# PROBLEM SET 8

DUE MARCH 16

(1) 3.10.2.5. Note that the contrapositive of this statement is what is often called the *divergence test*.

(2) 3.10.2.15

(3) 3.10.2.11 (ii) (You can use the previous problem.)

(4) Prove the *alternating series test*: Let $(b_n)$ be a non-increasing sequence where $b_n \geq 0$ for all $n \in \mathbb{N}$ and $\lim_{n\to\infty} b_n = 0$. Show that the series $\sum_{n=1}^{\infty}(-1)^{n+1}b_n$ converges.
   (a) Show that the subsequence $(S_{2k})_{k\in\mathbb{N}}$ of the sequence of the partial sums is non-decreasing.
   (b) Show that $(S_{2k})_{k\in\mathbb{N}}$ is bounded above.
   (c) Conclude that $S_{2k} \to S$ for some $S \in \mathbb{R}$.
   (d) Show that $S_k \to S$.

(5) Complete the proof that every rearrangement of the absolutely convergent series converges to the same sum. We defined

$$a_n^+ = \begin{cases} a_n & \text{if } a_n \geq 0 \\ 0 & \text{if } a_n < 0 \end{cases} \qquad a_n^- = \begin{cases} 0 & \text{if } a_n \geq 0 \\ -a_n & \text{if } a_n < 0 \end{cases}$$

and proved that $\sum_{n=1}^{\infty} a_n^+$ and $\sum_{n=1}^{\infty} a_n^-$ both converge. Show that

$$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} a_n^+ - \sum_{n=1}^{\infty} a_n^-$$

(Bonus) Show that $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = \ln 2$. (You can use all your calculus knowledge without proofs).
   (a) Show that $S_{2k} = \sum_{n=k+1}^{2k} \frac{1}{n}$.
   (b) Find appropriate integrals of the function $f(x) = \frac{1}{x}$ that bound $S_{2k}$ below and above.
   (c) Use the squeeze theorem to show that $(S_{2k}) \to \ln 2$.