Math 15910: Problem Set 7

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Exercise 1

Problem 3.2.9

1. *Show that any irrational number multiplied by any non-zero rational number is irrational.*

   Proof. Take $x \in \mathbb{R}\backslash\mathbb{Q}$ and $\frac{a}{b} \in \mathbb{Q}\backslash\{0\}$ where $a, b \in \mathbb{Z}$. Let's suppose that the product of x and $\frac{a}{b}$ is a rational number $\frac{c}{d} \in \mathbb{Q}\backslash\{0\}$ where $c, d \in \mathbb{Z}$, 0 being excluded because x is irrational and therefore non-zero and $\frac{a}{b}$ is non-zero by assumption. Then,

   $$x \cdot \frac{a}{b} = \frac{c}{d}$$

   Because $\frac{a}{b} \neq 0$, $a \neq 0$. Since $\mathbb{Q}$ is a field with a multiplicative inverse, we can arrange the above as follows.

   $$x = \frac{cb}{da}$$

   Multiplication is closed in $\mathbb{Z}$. Hence, $cb, da \in \mathbb{Z}$, and $\frac{cb}{da} \in \mathbb{Q}$. This is a contradiction, so any irrational number multiplied by any non-zero rational number is irrational.

   $\square$

2. *Show that the product of two irrational numbers may be rational or irrational.*

   Proof. We begin by proving that the multiplicative inverse of an irrational number is irrational. Take $x \in \mathbb{R}\backslash\mathbb{Q}$. x is non-zero. Suppose $\frac{1}{x} = \frac{a}{b} ... \exists a, b \in \mathbb{Z}$. Then, $x = \frac{b}{a}$, which contradicts our assumption. Thus, $\frac{1}{x}$ is irrational. Then, $x \cdot \frac{1}{x} = 1 \in \mathbb{Q}$, so the product of two irrational numbers can be rational. Next, take $\sqrt{2}$. We know that $\sqrt{2} \in \mathbb{R}\backslash\mathbb{Q}$. Now, take a look at $\sqrt{\sqrt{2}}$. $\sqrt{\sqrt{2}} \cdot \sqrt{\sqrt{2}} = \sqrt{2}$. If $\sqrt{\sqrt{2}} \in \mathbb{Q}$, $\sqrt{2}$ would be rational since the rationals are closed under mulitplication. However, this is a contradiction, so $\sqrt{\sqrt{2}} = 2^{\frac{1}{4}}$ is an irrational number, and we have shown that the product of two irrational numbers may be rational or irrational.

   □

Exercise 2

Show rigorously that $\sup\{1 - \frac{1}{n} | n \in \mathbb{N}\} = 1$.

Proof. Since $n \in \mathbb{N}$, $\frac{1}{n} > 0$. Thus, it is easy to see that 1 is an upper bound. Then, for every positive and real ϵ, from the Archimedean property we know that there exists $1/n < \epsilon$. Thus, for every ϵ, there exists $x_\epsilon \in \{1 - \frac{1}{n} | n \in \mathbb{N}\}$ such that $x_\epsilon > 1 - \epsilon$. Thus, all upper bounds must be at least as large as 1, and since 1 is an upper bound, we know that 1 is the supremum of this set.

□

Exercise 3

Let $S = \sup\{1 - \frac{(-1)^n}{n} | n \in \mathbb{N}\}$. *Find sup* S *and inf* S.

Proof. First, we note that $S = A \cup B$ where $A = \{1 - \frac{(-1)^{2m}}{2m} | m \in \mathbb{N}\}$ and $B = \{1 - \frac{(-1)^{2m-1}}{2m-1} | m \in \mathbb{N}\}$. $2m$ is always even, while $2m - 1$ is always odd. Thus, we reformulate the above as $A = \{1 - \frac{1}{2m} | m \in \mathbb{N}\}$, $B = \{1 + \frac{1}{2m-1} | m \in \mathbb{N}\}$. Furthermore, we make use of the fact we proved in PS 6 that $\sup(A \cup B) = \max\{\sup(A), \sup(B)\}$ and the analogous $\inf(A \cup B) = \min\{\inf(A), \inf(B)\}$. By substituting n from the previous exercise by $2m$, we can easily find that $\sup A = 1$. Since $2m \in \mathbb{N}$, $\frac{1}{2m} \geq \frac{1}{2}$, $\frac{1}{2}$ is a lower bound of A. Since $\frac{1}{2} \in A$, for any $\frac{1}{2} + \epsilon$, the element $\frac{1}{2}$ in A is lower and therefore all lower bounds must be at least as small as $1/2$. Thus, the infimium of A is $1/2$.

Next, we look at $B = \{1 + \frac{1}{2m-1} | m \in \mathbb{N}\}$. Since $m \in \mathbb{N}$, $\frac{1}{2m-1} \leq 1$. Thus, 2 is an upper bound of B. Since $2 \in B$, for any $2 - \epsilon$, the element 2 in B is greater and therefore all upper bounds must be at least as great as $1/2$. Thus, the supremum of B is 2. Since $m \in \mathbb{N}$, $\frac{1}{2m-1} > 0$. Thus, it is easy to see that 1 is a lower bound of B. Then, for every positive and real ϵ, from the Archimedean property we know that there exists $\frac{1}{2n-1} \leq 1/n < \epsilon$. Thus, for every ϵ, there exists $x_\epsilon \in B$ such that $x_\epsilon < 1 + \epsilon$. Thus, all lower bounds must be at least as small as 1, and since 1 is a lower bound, we know that 1 is the infimum of B.

Therefore, $\sup S = \max\{\sup(A), \sup(B)\} = \max\{1, 2\} = 2$ and $\inf S = \min\{\inf(A), \inf(B)\} = \min\{1/2, 1\} = 1/2$.

□

Exercise 4

Show that every bounded open interval $(a, b) \subseteq \mathbb{R}$ *can be described as* $\{x \in \mathbb{R} || x - x_0| < \epsilon\}$ *for some value* $x_0 \in \mathbb{R}$ *and some* $\epsilon > 0$. *What are the values of* x_0 *and* ϵ *in terms of* a *and* b? *And conversely, given* x_0 *and* ε, *what are the endpoints of the interval* $\{x \in \mathbb{R} || x - x_0| < \epsilon\}$?

Proof. In this definition, x_0 denotes the median point between a and b, or $\frac{a+b}{2}$, and ϵ denotes half of the length of the interval $\frac{|b-a|}{2}$. $\{x \in \mathbb{R} || x - x_0| < \epsilon\}$ says that for every $x \in R$ such that the distance between x and x_0, the median point of the interval (a, b), measured by $|x - x_0|$, must be less than ϵ which is the distance from this median point to a or b. Thus the endpoints of this interval can be described as $x_0 \pm \epsilon$, and the interval can be restated as $(x_0 - \epsilon, x_0 + \epsilon)$ or the symmetric neighborhood of x_0.

□

Exercise 5

Problem 3.4.3

Suppose that I is a subset of $\mathbb{R}$. Show that I is an interval if and only if for all $a, b \in I$, with $a \leq b$, the closed interval $[a, b] \subseteq I$.

Proof. Let x, y be the endpoints of I, where $x < y$. Suppose I is an interval. Take $a, b \in I$. Then, $x \leq a$ and $b \leq y$, or else $a, b \notin I$. Because, all values between x and y are in I, any value between a and b is also between x and y and is therefore inside I. From our supposition, $a, b \in I$. Thus, $[a, b] \subseteq I$.

Now, take the statement if for all $a, b \in I$ such that $a \leq b$, $[a, b] \subseteq I$, then I is an interval. We prove the converse. Suppose I is not an interval. This means I cannot be written in any of the 10 categories described in Definition 3.4.1 of Sally. This means there is a discontinuity between 2 values in I, or 2 values $c, d \in I$ where $c < d$ such that $\frac{c+d}{2} \notin I$. Then, it follows that for some $a, b \in I$ such that $a \leq b$, $[a, b] \nsubseteq I$. Thus, the converse is true, and we have proven the statement.

□

Exercise 6

Let $I_n = [a_n, b_n]$ for $n \in \mathbb{N}$ be a nested collection of intervals. Suppose that $\inf\{b_n - a_n | n \in \mathbb{N}\} = 0$. *Show that the number $\xi \in \cap_{n=1}^{\infty} I_n$ is unique.*

Proof. Suppose that $\inf\{b_n - a_n | n \in \mathbb{N}\} = 0$. Then, for one of the intervals I_ξ, $a_n = b_n$. Thus, the interval is a point, or unique value ξ such that $a_n = b_n = \xi$. By the Nested Interval Theorem, we know that $\cap_{n=1}^{\infty} I_n \neq \emptyset/$. Therefore, there must be at least one value that is found in every interval. However, since we know that one of these intervals only contains one unique value ξ, for any value $i \in I_i$ to be in $\cap_{n=1}^{\infty} I_n$, $i \in I_\xi \implies i = \xi$. Thus, $\xi \in \cap_{n=1}^{\infty} I_n$ is unique.

□

Exercise 7

Problem 3.6.5

Show that the limit of a convergent sequence is unique.

Proof. Let us denote this sequence by a_n and its limits L, M. Suppose $L \neq M$. Then, for every $\epsilon > 0$, there exists $N_L, N_M \in \mathbb{N}$ such that if $n > N_L$, $|a_n - L| < \epsilon$ and if $n > N_M$, $|a_n - M| < \epsilon$. Let $\epsilon = \frac{|L-M|}{4}$, and $n > \max\{N_L, N_M\}$. Then, from trianble inequality in theorem 3.6.2,

$$
\begin{aligned}
|L - M| &\leq |L - a_n| + |a_n - M| < 2\epsilon \\
\implies |L - M| &\leq |L - a_n| + |a_n - M| < \frac{|L - M|}{2} \\
\implies |L - M| &< \frac{|L - M|}{2}
\end{aligned}
$$

This is impossible, since $|L - M| > 0$. Thus, $L = M$, and the limit of a sequence is unique.

□

Exercise 8

Show that the sequence $a_n = (-1)^n$ is divergent

Proof. Take $a \in \mathbb{R}$ and $\epsilon = 1$. Then, for all $N \in \mathbb{N}$, take $n = 2N > N$. Then,

$$\begin{aligned} |a_n - a| &= |(-1)^n - a| \\ &= |(-1)^{2N} - a| \\ &= |1 - a| \end{aligned}$$

If $a \leq 0$ or $a \geq 2$, then $|1 - a| \geq \epsilon$, and a is not a limit. If $0 < a < 2$, we change n to be $2N + 1 > 2N > N$. Then,

$$\begin{aligned} |a_n - a| &= |(-1)^n - a| \\ &= |(-1)^{2N+1} - a| \\ &= |-1 - a| \geq 1 = \epsilon \end{aligned}$$

Thus, for all $a \in \mathbb{R}$, there exists $\epsilon > 0$ such that for all $N \in \mathbb{N}$, there is an $n > N$ such that $|a_n - a| \geq \epsilon$. This shows that the sequence is divergent.

□

Bonus

Use the nested interval theorem to obtain a new proof of the fact that R is uncountable.

Proof. We already know there is a bijection between $(0, 1)$ and $\mathbb{R}$, so we will prove that $(0, 1)$ is uncountable.

Suppose it is countable. Then, the elements of $(0, 1)$ are subscriptable as $\{a_1, a_2, \dots, a_n, \dots\}$. Since $(0, 1)$ is a bounded interval in $\mathbb{R}$, it contains a nested sequence of closed bounded intervals in $\mathbb{R}$. Let these intervals I_n be such that $a_n \notin I_n$. From the nested interval theorem, $\cap_{n=1}^{\infty} I_n$ is nonempty. Then, there exists x_i such that $x_i \in \cap_{n=1}^{\infty} I_n$, or x_i is in every single interval I_n. However, this is a contradiction since the set of $I_{n>i}$ do not include x_i by definition. Therefore, $(0, 1)$ is not countable, and concomitantly, $\mathbb{R}$ is uncountable.

□