Math 15910: Problem Set 8

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Contents

Exercise 1 **1**
Problem 3.2.9 1

Exercise 2 **2**

Exercise 3 **2**

Exercise 4 **2**

Exercise 5 **3**
Problem 3.6.13 3

Exercise 6 **3**

Exercise 7 **4**
Problem 3.6.18 4

Exercise 8 **5**
Problem 3.6.21 5

Exercise 1

Problem 3.2.9

Suppose $\lim_{n\to\infty} x_n = x$ *and* $\lim_{n\to\infty} y_n = y$

1. *Show that* $\lim_{n\to\infty}(x_n + y_n) = x + y$

   Proof. From the supposition, we know that for all $\epsilon > 0$ there exists an $N \in \mathbb{N}$ such that for all $n > N$,

   $$|x_n - x| < \epsilon$$
   $$|y_n - y| < \epsilon$$

   Let $\epsilon = \frac{\epsilon'}{2}$. Then,

   $$|x_n - x| < \frac{\epsilon'}{2}$$
   $$|y_n - y| < \frac{\epsilon'}{2}$$

   From triangle inequality,

   $$|x_n + y_n - x - y| \leq |x_n - x| + |y_n - y| < \frac{\epsilon'}{2} + \frac{\epsilon'}{2}$$
   $$|x_n + y_n - (x + y)| < \epsilon'$$

This holds for all $\epsilon' > 0$. Thus, $x + y$ is the limit.

□

2. *Show that if for all $n \in \mathbb{N} y_n \neq 0$ and $y \neq 0$, then* $\lim_{n\to\infty}(\frac{1}{y_n}) = \frac{1}{y}$.

   Proof. Since $\lim_{n\to\infty} y_n = y$, it is obvious that $\lim_{n\to\infty} |y_n| = |y|$. Then, for all $\epsilon > 0$, there exists an N such that for all $n > N$, $|y_n|$ becomes arbitrarily close to $|y|$. Thus, we can say with certainty that there exists N_1 such that for all $n > N_1$, $|y_n| > \frac{|y|}{2}$. This implies $\frac{1}{|y_n|} < \frac{2}{|y|}$. Further, let $\epsilon = \epsilon' \frac{|y|^2}{2}$. Then, there also exists N_2 such that for all $n > N_2$, $|y_n - y| < \epsilon$. Now, let us set $N = \max\{N_1, N_2\}$. Then,

$$
\begin{aligned}
&|\frac{1}{y_n} - \frac{1}{y}| \\
=&|\frac{y - y_n}{y_n y}| \\
=&\frac{|y_n - y|}{|y_n||y|} \\
<&\epsilon' \frac{|y|^2}{2} \cdot \frac{2}{|y|} \cdot \frac{1}{|y|} \\
=&\epsilon'
\end{aligned}
$$

   Therefore, for all $\epsilon' > 0$, there exists $N \in \mathbb{N}$ such that for all $n > N$, $|\frac{1}{y_n} - \frac{1}{y}| < \epsilon'$. Thus, $\lim_{n\to\infty}(\frac{1}{y_n}) = \frac{1}{y}$.

□

Exercise 2

Let $m \in \mathbb{N}$. Prove that $(x_n)_{n\in\mathbb{N}}$ converges iff $(x_{m+n})_{n\in\mathbb{N}}$ converges. Moreover, show that $\lim_{n\to\infty} x_n = \lim_{n\to\infty} x_{m+n}$

Proof. If $(x_n)_{n\in\mathbb{N}}$ converges to x, then we know that for all $\epsilon > 0$ there exists an $N \in \mathbb{N}$ such that for all $n > N$, $|x_n - x| < \epsilon$. Since $m + n > n$, (x_{m+n}) also converges to the same limit x.

If $(x_{m+n})_{n\in\mathbb{N}}$ converges to x, then for all $\epsilon > 0$ there exists an $N_1 \in \mathbb{N}$ such that for all $m + n > N_1$, $|x_{m+n} - x| < \epsilon$. Then, take $N_2 = N_1 + m$. For all $n + m > N_2, n + m > N_1 + m > N_1$. Furthermore, $n + m > N_1 + m \implies n > N_1$. Thus, $|x_n - x| < \epsilon$. Such an N_2 exists for all ϵ, given that $(x_{m+n})_{n\in\mathbb{N}}$ converges. Thus, $(x_n)_{n\in\mathbb{N}}$ converges iff $(x_{m+n})_{n\in\mathbb{N}}$, and their limits are equivalent.

□

Exercise 3

Show that $(x_n)_{n\in\mathbb{N}}$ converges to L if and only if every subsequence of $(x_n)_{n\in\mathbb{N}}$ converges to L

Proof. Let b_n be a subsequence of x_n. Note that for all x_n, $b_n = x_m$ for $m > n$. By definition of convergence, for all $\epsilon > 0$ there exists an $N \in \mathbb{N}$ such that for all $n > N$, $|x_n - L| < \epsilon$. Then, since $b_n = x_m$ where $m > n$, $|b_n - L| < \epsilon$, and thus b_n converges to L. If every subsequence of $(x_n)_{n\in\mathbb{N}}$ converges to L, since a sequence is a subsequence of itself, it follows that $(x_n)_{n\in\mathbb{N}}$ converges to L.

□

Exercise 4

Prove directly from the definition that $a_n = \frac{n+2}{2n+1}$ is Cauchy.

Proof. Let $N = \frac{3}{2\epsilon}$. Then, for $m, n > N$,

$$
\begin{aligned}
|\frac{n+2}{2n+1} - \frac{m+2}{2m+1}| &= |\frac{3m-3n}{(2n+1)(2m+1)}| \\
&= |\frac{3n-3m}{(2n+1)(2m+1)}| \\
&\leq |\frac{3n}{(2n+1)(2m+1)}| + |\frac{3m}{(2n+1)(2m+1)}| \\
&\leq |\frac{3n}{(2n2m)}| + |\frac{3m}{(2n2m)}| \\
&= |\frac{3}{4m}| + |\frac{3}{4n}| \\
&< |\frac{\epsilon}{2}| + |\frac{\epsilon}{2}| \\
&= \epsilon
\end{aligned}
$$

Thus, a_n is Cauchy.

□

Exercise 5

Problem 3.6.13

Prove that every Cauchy sequence in $\mathbb{R}$ is bounded.

Proof. Since the sequence is Cauchy, for all $\epsilon > 0$, there exists N such that for all $m, n > N$, $|a_n - a_m| < \epsilon$. Let $\epsilon = \epsilon_0$ and $N = N_0$ be the N such that for all $m, n > N_0$, $|a_n - a_m| < \epsilon_0$. Furthermore, let $m = m_0 > N_0$. Then, from triangle inequality,

$$
\begin{aligned}
&|a_n - a_{m_0}| < \epsilon_0 \\
\implies &|a_n - a_{m_0}| + |a_{m_0}| < \epsilon_0 + |a_{m_0}|
\end{aligned}
$$

From triangle inequality,

$$
\begin{aligned}
|a_n| = |a_n - a_{m_0} + a_{m_0}| \leq |a_n - a_{m_0}| + |a_{m_0}| < \epsilon_0 + |a_{m_0}| \\
\implies |a_n| < \epsilon_0 + |a_{m_0}|
\end{aligned}
$$

Thus, $|a_n|$ is bounded.

Theorem 3.6.14 shows that a sequence is convergent if and only if it is Cauchy. Since we know that Cauchy sequences are bounded, we can therefore deduce that all convergent sequences are bounded.

□

Exercise 6

Let $(a_n)_{n\in\mathbb{N}}$ be a sequence of strictly positive real numbers and suppose that $(a_n) \to a$

1. *Show that $a \geq 0$.* Proof. Suppose $a < 0$. Note that $a_n > 0$ for all n. Then, since a is the limit of a_n, for all $\epsilon > 0$, there exists N such that for all $n > N$, $|a_n - a| < \epsilon$. Let $\epsilon = \frac{|a|}{2} > 0$. Since $a < 0$ and $a_n > 0$,

$$
\begin{aligned}
|a_n - a| = a_n - a < \epsilon = -\frac{a}{2} \\
\implies a_n < -\frac{a}{2} + a = \frac{a}{2}
\end{aligned}
$$

But $\frac{a}{2} < 0$, and $a_n > 0$, which is a contradiction. Thus, $a \geq 0$.

□

2. *Show that* $(\sqrt{a_n}) \to \sqrt{a}$. Proof. Let $\epsilon = \frac{\epsilon'}{\sqrt{a}}$. Then, there exists $N \in \mathbb{N}$ such that for all $n > N$, the below holds.

$$\begin{aligned} |\sqrt{a_n} - \sqrt{a}| &= \frac{|\sqrt{a_n} - \sqrt{a}| \cdot |\sqrt{a_n} + \sqrt{a}|}{|\sqrt{a_n} + \sqrt{a}|} \\ &= \frac{|a_n - a|}{\sqrt{a_n} + \sqrt{a}} \\ &< \frac{|a_n - a|}{\sqrt{a}} \\ &< \frac{\epsilon}{\sqrt{a}} = \epsilon' \end{aligned}$$

Thus, the limit of $\sqrt{a_n}$ is $\sqrt{a}$.

□

Exercise 7

Problem 3.6.18

Find the accumulation points of the following sets in $\mathbb{R}$

1. $S = (0, 1)$;

   Since the interval $(0, 1)$ is continuous within its end points, it is clear that for any $x \in (0, 1)$, for all $\epsilon > 0$, $(x-\epsilon, x+\epsilon) \cap (0, 1) \backslash \{x\} \neq \emptyset$. Now, take 0 and 1. If $x = 0$, then there exists $x+\epsilon_0 \in (x-\epsilon, x+\epsilon)$ such that $x + \epsilon_0 \in (0, 1)$ where $\epsilon_0 < \epsilon, \epsilon_0 < 1$. Further, if $x = 1$, then there exists $x - \epsilon_0 \in (x-\epsilon, x+\epsilon)$ such that $x - \epsilon_0 \in (0, 1)$ where $\epsilon_0 < \epsilon, \epsilon_0 < 1$. Thus, 1 and 0 are included, and the accumulation points of $(0, 1)$ are $[0, 1]$.

2. $S = \{(-1)^n + \frac{1}{n}\}$;

   Let us observe the cases when n is even and odd. When n is even, we will denote this $n = 2m$ for $m \in \mathbb{N}$. Then, $(-1)^n + \frac{1}{n}$ gets arbitrarily close to 1, since the greater m is, the smaller $\frac{1}{n}$ gets, where $(-1)^n$ always equals 1. In other words, for every $\epsilon > 0$, there exists $m \in \mathbb{N}$ such that $2m > \frac{1}{\epsilon}$, or $\frac{1}{2m} < \epsilon$. Thus, $(1-\epsilon, 1+\epsilon) \cap S \backslash \{1\} \neq \emptyset$. So, 1 is an accumulation point. Further, suppose n is odd, and $n = 2m + 1$ for $m \in \mathbb{N}$. Then, $(-1)^n = -1$, and there exists $m \in \mathbb{N}$ such that $2m > \frac{1}{\epsilon}$, or $\frac{1}{2m} < \epsilon$. Thus, $(-1-\epsilon, -1+\epsilon) \cap S \backslash \{-1\} \neq \emptyset$. So, -1 is an accumulation point.For any other $x \in \mathbb{R}$, let s denote the closes number of the form $(-1)^n + \frac{1}{n}$. Then, for any $\epsilon < |x - s|$, there exists no number that can be represented as $(-1)^n + \frac{1}{n}$ within $(x - \epsilon, x + \epsilon)$, or else it would contradict our supposition that s is the closest of such numbers to x. Thus, x will not be an accumulation point, and the only accumulation points are $\{-1, 1\}$.

3. $S = \mathbb{Q}$;

   Take $x \in \mathbb{R}$. Suppose $x > 0$. Then, $x + \epsilon > 0$. Since $x, \epsilon \in \mathbb{R}$, ϵ has a decimal expansion, either finite or infinite. Take this decimal expansion until the first nonzero digit in the decimal expansion of ϵ and denote this ϵ_1. Then, $x < x + \epsilon_1 < x + \epsilon$. Now, take the decimal expansion of $x + \epsilon_1$ until the digit in the place after the place of the first digit of ϵ. Denote this $(x + \epsilon_1)'$. Then, $x < (x + \epsilon_1)' < x + \epsilon$, and $(x + \epsilon_1)'$ is rational. Similarly, we can find that for any $x \leq 0$, $x - \epsilon < (x - \epsilon_1)' < x$. Thus, there exists a rational number in the neighborhood of every $x \in \mathbb{R}$, and the accumulation point of $\mathbb{Q}$ is $\mathbb{R}$.

4. $S = \mathbb{Z}$;

   The integers do not have an accumulation point. Suppose they do. Then, for any $x \in \mathbb{R}$, take $s = \max\{|x - \lfloor x \rfloor|, |x - \lceil x \rceil|\}$ and let $\epsilon < s$. Then, for all $(x - \epsilon, x + \epsilon)$, there does not exist any integer. Thus, there is not accumulation point for $\mathbb{Z}$.

5. *S is the set of rational numbers whose denominators are prime.*

   We know that the sequence of primes is infinite and increasing, and therefore the sequence of fractions $\frac{1}{p_i}$, where p_i is a prime, is infinite and decreasing. For every $\epsilon > 0$, there exists $p_i > \epsilon$. Thus, for every $\epsilon = \frac{1}{\epsilon}$, there is a $\frac{1}{p_i} < \epsilon$. Therefore, $(-\epsilon, \epsilon) \cap S \neq \emptyset \implies \{0\}$ is an accumulation point. For any other $x \in \mathbb{R}$, take p_i such that $\frac{1}{p_i} < \epsilon$. Then, there exists at least one $\frac{k}{p_i} \in (x - \epsilon, x + \epsilon), k \in \mathbb{Z}$. Thus, all $x \in \mathbb{R}$ is an accumulation point of S.

   □

Exercise 8

Problem 3.6.21

1. *Find an infinite subset of $\mathbb{R}$ that does not have an accumulation point in $\mathbb{R}$*

   $\mathbb{Z}$ is an infinite subset of $\mathbb{R}$ that does not have an accumulation point, as we saw earlier.

2. *Find a bounded subset of $\mathbb{R}$ that does not have an accumulation point in $\mathbb{R}$*

   The set $\{1\}$ is a bounded subset but does not have an accumulation point. This is because for any $x \in \mathbb{R}$, there exists a small enough $\epsilon > 0$ such that $(x - \epsilon, x + \epsilon) \backslash \{x\}$.

3. *Find a bounded infinite subset of $\mathbb{Q}$ that does not have an accumulation point in $\mathbb{Q}$*

   The sequence

   $$x_n = \lfloor \sqrt{2} \cdot 10^{n-1} \rfloor \cdot \frac{1}{10^{n-1}}$$

   is bounded above by $\sqrt{2}$, below by 1, and is infinite. Its limit is $\sqrt{2}$, as x_n is just the decimal expansion of $\sqrt{2}$ until the $n-1$th digit after the decimal.

   □