Math 15910: Problem Set 9

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Exercise 1

Problem 3.10.25

Suppose that a series $\sum_{n=1}^{\infty} a_n$ *converges. Show that* $\lim_{n\to\infty} = 0$.

Proof. Since $\sum_{n=1}^{\infty} a_n$ converges, we know that it is Cauchy, and that $\forall \epsilon > 0, \exists N \in \mathbb{N}$ such that $\forall n, m > N$, the below holds:

$$|\sum_{k=m+1}^{n} a_k| < \epsilon$$

Now, let $\epsilon = \frac{\epsilon}{2}$ and $n, m > N$ such that the above holds. Then, the above also holds for $n-1, m$, assuming $n-1 > m > N$. Thus, we have

$$|\sum_{k=m+1}^{n} a_k| < \frac{\epsilon}{2}$$

$$|\sum_{k=m+1}^{n-1} a_k| < \frac{\epsilon}{2}$$

$$\implies |\sum_{k=m+1}^{n} a_k - \sum_{k=m+1}^{n-1} a_k| = |a_n| < |\sum_{k=m+1}^{n} a_k| + |\sum_{k=m+1}^{n-1} a_k| < \epsilon$$

$$|a_n| < \epsilon$$

Thus, $\lim_{n\to\infty} = 0$.

□

Exercise 2

1. *Prove the Comparison Test.*

   Proof. Since $a_n > 0$ for $n \in \mathbb{N}$ and $\sum_{n=1}^{\infty} a_n$ converges, from the monotone criterion, we know that $(A_k)_{k\in\mathbb{N}}$ is bounded, where A_k represents the partial sum of a_n. Then, since $|b_n| \leq a_n$ for all n,

   $$|B_k| = \sum_{n=1}^{k} |b_n| \leq \sum_{n=1}^{k} a_n = A_n \implies (|B_k|)_{k\in\mathbb{N}} \text{ is bounded}$$

   Thus, $\sum_{n=1}^{\infty} |b_n|$ converges and $\sum_{n=1}^{\infty} b_n$ does too.

2. *If the series $\sum_{n=1}^{\infty} a_n$ converges to s and c is any constant, show that the series $\sum_{n=1}^{\infty} ca_n$ converges to cs.*

   Proof. Since c is a constant,

   $$C_k = \sum_{n=1}^{k} ca_n = c\sum_{n=1}^{k} a_n = cA_k$$

   Thus, $(C_k)_{k\in\mathbb{n}} = (cA_k)_{k\in\mathbb{n}}$. Since $(A_k)_{k\in\mathbb{n}}$ is a convergent sequence whose limit is zero, following the easily proven limit law of multiplication by scalar.

3. *Suppose that $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ are infinite series. Suppose that $a/n > 0$ and $b_n > 0$ for $n \in \mathbb{N}$ and $\lim_{n\to\infty} a_n/b_n = c > 0$. Show that $\sum_{n=1}^{\infty} a_n$ converges iff $\sum_{n=1}^{\infty} b_n$ converges.*

   Proof. From $\lim_{n\to\infty} a_n/b_n = c > 0$, we have for all $\epsilon > 0$ there is $N \in \mathbb{N}$ such that for all $n > N$,

   $$\begin{aligned} &|\frac{a_n}{b_n} - c| < \epsilon \\ \implies &-\epsilon < \frac{a_n}{b_n} - c < \epsilon \\ \implies &-\epsilon + c < \frac{a_n}{b_n} < \epsilon + c \\ \implies &b_n(-\epsilon + c) < a_n < b_n(\epsilon + c) \end{aligned}$$

   Thus, when b_n converges, from (2.), $b_n(\epsilon + c)$ converges, and since that bounds a_n above, a_n converges. Similarly, when a_n converges, $b_n(-\epsilon + c)$ converges and hence b_n converges.

   □

Exercise 3

Problem 3.10.2.11 (ii)

if $p \in \mathbb{R}$ and $p < 1$, show that $\sum_{n=1}^{\infty} 1/n^p$ diverges.

Proof. When $p = 1$, we have the harmonic series. We also have that $1/n < 1/n^p$ for all n when $p < 1$. The harmonic series diverges, and from the comparison test, $\sum_{n=1}^{\infty} 1/n^p$ diverges as well.

□

Exercise 4

Prove the alternating series test: Let (b_n) be a non-increasing sequence where $b_n \geq 0$ for all $n \in \mathbb{N}$ and $\lim_{n\to\infty} b_n = 0$. *Show that the series $\sum_{n=1}^{\infty}(-1)^{n+1}b_n$ converges.*

Proof.

$$\begin{aligned} S_{2k} &= b_1 - b_2 + b_3 - b_4 ... b_{2k-1} - b_{2k} \\ &= (b_1 - b_2) + (b_3 - b_4) ... + (b_{2k-1} - b_{2k}) \end{aligned}$$

Since b_n is nondecreasing, all $(b_n - b_{n+1}) \geq 0$, and thus $(S_{2k})_{k\in\mathbb{N}}$ is non-decreasing.

Since $\lim_{n\to\infty} b_n = 0$,

$$\lim_{k\to\infty} b_{2k-1} - b_{2k} = 0 - 0 = 0$$

So $(S_{2k})_{k\in\mathbb{N}}$ is bounded above.

From the monotone criterion, $S_{2k} \to S$ for some $S \in \mathbb{R}$.

$$S_{2k+1} = S_{2k} + b_{2k+1}$$

$$\implies \lim_{k\to\infty} S_{2k+1} = \lim_{k\to\infty} S_{2k} + b_{2k+1} = S + 0 = S$$

Thus, for all S_k, $\lim_{k\to\infty} S_k = S$. Because the sum of partials converges, $\sum_{n=1}^{\infty}(-1)^{n+1}b_n$ converges as well.

$\square$

Exercise 5

Complete the proof that every rearrangement of the absolutely convergent series converges to the same sum.

Proof.

$$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} a_n^+ - \sum_{n=1}^{\infty} a_n^-$$

By factoring the negative sign in to the summation, we get

$$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} a_n^+ + \sum_{n=1}^{\infty} -a_n^-$$

Since $a_n^- = -a_n$ for $a_n < 0$,

$$\sum_{n=1}^{\infty} -a_n^- = \sum_{a_n<0} a_n$$

Similarly, by definition,

$$\sum a_n^+ = \sum_{a_n\geq 0} a_n$$

Since all three sums converge, we can treat them as finite sums and reorder them. From here, we have

$$\sum_{a_n\geq 0} a_n + \sum_{a_n<0} a_n$$

$$= \sum_{n=1}^{\infty} a_n$$

And thus,

$$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} a_n^+ - \sum_{n=1}^{\infty} a_n^-$$

$\square$

Bonus

Show that $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = \ln 2$

$$S_{2k} = \frac{1}{1} - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} ... + \frac{1}{2k-1} - \frac{1}{2k}$$

$$= \frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} ... + \frac{1}{2k-1} + \frac{1}{2k} - 2(\frac{1}{2} + \frac{1}{4} + \frac{1}{6} ... + \frac{1}{2k-1} + \frac{1}{2k})$$
$$= \frac{1}{k+1} + \frac{1}{k+2} + ... + \frac{1}{2k}$$
$$= \sum_{n=k+1}^{2k} \frac{1}{n}$$

Then,

$$\int_{n=2k}^{4k} \frac{1}{n} < \sum_{n=k+1}^{2k} \frac{1}{n} < \int_{n=k+1}^{2k+2} \frac{1}{n}$$

which evaluates to $ln2$?

Or, you could just say $\sum_{n=k+1}^{2k} \frac{1}{n} = \sum_{n=k+1}^{2k} \frac{k}{n} \frac{1}{k}$ and use the Riemann sum to conclude

$$\int_1^2 \frac{1}{x} dx = [\ln x]_1^2 = \ln 2$$