Chi-squared and t-distributions

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Contents

The Chi-squared Distribution

Let

$$Z_1, Z_2, \dots, Z_n \overset{i.i.d.}{\sim} N(0,1)$$

Then,

$$Y = Z_1^2 + Z_2^2 + \cdots + Z_n^2 \sim \chi_n^2$$

Expectation

First,

$$Var(Z) = E(Z^2) - E(Z)^2 = E(Z^2) - 0 = 1 \implies E(Z^2) = 1$$

Then,

$$\begin{aligned} E(Y) &= E(\sum_i^n Z_i^2) \\ &= \sum_i^n E(Z_i^2) \\ &= \sum_i^n 1 \\ &= n \end{aligned}$$

Variance

$$\begin{aligned}
Var(Y) &= \sum_i Var(Z_i^2) \ldots Independence \\
Var(Z_i^2) &= E(Z^4) - E(z^2)^2 \\
&= E(Z^4) - 1^2 \\
\Gamma(z) &= \int_0^\infty x^{z-1}e^{-x}dx;\ \Gamma(z+1) = z\Gamma(z) \\
\Gamma(s)\Gamma(1-s) &= \frac{\pi}{\sin \pi s} \implies \Gamma(\frac{1}{2})\Gamma(1-\frac{1}{2}) = \frac{\pi}{\sin\frac{\pi}{2}} \implies \Gamma(\frac{1}{2})^2 = \pi \implies \Gamma(\frac{1}{2}) = \sqrt{\pi} \\
E(Z^4) &= \int_{-\infty}^{\infty} z^4 \frac{1}{\sqrt{2\pi}} \exp(-\frac{z^2}{2})dz \\
&= \frac{2}{\sqrt{2\pi}} \int_0^\infty z^4 \exp(-\frac{z^2}{2})dz \,|_{u-sub\ u=z^2/2} \\
&= \frac{2}{\sqrt{2\pi}} \cdot 2\sqrt{2} \int_0^\infty u^{\frac{3}{2}} e^{-u} du \\
&= \frac{4}{\sqrt{\pi}} \int_0^\infty u^{\frac{3}{2}} e^{-u} du \\
&= \frac{4}{\sqrt{\pi}} \Gamma(\frac{5}{2}) \\
&= \frac{4}{\sqrt{\pi}} \cdot \underbrace{\frac{3}{2} \cdot \frac{1}{2} \cdot \Gamma(\frac{1}{2})}_{\text{recursiveness of } \Gamma()} \\
&= \frac{4}{\sqrt{\pi}} \cdot \frac{3}{2} \cdot \frac{1}{2} \cdot \Gamma(\frac{1}{2}) \\
&= \frac{4}{\sqrt{\pi}} \cdot \frac{3}{2} \cdot \frac{1}{2} \cdot \sqrt{\pi} \\
&= 3 \\
\implies Var(Y) &= \sum_i Var(Z_i^2) \\
&= \sum_i \big(E(Z^4) - 1\big) \\
&= \sum_i (3-1) \\
&= \sum_i 2 \\
&= 2n \ldots \square
\end{aligned}$$