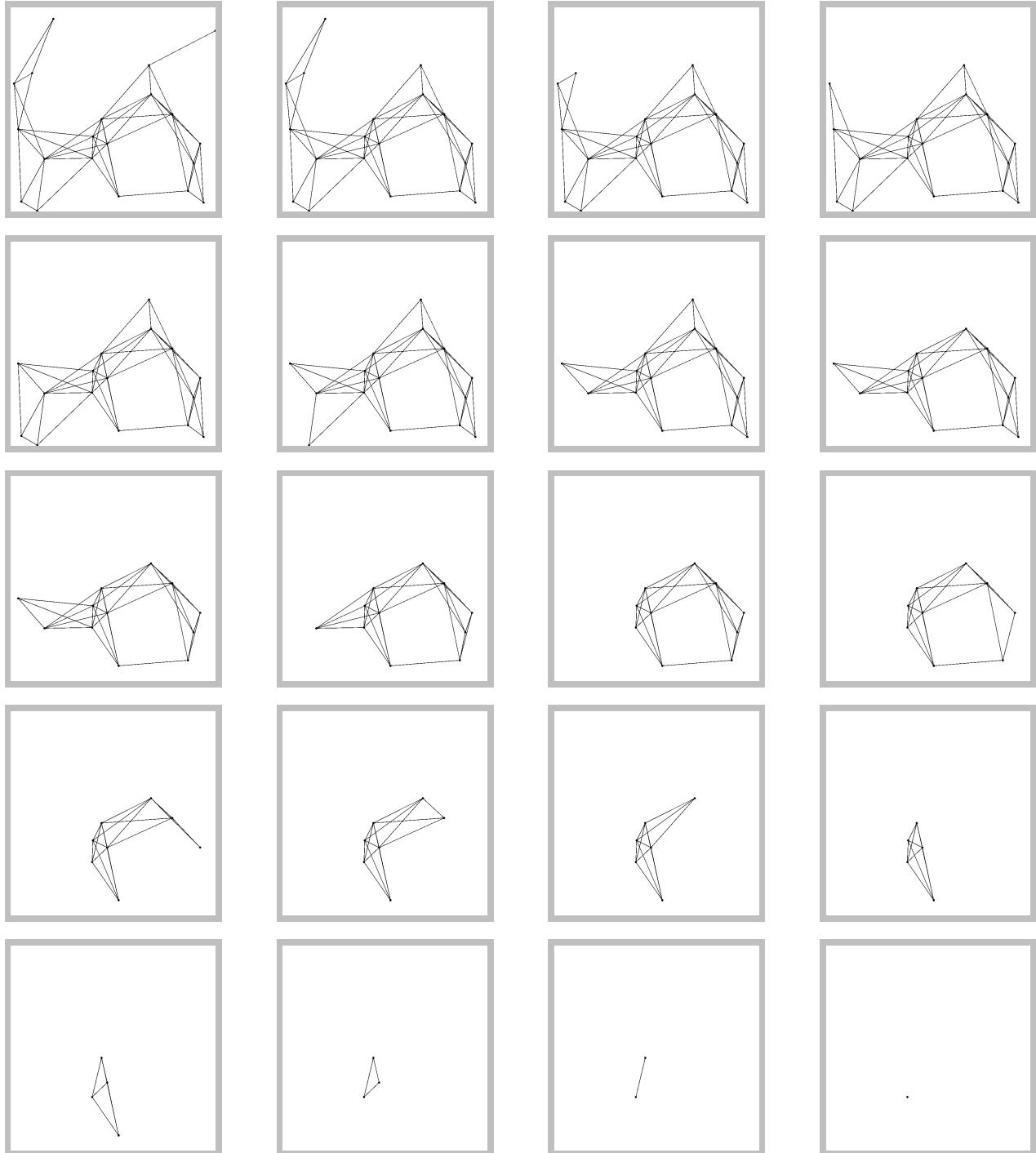


Figure 1: Smallest-last vertex ordering deletion process

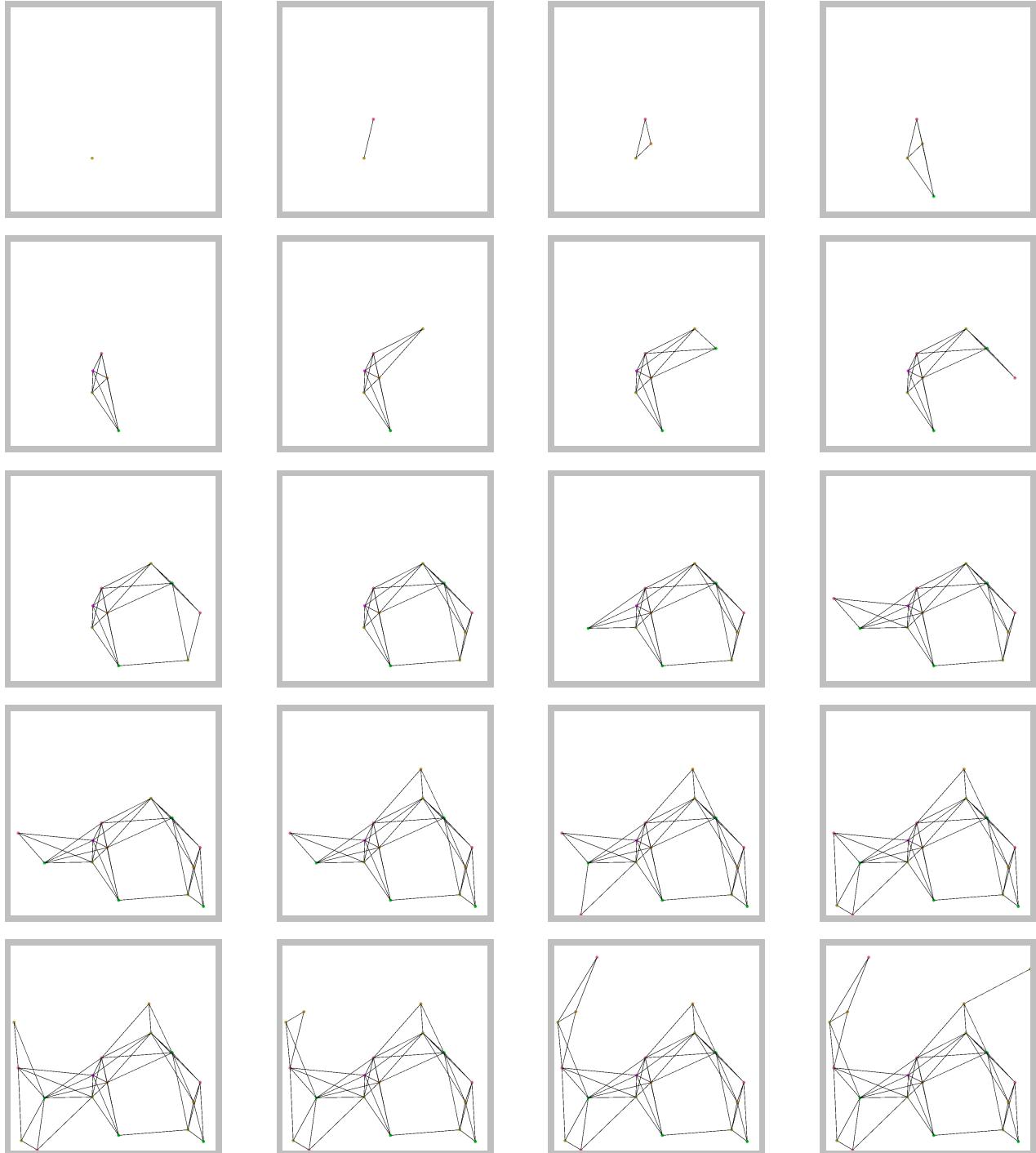


Figure 2: Smallest-last vertex ordering coloring process

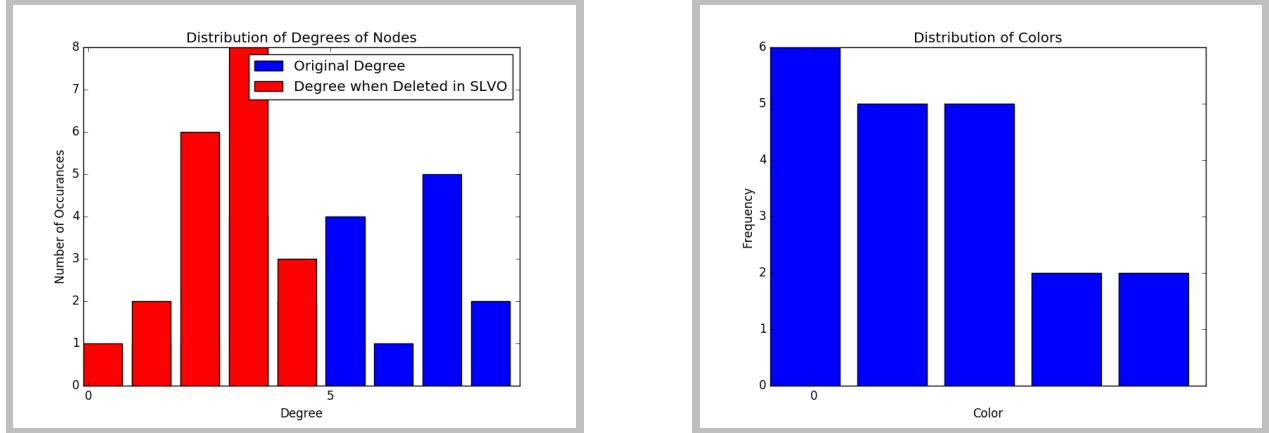


Figure 3: Distribution of degree when deleted and color set size for the 20 node walkthrough

together, tail removal takes $O(3|V|)$ space and $\Theta(4|V| + 4|E|)$ time.

The next part of the cleaning is selecting the major component, which is implemented using breadth-first search. Before starting BFS, some setup is needed. First, the bipartite is copied into a local list for iteration. Then, two dictionaries are created for indexing from the local list of bipartite nodes to the master list of nodes. Next, a list of integers is created for keeping track of which nodes have been visited during BFS. At this point, $O(4|V|)$ space has been used. Then, BFS starts and runs until every node has been visited. While nodes have not been visited, the first unvisited node is selected to be the root of the search tree. This root is put in the queue, added as the first item in a set to a list of sets representing the components in the graph, and the visit time is set to 1. Then, while the queue is not empty, an item is popped and all of its edges are checked to see if they have already been visited. Each one that has not been visited is pushed into the queue, marked as visited, added to the set representing the current component being searched, and the visit time is incremented. Once the queue is empty, the final visit time is saved as the number of nodes in the component. After all nodes have been visited, all that is needed is to return the component with the largest number of visits and the major component has been determined. This implementation of BFS requires $\Theta(|V| + 2|E|)$ time and $O(2|V|)$ space because the nodes are copied into their respective component sets, and the queue could grow to hold all nodes in the graph in the worst case.

The last step in preparing the backbones is to remove all of the bridges and minor blocks. Bridge removal uses depth-first search, however, some other data is needed to keep track of the visit time for nodes (*tin*) and the visit time of their ancestors (*fup*) in the DFS tree. First, a local copy of the bipartite is created to iterate over, and, similar to BFS, two dictionaries are created for indexing between the local list of nodes and the master list of nodes. A list is created to keep track of whether nodes have been visited or not, the visit time of the DFS algorithm at the node, and the minimum visit time of a node's descendants. All of these data structures together require $\Theta(6|V|)$ space and can be created in $\Theta(6|V|)$ time. Now, DFS can run until all of the nodes have been visited. The first node that hasn't been visited is selected as the root of the search tree for DFS. Each edge this node shares with another node in the major component is iterated over. *Fup* for the current node is calculated for each of the neighbor nodes that has not been visited as the minimum of *fup* for the current node and *tin* of the current edge. If the neighbor hasn't been visited, DFS is called recursively on the edge to search it. Once the search returns, *fup* for the current node is calculated as the minimum of *fup* for the current node and *fup* for the current edge. There is now enough information to determine if the current edge is a bridge. If *fup* for the current edge is greater than *tin* for the current node, then the neighbor must not have another path to any of the ancestors of the current node, so it is a bridge and the current nodes are saved to a list of bridges. DFS itself runs in $\Theta(|V| + 2|E|)$ time and uses $O(2|E|)$ space in the worst case which would be that all nodes in the graph are part of a bridge (however, this would never happen because tails have already been removed).

The final step of bridge removal is to use the list of nodes that are part of the bridges to determine the major component with the bridges removed. BFS is suitable for this because it is already implemented to return the major component of a graph. In order to make BFS skip the bridge nodes, each time an edge is visited that has both nodes in the set of bridge nodes, continue is called to skip the rest of the

iteration. This will prevent pushing that neighbor to the queue and will disconnect those components. BFS will then proceed and return the major component. All together, bridge removal uses $O(8|V| + 2|E|)$ space and runs in $\Theta(8|V| + 4|E|)$ time.

At this point, six potential backbones have been determined from the original six bipartite subgraphs. Now, the two largest backbones need to be determined. These are the backbones with the largest size, or the highest number of edges. To find the two largest backbones, two parallel lists are created that each have two elements. The first list is for the sizes of the backbones, and the second is for the backbones themselves. For each backbone, the size is calculated by iterating over all the nodes in the backbone and summing the number of edges each node shares with another node in the backbone. Because the backbones are represented as a set, it takes constant time to see if a node is in the backbone. Once the size has been calculated for a backbone, it is checked to see if it is larger than the saved backbone with the minimum size. If this is the case, it replaces that backbone in the list of results and its size is saved. This requires $\Theta(|V| + 2|E|)$ time for each backbone. After the two largest backbones have been determined, some metadata is calculated about them and returned as a parallel array to the list of backbones. This meta data is the order and size of each backbone, which is not dependent on the size of the backbones.

Finally, the domination of the two largest backbones needs to be calculated. This is done by initializing a search space with all of the nodes in the master list of nodes that are not in the backbone. This search space is then iterated over, and each edge is checked to see if the neighboring node is in the backbone. If a node does share an edge with the backbone, it is removed from the search space. Also, once it has been found that the current node shares an edge with a backbone node, the rest of the edges for the current node can be skipped. At the end of this, the search space will have all nodes that do not share an edge with a backbone node. It is then easy to calculate the domination of the backbone by subtracting this number from the total number of nodes and dividing by the total number of nodes. This runs in $\Theta(|V| + |E|)$ time and requires $\Theta(|V|)$ space to initialize the search space.

If the topology is a sphere, the number of faces of the backbone can be calculated using Euler's Polyhedral Formula. This formula operates under the assumption that a graph is connected and can be represented in planar form. The first is guaranteed because the backbone is the major connected component found in a bipartite subgraph. The second is true because the nodes comprising the backbone can be projected onto a plane and there will be no overlapping edges because the edges do not overlap in the original representation. Therefore, the number of faces can be calculated in constant time using the meta data of the backbone generated earlier.

To illustrate this further, the above walkthrough is extended to include the backbone determination stages based on the two largest color sets. These stages are given in Figure 4. With the selected color sets, removing the tails is sufficient for creating the backbone. The other steps yield the same graph, but are necessary for higher-order graphs.

2.4 Verification

2.4.1 Node Placement

The nodes can be verified to be distributed uniformly if the degrees follow a normal distribution. To show that the distribution of degrees for each of the geometries are following a normal distribution, the degree histograms are plotted for each of the benchmarks. The histograms for Square are given in Figure 6, Disk are given in Figure 7, and Sphere are given in Figure 8. These histograms clearly follow a normal distribution, so the nodes must be placed uniformly.

2.4.2 Edge Determination

The runtime for the edge detection methods can be verified by varying the number of nodes and measuring the runtime of each algorithm. By looking at how the runtime grows, we can calculate the trendline that best fits the growth rate. For the first comparison, the number of nodes is varied from 4,000 to 64,000 in steps of 4,000, while holding the desired average degree constant at 16. As we can see in Figure 5, the growth rates of the brute force and sweep methods are quadratic, while the growth rate of the cell method. The trendline functions are given on the graph.

For the second metric, the number of nodes is held constant at 32,000 and varied the desired average degree from 2 to 32 in steps of 2. The graph is given in Figure 5. The cell method clearly grows linearly, but the sweep method is harder to gauge. Since varying the desired average degree should only change the node radius, this should grow linearly as well. However, because each graph is randomly generated, some graphs can have nodes that are closer to sorted order than others. This can effect the measured runtime.

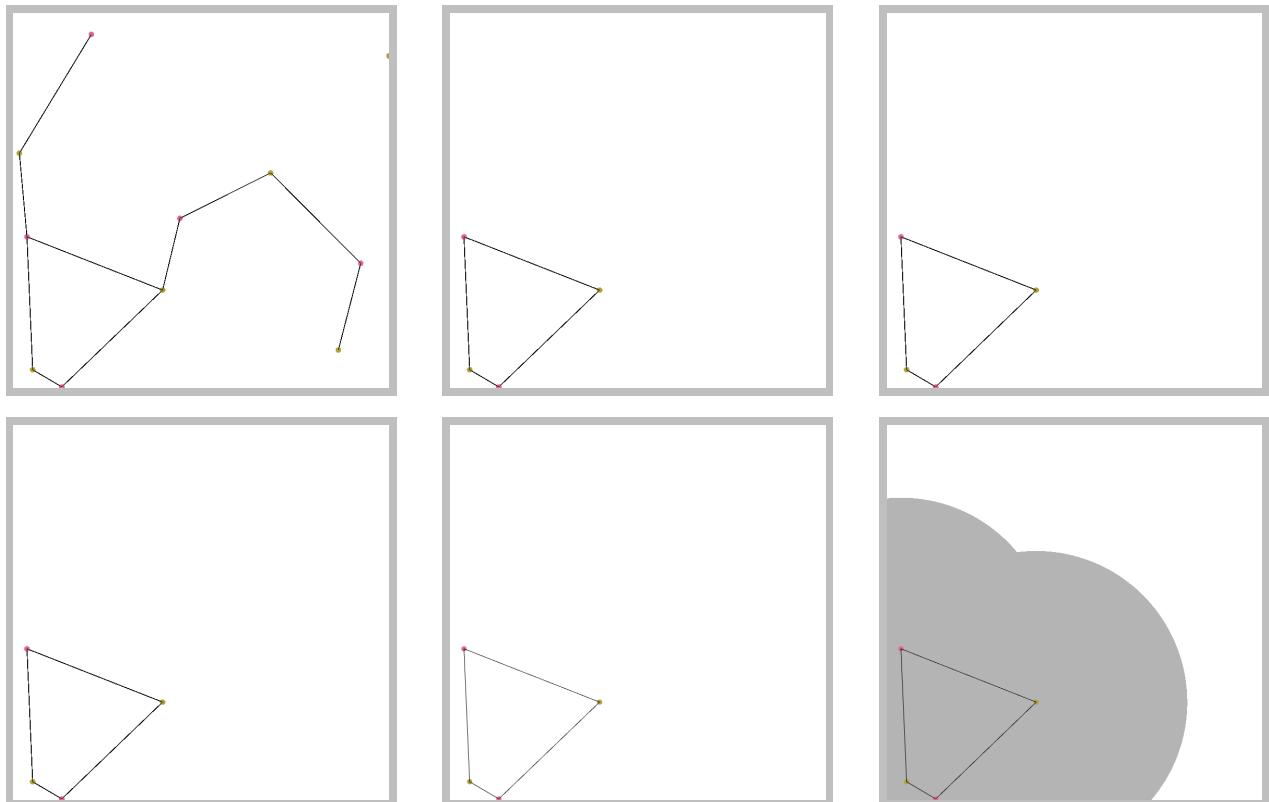


Figure 4: Backbone determination walkthrough for two largest color sets. top left: bipartite subgraph, top center: tails removed, top right: major component, bottom left: bridges and minor blocks removed, bottom center: final backbone, bottom right: coverage area

2.4.3 Graph Coloring

Smallest-last vertex ordering can be verified by looking at the distribution of the degrees of nodes when deleted. Since this algorithm repeatedly removes the node with the fewest connections, and because the removal of that node will cause the fewest number of nodes to move to the next lowest bucket, we would expect the bulk of the nodes to have a large degree when they are deleted. This would be indicated by a negative skew in the distribution of degrees when deleted. Additionally, since the nodes are only removed when they satisfy the criteria of being the node with the minimum degree, we should see the standard deviation of the distribution of nodes to be much smaller than in the original distribution of degrees. Both of these features can be found in Figures 9, 10, and 11 which plot the original distribution of degrees alongside the distribution of degrees when deleted. We see that the distribution of degrees when deleted follows a normal distribution with a negative skew and a relatively small standard deviation compared to the original distribution of degrees.

The color sets can be verified by looking at the distribution of colors used to color the graph. The number of items in each color should follow a trend where the first colors used have the most members, and the last colors have the fewest items because they are used to accommodate nodes where the earlier colors are all used by a node's neighbors. This trend is shown in Figures 12, 13, and 14.

To further verify the accuracy of the smallest-last coloring implementation additional code was used to verify that the coloring result was correct while running benchmarks. All of the nodes in the smallest-last vertex ordering are traversed, and for each node, the edges are visited to see if any adjacent nodes have the same color as the node being checked. If any of these neighbors have the same color, the coloring is not correct and our independent sets cannot be used for backbone determination. All of the benchmarks ran and returned valid colorings.

2.4.4 Backbone Determination

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3 Appendix A - Figures

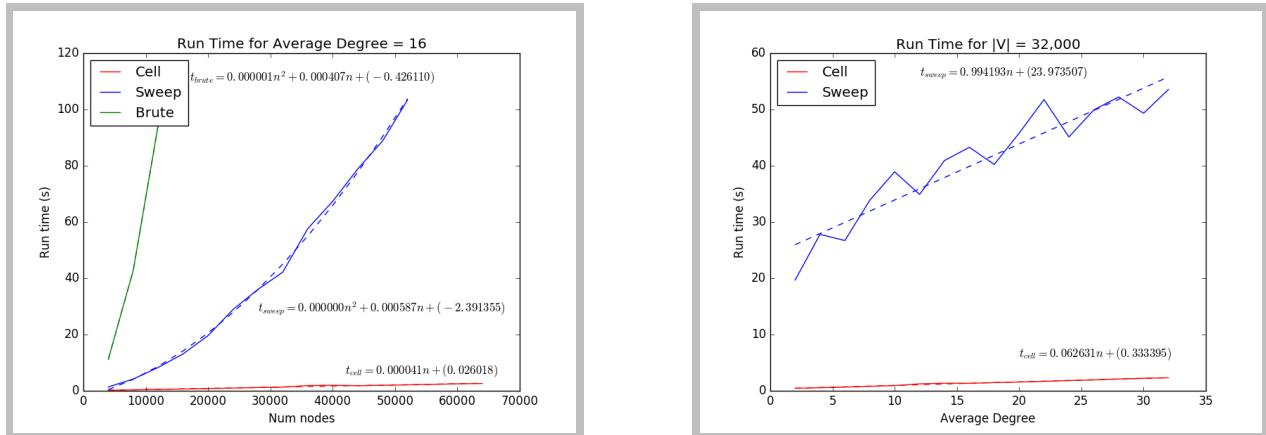


Figure 5: Runtime for edge detection methods. left: constant average degree of 16, right: variable average degree

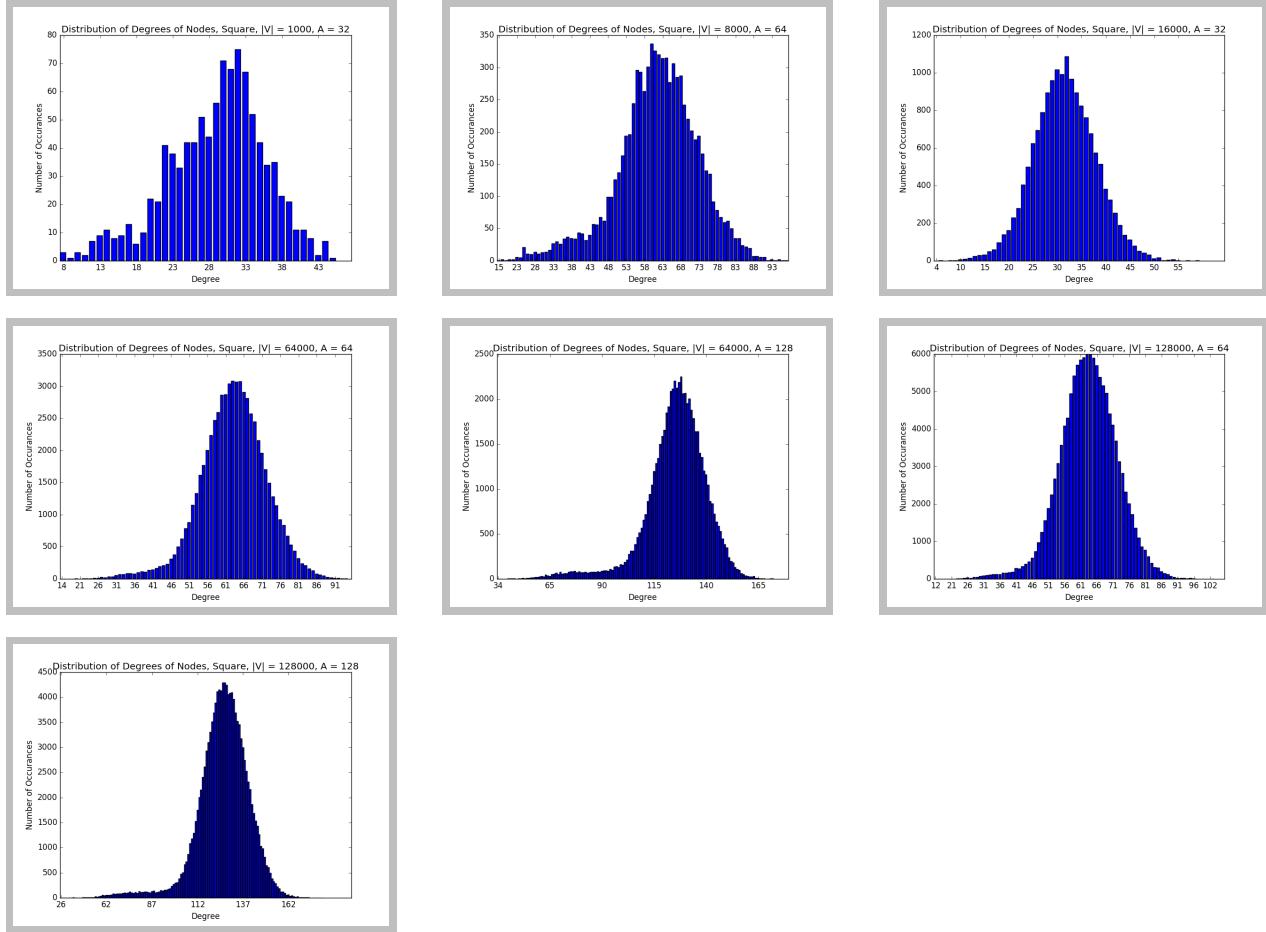


Figure 6: Square benchmarks distribution of degree graphs

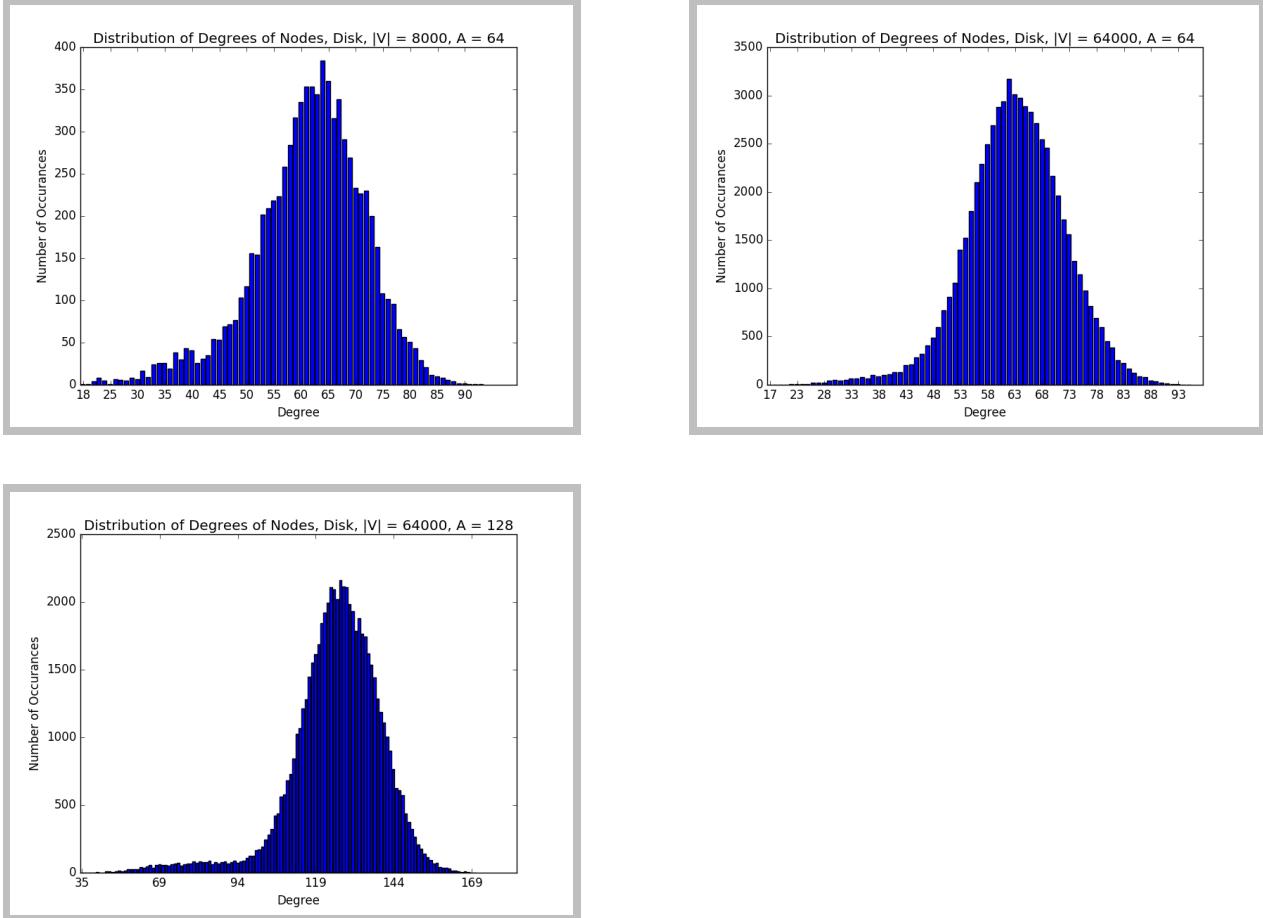


Figure 7: Disk benchmarks distribution of degree graphs

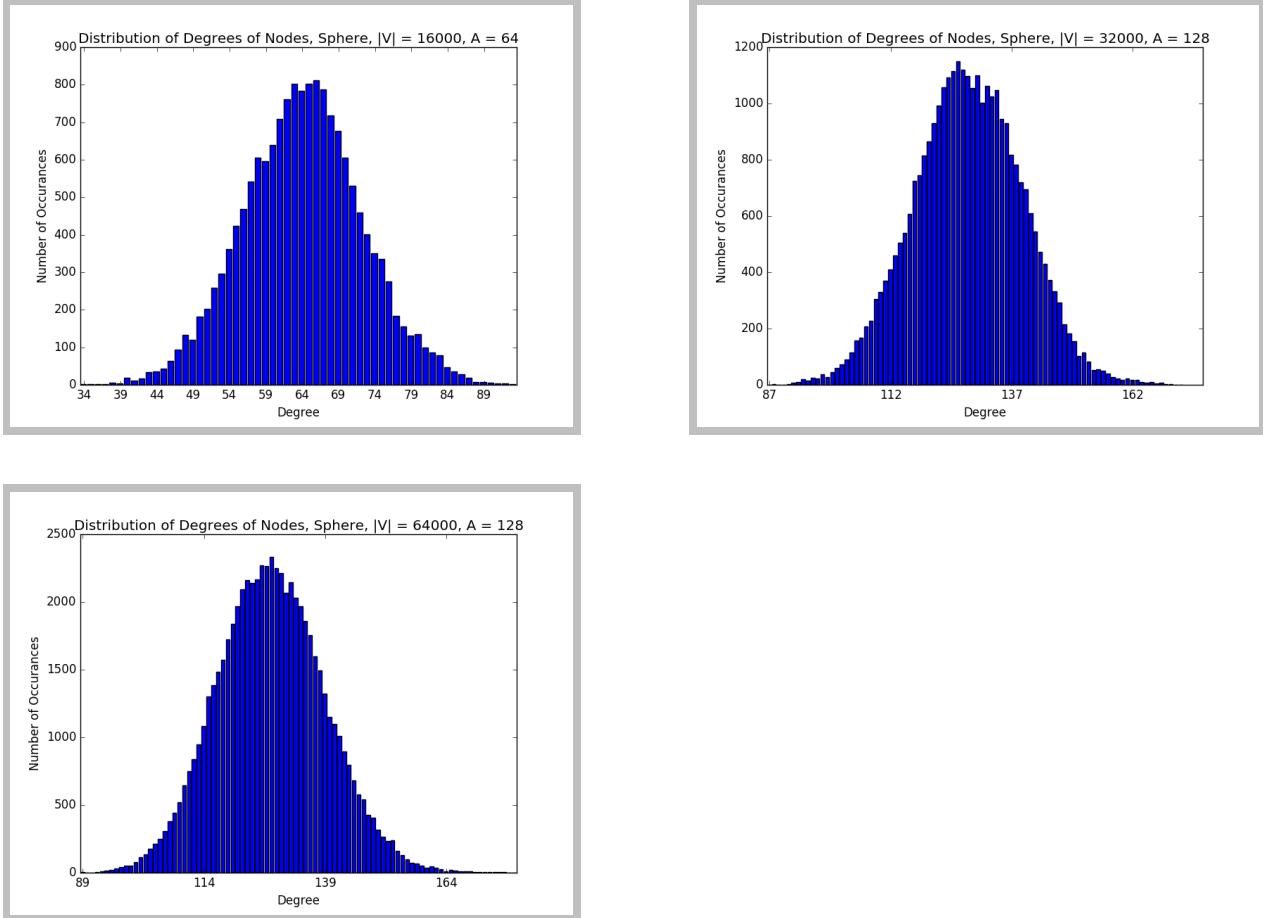


Figure 8: Sphere benchmarks distribution of degree graphs

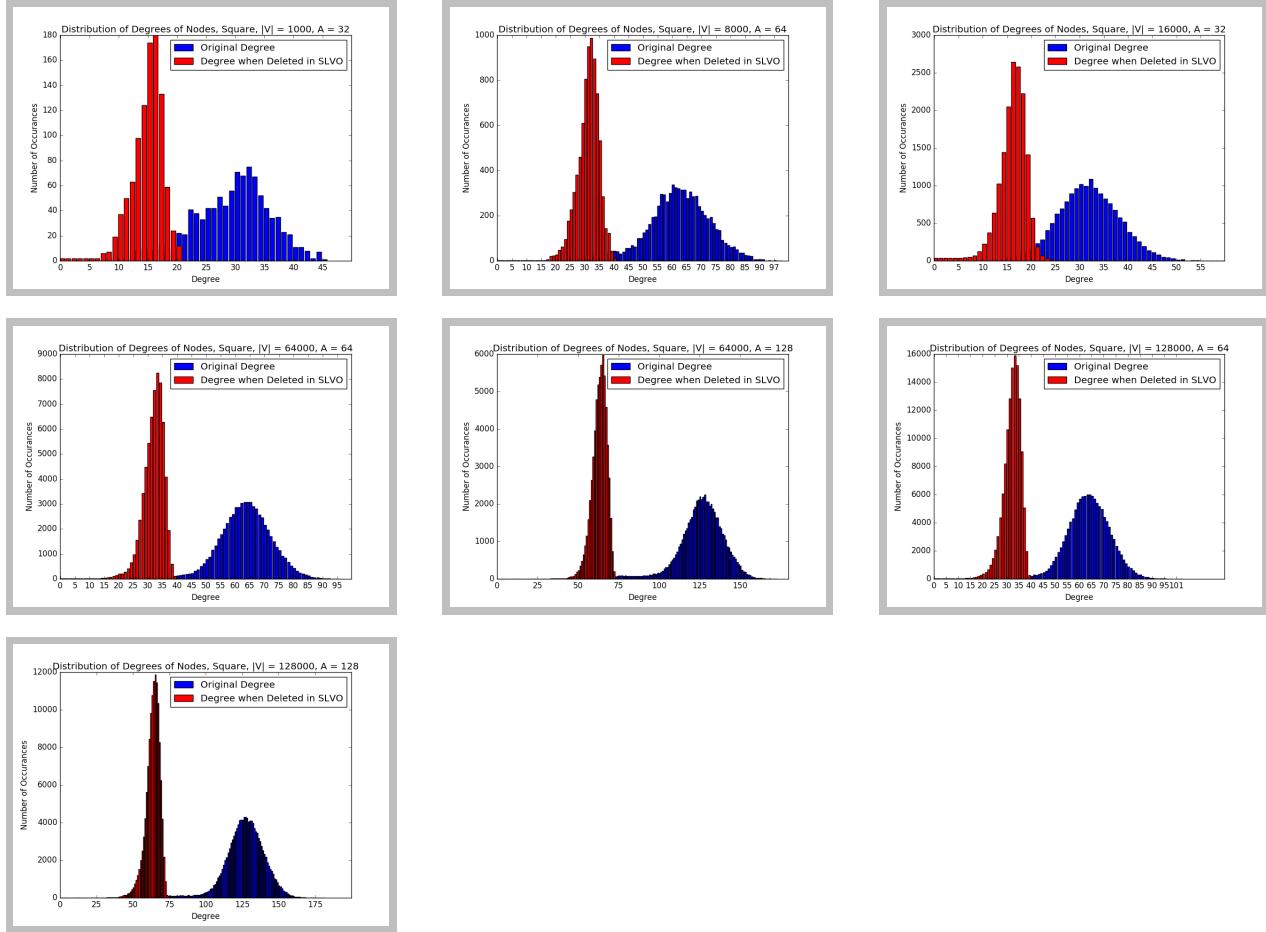


Figure 9: Square benchmarks distribution of degree when deleted graphs

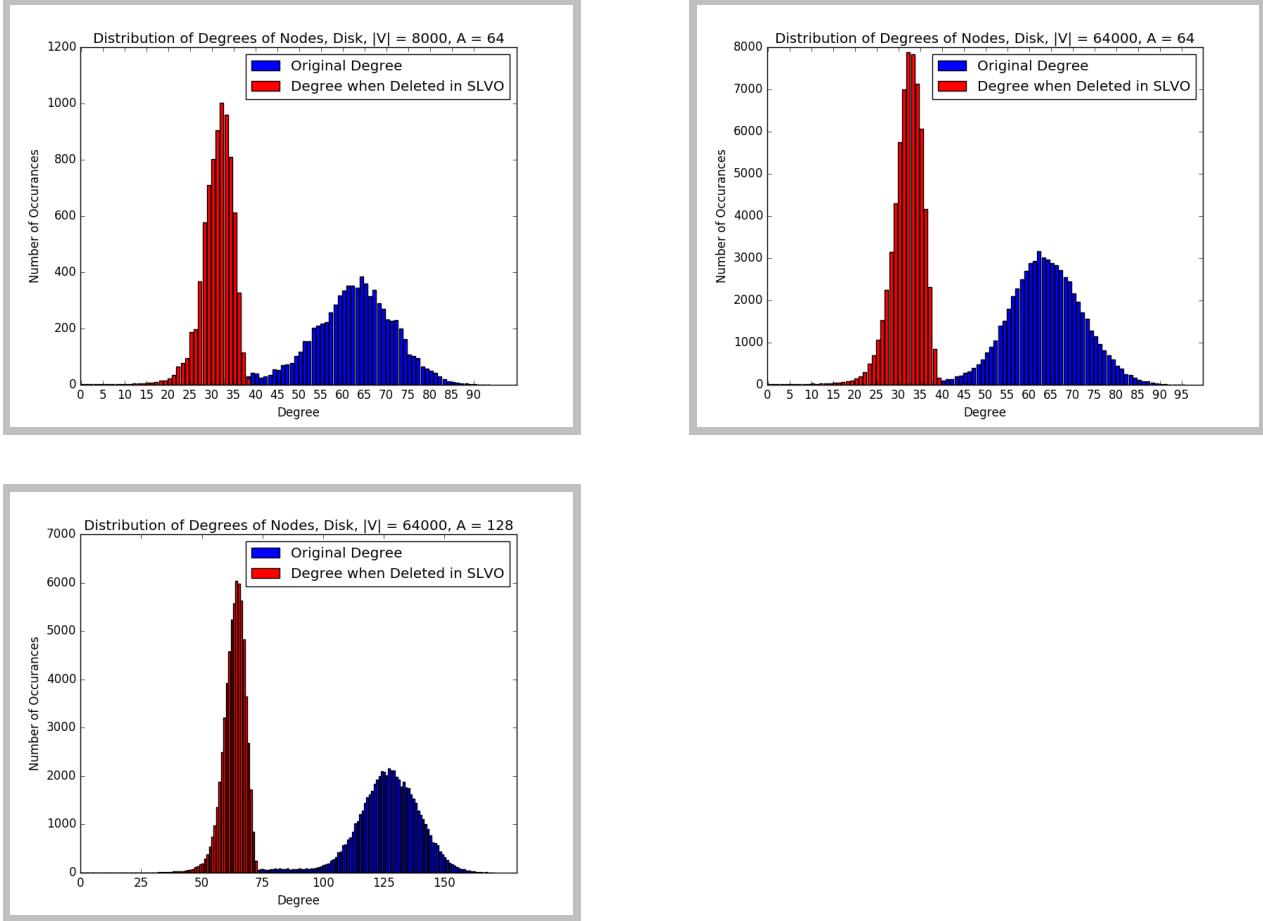


Figure 10: Disk benchmarks distribution of degree when deleted graphs

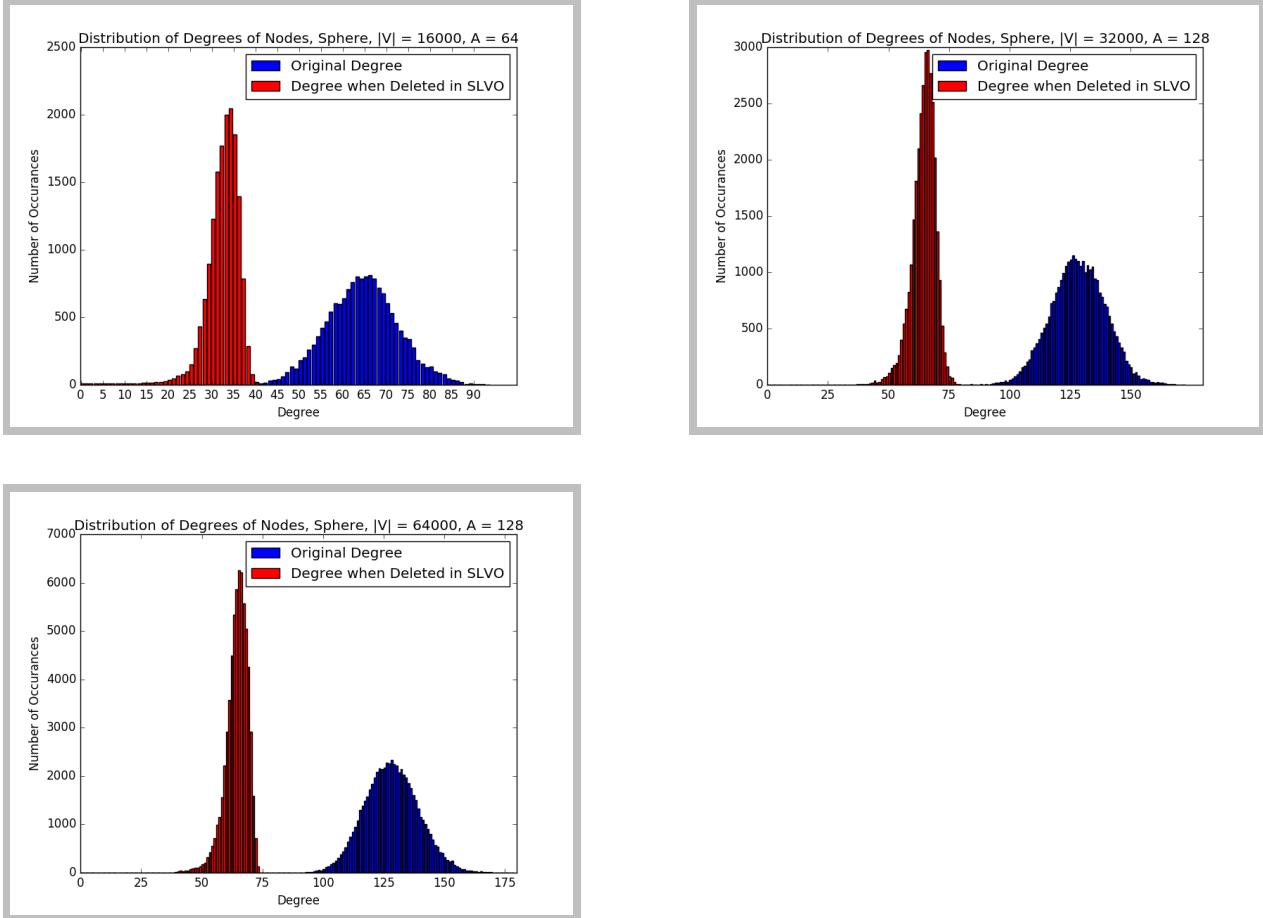


Figure 11: Sphere benchmarks distribution of degree when deleted graphs

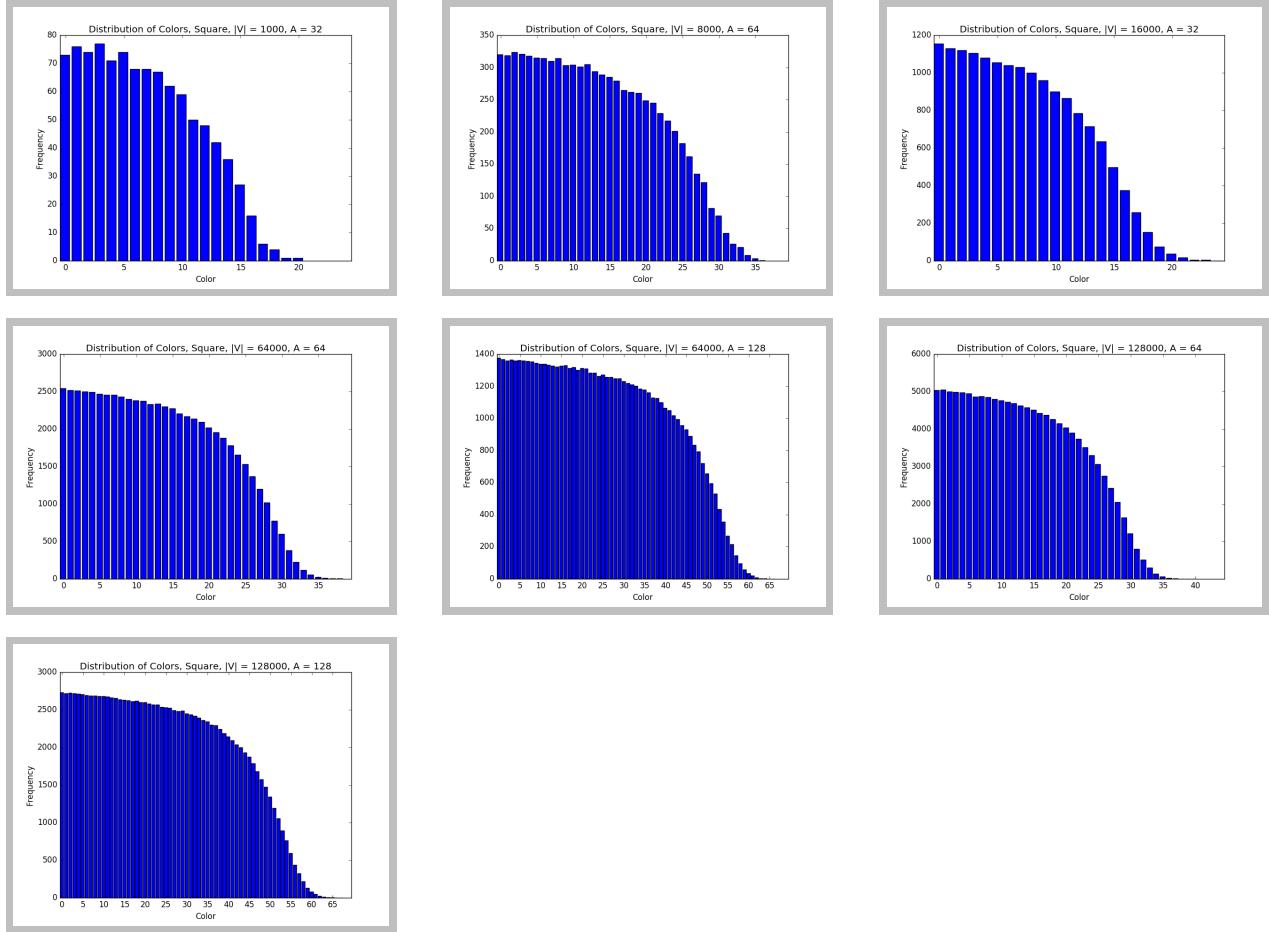


Figure 12: Square benchmarks distribution of colors graphs

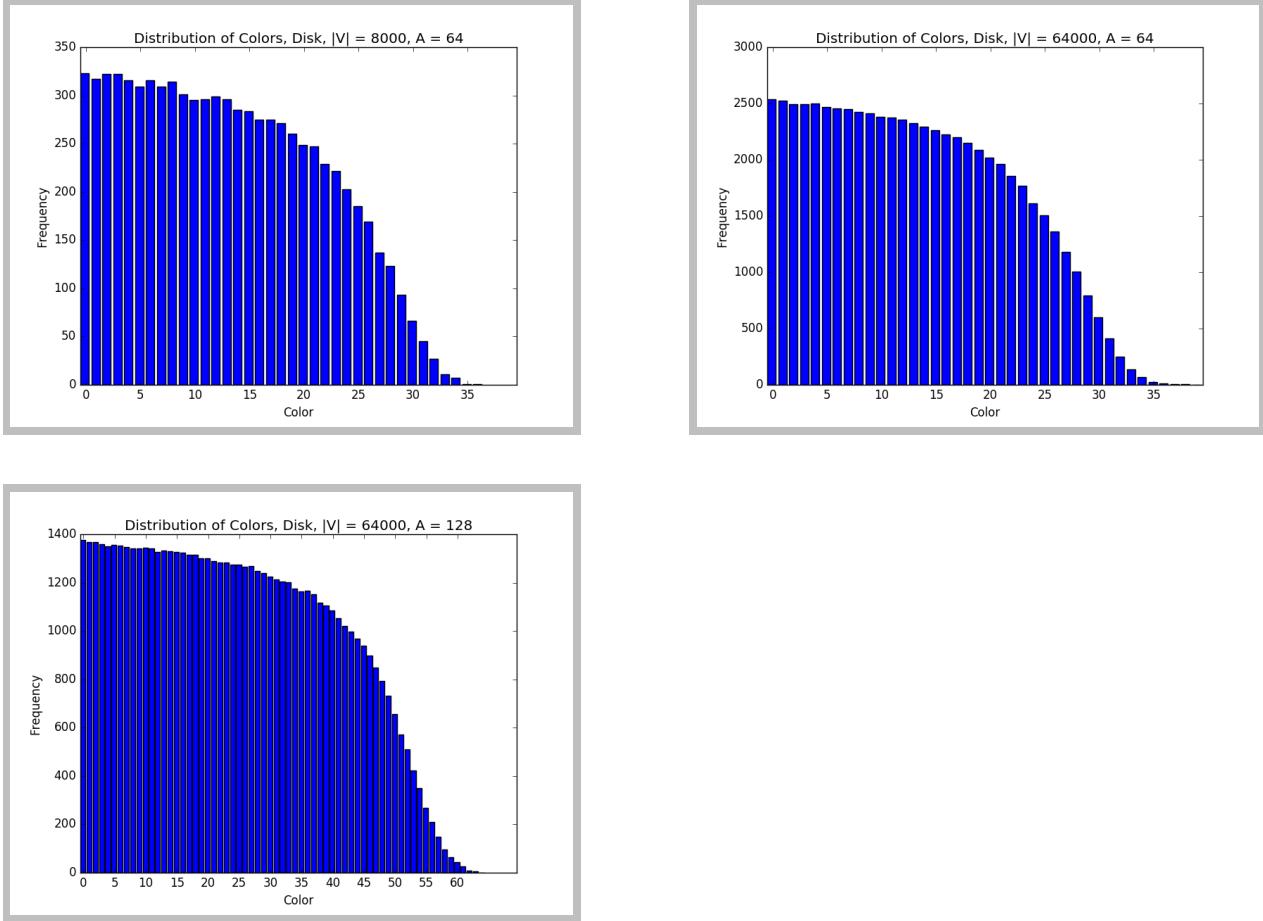


Figure 13: Disk benchmarks distribution of colors graphs

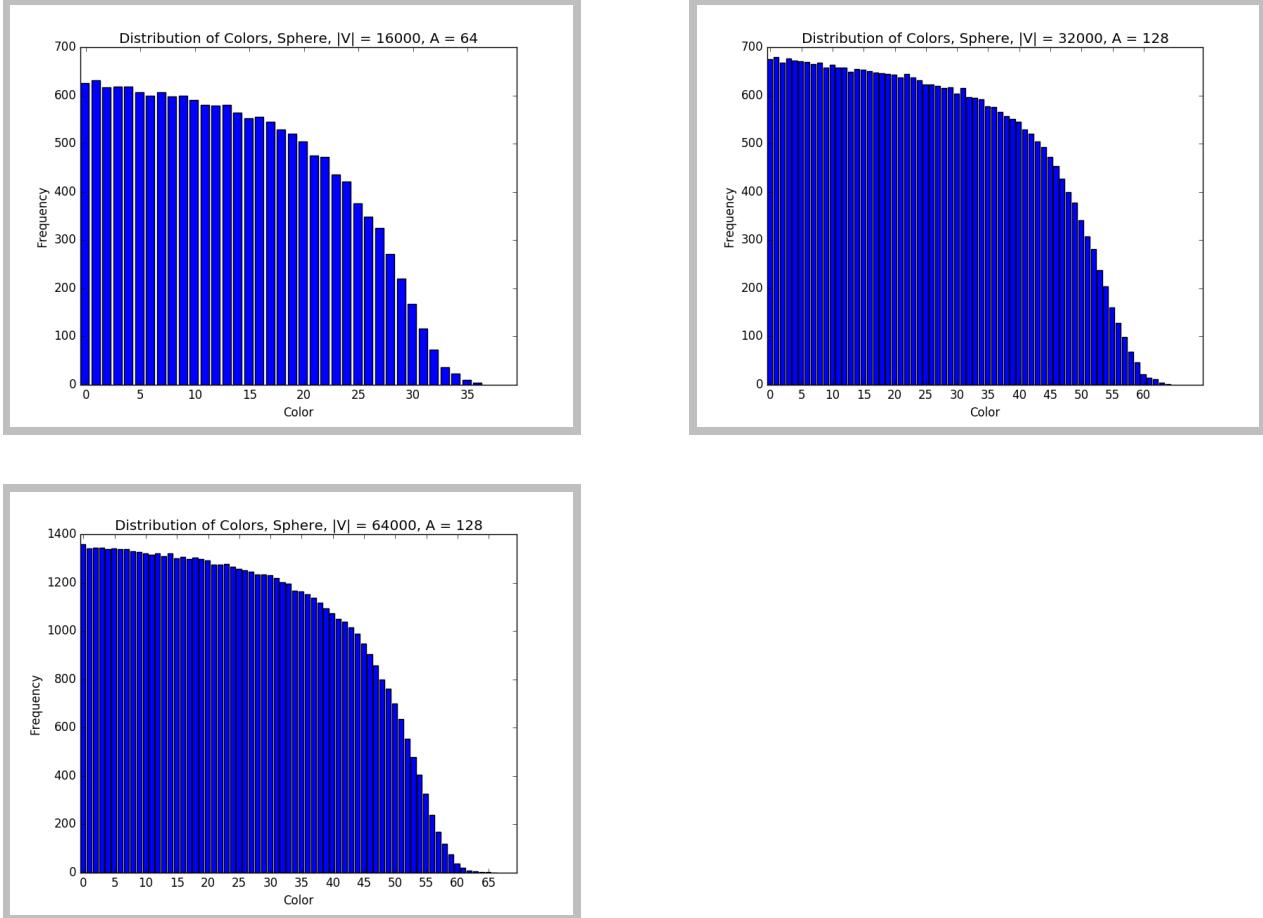


Figure 14: Sphere benchmarks distribution of colors graphs

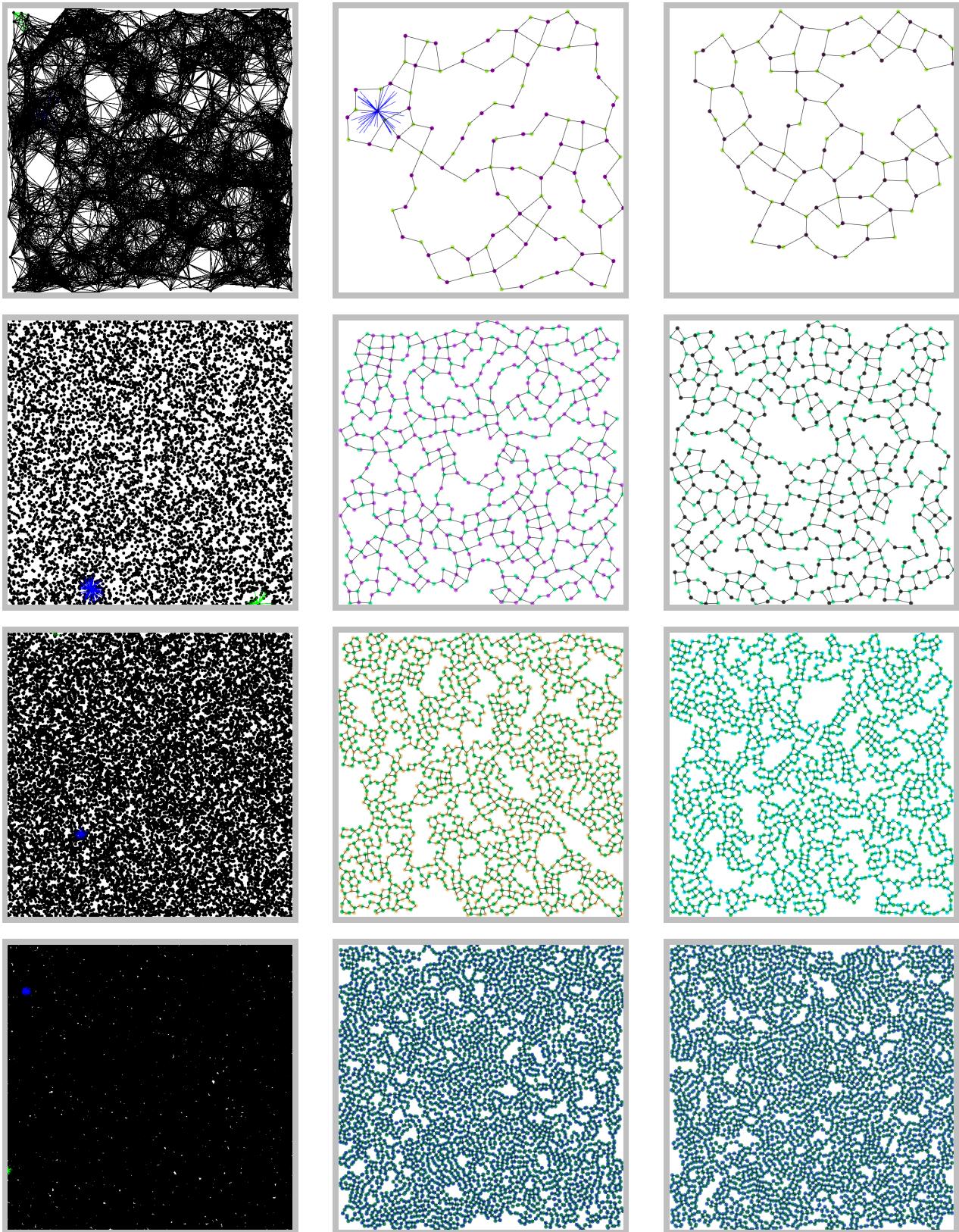


Figure 15: Square benchmark graphs and backbones

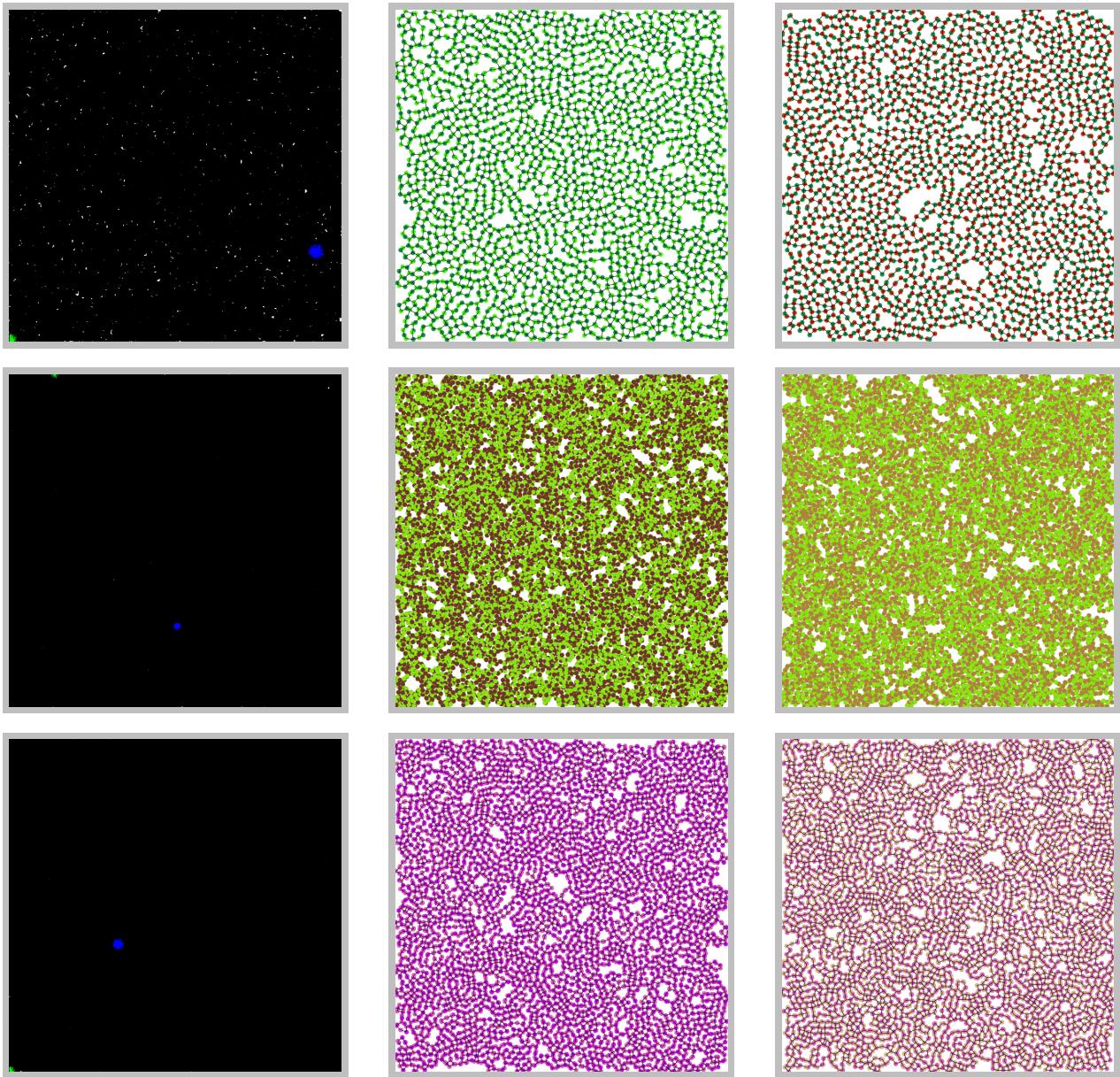


Figure 16: Square benchmark graphs and backbones

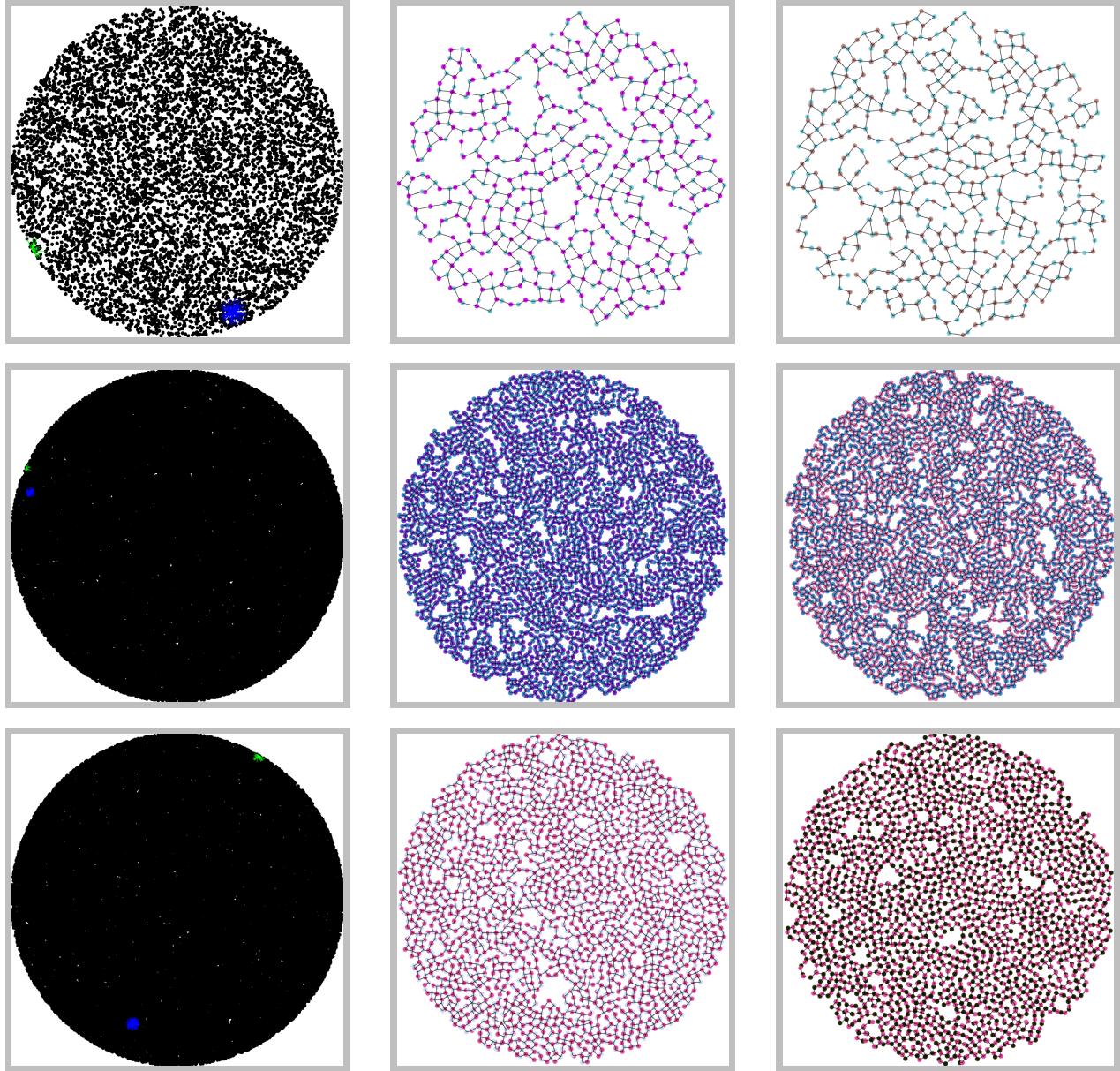


Figure 17: Disk benchmark graphs and backbones

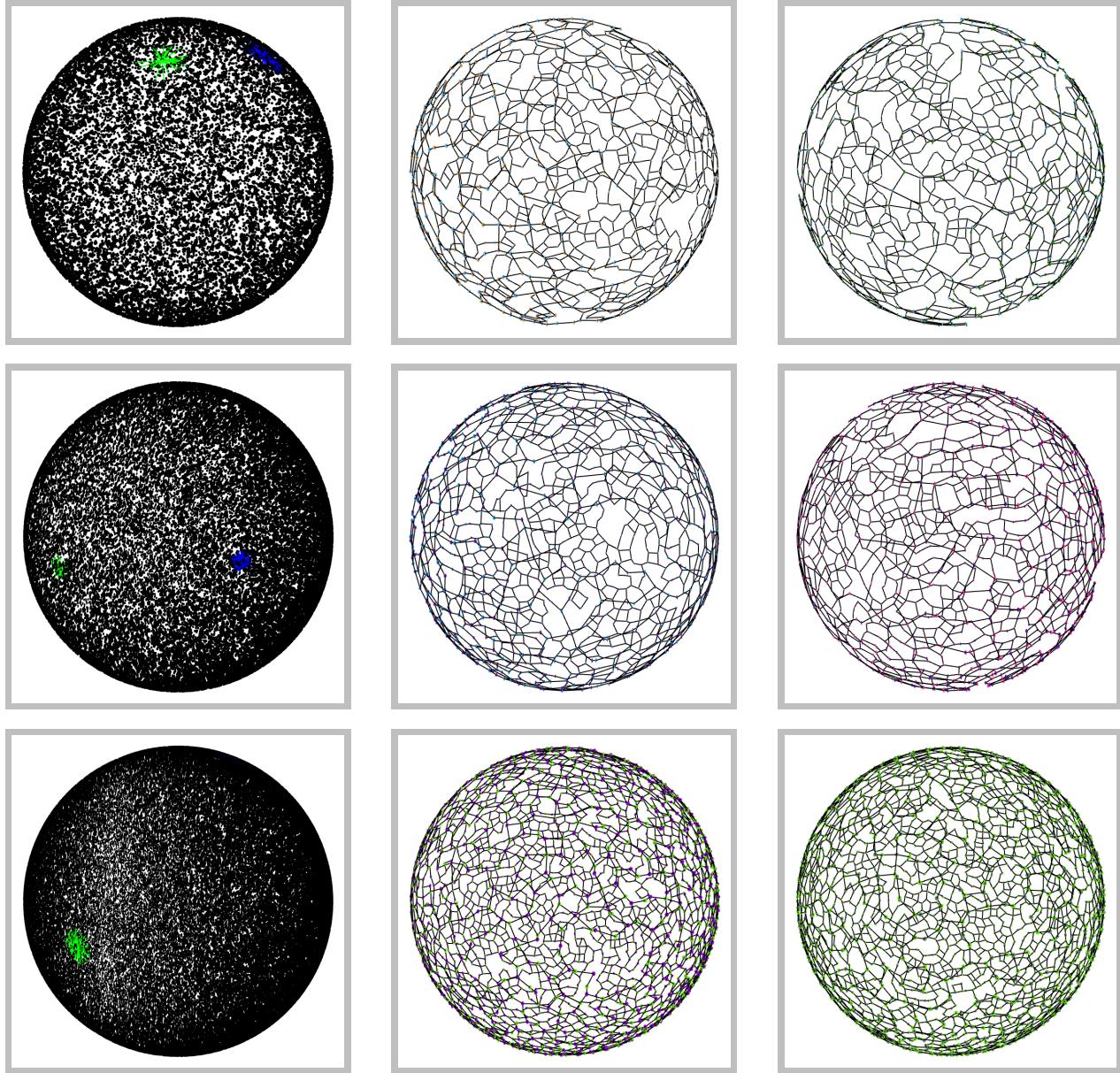


Figure 18: Sphere benchmark graphs and backbones

4 Appendix B - Code Listings

Listing 1: Processing driver

```

1 import random
2 import sys
3 import time
4 import math
5 from collections import Counter
6 from objects.topology import Square, Disk, Sphere
7
8 CANVAS_HEIGHT = 720
9 CANVAS_WIDTH = 720
10
11 NUM_NODES = 20
12 AVG_DEG = 10
13
14 MAX_NODES_TO_DRAW_EDGES = 8000
15
16 RUN_BENCHMARK = False
17
18 def setup():
19     size(CANVAS_WIDTH, CANVAS_HEIGHT, P3D)
20     background(0)
21
22 def draw():
23     global curr_vis
24     global draw_domination
25
26     if curr_vis == 0:
27         topology.drawGraph(MAX_NODES_TO_DRAW_EDGES)
28     elif curr_vis == 1:
29         topology.drawSlvo()
30     elif curr_vis == 2:
31         topology.drawColoring()
32     elif curr_vis == 3:
33         topology.drawPairs(0)
34     elif curr_vis == 4:
35         topology.drawPairs(1)
36     elif curr_vis == 5:
37         topology.drawPairs(2)
38     elif curr_vis == 6:
39         topology.drawPairs(3)
40     elif curr_vis == 7:
41         topology.drawBackbones(draw_domination)
42
43 def keyPressed():
44     global curr_vis
45     global step_size
46     global vis_names
47
48     if key == ' ':
49         toggleLooping()
50     elif key == 'c':
51         if curr_vis == 7:
52             toggleDrawDomination()
53     elif key == 'i':
54         topology.switchFgBg()
55     elif key == 'l':
56         incrementVis()
57         topology.mightResetCurrNode()
58         print vis_names[curr_vis]
59     elif key == 'h':
60         decrementVis()
61         topology.mightResetCurrNode()
62         print vis_names[curr_vis]
63     elif key == 'k':
64         if curr_vis > 2 and curr_vis < 7:

```

```

65         topology.incrementCurrPair()
66     elif curr_vis == 7:
67         topology.incrementCurrBackbone()
68     else:
69         topology.incrementCurrNode(step_size)
70     elif key == 'j':
71         if curr_vis > 2 and curr_vis < 7:
72             topology.decrementCurrPair()
73     elif curr_vis == 7:
74         topology.decrementCurrBackbone()
75     else:
76         topology.decrementCurrNode(step_size)
77     elif key == 'y':
78         saveFrame("../report/images/{}-#####.png".format(vis_names[curr_vis]))
79     elif key >= '0' and key <= '9':
80         step_size = 2**int(key)
81         print "New step size:", step_size
82     elif key == ']':
83         step_size = 2*step_size
84         print "New step size:", step_size
85     elif key == '[':
86         step_size = step_size/2
87         print "New step size:", step_size
88     elif key == 'm':
89         print "\n—— Help Menu ——"
90         print "Use 'hjkl' to move between visualizations"
91         print "Press 'i' to invert the color scheme"
92         print "Press 'y' to take a screenshot of the current frame"
93         print "Press 'c' to show the coverage of the backbone"
94         print "Entering a number n between 0 and 9 will set the step size to 2^n nodes"
95         print "]' will double the step size, '[' will half it"
96         print "Press space to pause rotation of the sphere"
97
98 def toggleLooping():
99     global is_looping
100    if is_looping:
101        noLoop()
102        is_looping = False
103    else:
104        loop()
105        is_looping = True
106
107 def toggleDrawDomination():
108     global draw_domination
109     if draw_domination:
110         draw_domination = False
111     else:
112         draw_domination = True
113
114 def incrementVis():
115     global curr_vis
116     global topology
117     if curr_vis < 7:
118         curr_vis += 1
119     background(topology.color_bg)
120
121 def decrementVis():
122     global curr_vis
123     global topology
124     if curr_vis > 0:
125         curr_vis -= 1
126     background(topology.color_bg)
127
128 def main():
129     sys.setrecursionlimit(8000)
130
131     global is_looping

```

```

132 global draw_domination
133 global curr_vis
134 global step_size
135 global vis_names
136 is_looping = True
137 draw_domination = False
138 curr_vis = 0
139 step_size = 1
140 vis_names = ["rgg", "slvo", "color", "bipartite", "no-tails",
141           "major-comp", "no-bridge", "backbone"]
142
143 global topology
144 # topology = Square()
145 # topology = Disk()
146 topology = Sphere()
147
148 topology.num_nodes = NUM_NODES
149 topology.avg_deg = AVG_DEG
150 topology.canvas_height = CANVAS_HEIGHT
151 topology.canvas_width = CANVAS_WIDTH
152
153 if RUN_BENCHMARK:
154     n_benchmark = 0
155     topology.prepBenchmark(n_benchmark)
156
157 run_time = time.clock()
158
159 topology.generateNodes()
160 topology.findEdges(method="cell")
161 topology.colorGraph()
162 topology.generateBackbones()
163
164 run_time = time.clock() - run_time
165
166 print "Average degree: {}".format(topology.findAvgDegree())
167 print "Min degree: {}".format(topology.getMinDegree())
168 print "Max degree: {}".format(topology.getMaxDegree())
169 print "Num edges: {}".format(topology.findNumEdges())
170 print "Node r: {:.3f}".format(topology.node_r)
171 print "Terminal clique size: {}".format(topology.term_clique_size)
172 print "Number of colors: {}".format(len(set(topology.node_colors)))
173 print "Max degree when deleted: {}".format(max(topology.deg_when_del.values()))
174
175 color_cnt = Counter(topology.node_colors)
176 print "Max color set size: {} \t color: {}".format(
177     color_cnt.most_common(1)[0][1], color_cnt.most_common(1)[0][0])
178 print "Backbone 1 order: {} \t size: {} \t coverage: {}".format(
179     topology.backbones_meta[0][0], topology.backbones_meta[0][1],
180     topology.backbones_meta[0][2])
181 print "Backbone 2 order: {} \t size: {} \t coverage: {}".format(
182     topology.backbones_meta[1][0], topology.backbones_meta[1][1],
183     topology.backbones_meta[1][2])
184 b1_colors = list(set(
185     [topology.node_colors[i] for i in list(topology.backbones[0])]))
186 print "Backbone 1 colors: {} {}".format(b1_colors[0], b1_colors[1])
187 b2_colors = list(set(
188     [topology.node_colors[i] for i in list(topology.backbones[1])]))
189 print "Backbone 2 colors: {} {}".format(b2_colors[0], b2_colors[1])
190
191 if isinstance(topology, Sphere):
192     print "Backbone 1 faces: {}".format(topology.num_faces[0])
193     print "Backbone 2 faces: {}".format(topology.num_faces[1])
194
195 # print "Run time: {:.3f} s".format(run_time)
196
197 print "\nPress 'm' for the menu"
198 main()

```

Listing 2: Topology class and subclasses

```

1 import random
2 import math
3 import time
4 import sys
5 from collections import deque
6
7 # increase recursion limit for DFS
8 sys.setrecursionlimit(8000)
9
10 # benchmarks (num_nodes, avg_deg)
11 SQUAREBENCHMARKS = [(1000,32), (8000,64), (16000,32), (64000,64), (64000,128),
12 (128000,64), (128000, 128)]
13 DISKBENCHMARKS = [(8000,64), (64000,64), (64000,128)]
14 SPHEREBENCHMARKS = [(16000,64), (32000,128), (64000,128)]
15 """
16 Topology - super class for the shape of the random geometric graph
17 """
18
19 class Topology(object):
20
21     num_nodes = 100
22     avg_deg = 0
23     canvas_height = 720
24     canvas_width = 720
25
26     def __init__(self):
27         self.nodes = []
28         self.edges = {}
29         self.node_r = 0.0
30         self.minDeg = ()
31         self.maxDeg = ()
32         self.slvo = []
33         self.deg_when_del = {}
34         self.node_colors = []
35         self.num_color_sets = 4
36         self.pairs = []
37         self.no_tails = []
38         self.major_comps = []
39         self.clean_pairs = []
40         self.backbones = []
41         self.backbones_meta = []
42         self.curr_node = 0
43         self.curr_pair = 0
44         self.curr_backbone = 0
45
46         # used to control _drawNodes functionality
47         self.n_limit = 8000
48         self.rot = (0,0,0)
49         self.color_bg = 0
50         self.color_fg = 255
51         self.color_fill = 220
52
53     # public function for generating nodes of the graph, must be subclassed
54     def generateNodes(self):
55         print "Method for generating nodes not subclassed"
56
57     # public function for finding edges
58     def findEdges(self, method="brute"):
59         self._getRadiusForAverageDegree()
60         self._addNodesAsEdgeKeys()
61
62         if method == "brute":
63             self._bruteForceFindEdges()
64         elif method == "sweep":
65             self._sweepFindEdges()
66         elif method == "cell":
67             self._cellFindEdges()

```

```

68         else:
69             print "Find edges method not defined: {}".format(method)
70
71     self._findMinAndMaxDegree()
72
73     # brute force edge detection
74     def _bruteForceFindEdges(self):
75         for i, n in enumerate(self.nodes):
76             for j, m in enumerate(self.nodes):
77                 if i != j and self._distance(n, m) <= self.node_r:
78                     self.edges[n].append(j)
79
80     # sweep edge detection
81     def _sweepFindEdges(self):
82         self.nodes.sort(key=lambda x: x[0])
83
84         for i, n in enumerate(self.nodes):
85             search_space = []
86             for j in range(1, self.num_nodes-i):
87                 if abs(n[0] - self.nodes[i+j][0]) <= self.node_r:
88                     search_space.append(i+j)
89                 else:
90                     break
91             for j in search_space:
92                 if self._distance(n, self.nodes[j]) <= self.node_r:
93                     self.edges[n].append(j)
94                     self.edges[self.nodes[j]].append(i)
95
96     # cell edge detection
97     def _cellFindEdges(self):
98         num_cells = int(1/self.node_r) + 1
99         cells = []
100        for i in range(num_cells):
101            cells.append([[[] for j in range(num_cells)]])
102
103        for i, n in enumerate(self.nodes):
104            cells[int(n[0]/self.node_r)][int(n[1]/self.node_r)].append(i)
105
106        for i in range(num_cells):
107            for j in range(num_cells):
108                for n_i in cells[i][j]:
109                    for c in self._findAdjCells(i, j, num_cells):
110                        for m_i in cells[c[0]][c[1]]:
111                            if self._distance(self.nodes[n_i], self.nodes[m_i]) <=
112                                self.node_r:
113                                self.edges[self.nodes[n_i]].append(m_i)
114                                self.edges[self.nodes[m_i]].append(n_i)
115                                for m_i in cells[i][j]:
116                                    if self._distance(self.nodes[n_i], self.nodes[m_i]) <=
117                                        self.node_r and n_i != m_i:
118                                        self.edges[self.nodes[n_i]].append(m_i)
119
120    # cell edge detection helper function
121    def _findAdjCells(self, i, j, n):
122        adj_cells = [(1,-1), (0,1), (1,1), (1,0)]
123        return (((i+x[0])%n,(j+x[1])%n) for x in adj_cells)
124
125    # function for finding the radius needed for the desired average degree
126    # must be subclassed
127    def _getRadiusForAverageDegree(self):
128        print "Method for finding necessary radius for average degree not
129        subclassed"
130
131    # helper function for findEdges, initializes edges dict
132    def _addNodesAsEdgeKeys(self):
133        self.edges = {n:[] for n in self.nodes}
134
135    # calculates the distance between two nodes (2D)

```

```

133     def _distance(self, n, m):
134         return math.sqrt((n[0] - m[0])**2+(n[1] - m[1])**2)
135
136     # public function for finding the number of edges
137     def findNumEdges(self):
138         sigma_edges = 0
139         for k in self.edges.keys():
140             sigma_edges += len(self.edges[k])
141
142         return sigma_edges/2
143
144     # public function for finding the average degree of nodes
145     def findAvgDegree(self):
146         return 2*self.findNumEdges()/self.num_nodes
147
148     # helper funciton for finding nodes with min and max degree
149     def _findMinAndMaxDegree(self):
150         self.minDeg = self.edges.keys()[0]
151         self.maxDeg = self.edges.keys()[0]
152
153         for k in self.edges.keys():
154             if len(self.edges[k]) < len(self.edges[self.minDeg]):
155                 self.minDeg = k
156             if len(self.edges[k]) > len(self.edges[self.maxDeg]):
157                 self.maxDeg = k
158
159     # public function for getting the minimum degree
160     def getMinDegree(self):
161         return len(self.edges[self.minDeg])
162
163     # public functino for getting the maximum degree
164     def getMaxDegree(self):
165         return len(self.edges[self.maxDeg])
166
167     # public function for setting up the benchmark to run, must be subclassed
168     def prepBenchmark(self, n):
169         print "Method for preparing benchmark not subclassed"
170
171     # public function for drawing the graph
172     def drawGraph(self, n_limit):
173         self.n_limit = n_limit
174         self._drawNodes(self.nodes)
175         self._drawEdges(self.nodes)
176
177     # responsible for drawing the nodes in the canvas
178     def _drawNodes(self, node_list):
179         strokeWeight(2)
180         stroke(self.color_fg)
181         fill(self.color_fg)
182
183         for n in node_list:
184             ellipse(n[0]*self.canvas_width, n[1]*self.canvas_height, 5, 5)
185
186     # responsible for drawing the edges in the canavas
187     def _drawEdges(self, node_list):
188         strokeWeight(1)
189         s = set(node_list)
190
191         for n in node_list:
192             stroke(self.color_fg)
193             fill(self.color_fg)
194
195             if len(node_list) < self.n_limit:
196                 for m_i in self.edges[n]:
197                     if self.nodes[m_i] in s:
198                         line(n[0]*self.canvas_width, n[1]*self.canvas_height, self.nodes[m_i][0]*self.canvas_width, self.nodes[m_i][1]*self.canvas_height)
199                     if n == self.minDeg:

```

```

200         stroke(0,255,0)
201         for n_i in self.edges[self.minDeg]:
202             line(self.minDeg[0]*self.canvas_width, self.minDeg[1]*self.
203 canvas_height, self.nodes[n_i][0]*self.canvas_width, self.nodes[n_i][1]*self.
204 canvas_height)
205         elif n == self.maxDeg:
206             stroke(0,0,255)
207             for n_i in self.edges[self.maxDeg]:
208                 line(self.maxDeg[0]*self.canvas_width, self.maxDeg[1]*self.
209 canvas_height, self.nodes[n_i][0]*self.canvas_width, self.nodes[n_i][1]*self.
210 canvas_height)
211
212     # uses smallest last vertex ordering to color the graph
213     def colorGraph(self):
214         self.slvo, self.deg_when_del = self._smallestLastVertexOrdering()
215         self.node_colors = self._assignNodeColors(self.slvo)
216         self.color_map = self._mapColorsToRGB(self.node_colors)
217
218     # constructs a degree structure and determines the smallest last vertex
219     # ordering
220     def _smallestLastVertexOrdering(self):
221         deg_sets = {l:set() for l in range(len(self.edges[self.maxDeg])+1)}
222         deg_when_del = {n:len(self.edges[n]) for n in self.nodes}
223
224         for i, n in enumerate(self.nodes):
225             deg_sets[deg_when_del[n]].add(i)
226
227         smallest_last_ordering = []
228
229         clique_found = False
230         j = len(self.nodes)
231         while j > 0:
232             # get the current smallest bucket
233             curr_bucket = 0
234             while len(deg_sets[curr_bucket]) == 0:
235                 curr_bucket += 1
236
237             # if all the remaining nodes are connected we have the terminal clique
238             if not clique_found and len(deg_sets[curr_bucket]) == j:
239                 clique_found = True
240                 self.term_clique_size = curr_bucket
241
242             # get node with smallest degree
243             v_i = deg_sets[curr_bucket].pop()
244             smallest_last_ordering.append(v_i)
245
246             # decrement position of nodes that shared an edge with v
247             for n_i in (n_i for n_i in self.edges[self.nodes[v_i]] if n_i in
248                         deg_sets[deg_when_del[self.nodes[n_i]]]):
249                 deg_sets[deg_when_del[self.nodes[n_i]]].remove(n_i)
250                 deg_when_del[self.nodes[n_i]] -= 1
251                 deg_sets[deg_when_del[self.nodes[n_i]]].add(n_i)
252
253             j -= 1
254
255             # reverse list since it was built shortest-first
256             return smallest_last_ordering[::-1], deg_when_del
257
258     # assigns the colors to nodes given in a smallest-last vertex ordering as a
259     # parallel array
260     def _assignNodeColors(self, slvo):
261         colors = [-1 for _ in range(len(slvo))]
262         for i in slvo:
263             adj_colors = set([colors[j] for j in self.edges[self.nodes[i]]])
264             color = 0
265             while color in adj_colors:
266                 color += 1
267             colors[i] = color

```

```

261     return colors
262
263
264     # generates random color codes for each color set and returns them in a
265     # dictionary
266     def _mapColorsToRGB(self, color_list):
267         s = set(color_list)
268         color_map = {}
269         while len(s) > 0:
270             c = s.pop()
271             color_map[c] = (random.randint(0,255), random.randint(0,255), random.
272                             randint(0,255))
273
274     return color_map
275
276
277     # draw nodes as they are removed in smallest-last vertex ordering
278     def drawSlvo(self):
279         l = [self.nodes[i] for i in self.slvo[0:self.num_nodes - self.curr_node]]
280         self._drawNodes(l)
281         self._drawEdges(l)
282
283
284     # increments curr_node, used to limit the number of nodes drawn
285     def incrementCurrNode(self, s):
286         if self.curr_node + s <= self.num_nodes:
287             self.curr_node += s
288             background(self.color_bg)
289         elif self.curr_node != self.num_nodes:
290             self.curr_node = self.num_nodes
291             background(self.color_bg)
292
293
294     # decrements curr_node, used to limit the number of nodes drawn
295     def decrementCurrNode(self, s):
296         if self.curr_node - s >= 0:
297             self.curr_node -= s
298             background(self.color_bg)
299         elif self.curr_node == 0:
300             self.curr_node = 0
301             background(self.color_bg)
302
303
304     # used to reset curr node if all nodes have been drawn and the method changes
305     def mightResetCurrNode(self):
306         if self.curr_node == self.num_nodes:
307             curr_node = 0
308             background(self.color_bg)
309
310
311     # increments curr_backbone, used to draw different backbones
312     def incrementCurrPair(self):
313         if self.curr_pair < len(self.pairs) - 1:
314             self.curr_pair += 1
315             background(self.color_bg)
316
317
318     # decrements curr_backbone, used to draw different backbones
319     def decrementCurrPair(self):
320         if self.curr_pair > 0:
321             self.curr_pair -= 1
322             background(self.color_bg)
323
324
325     # increments curr_backbone, used to draw different backbones
326     def incrementCurrBackbone(self):
327         if self.curr_backbone < len(self.backbones) - 1:
328             self.curr_backbone += 1
329             background(self.color_bg)
330
331
332     # decrements curr_backbone, used to draw different backbones
333     def decrementCurrBackbone(self):
334         if self.curr_backbone > 0:
335             self.curr_backbone -= 1
336             background(self.color_bg)

```

```

327
328     # switch foreground and background colors
329     def switchFgBg(self):
330         self.color_fg, self.color_bg = self.color_bg, self.color_fg
331         background(self.color_bg)
332
333     # used to draw the graph with the nodes colored
334     def drawColoring(self):
335         l = [self.nodes[i] for i in self.slvo[0:self.curr_node]]
336         self._drawNodes(l)
337         self._applyColors(self.slvo[0:self.curr_node])
338         self._drawEdges(l)
339
340     # places colors on the nodes
341     def _applyColors(self, node_i_list):
342         strokeWeight(5)
343
344         num_colors = max(self.node_colors)
345
346         for n_i in node_i_list:
347             c = self.color_map[self.node_colors[n_i]]
348             stroke(c[0], c[1], c[2])
349             fill(c[0], c[1], c[2])
350             ellipse(self.nodes[n_i][0]*self.canvas_width, self.nodes[n_i][1]*self.
351 canvas_height, 5, 5)
352
353     # public function for pairing the independent sets and picking the largest
354     # backbones
355     def generateBackbones(self):
356         # pair four largest independent sets
357         self.pairs = self._pairIndependentSets(self.node_colors)
358
359         # delete minor components and tails
360         self.no_tails, self.major_comps, self.clean_pairs = self._cleanPairs(self.
361 pairs)
362
363         # pick two backbones of largest size
364         self.backbones, self.backbones_meta = self._getLargestBackbones(self.
365 clean_pairs)
366
367         # calculate domination
368         self.backbones_meta = self._getDominations(self.backbones, self.
369 backbones_meta)
370
371     # pairs the four largest independent color sets
372     def _pairIndependentSets(self, color_list):
373         # the first four color sets should be the largest (slvo)
374         indep_sets = [set() for _ in range(self.num_color_sets)]
375
376         for i, n in enumerate(self.nodes):
377             if self.node_colors[i] < self.num_color_sets:
378                 indep_sets[self.node_colors[i]].add(i)
379
380         # return combinations of sets (union)
381         return [s1 | s2 for i, s1 in enumerate(indep_sets) for s2 in indep_sets[i
382 +1:]]
383
384     # removes the minor components and tails from the bipartite subgraphs
385     def _cleanPairs(self, bipartites):
386         no_tails = []
387         major_comps = []
388         results = []
389         for b in bipartites:
390             # remove the tails and save the graph for visualization
391             b = self._removeTails(b)
392             no_tails.append(b)
393
394             # use BFS to get the major component

```

```

389         major_comp = self._ bfs(b)
390         major_comps.append(major_comp)
391
392     # use DFS to remove bridges
393     backbone = self._removeBridges(major_comp)
394     results.append(backbone)
395
396     return no_tails, major_comps, results
397
398 # remove tails from bipartite, very similar to smallest-last vertex ordering
399 def _removeTails(self, bipartite):
400     bipartite = bipartite.copy()
401     # build graph representation
402     points = list(bipartite)
403     deg_sets = {1: set() for l in range(len(self.edges[self.maxDeg]) + 1)}
404     deg_map = {n_i: len([e_i for e_i in self.edges[self.nodes[n_i]] if e_i in
405     bipartite]) for n_i in points}
406
407     for i in points:
408         deg_sets[deg_map[i]].add(i)
409
410     # remove nodes with zero or one edge until there are no tails
411     while len(deg_sets[0]) > 0 or len(deg_sets[1]) > 0:
412         to_remove = deg_sets[0] | deg_sets[1]
413         deg_sets[0] = set()
414         deg_sets[1] = set()
415
416         for n_i in list(to_remove):
417             for e_i in [e_i for e_i in self.edges[self.nodes[n_i]] if e_i in
418             bipartite]:
419                 if e_i in deg_sets[deg_map[e_i]]:
420                     deg_sets[deg_map[e_i]].remove(e_i)
421                     deg_map[e_i] -= 1
422                     deg_sets[deg_map[e_i]].add(e_i)
423
424     bipartite.remove(n_i)
425
426     return bipartite
427
428 # use BFS to find the major component
429 def _ bfs(self, bipartite, rm_edges=None):
430     points = list(bipartite)
431     # used to index into the points array
432     index_to_local = {n_i: i for i, n_i in enumerate(points)}
433     # used to index into the nodes array
434     index_to_global = {i: n_i for i, n_i in enumerate(points)}
435     visited = [0 for _ in points]
436     visits = []
437     components = []
438
439     while 0 in visited:
440         visit = 1
441
442         queue = deque()
443         root = visited.index(0)
444         queue.append(root)
445         visited[root] = 1
446         # builds a set for the points in each component
447         components.append(set([index_to_global[root]]))
448
449         while len(queue) > 0:
450             curr = queue.pop()
451
452                 for e in [index_to_local[e] for e in self.edges[self.nodes[points[
453                 curr]]]] if e in bipartite]:
454                     if rm_edges != None and (e in rm_edges and curr in rm_edges):
455                         continue
456                     if visited[e] == 0:

```

```

454             visit += 1
455             queue.append(e)
456             components[-1].add(index_to_global[e])
457             visited[e] = 1
458
459         visits.append(visit)
460
461     if len(components) > 0:
462         return components[visits.index(max(visits))]
463     else:
464         return set()
465
466 # removes all bridges and minor blocks from major component
467 # algorithm: https://e-maxx-eng.appspot.com/graph/bridge-searching.html
468 def _removeBridges(self, major_comp):
469     points = list(major_comp)
470     # used to index into the points array
471     index_to_local = {n_i:i for i, n_i in enumerate(points)}
472     # used to index into the nodes array
473     index_to_global = {i:n_i for i, n_i in enumerate(points)}
474     visited = [0 for _ in points]
475     bridge_nodes = set()
476     tin = [-1 for _ in points]
477     fup = [-1 for _ in points]
478     visit = 0
479
480     for i, p in enumerate(points):
481         if visited[i] == 0:
482             self._dfs(major_comp, points, i, p, index_to_local, visited,
483             bridge_nodes, tin, fup, visit)
484
485     return self._bfs(major_comp, bridge_nodes)
486
487 # use DFS to find bridges
488 def _dfs(self, comp, points, i, p, index_to_local, visited, bridge_nodes, tin,
489          fup, visit, to=-1):
490     visited[i] = 1
491     tin[i] = visit
492     fup[i] = visit
493     visit += 1
494     for e in [index_to_local[e] for e in self.edges[self.nodes[p]] if e in
495               comp]:
496         if e == to:
497             continue
498         if visited[e] == 1:
499             fup[i] = min(fup[i], tin[e])
500         else:
501             self._dfs(comp, points, e, points[e], index_to_local, visited,
502             bridge_nodes, tin, fup, visit, to=e)
503             fup[i] = min(fup[i], fup[e])
504             if fup[e] > tin[i]:
505                 if i not in bridge_nodes:
506                     bridge_nodes.add(i)
507                 if e not in bridge_nodes:
508                     bridge_nodes.add(e)
509
510 # public function for drawing the color set pairs
511 def drawPairs(self, mode=0):
512     l_i = []
513     if mode == 0:
514         l_i = list(self.pairs[self.curr_pair])
515     elif mode == 1:
516         l_i = list(self.no_tails[self.curr_pair])
517     elif mode == 2:
518         l_i = list(self.major_comps[self.curr_pair])
519     elif mode == 3:
520         l_i = list(self.clean_pairs[self.curr_pair])

```

```

518     l_n = [self.nodes[i] for i in l_i]
519     self._drawNodes(l_n)
520     self._applyColors(l_i)
521     self._drawEdges(l_n)
522
523     # returns the two major components with the largest size
524     def _getLargestBackbones(self, c_pairs):
525         # sizes = [-1]
526         # result = [None]
527         sizes = [-1, -1]
528         result = [None, None]
529         for p in c_pairs:
530             size = self._calcSize(p)
531
532             if size > min(sizes):
533                 min_i = sizes.index(min(sizes))
534                 sizes[min_i] = size
535                 result[min_i] = p
536
537         # saves backbone meta data (order, size)
538         meta = [(len(result[i]), sizes[i]) for i in range(len(result))]
539         if len(result) > 1 and sizes[1] > sizes[0]:
540             return result[::-1], meta[::-1]
541
542     return result, meta
543
544     # calculates the size of a graph
545     def _calcSize(self, graph):
546         size = 0
547         for n_i in list(graph):
548             size += len([e for e in self.edges[self.nodes[n_i]] if e in graph])
549
550         return size
551
552     # calculates the percentage of nodes covered by each backbone
553     def _getDominations(self, b_bones, meta):
554         for i, b in enumerate(b_bones):
555             # find the number of nodes that do not share an edge with a backbone
556             node
557                 # search all nodes not in backbone
558                 search_space = set(range(self.num_nodes)) - b
559                 for n_i in list(search_space):
560                     for e in self.edges[self.nodes[n_i]]:
561                         if e in b:
562                             search_space.remove(n_i)
563                             break
564
565             meta[i] = (meta[i][0], meta[i][1], (self.num_nodes - len(search_space)
566             + 0.0)/self.num_nodes)
567
568     return meta
569
570     # public function for drawing the backbones
571     def drawBackbones(self, draw_domination=False):
572         l_i = list(self.backbones[self.curr_backbone])
573         l_n = [self.nodes[i] for i in l_i]
574         if draw_domination:
575             self._drawDomination(l_i)
576         else:
577             background(self.color_bg)
578             self._drawNodes(l_n)
579             self._applyColors(l_i)
580             self._drawEdges(l_n)
581
582     # draws connection radius around backbone nodes
583     def _drawDomination(self, node_i_list):
584         strokeWeight(5)
585         stroke(self.color_fill)

```

```

584         fill(self.color_fill)
585
586     for n_i in node_i_list:
587         ellipse(self.nodes[n_i][0]*self.canvas_width, self.nodes[n_i][1]*self.
588         canvas_height, 2*self.node_r*self.canvas_width, 2*self.node_r*self.
589         canvas_height)
590
591     """
592     Square – inherits from Topology, overloads generateNodes and
593     -getRadiusForAverageDegree
594     for a unit square topology
595     """
596     class Square(Topology):
597
598         def __init__(self):
599             super(Square, self).__init__()
600
601             # places nodes uniformly in a unit square
602             def generateNodes(self):
603                 for i in range(self.num_nodes):
604                     self.nodes.append((random.uniform(0,1), random.uniform(0,1)))
605
606             # calculates the radius needed for the requested average degree in a unit
607             # square
608             def _getRadiusForAverageDegree(self):
609                 self.node_r = math.sqrt((self.avg_deg / (self.num_nodes * math.pi)))
610
611             # gets benchmark setting for square
612             def prepBenchmark(self, n):
613                 self.num_nodes = SQUARE_BENCHMARKS[n][0]
614                 self.avg_deg = SQUARE_BENCHMARKS[n][1]
615
616     """
617     Disk – inherits from Topology, overloads generateNodes and
618     -getRadiusForAverageDegree
619     for a unit circle topology
620     """
621     class Disk(Topology):
622
623         def __init__(self):
624             super(Disk, self).__init__()
625
626             # places nodes uniformly in a unit square and regenerates the node if it falls
627             # outside of the circle
628             def generateNodes(self):
629                 for i in range(self.num_nodes):
630                     p = (random.uniform(0,1), random.uniform(0,1))
631                     while self._distance(p, (0.5,0.5)) > 0.5:
632                         p = (random.uniform(0,1), random.uniform(0,1))
633                     self.nodes.append(p)
634
635             # calculates the radius needed for the requested average degree in a unit
636             # circle
637             def _getRadiusForAverageDegree(self):
638                 self.node_r = math.sqrt((self.avg_deg + 0.0) / self.num_nodes) / 2
639
640             # gets benchmark setting for disk
641             def prepBenchmark(self, n):
642                 self.num_nodes = DISK_BENCHMARKS[n][0]
643                 self.avg_deg = DISK_BENCHMARKS[n][1]
644
645     """
646     Sphere – inherits from Topology, overloads generateNodes,
647     -getRadiusForAverageDegree,
648     and _distance for a unit sphere topology. Also updates the drawGraph function for
649     a 3D canvas
650     """
651     class Sphere(Topology):

```

```

645
646     # adds rotation and node limit variables
647     def __init__(self):
648         super(Sphere, self).__init__()
649         self.rot = (0,math.pi/4,0) # this may move to Topology if rotation is
650         given to the 2D shapes
651         self.num_faces = []
652
653     # places nodes in a unit cube and projects them onto the surface of the sphere
654     def generateNodes(self):
655         for i in range(self.num_nodes):
656             # equations for uniformly distributing nodes on the surface area of
657             # a sphere: http://mathworld.wolfram.com/SpherePointPicking.html
658             u = random.uniform(-1,1)
659             theta = random.uniform(0, 2*math.pi)
660             p = (
661                 math.sqrt(1 - u**2) * math.cos(theta),
662                 math.sqrt(1 - u**2) * math.sin(theta),
663                 u
664             )
665             self.nodes.append(p)
666
667     # calculates the radius needed for the requested average degree in a unit
668     # sphere
669     def _getRadiusForAverageDegree(self):
670         self.node_r = math.sqrt((self.avg_deg + 0.0)/self.num_nodes)*2
671
672     # calculates the distance between two nodes (3D)
673     def _distance(self, n, m):
674         return math.sqrt((n[0] - m[0])**2+(n[1] - m[1])**2+(n[2] - m[2])**2)
675
676     # gets benchmark setting for sphere
677     def prepBenchmark(self, n):
678         self.num_nodes = SPHERE_BENCHMARKS[n][0]
679         self.avg_deg = SPHERE_BENCHMARKS[n][1]
680
681     # public function for drawing graph, updates node limit if necessary
682     def drawGraph(self, n_limit):
683         self.n_limit = n_limit
684         self._drawNodesAndEdges(self.nodes)
685
686     # responsible for drawing nodes and edges in 3D space
687     def _drawNodesAndEdges(self, node_list):
688         # positions camera
689         camera(self.canvas_width/2, self.canvas_height/2, self.canvas_width*-2,
690                0.5,0.5,0, 0,1,0)
691
692         # updates rotation
693         self.rot = (self.rot[0], self.rot[1]-math.pi/100, self.rot[2])
694
695         background(self.color_bg)
696         strokeWeight(2)
697         stroke(self.color_fg)
698         fill(self.color_fg)
699
700         s = set(node_list)
701
702         for n in node_list:
703             pushMatrix()
704
705             # sets new rotation
706             rotateZ(self.rot[2])
707             rotateY(-1*self.rot[1])
708
709             # sets drawing origin to current node
710             translate(n[0]*self.canvas_width, n[1]*self.canvas_height, n[2]*self.
711 canvas_width)

```

```

709         # places ellipse at origin
710         ellipse(0, 0, 10, 10)
711
712         # draw all edges
713         if len(node_list) <= self.n_limit:
714             for e_i in self.edges[n]:
715                 if self.nodes[e_i] in s:
716                     e = self.nodes[e_i]
717                     # draws line from origin to neighboring node
718                     line(0,0,0, (e[0] - n[0])*self.canvas_width, (e[1] - n[1])*
719                         self.canvas_height, (e[2] - n[2])*self.canvas_width)
720                     # draw edges for min degree node
721                     if n == self.minDeg:
722                         stroke(0,255,0)
723                         for e_i in self.edges[n]:
724                             e = self.nodes[e_i]
725                             # draws line from origin to neighboring node
726                             line(0,0,0, (e[0] - n[0])*self.canvas_width, (e[1] - n[1])*
727                                 self.canvas_height, (e[2] - n[2])*self.canvas_width)
728                             stroke(self.color_fg)
729                         # draw edges for max degree node
730                         elif n == self.maxDeg:
731                             stroke(0,0,255)
732                             for e_i in self.edges[n]:
733                                 e = self.nodes[e_i]
734                                 # draws line from origin to neighboring node
735                                 line(0,0,0, (e[0] - n[0])*self.canvas_width, (e[1] - n[1])*
736                                     self.canvas_height, (e[2] - n[2])*self.canvas_width)
737
738         popMatrix()
739
740         # draw nodes as they are removed in smallest-last vertex ordering
741         def drawSlvo(self):
742             l = [self.nodes[i] for i in self.slvo[0:self.num_nodes - self.curr_node]]
743             self._drawNodesAndEdges(l)
744
745         # used to draw the graph with the nodes colored
746         def drawColoring(self):
747             l = [self.nodes[i] for i in self.slvo[0:self.curr_node]]
748             self._drawNodesAndEdges(l)
749             self._applyColors(self.slvo[0:self.curr_node])
750
751         # places colors on the nodes
752         def _applyColors(self, node_i_list, draw_domination=False):
753             strokeWeight(2)
754
755             num_colors = max(self.node_colors)
756
757             for n_i in node_i_list:
758                 c = self.color_map[self.node_colors[n_i]]
759                 stroke(c[0], c[1], c[2])
760                 fill(c[0], c[1], c[2])
761
762                 pushMatrix()
763
764                 # sets new rotation
765                 rotateZ(self.rot[2])
766                 rotateY(-1*self.rot[1])
767
768                 # sets drawing origin to current node
769                 translate(self.nodes[n_i][0]*self.canvas_width, self.nodes[n_i][1]*
770                         self.canvas_height, self.nodes[n_i][2]*self.canvas_width)
771
772                 if draw_domination:
773                     stroke(self.color_fill)
774                     fill(self.color_fill, 0.2)
775                     # places sphere at origin

```

```

773             sphere(self.node_r * self.canvas_width)
774
775         # places ellipse at origin
776         ellipse(0, 0, 10, 10)
777
778     popMatrix()
779
780     # public function for pairing the independent sets and picking the largest
781     # backbones
782     def generateBackbones(self):
783         # uses base class method for generating backbones and meta data
784         super(Sphere, self).generateBackbones()
785
786         # calculate faces
787         self.num_faces = self._countFaces(self.backbones_meta)
788
789     # calcualtes the number of faces in the backbones of sphere topology
790     def _countFaces(self, b_meta):
791         # Euler's polyhedral formula
792         # http://mathworld.wolfram.com/PolyhedralFormula.html
793         return [2 - m[0] + m[1] for m in b_meta]
794
795     # public function for drawing the color set pairs
796     def drawPairs(self, mode=0):
797         l_i = []
798         if mode == 0:
799             l_i = list(self.pairs[self.curr_pair])
800         elif mode == 1:
801             l_i = list(self.no_tails[self.curr_pair])
802         elif mode == 2:
803             l_i = list(self.major_comps[self.curr_pair])
804         elif mode == 3:
805             l_i = list(self.clean_pairs[self.curr_pair])
806
807         l_n = [self.nodes[i] for i in l_i]
808         self._drawNodesAndEdges(l_n)
809         self._applyColors(l_i)
810
811     # public function for drawing the backbones
812     def drawBackbones(self, draw_domination=False):
813         l_i = list(self.backbones[self.curr_backbone])
814         l_n = [self.nodes[i] for i in l_i]
815         self._drawNodesAndEdges(l_n)
816         self._applyColors(l_i, draw_domination)

```