Math 15910: Problem Set 7

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Exercise 1

Problem 3.2.9

Suppose $\lim_{n\to\infty} x_n = x$ *and* $\lim_{n\to\infty} y_n = y$

1. *Show that* $\lim_{n\to\infty}(x_n + y_n) = x + y$

   Proof. From the supposition, we know that for all $\epsilon > 0$ there exists an $N \in \mathbb{N}$ such that for all $n > N$,

   $$|x_n - x| < \epsilon$$
   $$|y_n - y| < \epsilon$$

   Let $\epsilon = \frac{\epsilon'}{2}$. Then,

   $$|x_n - x| < \frac{\epsilon'}{2}$$
   $$|y_n - y| < \frac{\epsilon'}{2}$$

From triangle inequality,

$$|x_n + y_n - x - y| \leq |x_n - x| + |y_n - y| < \frac{\epsilon'}{2} + \frac{\epsilon'}{2}$$
$$|x_n + y_n - (x + y)| < \epsilon'$$

This holds for all $\epsilon' > 0$. Thus, $x + y$ is the limit.

□

2. *Show that if for all $n \in \mathbb{N} y_n \neq 0$ and $y \neq 0$, then* $\lim_{n\to\infty}(\frac{1}{y_n}) = \frac{1}{y}$.

   Proof. Since $\lim_{n\to\infty} y_n = y$, it is obvious that $\lim_{n\to\infty} |y_n| = |y|$. Then, for all $\epsilon > 0$, there exists an N such that for all $n > N$, $|y_n|$ becomes arbitrarily close to $|y|$. Thus, we can say with certainty that there exists N_1 such that for all $n > N_1$, $|y_n| > \frac{|y|}{2}$. This implies $\frac{1}{|y_n|} < \frac{2}{|y|}$. Further, let $\epsilon = \epsilon' \frac{|y|^2}{2}$. Then, there also exists N_2 such that for all $n > N_2$, $|y_n - y| < \epsilon$. Now, let us set $N = \max\{N_1, N_2\}$. Then,

$$\begin{aligned} &|\frac{1}{y_n} - \frac{1}{y}| \\ =&|\frac{y - y_n}{y_n y}| \\ =&\frac{|y_n - y|}{|y_n||y|} \\ <&\epsilon' \frac{|y|^2}{2} \cdot \frac{2}{|y|} \cdot \frac{1}{|y|} \\ =&\epsilon' \end{aligned}$$

   Therefore, for all $\epsilon' > 0$, there exists $N \in \mathbb{N}$ such that for all $n > N$, $|\frac{1}{y_n} - \frac{1}{y}| < \epsilon'$. Thus, $\lim_{n\to\infty}(\frac{1}{y_n}) = \frac{1}{y}$.

□

Exercise 2

Let $m \in \mathbb{N}$. Prove that $(x_n)_{n\in\mathbb{N}}$ converges iff $(x_{m+n})_{n\in\mathbb{N}}$ converges. Moreover, show that $\lim_{n\to\infty} x_n = \lim_{n\to\infty} x_{m+n}$

Proof. If $(x_n)_{n\in\mathbb{N}}$ converges to x, then we know that for all $\epsilon > 0$ there exists an $N \in \mathbb{N}$ such that for all $n > N$, $|x_n - x| < \epsilon$. Since $m + n > n$, (x_{m+n}) also converges to the same limit x.

If $(x_{m+n})_{n\in\mathbb{N}}$ converges to x, then for all $\epsilon > 0$ there exists an $N_1 \in \mathbb{N}$ such that for all $m + n > N_1$, $|x_{m+n} - x| < \epsilon$. Then, take $N_2 = N_1 + m$. For all $n + m > N_2, n + m > N_1 + m > N_1$. Furthermore, $n + m > N_1 + m \implies n > N_1$. Thus, $|x_n - x| < \epsilon$. Such an N_2 exists for all ϵ, given that $(x_{m+n})_{n\in\mathbb{N}}$ converges. Thus, $(x_n)_{n\in\mathbb{N}}$ converges iff $(x_{m+n})_{n\in\mathbb{N}}$, and their limits are equivalent.

□

Exercise 3

Show that $(x_n)_{n\in\mathbb{N}}$ converges to L if and only if every subsequence of $(x_n)_{n\in\mathbb{N}}$ converges to L

Proof. Let b_n be a subsequence of x_n. If

□

Exercise 4

Show that every bounded open interval $(a, b) \subseteq \mathbb{R}$ can be described as $\{x \in \mathbb{R} | |x - x_0| < \epsilon\}$ for some value $x_0 \in \mathbb{R}$ and some $\epsilon > 0$. What are the values of x_0 and ϵ in terms of a and b? And conversely, given x_0 and ε, what are the endpoints of the interval $\{x \in \mathbb{R} | |x - x_0| < \epsilon\}$?

Proof. In this definition, x_0 denotes the median point between a and b, or $\frac{a+b}{2}$, and ϵ denotes half of the length of the interval $\frac{|b-a|}{2}$. $\{x \in \mathbb{R} | |x - x_0| < \epsilon\}$ says that for every $x \in R$ such that the distance between x and x_0, the median point of the interval (a, b), measured by $|x - x_0|$, must be less than ϵ which is the distance from this median point to a or b. Thus the endpoints of this interval can be described as $x_0 \pm \epsilon$, and the interval can be restated as $(x_0 - \epsilon, x_0 + \epsilon)$ or the symmetric neighborhood of x_0.

□

Exercise 5

Problem 3.4.3

Suppose that I is a subset of $\mathbb{R}$. Show that I is an interval if and only if for all $a, b \in I$, with $a \leq b$, the closed interval $[a, b] \subseteq I$.

Proof. Let x, y be the endpoints of I, where $x < y$. Suppose I is an interval. Take $a, b \in I$. Then, $x \leq a$ and $b \leq y$, or else $a, b \notin I$. Because, all values between x and y are in I, any value between a and b is also between x and y and is therefore inside I. From our supposition, $a, b \in I$. Thus, $[a, b] \subseteq I$.

Now, take the statement if for all $a, b \in I$ such that $a \leq b$, $[a, b] \subseteq I$, then I is an interval. We prove the converse. Suppose I is not an interval. This means I cannot be written in any of the 10 categories described in Definition 3.4.1 of Sally. This means there is a discontinuity between 2 values in I, or 2 values $c, d \in I$ where $c < d$ such that $\frac{c+d}{2} \notin I$. Then, it follows that for some $a, b \in I$ such that $a \leq b$, $[a, b] \nsubseteq I$. Thus, the converse is true, and we have proven the statement.

□

Exercise 6

Let $I_n = [a_n, b_n]$ for $n \in \mathbb{N}$ be a nested collection of intervals. Suppose that $\inf\{b_n - a_n | n \in \mathbb{N}\} = 0$. *Show that the number $\xi \in \cap_{n=1}^{\infty} I_n$ is unique.*

Proof. Suppose that $\inf\{b_n - a_n | n \in \mathbb{N}\} = 0$. Then, for one of the intervals I_ξ, $a_n = b_n$. Thus, the interval is a point, or unique value ξ such that $a_n = b_n = \xi$. By the Nested Interval Theorem, we know that $\cap_{n=1}^{\infty} I_n \neq \emptyset/$. Therefore, there must be at least one value that is found in every interval. However, since we know that one of these intervals only contains one unique value ξ, for any value $i \in I_i$ to be in $\cap_{n=1}^{\infty} I_n$, $i \in I_\xi \implies i = \xi$. Thus, $\xi \in \cap_{n=1}^{\infty} I_n$ is unique.

□

Exercise 7

Problem 3.6.5

Show that the limit of a convergent sequence is unique.

Proof. Let us denote this sequence by a_n and its limits L, M. Suppose $L \neq M$. Then, for every $\epsilon > 0$, there exists $N_L, N_M \in \mathbb{N}$ such that if $n > N_L$, $|a_n - L| < \epsilon$ and if $n > N_M$, $|a_n - M| < \epsilon$. Let $\epsilon = \frac{|L-M|}{4}$, and $n > \max\{N_L, N_M\}$. Then, from trianble inequality in theorem 3.6.2,

$$\begin{aligned} |L-M| &\leq |L-a_n| + |a_n - M| < 2\epsilon \\ \implies |L-M| &\leq |L-a_n| + |a_n - M| < \frac{|L-M|}{2} \\ \implies |L-M| &< \frac{|L-M|}{2} \end{aligned}$$

This is impossible, since $|L-M| > 0$. Thus, $L = M$, and the limit of a sequence is unique.

$\square$

Exercise 8

Show that the sequence $a_n = (-1)^n$ is divergent

Proof. Take $a \in \mathbb{R}$ and $\epsilon = 1$. Then, for all $N \in \mathbb{N}$, take $n = 2N > N$. Then,

$$\begin{aligned} |a_n - a| &= |(-1)^n - a| \\ &= |(-1)^{2N} - a| \\ &= |1-a| \end{aligned}$$

If $a \leq 0$ or $a \geq 2$, then $|1-a| \geq \epsilon$, and a is not a limit. If $0 < a < 2$, we change n to be $2N+1 > 2N > N$. Then,

$$\begin{aligned} |a_n - a| &= |(-1)^n - a| \\ &= |(-1)^{2N+1} - a| \\ &= |-1-a| \geq 1 = \epsilon \end{aligned}$$

Thus, for all $a \in \mathbb{R}$, there exists $\epsilon > 0$ such that for all $N \in \mathbb{N}$, there is an $n > N$ such that $|a_n - a| \geq \epsilon$. This shows that the sequence is divergent.

$\square$

Bonus

Use the nested interval theorem to obtain a new proof of the fact that R is uncountable.

Proof. We already know there is a bijection between $(0,1)$ and $\mathbb{R}$, so we will prove that $(0,1)$ is uncountable.

Suppose it is countable. Then, the elements of $(0,1)$ are subscriptable as $\{a_1, a_2, \dots, a_n, \dots\}$. Since $(0,1)$ is a bounded interval in $\mathbb{R}$, it contains a nested sequence of closed bounded intervals in $\mathbb{R}$. Let these intervals I_n be such that $a_n \notin I_n$. From the nested interval theorem, $\cap_{n=1}^{\infty} I_n$ is nonempty. Then, there exists x_i such that $x_i \in \cap_{n=1}^{\infty} I_n$, or x_i is in every single interval I_n. However, this is a contradiction since the set of $I_{n>i}$ do not include x_i by definition. Therefore, $(0,1)$ is not countable, and concomitantly, $\mathbb{R}$ is uncountable.

$\square$