P-set 3

(⇒)

1). First, assume that $g: A \to B$ is injective

Then, it follows that A and B are nonempty.

Define $C = g(A)$, the image of g. Then, $g: A \to C$ is a bijection, meaning there exists $g^{-1}: C \to A$ which is surjective.

Define $h: B \to A$

$$h(b) = \begin{cases} g^{-1}(b) & \text{for } b \in C \\ a^* & \text{for } b \in B \setminus C \end{cases}$$

$\overset{*}{a}$ is any where $\overset{\vee}{a^*} \in A$.

Now we want to find $b \in B$ such that

$$h(b) = a \quad \text{for any } a \in A.$$

Then,

$$g^{-1}(b) = a \qquad \text{for}$$

$$\Longrightarrow \quad b = g(a)$$

Thus, given $a \in A$, we have $a = h(g(a))$, and have found a surjective function h.

(⇐) Now assume $h: B \to A$ is surjective. That means, for every $a_i \in A$ there corresponds a set of at least one $b_i \in B$ s.t. $h(b_i) = a_i$, and $a_i \neq a_j \Rightarrow b_i \neq b_j$.

Now, choose one of these b_i s for each a_i. Define g as follows:

$$g(a_i) = b_i \quad \text{for } b_i \in B \text{ such that } h(b_i) = a_i$$

Then, for $a_i, a_j \in A$, if

$$g(a_i) = g(a_j)$$

$$b_i = b_j, \text{ From the contrapositive of}$$

$$a_i \neq a_j \Rightarrow b_i \neq b_j,$$

$\Rightarrow a_i = a_j$. Thus, there exists an injective function $g: A \to B$.

□

(2) Take $a_1, a_2 \in A$. Then,

$$g \circ f(a_1) = g \circ f(a_2)$$

$$\Rightarrow g(f(a_1)) = g(f(a_2))$$

Since $g : B \to C$ is bijective, there exists g^{-1} such that

$g^{-1} \circ g = I_B$.

$$\Rightarrow f(a_1) = f(a_2)$$

Since $f : A \to B$ is bijective, it follows that there exists f^{-1} such that $f^{-1} \circ f = I_A$

$$\Rightarrow a_1 = a_2$$

Thus, $g \circ f$ is injective.

Now, let $c \in C$. $c = g \circ f(a)$

$$\Rightarrow g^{-1}(c) = f(a)$$

$$\Rightarrow f^{-1}(g^{-1}(c)) = a$$

Thus, given any $c \in C$, there exists $g \circ f(f^{-1}(g^{-1}(c))) = c$, and $g \circ f$ is surjective.

Therefore, $g \circ f$ is bijective.

From $g \circ f(f^{-1}(g^{-1}(c))) = c$

$$\Rightarrow g \circ f \circ f^{-1}(g^{-1}(c)) = c$$

$$\Rightarrow (g \circ f) \circ f^{-1} \circ g^{-1}(c) = c \Rightarrow \underline{(g \circ f)^{-1} c = f^{-1} \circ g^{-1}(c)}$$

Therefore, $(g \circ f)^{-1} : C \to A = f^{-1} \circ g^{-1} : C \to A$. $\square$

(3) Since f is a bijection, all $z \in \mathbb{Z}$ can be expressed

as
$$z = \frac{n}{2} \text{ for an even } n \in \mathbb{N}$$
$$\text{or } z = \frac{1-n}{2} \text{ for an odd } n \in \mathbb{N}.$$

In other words,
$$z = \begin{cases} \frac{2m}{2} \\ \frac{1-(2m-1)}{2} \end{cases} \text{ for } m \in \mathbb{N}.$$

Since $m \geq 1$, $\frac{2m}{2} > 0$, $\frac{1-(2m-1)}{2} \leq 0$. Thus,
$$z = \begin{cases} \frac{n}{2} & \Rightarrow z > 0 \\ \frac{1-n}{2} & \Rightarrow z \leq 0 \end{cases}$$

Since $f^{-1} : \mathbb{Z} \to \mathbb{N}$, we want to derive n from z.

For any $z \in \mathbb{Z}$,

if $z > 0$, $f^{-1}(z) = n$.
$$\Rightarrow f^{-1}\left(\frac{n}{2}\right) = n.$$

if $z \leq 0$ $f^{-1}(z) = n$
$$\Rightarrow f^{-1}\left(\frac{1-n}{2}\right) = n$$

From the above,
$$f^{-1}(z) = \begin{cases} 2z & \text{if } z > 0 \\ -2z+1 & \text{if } z \leq 0 \end{cases}$$

$\square$

(4) (ii) Take $x \in f(A_1 \cap A_2)$. For every $x \in f(A_1 \cap A_2)$, there must exist at least one $y \in A_1 \cap A_2$ such that $f(y) = x$. From here, we can deduce that $y \in A_1$ and $y \in A_2$.

$\Rightarrow \quad f(y) \in f(A_1)$ and $f(y) \in f(A_2) \Rightarrow x \in f(A_1)$ and $x \in f(A_2)$

$\Rightarrow \quad x \in f(A_1) \cap f(A_2)$ Thus, we know that $f(A_1 \cap A_2) \subseteq f(A_1) \cap f(A_2)$. However, the converse is not necessarily true. Take $A_1 = \{0.9, 0.8\}$, $A_2 = \{0.7, 0.6\}$ and $f = \lceil \, \rceil$ (the ceiling function). Then, $f(A \cap B) = \varnothing$, but

$$f(A) \cap f(B) = \{1, 1.\} \cap \{1, 1\} = \{1, 1\}.$$

and $f(A) \cap f(B) \not\subseteq f(A \cap B)$ $\square$

(iv) Take $x \in f^{-1}(B_1 \cap B_2)$. Then, $f(x) \in B_1 \cap B_2$, so $f(x) \in B_1$ and $f(x) \in B_2$. Hence, $x \in f^{-1}(B_1)$ and $x \in f^{-1}(B_2)$. That is, $x \in f^{-1}(B_1) \cap f^{-1}(B_2)$. Thus, $f^{-1}(B_1 \cap B_2) \subseteq f^{-1}(B_1) \cap f^{-1}(B_2)$. Read this backwards to get $f^{-1}(B_1) \cap f^{-1}(B_2) \subseteq f^{-1}(B_1 \cap B_2)$.

Therefore, $f^{-1}(B_1 \cap B_2) = f^{-1}(B_2) \cap f^{-1}(B_1)$ $\square$

(5) See the end of (4) (iii)

(6) The Cartesian Product

$$A_1 \times A_2 = \{(a_1, a_2) \mid a_1 \in A_1, a_2 \in A_2\}$$

Now, take the set of all functions $f : \{1,2\} \to X$ such that $f(1) \in A_1$, $f(2) \in A_2$. We denote a function in this set as f_{ij} if $f_{ij}(1) = a_{1i}$, $f_{ij}(2) = a_{2j}$ for $a_{1i} \in A_1$, $a_{2j} \in A_2$. Now, we define $g : X \to f$

as $\quad g((a_{1i}, a_{2j})) = f_{ij}$.

Suppose $\quad g((a_{1i}, a_{2j})) = g((a_{1k}, a_{2l}))$

$$\Rightarrow \quad f_{ij} = f_{k,l}$$

By definition, $\quad f_{ij}(1) = f_{k,l}(1) = a_{1,i} = a_{1,l}$

$$f_{ij}(2) = f_{kl}(2) = a_{2j} = a_{2l}$$

Thus, $(a_{1i}, a_{2j}) = (a_{1k}, a_{2l})$,

and g is injective.

Now, take $f_{ij} \in f$ such that $g((a_{1i}, a_{2j})) = f_{ij}$ for some $(a_{1i}, a_{2j}) \in A_1 \times A_2$.

$$g((a_{1i}, a_{2j})) = f_{ij}$$

From the definition of f

$$(a_{1i}, a_{2j}) = (f_{ij}(1), f_{ij}(2))$$

Thus, $g((f_{ij}(1), f_{ij}(2))) = f_{ij}$

and g is surjective.

We have thus proven a bijection between the Cartesian product of A_1, A_2 and the set of all functions $f: \{1,2\} \to X$ such that $f(1) = a_1$, $f(2) = a_2$ for all $a_1 \in A_1$, $a_2 \in A_2$.

$\square$

(7)

If $A \subseteq B$, the identity function $I: A \to B$, $I(x) = x$, is an injection. By definition of injection, for all $a \in A$, there is a distinct element $b \in B$. Thus, we know that

$$|A| \leq |B|. \quad \text{(Proposition 2)}$$

Following the same logic, we also know that

$$B \subseteq C$$
$$\Rightarrow |B| \leq |C|.$$

Since $|A| = |C|$, the only value $|B|$ that satisfies the above is $|A| = |B|$

$\square$

(8) (a) Since $\mathbb{N} \subseteq \mathbb{Q}_+$, the identity function

$I : \mathbb{N} \to \mathbb{Q}_+$, $I(n) = n$ is an injection.

if $I(n_1) = I(n_2)$

$\Rightarrow n_1 = n_2$.

b) Since the set of positive rational numbers are expressed by $\frac{m}{n}$ where $m, n \in \mathbb{N}$ and m, n are co-prime to each other. Then, for every $\frac{m}{n}$, the function $h : \mathbb{Q}_+ \to \mathbb{N} \times \mathbb{N}$

is defined as $h(\frac{m}{n}) = (m, n)$.

if $h(\frac{m}{n}) = g(\frac{m'}{n'})$

$$\Rightarrow (m, n) = (m', n')$$

$$\Rightarrow m = m', \ n = n'$$

$$\Rightarrow \frac{m}{n} = \frac{m'}{n'}$$

Thus, h is an injection.

(c) From proposition 2, we know that $\#A \leq \#B$ iff there exists an interjection $A \to B$. From (a), we can deduce that $|\mathbb{N}| \leq |\mathbb{Q}_+|$. Furthermore, from (b), $|\mathbb{Q}_+| < |\mathbb{N} \times \mathbb{N}|$. As proven in the video, $|\mathbb{N}| = |\mathbb{N} \times \mathbb{N}|$. Thus, we know from Exercise 1.8.12 that since

$$|\mathbb{N}| \leq |\mathbb{Q}_+| \leq |\mathbb{N} \times \mathbb{N}| = |\mathbb{N}|$$

$|\mathbb{N}| = |\mathbb{Q}_+|$. By definition of cardinality, this implies a bijection between $\mathbb{Q}_+$ and $\mathbb{N}$, meaning $\mathbb{Q}_+$ is countably infinite.

(8) (d) Similarly for $\mathbb{Q}_-$:

1. There exists an interjection $g: \mathbb{N} \to \mathbb{Q}_-$ defined by

$$g(n) = -n.$$

If $g(n) = g(m)$

$$\Rightarrow -n = -m$$
$$\underline{n = m}$$

2. We define $h: \mathbb{Q}_- \to \mathbb{N} \times \mathbb{N}$ as

$$h\left(-\frac{m}{n}\right) = (m, n)$$

for $-\frac{m}{n} \in \mathbb{Q}_-$, where $m, n \in \mathbb{N}$ and m, n are co-prime.

If $h\left(-\frac{m_1}{n_1}\right) = h\left(-\frac{m_2}{n_2}\right)$,

$$\Rightarrow (m_1, n_1) = (m_2, n_2)$$

$$\Rightarrow m_1 = m_2, \ n_1 = n_2$$

$$\Rightarrow \frac{m_1}{n_1} = \frac{m_2}{n_2}$$

$$\Rightarrow -\frac{m_1}{n_1} = -\frac{m_2}{n_2}.$$

Thus, h is an injection.

3. From the same reasoning as before,

$$|\mathbb{N}| \leq |\mathbb{Q}_-| \leq |\mathbb{N}|$$

$\Rightarrow |\mathbb{N}| = |\mathbb{Q}_-| \Rightarrow$ there is a bijection between $\mathbb{Q}_-$ and $\mathbb{N}$, making $\mathbb{Q}_-$ countably infinite. From Fact 1.8.25,

$\mathbb{Q} = \mathbb{Q}_- \cup \mathbb{Q}_+$ is countably infinite. $\square$

(9)

Define $f : (0,1) \to \mathbb{R}$ as.

$$f(x) = \log_2 \frac{1-x}{x}$$

Since $x \in (0,1)$, $x \in \mathbb{R}$. Thus, the above function yields a number in $\mathbb{R}$.

If

$$f(x_1) = f(x_2)$$

$$\Rightarrow \log_2 \frac{1-x_1}{x_1} = \log_2 \frac{1-x_2}{x_2}$$

$$\Rightarrow \frac{1-x_1}{x_1} = \frac{1-x_2}{x_2}$$

$$\Rightarrow (1-x_1)x_2 = (1-x_2)x_1$$

$$\Rightarrow x_2 - x_1x_2 = x_1 - x_1x_2$$

$$\Rightarrow x_2 = x_1$$

Thus, f is injective.

Now, take $y \in \mathbb{R}$. We want $x \in (0,1)$ such that $f(x) = y$.

$$\log_2 \frac{1-x}{x} = y$$

$$\Rightarrow \frac{1-x}{x} = 2^y$$

$$\Rightarrow 1 = 2^y x + x$$

$$\Rightarrow \frac{1}{2^y+1} = x.$$

Thus, given any $y \in \mathbb{R}$, $f\left(\frac{1}{2^y+1}\right) = y$. Thus, f is surjective.

contin. →

(9)

It follows that f is bijective.

From here, we get that $\mathbb{R}$ and $(0,1)$ have the same cardinalities. However, from the video we know that $(0,1)$ and $\mathbb{N}$ do not have a bijection, and thus do not have equivalent cardinalities. Hence, $\mathbb{R}$ and $\mathbb{N}$ have differing cardinalities so there can't be a bijection between them. Therefore, $\mathbb{R}$ is not countable.

$\square$

(10) The function $f : [0,1) \to (0,1)$

$$f(x) = \begin{cases} \frac{n+1}{n+2} & \text{if } x = \frac{n}{n+1} \text{ for } n \in \mathbb{N} \cup \{0\} \\ x & \text{for all else.} \end{cases}$$

is bijective.

$[0,1) \to$

$0 + \frac{1}{2^n} \quad \frac{1}{2^{1+n}}$ $\quad (0,1) \to [0,1]$