Midterm

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1\.

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Take $(x_1, y_1) \in X$

It is obvious that $x_1 - 3y_1 = x_1 - 3y_1$. Thus, $(x_1, y_1) \sim (x_1, y_1)$ and $\sim$ is reflexive. ✓

Now take $(x_1, y_1), (x_2, y_2) \in X$.

If $(x_1, y_1) \sim (x_2, y_2)$, then

$$x_1 - 3y_1 = x_2 - 3y_2$$

$$\Rightarrow 3y_2 + x_1 = 3y_1 + x_2$$

$$\Rightarrow 3y_2 - x_2 = 3y_1 - x_1$$

$$\Rightarrow x_2 - 3y_2 = x_1 - 3y_1$$

$$\Rightarrow (x_2, y_2) \sim (x_1, y_1).$$ ✓

Thus, $(x_1, y_1) \sim (x_2, y_2) \Rightarrow (x_2, y_2) \sim (x_1, y_1)$ and $\sim$ has symmetry.

Now, take $(x_1, y_1), (x_2, y_2), (x_3, y_3) \in X$.

If $(x_1, y_1) \sim (x_2, y_2) \Rightarrow x_1 - 3y_1 = x_2 - 3y_2 \quad \ldots ①$

If $(x_2, y_2) \sim (x_3, y_3) \Rightarrow x_2 - 3y_2 = x_3 - 3y_3 \quad \ldots ②$

Plug ① into ② to yield $x_1 - 3y_1 = x_3 - 3y_3$ ✓

$$\Rightarrow (x_1, y_1) \sim (x_3, y_3).$$

Thus, $\sim$ is transitive and is an equivalence relation.

$\longrightarrow$

1.

An example of an equivalence class for this $\sim$ would be

$C(0) = \{z \in \mathbb{Z} \mid (z, 3z)\}$ $\{(z,3z) \mid z \in \mathbb{Z}\}$ ✓

Since $\sim$ is defined over a set of integers, and there exists an equivalence class per unique outcome of the computation of the set of integers $3x_1 - y_1$, each unique outcome will also be an integer, since integers form a ring.

Thus, there exists a bijection between $3x_1 - y_1$ and the set of all integers, and the cardinality of the equivalence classes is equal to the cardinality of $\mathbb{Z}$, or countably infinite. ✓

□

7.

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We prove this by induction.

When $n=1$, then

$$\frac{1}{1\cdot 3} = \frac{1}{2+1}$$

$$\Rightarrow \frac{1}{3} = \frac{1}{3}$$

Thus, the above holds.

Now, suppose it holds for $n=k$. Then, for $n=k+1$.

$$\frac{1}{1\cdot 3} + \frac{1}{3\cdot 5} + \cdots + \frac{1}{(2k-1)(2k+1)} + \frac{1}{(2k+1)(2k+3)}$$

Since our assumption holds at $n=k$,

$$= \frac{k}{2k+1} + \frac{1}{(2k+1)(2k+3)}$$

$$= \frac{2k^2+3k+1}{(2k+1)(2k+3)}$$

$(2k+1)(k+1)$

$$= \frac{(2k+1)(k+1)}{(2k+1)(2k+3)}$$

$$= \frac{k+1}{2k+3}$$

Therefore, it holds for $n=k+1$.

Thus, we have proven the given formula $\square$

3. Here I assume f^{-1} denotes the preimage and does not
7/15 imply a bijection between A and B.

False

(a) If $f^{-1}(U)$ is countable, then U is countable.
Since $f: A \to B$, the preimage of $U \subseteq B$ is the set of $a \in A$ such that $f(a) \subseteq U$.

Then, U is the image of $f^{-1}(U)$ under f, or $f(f^{-1}(U))$.
Now take $f: f^{-1}(U) \to U$. A function is always surjective onto its image. We can thus infer that $|f^{-1}(U)| \geq U$. Thus, if $|f^{-1}(U)|$ is finite, then $|U|$ is finite and countable. If $f^{-1}(U)$ is countably infinite there exists an injection $g: f^{-1}(U) \to \mathbb{N}$. Since $|U| \leq |f^{-1}(U)|$ there exists an injection $h: U \to f^{-1}(U)$. Thus, there exists an injection $g \circ h: U \to \mathbb{N}$ and U is countable.

(b) False. Take $f: \mathbb{R} \to \mathbb{R}$
where $f(x) = 0$ for all $x \in \mathbb{R}$. ✓

Then, take $U = \{0\} \subseteq \mathbb{R}$. U would be countable since it is finite. However, $f^{-1}(U)$ would be ~~countable~~, since it is the set of all $\mathbb{R}$. Thus, the statement does not hold.

(c). If f is injective, since a function can be defined as a subset of $A \times B$ such that each element of a occurs exactly once, injectivity would imply that there exists only a single element of B that corresponds to a distinct element of A. From the contrapositive of what we have proven in PS2, Exercise 6 (Problem 1.7.9 from Sally), we can conclude that
Pigeonhole Principle
$|A| \leq |B|$ if A and B are finite.

Since $P(A)$ is defined as the set of all subsets of A, it follows from the definition that

$$|P(A)| \leq |P(B)| .$$

Then, there exists an injection between $P(A)$, $P(B)$, since for every element in $P(A)$ there can be assigned a unique element of $P(B)$ not previously assigned to another element.

Now suppose A and B are countably infinite.

From Exercise 1 of PS4, this means there is a bijection between $P(A)$ and $P(B)$, so there certainly is an injection.

What if A, B are uncountable?

4. Since m is finite,

10/10 Let us fix $m \in \mathbb{N}$

For every $m \in \mathbb{N}$, there exists a subset $Y_m \subseteq X$ such that

$$Y_m = \left\{ y \in (0,1) \,\middle|\, y = \sum_{i=1}^{m} \frac{y_i}{2^i} \quad \text{where } y_i \in \{0,1\} \right\}$$

y can be expanded as

$$y = \frac{y_1}{2} + \frac{y_2}{2^2} + \cdots + \frac{y_m}{2^m}$$

where for each y_i, there are two options, $y_i = 1$ or $y_i = 0$.

Thus, Y_m, which is the set of all ys, will have cardinality

$$\underbrace{2 \times 2 \times \cdots \times 2}_{m} = 2^m . \quad ✓$$

Thus, for all $m \in \mathbb{N}$ there is $Y_m \subseteq X$ with 2^m finite binary expansions of numbers between $(0,1)$.

Now, we construct a bijection between X and $\mathbb{N}$. For all $m \in \mathbb{N}$, we map every finite decimal expansion $x = \sum_{i=1}^{m} \frac{x_i}{2^i}$, $x_i \in \{0,1\}$ to the next 2^m numbers in $\mathbb{N}$, not previously mapped to by any x. In ✓ other words, starting from $\sum_{n=1}^{m} 2^{n-1}$, we map the 2^m number of xs to $\mathbb{N}$.
Assume that all $x \in Y_m$ are ordered in ascending order.
Let's give this function the name $g : X \to \mathbb{N}$. Take $x_1, x_2 \in X$.

Then, g is an injection because every $x \in X$ is uniquely mapped to the next $n \in \mathbb{N}$ in order of what Y_m they are in, and their order within Y_m.

4.

g is also surjective. Take $n \in \mathbb{N}$..

$$n = g(\text{some } x \in X).$$

Now take the smallest m such that $n \leq \sum_{i=1}^{m} 2^i$

Then, we know that there exists $Y_m \subseteq X$ corresponding to that m. Then,

$$n = g\left(n - \sum_{i=1}^{m-1} 2^i \text{ th } x \text{ in } Y_m\right).$$

Thus, given any $n \in \mathbb{N}$, $g\left(n - \sum_{i=1}^{m-1} 2^i \text{ th } x \text{ in } Y_m\right) = n$, and g is surjective.

Therefore, since there is a bijection between X and $\mathbb{N}$, X is countably infinite. $\square$