MIDTERM EXAM
MATH 15910, WINTER 2021

Problem 1. Let $X = \mathbb{Z} \times \mathbb{Z}$. Given $(x_1, y_1) \in X$ and $(x_2, y_2) \in X$ we define

$$(x_1, y_1) \sim (x_2, y_2) \quad \text{if} \quad x_1 - 3y_1 = x_2 - 3y_2.$$

Prove that $\sim$ is an equivalence relation. Describe its equivalence classes. What is the cardinality of the set of the equivalence classes of $\sim$?

Solution.

(1) We verify that $\sim$ is reflexive: For every $(x, y) \in \mathbb{Z}\times\mathbb{Z}$ we have $x-3y = x-3y$, so $(x, y) \sim (x, y)$.
(2) We verify that $\sim$ is symmetric: Let $(x_1, y_1), (x_2, y_2) \in \mathbb{Z} \times \mathbb{Z}$ such that $(x_1, y_1) \sim (x_2, y_2)$. That means $x_1 - 3y_1 = x_2 - 3y_2$, so also $x_2 - 3y_2 = x_1 - 3y_1$. Therefore $(x_2, y_2) \sim (x_1, y_1)$.
(3) We verify that $\sim$ is transitive: Let $(x_1, y_1), (x_2, y_2), (x_3, y_3) \in \mathbb{Z} \times \mathbb{Z}$ such that $(x_1, y_1) \sim (x_2, y_2)$ and $(x_2, y_2) \sim (x_3, y_3)$. That means $x_1 - 3y_1 = x_2 - 3y_2$ and $x_2 - 3y_2 = x_3 - 3y_3$, so $x_1 - 3y_1 = x_3 - 3y_3$. Therefore $(x_1, y_1) \sim (x_3, y_3)$.

We have shown that $\sim$ is an equivalence relation. For every $k \in \mathbb{Z}$, there is an equivalence class of the form

$$\{(x, y) \in \mathbb{Z} \times \mathbb{Z} : x - 3y = k\}.$$

Thus the cardinality of the set of equivalence classes is in one-to-one correspondence with the set $\mathbb{Z}$, and therefore it is countable infinite. $\square$

Problem 2. Prove that for every $n \in \mathbb{N}$, we have

$$\frac{1}{1\cdot 3}+\frac{1}{3\cdot 5}+\frac{1}{5\cdot 7}+\cdots+\frac{1}{(2n-1)\cdot(2n+1)}=\frac{n}{2n+1}.$$

Solution. We prove it by induction.

(1) Base case: For $n=1$, the left hand side is equal $\frac{1}{1\cdot 3}=\frac{1}{3}$. The right hand side is equal to $\frac{1}{2\cdot 1+1}=\frac{1}{3}$, so both sides are equal.

(2) Inductive step: Suppose the statement holds for $n \geq 1$, i.e. $\frac{1}{1\cdot 3}+\frac{1}{3\cdot 5}+\frac{1}{5\cdot 7}+\cdots+\frac{1}{(2n-1)\cdot(2n+1)}=\frac{n}{2n+1}$. We want to prove that it holds for $n+1$:

$$\frac{1}{1\cdot 3}+\frac{1}{3\cdot 5}+\frac{1}{5\cdot 7}+\cdots+\frac{1}{(2n-1)\cdot(2n+1)}+\frac{1}{(2(n+1)-1)\cdot(2(n+1)+1)}=$$
$$=\frac{n}{2n+1}+\frac{1}{(2n+1)(2n+3)}=\frac{n(2n+3)+1}{(2n+1)(2n+3)}=\frac{(2n+1)(n+1)}{(2n+1)(2n+3)}=\frac{n+1}{2n+3}$$

$\square$

Problem 3. Let $f : A \to B$ be any function, and let $U \subseteq B$. Prove or disprove (e.g. by showing an example) the following statements.

(a) If $f^{-1}(U)$ is countable, then U is countable.
(b) If U is countable, then $f^{-1}(U)$ is countable.
(c) If f is injective, then there exists an injective function $P(A) \to P(B)$.

Solution.

(a) This statement is false. Here is an example where it does not hold. Let $A = \{1\}$, $B = \mathbb{R}$, and let $f(1) = 1$. Let $U = B = \mathbb{R}$ (or any other uncountable subset of B). Then U is uncountable, but $f^{-1}(U) = \{1\}$ is finite, so countable.
(b) This statement is false. Here is an example where it does not hold. Let $A = \mathbb{R}$, and $B = \{1\}$, and let $f(x) = 1$ for all $x \in A$. Let $U = B = \{1\}$. Then U is finite, so countable, but $f^{-1}(U) = A = \mathbb{R}$ is uncountable.
(c) This statement is true. We can define a function $F : P(A) \to P(B)$ by $U \mapsto f(U)$ for every $U \in P(A)$. We verify taht F is injective. Take $U_1, U_2 \subseteq A$ such that $f(U_1) = f(U_2)$. Since f is injective, it follows that $U_1 = U_2$.

□

Problem 4. Let X be the set of all numbers in interval $(0, 1)$ that have a finite decimal expansion, i.e.

$$X = \{x \in (0,1) \mid \exists m \geq 1 \text{ such that } x = \sum_{i=1}^{m} \frac{x_i}{2^i} \text{ where } x_i \in \{0,1\} \text{ for all } i \in \{1,\ldots,m\}\}$$

Show that X is countably infinite.

Solution. Given $m \in \mathbb{N}$, let $X_m = \{x \in (0,1) \mid x = \sum_{i=1}^{m} \frac{x_i}{2^i}$ where $x_i \in \{0,1\}$ for all $i \in \{1,\ldots,m\}\}$. The cardinality of X_m is finite, equal to $2^m - 1$. Since $X = \bigcup_{m \in \mathbb{N}} X_m$, i.e. X is a countable union of countable sets, we conclude that X is countable (see 1.8.25(3)). It remains to show that X is not finite. Here is an example of an infinite subset of X: $\{\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \ldots\}$. $\square$