CSc 225 Assignment 1:
Discrete Mathematics Unit

Due date:

The submission deadline is 11:55pm on Monday, May 18th, 2020.

How to hand it in:

Submit your assignment.pdf file through the Assignment 1 link on the CSC225 conneX page.
IMPORTANT: the file submitted **must** have a **.pdf** extension.

Exercises:

1. Given the word UNDERGRADUATE

   a. How many arrangements of the letters are there?

   2 U's + 1 N + 2 D's + 2 E's + 2 R's + 1 G + 2 A's + 1 T = 13 total letters

   Answer: $\frac{13!}{2!2!2!2!2!}$

   b. How many arrangements are there with all A's adjacent to one another?

   Combine the A's into a single letter X, we have XUNDERGRDUTE

   1 X + 2 U's + 1 N + 2 D's + 2 E's + 2 R's + 1 G + 1 T = 12 total letters

   Answer: $\frac{12!}{2!2!2!2!}$

   c. How many arrangements are there with none of the A's adjacent to one another?

   Arrange all the letters without A in $\frac{11!}{2!2!2!2!}$ ways

   With the non-A letters we have: U N D E R G R D U T E , and we see there are 12 spaces that we can place the A's. We need to choose 2 of those 12 spaces to the place As: $\binom{12}{2}$

   The product rule gives our final answer as: $\left(\frac{11!}{2!2!2!2!}\right) * \binom{12}{2}$

   d. How many arrangements are there with all of the vowels adjacent to one another?

   Combine the vowels into a single letter X, we have XNDRGRDT

   1 X + 1 N + 2 D's + 2 R's + 1 G + 1 T = 8 total letters

   We can arrange these letters $\frac{8!}{2!2!}$ ways

   But, we can arrange the vowels in the X a number of ways as well:

   2 U's + 2 E's + 2 A's = 6 total letters, arranged in $\frac{6!}{2!2!2!}$ ways

   The product rule gives our final answer as: $\left(\frac{8!}{2!2!}\right) * \left(\frac{6!}{2!2!2!}\right)$

2. Suppose you draw 5 cards from a standard deck of 52.

   a. How many ways can you draw exactly 3 clubs?

      Answer: $\binom{13}{3} * \binom{39}{2}$

   b. How many ways can you draw at least 2 hearts?

      Cases to consider: draw 2 hearts, 3 hearts, 4 hearts, or all 5 hearts:

      2 hearts: $\binom{13}{2} * \binom{39}{3}$

      3 hearts: $\binom{13}{3} * \binom{39}{2}$

      4 hearts: $\binom{13}{4} * \binom{39}{1}$

      5 hearts: $\binom{13}{5}$

      Answer: : $\binom{13}{2} * \binom{39}{3} + \binom{13}{3} * \binom{39}{2} + \binom{13}{4} * \binom{39}{1} + \binom{13}{5}$

   c. How many ways can you draw 3 clubs and 2 hearts?

      Answer: : $\binom{13}{3} * \binom{13}{2}$

3. Determine the coefficient of x^7y^5 in the following expansions:

   a. $(x+y)^{12}$

      Answer: $\binom{12}{7}$

   b. $(-4x+3y)^{12}$

      Replace a with $-4x$ and b with $3y$ and we get $(a+b)^{12}$

      Answer: $\binom{12}{7} * (-4)^7 * (3)^5$

   c. $(12x-2y)^{12}$

      Replace a with $12x$ and b with $-2y$ and we get $(a+b)^{12}$

      Answer: $\binom{12}{7} * (12)^8 * (-2)^5$

4. Determine the number of integer solutions of $x_1 + x_2 + x_3 + x_4 = 16$, where

a. $x_i \geq 0,\ \ 1 \leq i \leq 4$

Here, we have n = 4 distinct objects and we are choosing r = 16 of them, with repetition.

So, $\binom{n+r-1}{r} = \binom{4+16-1}{16} = \binom{19}{16} = 969$

b. $x_1, x_2 \geq 1,\ \ x_3, x_4 \geq 3$

Here, we start with $x_1, x_2 = 1$ and $x_3, x_4 = 3$. We now need to distribute the remaining 8 integers among the n = 4 distinct variables.

So, $\binom{n+r-1}{r} = \binom{4+8-1}{8} = \binom{11}{8} = 165$

c. $x_i \geq -1,\ \ 1 \leq i \leq 4$

We start with $x_i \geq -1$.
Adding 1 to both sides gives us $x_i + 1 \geq 0$.
Next, let $y_i = x_i + 1$
Now, let's make some substitutions to the original expression:

$$\begin{aligned} x_1 + x_2 + x_3 + x_4 &= 16 \\ (x_1 + x_2 + x_3 + x_4) + 4 &= (16) + 4 \\ (x_1 + 1) + (x_2 + 1) + (x_3 + 1) + (x_4 + 1) &= 20 \\ y_1 + y_2 + y_3 + y_4 &= 20 \end{aligned}$$

Now, this expression looks the same as (a), and so we have:

$\binom{4+20-1}{20} = \binom{23}{20} = 1771$

d. $x_i \geq 1,\ \ 1 \leq i \leq 3,\ \ 5 \leq x_4 \leq 7$

There are 3 cases to consider, when $x_4 = 5, x_4 = 6$, and $x_4 = 7$

With $x_1, x_2, x_3 \geq 1$ and $x_4 = 5$, we need to distribute the remaining 8 integers among the 3 distinct variables, so for this case we have $\binom{3+8-1}{8} = \binom{10}{8}$

With $x_1, x_2, x_3 \geq 1$ and $x_4 = 6$, we need to distribute the remaining 7 integers among the 3 distinct variables, so for this case we have $\binom{3+7-1}{7} = \binom{9}{7}$

With $x_1, x_2, x_3 \geq 1$ and $x_4 = 7$, we need to distribute the remaining 6 integers among the 3 distinct variables, so for this case we have $\binom{3+6-1}{6} = \binom{8}{6}$

Answer: $\binom{10}{8} + \binom{9}{7} + \binom{8}{6}$

5. As a New Year's Resolution, Ali decides to go for a run at least once a day for the first 5 weeks of the year. To not overdo it, Ali makes sure to not run more than 50 times during this 5-week time period. Show that there must be a period of consecutive days for which Ali goes on exactly 19 runs.

Let x_i be the total number of runs Ali has completed by the end of day i, for $1 \leq i \leq 35$. By the original assumptions, we have:

$$1 \leq x_1 < x_2 < \cdots < x_{35} \leq 50$$

Here, we have 35 distinct integers between 1 and 50, inclusive. Now, we want to show that there is a span of consecutive days where Ali jogs exactly 19 times. In other words, show that there exists $i > j$ such that $x_i - x_j = 19$ (or $x_i = x_j + 19$)

So, lets add 19 to each integer of the above values to get another 35 distinct values. Thus,

$$20 \leq x_1 + 19 < x_2 + 19 < \cdots < x_{35} + 19 \leq 69$$

Now, all told we have 70 integers in the range between 1 and 69, inclusive. By the pigeonhole principle, at least two of these 70 integers are equal. But each group of 35 integers are distinct, thus one of the x_i in the first group of 35 must be equal to one of the x_i+19 in the second group. Therefore, we know there is a consecutive span of days where Ali will go on exactly 19 runs.