Math 15910: Problem Set 4

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Contents

Exercise 1

Show that if A and B are countably infinite sets, then there exists a bijection between their power sets $P(A)$ and $P(B)$.

Since A and B are countably infinite, there exist bijective functions $f : A \to \mathbb{N}$, $g : B \to \mathbb{N}$. Then, $g^{-1} \circ f$ is a bijective function between A and B. Thus, we know that A and B are bijective. Let us denote $h : A \to B$ to be an arbitrary bijection. Now think of the function $q : P(A) \to P(B)$ which takes any element of $P(A)$, $\{a_1, a_2, ...a_n\} \subseteq A$, and returns $\{h(a_1), h(a_2), ..., h(a_n)\} = \{b_1, b_2, ..., b_n\} \subseteq B$.

Take $\{a_1, ..., a_n\}, \{a'_1, ..., a'_n\} \in P(A)$.
If,

$$\begin{aligned} q(\{a_1, a_2, ..., a_n\}) &= q(\{a'_1, a'_2, ..., a'_n\}) \\ \implies \{h(a_1), h(a_2), ..., h(a_n)\}) &= \{h(a'_1), h(a'_2), ..., h(a'_n)\} \end{aligned}$$

From the definition of equality of sets, and without loss of generality,

$$h(a_1) = h(a'_1), h(a_2) = h(a'_2)..., h(a_n) = h(a'_n)$$

Since h is bijective,

$$\implies \{h^{-1}(h(a_1)), h^{-1}(h(a_2)), ..., h^{-1}(h(a_n))\}) = \{h^{-1}(h(a_1')), h^{-1}(h(a_2')), ..., h^{-1}(h(a_n'))\}$$
$$\implies \{a_1, a_2, ..., a_n\} = \{a_1', a_2', ..., a_n'\}$$

From the definition of equality of sets, and without loss of generality,

$$a_1 = a_1', a_2 = a_2' ..., a_n = a_n'$$

Thus, q is injective.

Now, take $\{b_1, \dots, b_n\} \in P(B)$. We want $\{a_1, \dots, a_n\} \in P(A)$ such that $q(\{a_1, \dots, a_n\}) = \{b_1, \dots, b_n\}$.

$$q(\{a_1, ..., a_n\}) = \{b_1, \dots, b_n\}$$
$$\implies \{h(a_1), ..., h(a_n)\}) = \{b_1, \dots, b_n\}$$
$$\implies \{a_1, \dots, a_n\} = \{h^{-1}(b_1), \dots, h^{-1}(b_n)\}$$

Thus, given $\{b_1, \dots, b_n\} \in P(B)$, $q(\{h^{-1}(b_1), \dots, h^{-1}(b_n)\}) = \{b_1, \dots, b_n\}$. Hence, q is surjective.

It follows that q is bijective. With this, we have shown that there exists a bijection between $P(A)$ and $P(B)$ given two countably infinite sets A and B.

$\square$

Exercise 2

Problem 1.8.31

Suppose that A is a non-empty set. Show that $P(A)$ is in one to one correspondence with the set of all functions from A to $\{0, 1\}$

For every subset $X \subseteq A$, we define a function $f_X : X \to \{0, 1\}$ such that for every $a \in A$,

$$f_X(a) = \begin{cases} 1, & \text{if } a \in X \\ 0, & \text{if } a \notin X \end{cases}$$

Now, we define the function $F : P(A) \to \{f | f : A \to \{0, 1\}\}$ that takes a subset $B \in P(A)$ and returns f_B from the set of functions $\{f | f : A \to \{0, 1\}\}$.

Suppose $B, C \subseteq A$ and $F(B) = F(C)$. Then,

$$f_B = f_C$$
$$\implies f_B(a) = f_C(a) \text{ for every } a \in A$$

Since $B, C \subseteq A$, there does not exist any element within them that is not also in A. If $f_B(a) = f_C(a) = 1$, then $a \in B, C$. If $f_B(a) = f_C(a) = 0$, then $a \notin B, C$. By the definition of set equality, $B = C$. Thus, F is injective.

Now, for every $f_X \in \{f | f : A \to \{0, 1\}\}$ there exists $X \subseteq A$ such that $F(X) = f_X$.

$$F(X) = f_X$$

$$\implies X = \{a \in A | f_X(a) = 1\}$$

Thus, we have that for every $f_X \in \{f | f : A \to \{0, 1\}\}$, $F(\{a \in A | f_X(a) = 1\}) = f_X$, and F is surjective.

It follows that F is bijective, and $P(A)$ is in one to one correspondence with the set of all functions from A to $\{0, 1\}$.

$\square$

Exercise 3

Prove that a union $\cup_{n\in\mathbb{N}} A_n$ where sets A_n have the cardinality of the set of real numbers, has the cardinality of the set of real numbers.

We know that there is a bijection between $(0,1)$ and $\mathbb{R}$, so $(0,1)$ has the same cardinality as $\mathbb{R}$. In the last homework, I showed that there exists a bijection $f:[0,1)\to(0,1)$, defined as

$$f(x)=\begin{cases}\frac{n+1}{n+2} \text{ if } x=\frac{n}{n+1} \text{ for } n\in\mathbb{N}_0\\ x \text{ for all other } x\in[0,1)\end{cases}$$

Thus, $[0,1)$ has the same cardinality as the set of real numbers. Similarly, we can define functions for $[0,1),[1,2),...,[n-1,n)$ that take the form

$$f_n(x)=\begin{cases}\frac{m+1}{m+2} \text{ if } x=n-1+\frac{m}{m+1} \text{ for } m\in\mathbb{N}_0\\ x-n+1 \text{ for all other } x\in[n-1,n)\end{cases}$$

which are all bijections with $(0,1)$, and thus have equal cardinality to the set of real numbers. Since each A_n has the cardinality of the set of real numbers, and the sets $[0,1),[1,2),...,[n-1,n)$ also have the same cardinality, there exists a bijection between A_n and $[n-1,n)$. We now show that there is an injection from $\cup_{n\in\mathbb{N}} A_n$ to $[0,n)$ and an injection from $[0,n)$ to A_n.

Since for all A_i for $1\leq i\leq n$, A_i is bijective with $[i-1,i)$, there exists an injection in this direction. If all A_i are mutually disjoint, they would each have an injection from A_i to $[i-1,i)$, and thus the union $\cup_{n\in\mathbb{N}} A_n$ will inject to $[0,n)$, which is the union of all $[i,i-1)$. If the sets are not mutually disjoint, then each $a\in\cup_{n\in\mathbb{N}} A_n$ will inject to the $[i,i-1)$ corresponding to the first A_i in which a appears. It is clear that this is an injection, and thus there is an injection from $\cup_{n\in\mathbb{N}} A_n$ to $[0,n)$.

Next, since $[0,1)$ is bijective to A_1, there exists an interjection from $[0,1)$ to A_1. Then, by multiplying all elements in $[0,1)$ by n, we can create an injection from $[0,n)$ to A_1. Since $A_1\in\cup_{n\in\mathbb{N}} A_n$, it follows that there exists an injection from $[0,n)$ to A_n.

Since there is an injection from $\cup_{n\in\mathbb{N}} A_n$ to $[0,n)$ and an injection from $[0,n)$ to A_n, we can invoke the Schroeder-Bernstein theorem to conclude that there exists a bijection between $\cup_{n\in\mathbb{N}} A_n$ and $[0,n)$. It is easy to see that there exists a bijective function $f:[0,1)\to[0,n)$ such that

$$f(x)=nx$$

It is trivial to prove that this function is bijective. Thus, we know that [0, n) has the same cardinality as $\mathbb{R}$.

From here, we can conclude that a union $\cup_{n\in\mathbb{N}} A_n$ where sets A_n have the cardinality of the set of real numbers, has the cardinality of the set of real numbers.

□

Exercise 4

Prove that the set of irrational numbers has the same cardinality as the set of real numbers.

Consider the function

$$f(x)=\begin{cases}\arctan x \text{ when } \arctan x\in\mathbb{R}\backslash\mathbb{Q}\\ \arctan x+10\sqrt{2} \text{ when } \arctan x\in\mathbb{Q}\end{cases}$$

Given $x\in\mathbb{R}$, this function produces an irrational number, so this function is well defined.

Now, take $x_1,x_2\in\mathbb{R}$. Suppose $f(x_1)=f(x_2)$. Then, either $\arctan x_1$ and $\arctan x_2$ are both rational or both irrational. If x_1 was rational and $x_2$was irrational, then $\arctan x_1+10\sqrt{2}\neq\arctan x_2$ since the maximum possible value of $\arctan x$ is $\frac{\pi}{2}$ and the minimum is $-\frac{\pi}{2}$, making $\arctan x_1+10\sqrt{2}>\arctan x_2$.

When $\arctan x_1$ and $\arctan x_2$ are both rational,

$$f(x_1)=f(x_2)$$

$$
\begin{aligned}
&\implies \arctan x_1 + 10\sqrt{2} = \arctan x_2 + 10\sqrt{2} \\
&\implies \arctan x_1 = \arctan x_2 \\
&\implies \tan(\arctan x_1) = \tan(\arctan x_2) \\
&\implies x_1 = x_2
\end{aligned}
$$

When $\arctan x_1$ and $\arctan x_2$ are both rational,

$$
\begin{aligned}
f(x_1) &= f(x_2) \\
\implies \arctan x_1 &= \arctan x_2 \\
\implies \tan(\arctan x_1) &= \tan(\arctan x_2) \\
\implies x_1 &= x_2
\end{aligned}
$$

Thus, f is an interjection. Since $\mathbb{R}\backslash\mathbb{Q} \subseteq \mathbb{R}$, it is trivial to prove that there exists an injection $g : \mathbb{R}\backslash\mathbb{Q} \to \mathbb{R}$ (the identity function would be such an injection).

Hence, we can invoke the Schroeder-Bernstein theorem to conclude that there exists a bijection between the set of irrational numbers and the set of real numbers, and that they have the same cardinality.

$\square$

Exercise 5

Problem 1.6.2

Let $\mathcal{R}$ be a relation on X that satisfies
(a.) *For all $a \in X, (a, a) \in \mathcal{R}$, and*
(b.) *for $a, b, c \in X$, if $(a, b), (b, c) \in \mathcal{R}$, then $(c, a) \in \mathcal{R}$*

From the assumption (a.), we already have reflexivity. Now, take $(a, b) \in \mathcal{R}$. From (a.), we have $(b, b) \in \mathcal{R}$. From (b.), if $(a, b) \in \mathcal{R}$, we can say

$$
\begin{aligned}
&(a, b), (b, b) \in \mathcal{R} \\
&\implies (b, a) \in \mathcal{R}
\end{aligned}
$$

Thus, we obtain symmetry. With symmetry, it can be easily shown from (b.) that

$$(a, b), (b, c) \in \mathcal{R} \implies (c, a) \in \mathcal{R}$$

$$(c, a) \in \mathcal{R} \implies (a, c) \in \mathcal{R}$$

Thus,

$$(a, b), (b, c) \in \mathcal{R} \implies (a, c) \in \mathcal{R}$$

Therefore, we have established reflexivity, symmetry, and transitivity in $\mathcal{R}$, and have proven that $\mathcal{R}$ is an equivalence relation.

$\square$

Exercise 6

Problem 1.6.14

Take a set X and break it up into pairwise disjoint non-empty subsets whose union is all of X. Then, for $a, b \in X$, define $a \sim b$ if a and b are in the same subset. Prove that this is an equivalence relation

We denote these subsets as Y_i. Suppose $a \in Y_a$. Then, since $a \in Y_a$, $a \sim a$ and we have reflexivity. If $a \sim b$, then $a, b \in Y_a$. However, this would also imply $b \sim a$. Thus, we have symmetry. If $a \sim b$, then $a, b \in Y_a$. Furthermore, if $b \sim c$, then $b, c \in Y_b$. However, since Y_i's are pairwise disjoint, it follows from $b \in Y_a, b \in Y_b$ that $Y_a = Y_b$. thus, $a, c \in Y_a$ and $a \sim c$. From here, we have transitivity. Thus, the above is an equivalence relation.

$\square$

Exercise 7

Let A be a set, and $P(A)$ its power set. For $x, y \in P(A)$ let $x \sim y$ if x and y have the same cardinality. Prove that $\sim$ is an equivalence relation.
Compute the equivalence classes when $A = \{1, 2, 3\}$.

Take $x \in P(A)$. Then, since $|x| = |x|$, $x \sim x$. Thus, $\sim$ has reflexivity. For $x, y \in P(A)$, if $x \sim y$, then $|x| = |y|$. If $y \sim x$, then $|y| = |x|$. Thus, $x \sim y \implies y \sim x$. Hence, we have symmetry. If $x \sim y$, then $|x| = |y|$. Furthermore, if $y \sim z$, then $|y| = |z|$. It follows that $|x| = |z|$, and $x \sim z$. From here, we have transitivity. Thus, the above is an equivalence relation.

The equivalence classes for when $A = \{1, 2, 3\}$ are as follows:

- $\{x \in P(A) || x| = 0\} = \{\emptyset\}$
- $\{x \in P(A) || x| = 1\} = \{\{1\}, \{2\}, \{3\}\}$
- $\{x \in P(A) || x| = 2\} = \{\{1, 2\}, \{1, 3\}, \{2, 3\}\}$
- $\{x \in P(A) || x| = 3\} = \{\{1, 2, 3\}\}$

□

Exercise 8

Problem 1.6.15

We consider the set $F = \{\{a, b\} | a, b \in \mathbb{Z}$ and $b \neq 0\}$. For $(a, b), (c, d) \in F$, we define $(a, b) \sim (c, d)$ if $ad = bc$. Thus, for instance, $(2, 3) \sim (8, 12) \sim (-6, -9)$.
Show that $\sim$ is an equivalence relation on F

Take $(x, y) \in F$. Then, it is obvious that $xy = xy$, so $(x, y) \sim (x, y)$. So, $\sim$ is reflexive. Now, take $(x_1, y_1), (x_2, y_2) \in F$.

$$(x_1, y_1) \sim (x_2, y_2) \implies x_1 y_2 = x_2 y_1$$

Since equality $=$ is an equivalence relation,

$$\implies x_2 y_1 = x_1 y_2 \implies (x_2, y_2) \sim (x_1, y_1)$$

Hence, we have symmetry. Assume $(x, y) \sim (q, r)$ and hence $xr = yq$. Furthermore, assume $(q, r) \sim (s, t)$ and hence $qt = rs$. Then,

$$\begin{gathered} xr = yq \\ \implies xr * t = yq * t \\ \implies xrt = yqt \end{gathered}$$

From $qt = rs$,

$$\implies xrt = yrs$$

Since $r \neq 0$,

$$\implies xt = ys$$

$$\implies (x, y) \sim (s, t)$$

From here, we have $(x, y) \sim (q, r), (q, r) \sim (s, t) \implies (x, y) \sim (s, t)$. Thus, $\sim$ is transitive, and is an equivalence relation.

□