Homework Assignment 05

Numerical Statistics Fall, 2022

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2022-01-27

1.

Let $X_1, X_2, \ldots, X_n$ be independent random variables distributed according to the Bernoulli distribution with an unknown parameter $p(0 < p < 1)$, i.e., suppose that X_i 's are independent, $P(X_i = 1) = p$ and $P(X_i = 0) = 1 - p$ $(i = 1, 2, \ldots, n)$. Then the 95% confidence interval for p is

$$\frac{\bar{X}_n + \frac{(1.96)^2}{2n} \pm (1.96)\sqrt{\frac{\bar{X}_n(1-\bar{X}_n)}{n} + \frac{(1.96)^2}{4n^2}}}{1 + \frac{(1.96)^2}{n}} \quad \cdots \quad (CI_1)$$

derived from the quadratic equation

$$\left\{n + (1.96)^2\right\} p^2 - \left\{2n\bar{X}_n + (1.96)^2\right\} p + n\left(\bar{X}_n\right)^2 = 0 \quad \cdots (QE_1)$$

Denote the upper and lower limits of the 95% confidence interval (CI_1) by β and $\alpha(\alpha \leq \beta)$, respectively. That is, let β and α be

$$\frac{\bar{X}_n + \frac{(1.96)^2}{2n} \pm (1.96)\sqrt{\frac{\bar{X}_n(1-\bar{X}_n)}{n} + \frac{(1.96)^2}{4n^2}}}{1 + \frac{(1.96)^2}{n}}, \text{ respectively.}$$

Since the parameter p is assumed to lie between 0 and 1 , we might expect both β and α to meet the condition $0 < \alpha \leq \beta < 1$.

(1-2) Find a necessary and sufficient condition for $\beta < 1$.

Solutions.

(1-2) The necessary and sufficient condition for $\beta < 1$ is equivalent to that of $1 - \beta > 0$. We show the latter.

$$\begin{aligned}
1 - \beta &= 1 - \frac{\bar{X}_n + \frac{(1.96)^2}{2n} + (1.96)\sqrt{\frac{\bar{X}_n(1-\bar{X}_n)}{n} + \frac{(1.96)^2}{4n^2}}}{1 + \frac{(1.96)^2}{n}} \\
&= \frac{1 + \frac{(1.96)^2}{n} - \bar{X}_n - \frac{(1.96)^2}{2n} - (1.96)\sqrt{\frac{\bar{X}_n(1-\bar{X}_n)}{n} + \frac{(1.96)^2}{4n^2}}}{1 + \frac{(1.96)^2}{n}} \\
&= \frac{1 + \frac{(1.96)^2}{2n} - \bar{X}_n - (1.96)\sqrt{\frac{\bar{X}_n(1-\bar{X}_n)}{n} + \frac{(1.96)^2}{4n^2}}}{1 + \frac{(1.96)^2}{n}}
\end{aligned}$$

Since $n > 0$, $1 + \frac{(1.96)^2}{n} > 0$. So, to find the necessary and sufficient condition such that the above is positive, it suffices to show the condition for

$$1+\frac{(1.96)^2}{2n}-\bar{X}_n-(1.96)\sqrt{\frac{\bar{X}_n\left(1-\bar{X}_n\right)}{n}+\frac{(1.96)^2}{4n^2}}>0 \tag{1}$$

$$\iff 1+\frac{(1.96)^2}{2n}-\bar{X}_n>(1.96)\sqrt{\frac{\bar{X}_n\left(1-\bar{X}_n\right)}{n}+\frac{(1.96)^2}{4n^2}} \tag{2}$$

Because $0 \leq \bar{X}_n \leq 1$, Both LHS and RHS of (2) are positive. From here it follows that (2) is satisfied if and only if the LHS squared is greater than the RHS squared.

$$\iff \left(1-\bar{X}_n+\frac{(1.96)^2}{2n}\right)^2>\left((1.96)\sqrt{\frac{\bar{X}_n\left(1-\bar{X}_n\right)}{n}+\frac{(1.96)^2}{4n^2}}\right)^2 \tag{3}$$

$$\iff (1-\bar{X}_n)^2+\frac{(1.96)^2(1-\bar{X}_n)}{n}+\frac{(1.96)^4}{4n^2}>(1.96)^2\left(\frac{\bar{X}_n\left(1-\bar{X}_n\right)}{n}+\frac{(1.96)^2}{4n^2}\right) \tag{4}$$

$$\iff (1-\bar{X}_n)^2+\frac{(1.96)^2(1-\bar{X}_n)}{n}+\frac{(1.96)^4}{4n^2}>\frac{(1.96)^2\bar{X}_n\left(1-\bar{X}_n\right)}{n}+\frac{(1.96)^4}{4n^2} \tag{5}$$

$$\iff (1-\bar{X}_n)^2+\frac{1}{n}(1.96)^2(1-\bar{X}_n)(1-\bar{X}_n)>0 \tag{6}$$

$$\iff \underbrace{(1-\bar{X}_n)^2}_{\geq 0}+\underbrace{\frac{1}{n}(1.96)^2(1-\bar{X}_n)^2}_{\geq 0}>0 \tag{7}$$

Recall that $0 \leq \bar{X}_n \leq 1$. For the inequality in (7) to be strict, it must be that $\bar{X}_n \neq 1$. Thus, the necessary and sufficient condition for the above to be strictly positive is

$$\bar{X}_n < 1$$

$\square$

2.

(2) Let X_1 and X_2 be independent random variables with $E\left(X_1\right)=3\mu$ and $E\left(X_2\right)=-4\mu$, where the parameter μ is unknown. In addition, let $Var\left(X_1\right)=2$ and $Var\left(X_2\right)=5$. Then answer the questions below.

- (2-1) Find a condition on w_1 and w_2 such that $\widehat{\mu}_w=w_1X_1+w_2X_2$ is an unbiased estimator of μ. (2 points)
- (2-2) Find the set of numbers (w_1, w_2) that gives the minimum value of $Var\left(\widehat{\mu}_w\right)$ provided $\widehat{\mu}_w$ is an unbiased estimator of μ.

Solutions.

(2-1) For $\widehat{\mu}_w=w_1X_1+w_2X_2$ to be an unbiased estimator of μ, we must have $E(w_1X_1+w_2X_2)=\mu$.

$$\begin{aligned}
E(w_1X_1+w_2X_2) &= E(w_1X_1)+E(w_2X_2)\\
&= w_1E(X_1)+w_2E(X_2)\\
&= w_1(3\mu)+w_2(-4\mu)\\
&= (3w_1-4w_2)\mu=\mu\\
\iff& \mu=0 \text{ or } 3w_1-4w_2=1
\end{aligned}$$

Thus, the condition for w_1, w_2 is $3w_1 - 4w_2 = 1 \ldots \square$

(2-2) We can state the problem as

$$\underset{w_1, w_2}{\operatorname{argmin}} Var(\widehat{\mu}_w) \quad \text{s.t.} \quad E(\widehat{\mu}_w) = \mu$$

As we have seen from (2-1), the constraint evaluates to $3w_1 - 4w_2 = 1$. The objective function can be simplified thus:

$$Var(\widehat{\mu}_w) = Var(w_1 X_1 + w_2 X_2)$$

Since $X_1 \perp\!\!\!\perp X_2$, $Cov(X_1, X_2) = 0$ and thus,

$$\begin{aligned} Var(w_1 X_1 + w_2 X_2) &= Var(w_1 X_1) + Var(w_2 X_2) \\ &= w_1^2 Var(X_1) + w_2^2 Var(X_2) \\ &= 2w_1^2 + 5w_2^2 \end{aligned}$$

We want to minimize this subject to the constraint $3w_1 - 4w_2 = 1$. The Lagrangian of the problem can be formulated as thus:

$$\min_{w_1, w_2, \lambda} \mathcal{L} = 2w_1^2 + 5w_2^2 + \lambda(1 - 3w_1 + 4w_2)$$

FOCS:

$$[w_1]: \ 4w_1 - 3\lambda = 0 \tag{8}$$
$$[w_2]: \ 10w_2 + 4\lambda = 0 \tag{9}$$
$$[\lambda]: \ 1 - 3w_1 + 4w_2 = 0 \tag{10}$$

From (1) and (2), $w_1 = -\frac{15}{8} w_2$. Plugging this into (3), we get

$$\begin{aligned} 1 + \frac{45}{8} w_2 + 4w_2 &= 0 \\ \iff \frac{77}{8} w_2 &= -1 \\ \iff w_2 &= -\frac{8}{77} \\ w_1 &= -\frac{15}{8} \cdot \left(-\frac{8}{77}\right) = \frac{15}{77} \end{aligned}$$

Since the objective function is convex, this gives us the global minimum. Hence,

$$\begin{cases} w_1 &= \frac{15}{77} \\ w_2 &= -\frac{8}{77} \end{cases} \ldots \square$$