Math 15910: Problem Set 2

Underland, Jake

2021-01-25

Contents

Exercise 1

Problem 1.4.5

Prove that $A \times \emptyset = \emptyset \times A = \emptyset$

Let A and B be sets. Suppose $B = \emptyset$. Then, by definition, there does not exist any $b \in B$. Since the Cartesian product of A and B, $A \times B$ is defined as the set of the ordered pairs (a, b) such that $a \in A$ and $b \in B$, if there does not exist any $b \in B$, then there cannot exist any ordered pair (a, b). Therefore, there exists no $(a, b) \in A \times B$, implying that $A \times B = A \times \emptyset = \emptyset$. Similarly, $B \times A$ is defined as the set of the ordered pairs (b, a) such that $b \in B$ and $a \in A$. If there does not exist any $b \in B$, then there cannot exist any ordered pair (b, a), hence $B \times A$ would be empty. Thus, $B \times A = \emptyset \times A = \emptyset$.
From the above, we obtain $A \times \emptyset = \emptyset \times A = \emptyset$. $\square$

Exercise 2

Prove Theorem 1.4.7

If A has m elements and B has n elements, then $A \times B$ has mn elements

Let m be fixed, and let A be a set with m elements. B is a set with n elements. $m, n \in \mathbb{N}$.
If $n = 1$, there is one element $b \in B$. Then, $A \times B$ would be the set of all ordered pairs (a_i, b) for $1 \leq i \leq m$ and $a \in A, b \in B$.

$$\underbrace{(a_1, b), (a_2, b), \dots, (a_m, b)}_{m \text{ pairs}}$$

The total number of ordered pairs would be m, and since $n = 1$, the statement "$A \times B$ has mn elements" holds true.
Suppose the statement is true for $n = k$. Now assume $n = k + 1$. Then, the elements in $A \times B$ would be the combination of pairs created by (a_i, b_j) where $1 \leq i \leq m$ and $1 \leq j \leq k$, which is equivalent to $A \times B$ when $n = k$, plus the combination of pairs created by (a_i, b_{k+1}) where $1 \leq i \leq m$. From our supposition, the number of pairs created by (a_i, b_j) is mk. Furthermore, the number of pairs created by (a_{k+1}, b_j) is m, as can be seen below:

$$\underbrace{(a_1, b_{k+1}), (a_2, b_{k+1}), \dots, (a_m, b_{k+1})}_{m \text{ pairs}}$$

Thus, the total number of elements in $A \times B$ for $n = k + 1$ would be

$$mk + m = m(k + 1)$$

It follows that when we suppose the statement to be true for $n = k$, it holds true for $n = k + 1$. Therefore, we have shown by induction that If A has m elements and B has n elements, then $A \times B$ has mn elements.

□

Exercise 3

Problem 1.4.9

Prove that if A_1 has k_1 elements, A_2 has k_2 elements, ..., A_n has k_n elements, show that $|A_1 \times A_2 \times \dots \times A_n| = |A_1||A_2| \dots |A_n| = k_1 k_2 \dots k_n$.

When $n = 2$, we have already proven in the previous exercise that $|(A_1 \times A_2)| = |A_1||A_2| = k_1 k_2$. Thus, the above statement holds.
Now suppose that the statement holds for $n = j$. When $n = j + 1$,

$$A_1 \times A_2 \times \dots \times A_j \times A_{j+1}$$

$$= (A_1 \times A_2 \times \dots \times A_j) \times A_{j+1}$$

Here, we substitute $B = A_1 \times A_2 \times \dots \times A_j$ and treat B as a single set. From our supposition, $|B| = |A_1 \times A_2 \times \dots \times A_j| = |A_1||A_2| \dots |A_j| = k_1 k_2 \dots k_j$. Then,

$$|A_1 \times A_2 \times \dots \times A_j \times A_{j+1}|$$

$$= |B \times A_{j+1}|$$

From the previous exercise,

$$|B \times A_{j+1}|$$

$$= |B||A_{j+1}|$$

$$= k_1 k_2 \dots k_j k_{j+1}$$

Thus, the statement holds.
Therefore, we have proven by induction that if A_1 has k_1 elements, A_2 has k_2 elements, ..., A_n has k_n elements, then $|A_1 \times A_2 \times \cdots \times A_n| = |A_1||A_2| \dots |A_n| = k_1 k_2 \dots k_n$.

□

Exercise 4

Problem 1.4.10

i. If A and B are finite sets and $A \cap B = \emptyset$, show that $|A \cup B| = |A| + |B|$

Suppose that $|A| = m$, $|B| = n$ and $A \cap B = \emptyset$. Then, the elements of A can be written as $a_1, a_2, \dots, a_m$ and the elements of B can be written as $b_1, b_2, \dots, b_n$. If $A \cap B = \emptyset$, then by definition, $a_i \neq b_j$ for all $1 \leq i \leq m$, $1 \leq j \leq n$. It follows that the union of the two sets would contain the elements

$$\underbrace{a_1, a_2, \dots, a_m}_{m}, \underbrace{b_1, b_2, \dots, b_n}_{n}$$

without overlap. The number of elements in the union would be sum of the number of elements in A which is $|A| = m$ and the number of elements in B which is $|B| = n$.
Therefore, if A and B are finite sets and $A \cap B = \emptyset$, then $|A \cup B| = |A| + |B|$. □

ii. If A and B are finite sets, show that $|A \cup B| = |A| + |B| - |A \cap B|$

Suppose that $|A| = m$, $|B| = n$, and $|A \cap B| = x$. Then, the elements of A can be written as $\{a_1, a_2, \dots, a_{m-x}, c_1, c_2, ..., c_x\}$ and the elements of B can be written as $\{b_1, b_2, \dots, b_{n-x}, c_1, c_2, ..., c_x\}$ where $a \in A \backslash B$, $b \in B \backslash A$, $c \in A \cap B$. Then,

$$A \cup B = \{\underbrace{a_1, a_2, \dots, a_{m-x}}_{m-x}, \underbrace{b_1, b_2, \dots, b_{n-x}}_{n-x}, \underbrace{c_1, c_2, ..., c_x}_{x}\}$$

$$\Rightarrow |A \cup B| = (m - x) + (n - x) + x$$

$$= m + n - x$$

$$= |A| + |B| - |A \cap B|$$

Thus, we have shown that if A and B are finite sets, $|A \cup B| = |A| + |B| - |A \cap B|$. □

Exercise 5

Prove that, for all $n \in \mathbb{N}$,

$$1^2 + 2^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}$$

Start with $n = 1$. Then,

$$1^2 = 1, \frac{1(1+1)(2*1+1)}{6} = \frac{6}{6} = 1$$

$$\Rightarrow 1^2 = \frac{1(1+1)(2*1+1)}{6}$$

Thus, the statement holds.
Next, suppose the statement is true for $n = k$. Now, let's consider $n = k + 1$. We begin with the left hand side of the equation.

$$1^2 + 2^2 + \cdots + k^2 + (k+1)^2$$

$$= (1^2 + 2^2 + \cdots + k^2) + (k+1)^2$$

From our assumption,

$$
\begin{aligned}
&= \frac{k(k+1)(2k+1)}{6} + (k+1)^2 \\
&= \frac{2k^3+3k^2+k}{6} + \frac{6k^2+12k+6}{6} \\
&= \frac{2k^3+9k^2+13k+6}{6} \\
&= \frac{(k+1)(2k^2+7k+6)}{6} \\
&= \frac{(k+1)(k+2)(2k+3)}{6} \\
&= \frac{(k+1)\{(k+1)+1\}\{2(k+1)+1\}}{6}
\end{aligned}
$$

Above, we have derived the right hand side from the left hand side of the equation. Hence, the statement is true for $n = k+1$ given that the statement is true for $n = k$.
Therefore, we have proven by induction that

$$1^2 + 2^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}$$

□

Exercise 6

Problem 1.7.9

Suppose that A is a set with n elements, B is a set with m elements, and $n > m$. If $f : A \to B$ is a function, there are at least two distinct elements of A that correspond to the same element of B.

We take $|A| = n$, $|B| = m$ and $n > m$ as given. If $f : A \to B$ is a function, suppose that there is at most one element of A that corresponds to one element of B. Since a function that maps from A to B is defined as a subset of $A \times B$ such that each element of A occurs exactly once as the first coordinate, if there is at most one element of A that corresponds to one element of B, the subset can be written as (a_i, b_i) for $1 \leq i \leq m$. It is clear that the number of elements in this subset is m. By definition of a function, each element of A occurs exactly once as the first coordinate, so the number of elements in A is equal to the number of elements in the subset, and $n = m$. This contradicts $n > m$. Thus, if $n > m$ and if $f : A \to B$ is a function, there are at least two distinct elements of A that correspond to the same element of B.

□

Exercise 7

Problem 1.7.15

i. $f : \mathbb{N} \to \mathbb{N}, f(n) = 2n.$

The above function is injective
Proof: Take $x_1, x_2 \in \mathbb{N}$. Suppose that $f(x_1) = f(x_2)$. Then,

$$2x_1 = 2x_2$$

$$\implies x_1 = x_2$$

Since $f(x_1) = f(x_2) \implies x_1 = x_2$, f is injective.

Next, take $y \in \mathbb{N}$. We want to show that $y = f(x)$ for some $x \in \mathbb{N}$.

$$y = f(x)$$
$$\implies y = 2x$$
$$\implies x = \frac{y}{2}$$

However, if y is an odd number, there is no $x \in \mathbb{N}$ that satisfies the above. Therefore, f is not surjective.
Hence, f is injective.

ii. $f : \mathbb{Z} \to \mathbb{Z}, f(n) = n + 6$.

The above function is bijective.
Proof: Take $x_1, x_2 \in \mathbb{Z}$. Suppose that $f(x_1) = f(x_2)$. Then,

$$x_1 + 6 = x_2 + 6$$
$$\implies x_1 = x_2$$

Since $f(x_1) = f(x_2) \implies x_1 = x_2$, f is injective.
Next, take $y \in \mathbb{Z}$. We want to show that $y = f(x)$ for some $x \in \mathbb{N}$.

$$y = f(x)$$
$$\implies y = x + 6$$
$$\implies x = y - 6$$

Since $\mathbb{Z}$ is a ring and $y, 6 \in \mathbb{Z}$, $y - 6 \in \mathbb{Z}$. Thus, we have found that for all $y \in \mathbb{Z}$, $y = f(x)$ for some $x \in \mathbb{Z}$.
Hence, f is bijective.

iii. $f : \mathbb{N} \to \mathbb{Q}, f(n) = n$.

The above function is injective.
Proof: Take $x_1, x_2 \in \mathbb{N}$. Suppose that $f(x_1) = f(x_2)$. Then,

$$x_1 = x_2$$

Since $f(x_1) = f(x_2) \implies x_1 = x_2$, f is injective.
Next, take $y \in \mathbb{Q}$. We want to show that $y = f(x)$ for some $x \in \mathbb{N}$.

$$y = f(x)$$
$$\implies y = x$$
$$\implies x = y$$

However, the above does not hold for any $y \in \mathbb{Q} \backslash \mathbb{N}$. Therefore, f is not surjective.
Hence, f is injective.

v. $f : \mathbb{R} \to \mathbb{N}, f(x) =$ **the third digit of** x **after the decimal.**

The above function is neither injective nor surjective.
Proof: Take $x_1 = 0.125$ and $x_2 = 5.5555$. Then, $f(x_1) = f(x_2)$. However, $x_1 \neq x_2$. Thus, $f(x_1) = f(x_2) \nRightarrow x_1 = x_2$, and f is not injective.
Next, take $y \in \mathbb{N}$. We want to show that $y = f(x)$ for some $x \in \mathbb{N}$.

$$y = f(x)$$
$$\implies y = \text{the third digit of } x \text{ after the decimal.}$$
$$\implies x = \text{Any real number whose decimal expansion has } y \text{ in the third digit after the decimal}$$

However, the above does not hold for any $y > 9$, as the digit must be a natural number from 1 to 9. Therefore, f is not surjective.
Hence, f is neither injective nor surjetive.

Exercise 8

Problem 1.7.22

Show that $f : \mathbb{N} \to \mathbb{Z}$

$$f(n) = \begin{cases} \frac{n}{2}, & \text{if } n \text{ is even} \\ \frac{1-n}{2}, & \text{if } n \text{ is odd.} \end{cases}$$

is a bijection

Take $x_1, x_2 \in \mathbb{N}$. Suppose that $f(x_1) = f(x_2)$. We consider the following three cases:
1. If both x_1 and x_2 are even. Then, x_1 and x_2 can be expressed as $2m_1, 2m_2$ for some $m_1, m_2 \in \mathbb{N}$. Thus,

$$f(x_1) = f(x_2)$$

$$\implies \frac{2m_1}{2} = \frac{2m_2}{2}$$
$$\implies m_1 = m_2$$
$$\implies x_1 = x_2$$

2. If both x_1 and x_2 are odd. Then, x_1 and x_2 can be expressed as $2m_1 - 1, 2m_2 - 1$ for some $m_1, m_2 \in \mathbb{N}$. Thus,

$$f(x_1) = f(x_2)$$

$$\implies \frac{1-(2m_1-1)}{2} = \frac{1-(2m_2-1)}{2}$$
$$\implies \frac{2-2m_1}{2} = \frac{2-2m_2}{2}$$
$$\implies 1 - m_1 = 1 - m_2$$
$$\implies m_1 = m_2$$
$$\implies x_1 = x_2$$

3. If x_1 is even and x_2 is odd. Then, x_1 and x_2 can be expressed as $2m_1, 2m_2 - 1$ for some $m_1, m_2 \in \mathbb{N}$. Thus,

$$f(x_1) = f(x_2)$$

$$\implies \frac{2m_1}{2} = \frac{1-(2m_2-1)}{2}$$
$$\implies m_1 = 1 - m_2$$

However, there are no $m_1, m_2 \in \mathbb{N}$ that satisfy the above expression, since $m_1 \geq 1, 1-m_2 \leq 0$. Similarly, if x_1 is odd and x_2 is even, $f(x_1) \neq f(x_2)$.

From 3., we know that if $f(x_1) = f(x_2)$, then either both x_1 and x_2 are odd or both are even. From 1. and 2., we know that if are odd or even, $f(x_1) = f(x_2)$ implies that $x_1 = x_2$. Thus, f is an injection.

Now, take $y \in \mathbb{Z}$. We want to show that $y = f(x)$ for some $x \in \mathbb{N}$.
If $y > 0$:

$$y = f(x)$$
$$\implies y = \frac{x}{2}$$
$$\implies x = 2y$$

Since $y \in \mathbb{Z}, y > 0$, we get that $2y \in \mathbb{N}$ and $2y$ is even. From here, we obtain that for all $y > 0$, $y = f(x)$ for some even $x \in \mathbb{N}$.
If $y \leq 0$:

$$y = f(x)$$
$$\implies y = \frac{1-x}{2}$$
$$\implies x = 1 - 2y$$

Since $y \in \mathbb{Z}, y \leq 0$, we get that $1-2y \in \mathbb{N}$, and $1-2y$ is odd. From here, we obtain that for all $y \leq 0$, $y = f(x)$ for some odd $x \in \mathbb{N}$.

Together, we have shown that for all $y \in \mathbb{Z}$, $y = f(x)$ for some $x \in \mathbb{N}$. Thus, f is surjective.

Since f is both injective and surjective, f is bijective.

$\square$