4 Ramsey model

4.1 Introduction

What we will learn

- the simplest Dynamic General Equilibrium (DGE) model
- A technique called *Representative Agent* (代表的個人)
- Analytical exercise
- Quantitative exercise using Dynare

Introduction

- We will extend the household decision model to the general equilibrium.
- Households optimize the dynamic problem given the interest rate R_t.
- R_t is determined by the loan demand/supply in the market.
- DSGE stnads for the Dyanmic Stochastic General Equlibrium (動学的一般均衡モデル).
- Ramsey model (or Ramsey-Cass-Koopmans model) is the first one. But it is Deterministic, not Stochastic.
- It is also called optimal growth model (or neoclassical growth model, 最適成長モデル、新古典派成長モデル). In the Solow model, the saving rate s is a fixed parameter. In the optimal growth model, s is chosen by the household utility maximization problem under the market equilibrium.
- Miao, chapter 14. Niepelt Chapter 3.

4.2 Sequential problem

4.2.1 Setup

There are many households and firms. Precisely,

- there is a unit mass of continuous households.
- there is also a unit mass of a continuum of firms.

■Household The household problem is almost the same.

$$\max \sum_{t=0}^{\infty} \beta^t u(c_t)$$

$$\begin{aligned} \text{s.t. } k_{t+1} + c_t &= R_t k_t + w_t + \Pi_t \\ R_t &= 1 + r_t - \delta \end{aligned} \tag{1}$$

- For simplicity, all households are symmetric.

- Typically, $u(c_t) = \ln c_t$ or $u(c_t) = \frac{c_t^{1-\sigma}}{1-\sigma}$.
- Household asset a_t is replaced by capital k_t. We assume that each household directly own capital k_t. In the real world, capital is owned by firms, and the firms are owned by stock holders.
- labor supply is fixed as 1 unit. The wage w_t is labor income.
- Π_t is the financial income. As a stock holder, households receive dividends. (配当)
- R_t is called (gross) rental rate. It is slightly different from the (net) interest rate r_t. Since the household owns capital, it also needs to pay the depreciation costs.
- We also need the transversality condition to exclude infinite borrowing.

$$\lim_{T\to\infty} \beta^T u'(c_T) k_{T+1} = 0 \tag{2}$$

■Firm Firm's problem is simple. The capital is owned by households, so each firm cares only one period profit.

$$\Pi_t = \max_{K_t, N_t} F(K_t, N_t) - r_t K_t - w_t N_t \tag{3}$$

- K_t is the capital demand. N_t is the labor demand. The demands are denoted by large letters.
- F is the production function. homogeneous of degree one.

■Market There are capital, labor, and good markets.

- Households are symmetric. Total population is 1. The total supply of capital and labor are k_t and 1.
- Firms are also symmetric and population 1. Total demand is K_t and N_t

$$\text{labor market clearing: } 1 = N_t \tag{4}$$
$$\text{capital market clearing: } k_t = K_t \tag{5}$$

The good market condition is implicitly defined by the Walras law. The household budget constraint is written as

$$k_{t+1} + c_t = (1 + r_t - \delta)k_t + w_t + \Pi_t = (1-\delta)k_t + r_t k_t + w_t \times 1 + \Pi_t$$

Then, by the firm's problem,

$$k_{t+1} + c_t = (1-\delta)k_t + f(k_t) \tag{6}$$

Note that $f(k_t) \equiv F(K_t, 1)$. Equation (6) is the good market clearing condition. It is also called the resource constraint. (資源制約) A kind of budget constraint of macroeconomy itself. Each period, $(1-\delta)k_t$ capital is carried over. Additional $f(k_t)$ is added. Then, it is divided into the future k_{t+1} or consumption c_t.

4.2.2 Definition of Equilibrium

Two components.

- Variables: prices and allocations
- Conditions: Optimized conditions and equilibrium conditions

Definition 1. *The dynamic competitive equilibrium consists of*

(a) a sequence of prices $\{r_t, w_t\}_{t=0}^{\infty}$
(b) a sequence of allocations $\{k_t, K_t, N_t, \Pi_t\}_{t=0}^{\infty}$

such that

(i) Given $\{r_t, w_t, \Pi_t\}_{t=0}^{\infty}$*, households solve the problem* (1) *and derive* $\{c_t, k_{t+1}\}_{t=0}^{\infty}$
(ii) Given (r_t, w_t)*, firms solve the problem* (3) *and derive* (K_t, N_t, Π_t) *for each period* t
(iii) The transversality condition (2) *is satisfied.*
(iv) Labor market clearing condition (4) *is satisfied.*
(v) Capital market clearing condition (5) *is satisfied.*
(vi) Good market clearing condition (6) *is satisfied.*

4.2.3 Equilibrium

First, think about the firm's problem. I will show that $\Pi_t = 0$.

$$\Pi_t = \max_{K_t, N_t} F(K_t, N_t) - r_t K_t - w_t N_t$$

Remember, $F(K_t, N_t)$ is homogeneous of degree one (一次同次) or constant returns to scale. (規模に関して収穫一定) That is, for any $z \in \mathbb{R}_+$,

$$F(zK, zN) = zF(K, N)$$

Suppose $\Pi_t > 0$. Then, for any $z > 1$,

$$\Pi_t = F(K_t, N_t) - r_t K_t - w_t N_t < F(zK_t, zN_t) - r_t zK_t - w_t zN_t = z\Pi_t.$$

Hence, firms will increase production and $K_t \to \infty$ and $N_t \to \infty$ in the equilibrium. It will never happens. Hence, in the equilibrium, r_t and w_t increase so that $\Pi_t = 0$. Indeed, if $\Pi_t < 0$, no demand for capital and labor. Then, the equilibrium condition is also violated.

The firm's optimality conditions are

$$\frac{\partial F(K_t, N_t)}{\partial K_t} = r_t \tag{7}$$
$$\frac{\partial F(K_t, N_t)}{\partial N_t} = w_t \tag{8}$$

Let's solve household problems. Given $\Pi_t = 0$,

$$\begin{aligned} \text{Euler: } & u'(c_t) = \beta(1 + r_{t+1} - \delta)u'(c_{t+1}) & (9) \\ \text{budget: } & k_{t+1} + c_t = (1 + r_t - \delta)k_t + w_t & (10) \\ \text{ransversality: } & \lim_{T\to\infty} \beta^T u'(c_T)k_{T+1} = 0 & (11) \end{aligned}$$

Then, we can redefine the equilibrium.

Definition 2. *The dynamic competitive equilibrium consists of a sequence of prices* $\{r_t, w_t\}_{t=0}^{\infty}$ *and a sequence of allocations* $\{k_t, K_t, N_t\}_{t=0}^{\infty}$ *such that optimality conditions,* (7), (8), (9), (10), (11), *and market conditions,* (4), (5) *and* (6).

Let's eliminate prices to simplify the system of dynamic equations. The economy can be described by the only three conditions.

Definition 3. *The system of equations of the Ramsey model are*

$$\begin{aligned} \textit{Euler: } & u'(c_t) = \beta(1 + f'(k_{t+1}) - \delta)u'(c_{t+1}) & (12) \\ \textit{Resource: } & k_{t+1} + c_t = (1 - \delta)k_t + f(k_t) & (13) \\ \textit{Transversality: } & \lim_{T\to\infty} \beta^T u'(c_T)k_{T+1} = 0 & (14) \end{aligned}$$

4.2.4 Recursive problem

Let's redefine the problem using dynamic programming. It is important to distinguish the state variables (状態変数) and choices. For households, state variables are k. The choices are c and k'.

The household problem also depends on the market prices r and w. These are also state variables. By the static firm's problem, they are defined as the price functions.

$$r(K) = \frac{\partial F(K, 1)}{\partial K}, \quad (15)$$

$$w(K) = \frac{\partial F(K, 1)}{\partial N} \quad (16)$$

Then, the household has two state variables. The individual state variable k and the aggregate state variale K. Here, the aggregate variables are represented by capital letters.

$$\begin{aligned} & V(k, K) = \max u(c) + \beta V(k', K') \\ & \text{ s.t. } k' + c = (1 + r(K) - \delta)k + w(K) \quad (17) \end{aligned}$$

This solution is policy functions (decision rules), $c(k, K)$ and $k'(k, K)$.

The resource constraint is

$$K' + C = (1 - \delta)K + F(K, 1) \quad (18)$$

Let's define another concept of equilibrium. In the sequential problem, we specify the sequence of variables. Here, we define the functions.

Definition 4. *The recursive competitive equilibrium consists of the value function $V(k,K)$, policy functions $c(k,K)$ and $k'(k,K)$, and price functions $r(K)$ and $w(K)$ such that*

(i) Given $r(K)$, $w(K)$ and $V(k,K)$, households solve the problem (17) and derive policy functions $c(k,K)$ and $k'(k,K)$,

(ii) The firm's problem derives the price functions $r(K)$ and $w(K)$ as (15) and (16)

(iii) markets are cleared by $k=K$, $k'=K'$, $c=C$, and the resource constraint (18).

As you can imagine, The household problem (17) derives the same Euler condition. Then, the system of equations are the same. The recursive competitive equilibrium is useful for heterogeneous agents models. In such as model, k is different for each agent.

4.2.5 Planner's problem

There is a social planner (社会計画者) that maximizes people's utility given the resource constraint. This person is also called a representative agent. (代表的個人) It is interpreted as a benevolent Soviet Union.

$$\max \sum_{t=0}^{\infty} \beta^t u(c_t)$$
$$\text{s.t. } c_t + k_{t+1} = f(k_t) + (1-\delta)k_t$$

That's it! Lagrangean is

$$\sum_{t=0}^{\infty} \beta^t \big[u(c_t) + \lambda_t\big(f(k_t) + (1-\delta)k_t - c_t - k_{t+1}\big)\big]$$

FOC

$$u'(c_t) - \lambda_t = 0$$
$$-\lambda_t + \lambda_{t+1}\big(f'(k_{t+1}) + 1 - \delta\big) = 0$$

Then, we get the same Euler condition

$$u'(c_t) = \big(f'(k_{t+1}) + 1 - \delta\big)u'(c_{t+1})$$

The resource constraint is the same. The transversality condition is also necessary. We get the same system of equations as Definition 3.

We can also think about the recursive social planner's problem.

$$\max V(K) = u(C) + \beta V(K') \qquad \text{s.t. } C + K' = f(K) + (1-\delta)K$$

It is defined by capital letter variables because the social planner considers aggregate level. Just substitute the resource constraint into the utility function

$$\max V(K) = u\big(f(K) + (1-\delta)K - K'\big) + \beta V(K')$$

FOC

$$-u'(C) + \beta V(K') = 0$$

Envelope condition

$$V'(K) = \big(f'(K) + 1 - \delta\big)u'(C)$$

Then, rewrite the envelope as K'

$$V'(K') = \big(f'(K') + 1 - \delta\big)u'(C')$$

Substitute it to the FOC.

$$u'(C) = \beta\big(f'(K') + 1 - \delta\big)u'(C')$$

It is the same Euler equation!

■Interpretation Why the social planner's solution coincides with the market equilibrium? It is because of the second fundamental theorem of welfare economics. (厚生経済学の第 2 基本定理) That is, for any Pareto optimal allocation, there exists a price such that the Walrasian equilibrium under this price coincides with this Pareto optimal allocation. The social planner's problem derives the Pareto optimal allocation. Hence, the market equilibrium is the same. Indeed, there are many possible Pareto optimal allocations, but their differences are only about distributional issue. By symmetry, we exclude it. Then, the symmetric and Pareto optimal allocation is unique.

4.2.6 Phase diagram

Let's analyze the economic dynamics using a hand-drawn diagram. It is called phase diagram. (位相図)

$$\text{Euler: } u'(c_t) = \beta(1 + f'(k_{t+1}) - \delta)u'(c_{t+1}) \tag{19}$$

$$\text{Resource: } k_{t+1} + c_t = (1-\delta)k_t + f(k_t) \tag{20}$$

$$\text{Transversality: } \lim_{T\to\infty} \beta^T u'(c_T)k_{T+1} = 0 \tag{21}$$

First, derive the steady state equations

$$\Delta c = 0 \text{ line: } \; 1 = \beta\big(f'(k^{ss}) + 1 - \delta\big) \tag{22}$$

$$\Delta k = 0 \text{ line: } \; c^{ss} = f(k^{ss}) - \delta k^{ss} \tag{23}$$

Draw them on the $k - c$ diagram. Shape?

The economy is divided into four areas. Consider which direction the economy moves.

- LHS of $\Delta c = 0$ line: k_t is smaller than k^{ss}. Then $f'(k_t)$ is larger than $f'(k^{ss})$. Then, $u'(c_t)$ is larger than $u'(c_{t+1})$. It means c_t is smaller than c_{t+1}. c_t is growing.
- RHS of $\Delta c = 0$ line: by the similar logic, c_t is decreasing.

- Upper area of $\Delta k = 0$ line: c_t is larger than c^{ss}. Then, k^{t+1} is smaller than k_t. k_t is decreasing.
- Lower area of $\Delta k = 0$ line: k_t is increasing.

Draw two arrows for each area to show the direction.

It is a growth model. Suppose that $k_0 < k^{ss}$. Then, the choice of the social planner at the initial period is c_0 and k_1.

Two types of variables in the dynamic system.

- Predetermined variable (先決変数): It cannot immediately move. Usually, it is a stock variable. k_t is an example. It is gradually changed by investments.
- Jump variable: It can immediately change. Usually, it is a flow variable. c_t is an example. You may take 3000 kcal today, and then you may eat nothing tomorrow.

- There is a unique c_0^* such that the economy converges to the steady state.
- If $c_0 > c_0^*$? The economy crosses $\Delta k = 0$ line and moves to the upper left area and goes to the left. The speed is accelerated. Eventually, at a finite period T_U, hit $k_{T_U} = 0$. By no production, c_{T_U} jumps to 0. It is collapse of the economy due to over consumption path. By $u'(c_{T_U}) = +\infty$ and $f'(k_{T_U}) = +\infty$. Then, the Euler condition is violated: $u'(c_{T_U - 1}) < \big(f'(k_{T_U}) + 1 - \delta\big)u'(c_{T_U})$.
- If $c_0 < c_0^*$? The economy crosses $\Delta c = 0$ line and moves to the bottom right area. The decline in c_t is accelarated, and at a finte period T_B, $c_{T_B} = 0$. After that, $c_t = 0$ for all $t > T_B$. Then, the transversality condition $\lim_{T\to\infty} \beta^T u'(c_T)k_T$ is violated. It is an over-accumulation case.

Therefore, the dynamic equilibrium path is unique.

■Exercise

- The problems are just food for thought. If you have any questions, ask to our TA.
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- Suppose that the economy is at the steady state. What happens if the discount factor β suddenly and permanently declines? Analyze the dynamic path of k_{t+1} and c_t

- Introduce productivity as A. Now the production function is $Af(k)$. The economy is initially at the steady state. Then, by the introduction of IT technology, A suddenly and permanently increases at period T_A. What happens? Analyze the dynamic path of k_{t+1} and c_t. You must answer whether c_{T_A} jumps up or down.

- Introduce consumption tax. The Japanese government increased the consumption tax rate several times. For example, in October 2019, the government raised the tax rate from 8% to 10%. Notably, it had been already announced before October 2019.
  Suppose that the economy is initially at the steady state with 0% consumption tax rate. Then, at period T_A, the government announces that the tax rate is increased in future, at period T_B. Between T_A and T_B, the tax rate is 0. Then from T_B, it increases to $\tau > 0$. This type of shock is called news shock. Given $\tau > 0$, the household budget constraint is now written as

  $$(1+\tau)c_t + k_{t+1} = R_t k_t + w_t,$$

  Assume that the government uses the tax revenue for a completely useless project. It has no effect on the economy for simplicity. Analyze the dynamic path of k_t and c_t.
  Hint: the economy starts to move at T_A and changes the direction at T_B.

4.2.7 Numerical solution using Dynare

■Method

- Ramsey model can be solved and simulated as so called *perfect foresight* (完全予見) or *deterministic* simulation.
- In this simulation, an infinite period model is approximated as a finite period model.
- Justification: The economy converges to the steady state. The transition path is approximated by finite grids.
- Suppose a 50 period model.
- There are 100 variables
  - $c_0, c_1, c_2, \cdots, c_{48}, c_{49}, c^*$
  - $k_1, k_2, k_3, \cdots, k_{49}, k_{50}, k^*$
  - k_0 is given
  - c_{50} and k_{51} are the steady state values.
- There are 100 variables
  - $c_0, c_1, c_2, \cdots, c_{48}, c_{49}, c^*$
  - $k_1, k_2, k_3, \cdots, k_{49}, k_{50}, k^*$
  - k_0 is given
  - c_{50} and k_{51} are the steady state values, c^* and k^*. These are separately calculated.
- There are 100 equations.

$$
\begin{aligned}
u'(c_0) &= \beta\big(Af'(k_1) + 1 - \delta\big)u'(c_1) \\
u'(c_1) &= \beta\big(Af'(k_2) + 1 - \delta\big)u'(c_2) \\
&\vdots \\
u'(c_{48}) &= \beta\big(Af'(k_{49}) + 1 - \delta\big)u'(c_{49}) \\
u'(c_{49}) &= \beta\big(Af'(k_{50}) + 1 - \delta\big)u'(c_{50})
\end{aligned}
$$

and

$$
\begin{aligned}
c_0 + k_1 &= Af(k_0) + (1-\delta)k_0 \\
c_1 + k_2 &= Af(k_1) + (1-\delta)k_1 \\
&\vdots \\
c_{48} + k_{49} &= Af(k_{48}) + (1-\delta)k_{48} \\
c_{49} + k_{50} &= Af(k_{50}) + (1-\delta)k_{50}
\end{aligned}
$$

- Usually, Dynare assumes that the model is at the steady state at period 0. Then, an unexpected and permanent shock (called MIT shock) hits. Then, Dynare calculates the transition path to the new steady state.
- Let's consider an example that β increases from 0.8 to 0.95
- Specifications
  - I will explain the reasons of parameter settings in the RBC section.
  - $f(k) = k^\alpha$. Cobb-Douglas is assumed. $\alpha = 0.33$
  - $u(c) = c^{1-\sigma}/(1-\sigma)$, where $\sigma = 2$.

- $\delta = 0.2$.
- $A = 1$ for normalization.

■Dynare Dynare is a toolkit running on Matlab. It is a little confusing

- Programming language is slightly different from Matlab
  - \\ for comments instead of `%`
  - Summarized in several sections. Specialized for DSGE model
- file name is `xyz.mod` (Matlab uses `xyz.m`)
- run the program by a command.
  `dynare xyz.mod`
  (On Matlab, you click `Run`)
- You need to specify where is Dynare on your computer. You need to run the following command at the beginning
  - `addpath /Applications/Dynare/4.6.4/matlab` for Mac
  - `addpath C:\dynare\4.6.4\matlab` for windows

  or you can set its path with Matlab setting.

```
// Ramsey (Neoclassical Grwoth Model), So Kubota
// Waseda, econ research, DSGE
// In DYNARE, we use "//" to comment out instead of "%"
// simulation. beta changes from 0.8 to 0.95

// To let MATLAB know dynare, automatic setting at the start.
// Environment Menu --> Set Path --> add following folder
// or you can run the following command ate
// Mac:  /Applications/Dynare/4.5.7/matlab
// Win:  \dynare\4.5.7\matlab

// Dynare codes are devided into four blocks.
// 1 variables
// 2 parameters
// 3 model
// 4 stady state
// 5 simulation
// 6 plot

// 0 preparation //////////////////////////
clc;
close all;
// You should not write "clear" It eliminates necessary files too.

// 1 variables //////////////////////////
```

```
var c k;  // endogeneous variables
varexo bet;  // exogeneous variables (shocks), discount factor beta

// 2 parameters ////////////////////////
parameters A gam del alp;  // definitions
A=1; // Productivity A
gam=2; // CRRA parameter, gamma
del=.2; // depreciation rate, delta
alp=.33; // capital's share, alpha

// 3 model ////////////////////////
// k_t is just k, k_{t-1} is represented as k(-1).
// Dynare defines k_t as the amount of capital at the end of period t.
// From the original notation, push back all k by one period. c_t is OK.
model;
k=A*k(-1)^alp+(1-del)*k(-1)-c;  // time t+1 variables are indicated as (+1)
c^(-gam) = bet*(c(+1)^(-gam))*(A*alp*k^(alp-1)+1-del);
end;

// 4 steady state /////////////////////
initval;
bet=.8;
k=0.5;
c=0.5;
end;
steady;
// put this in if you want to start from the initial steady state,
// comment it out to start from the indicated values

endval;
bet=.95;
k=1;
c=1;
end;
steady;


// 5 simulation //////////////////////
perfect_foresight_setup(periods = 50);
perfect_foresight_solver;

%% 6 plot %%%%%%%%%%%%%%%%%%%%%
%% Matlab program from here %%%%%%%%%%%%%%%
```

```
% simple plot, just to check the result.
figure;
plot(k);
figure;
plot(c);
figure;
plot(k,c);


% A little fancy plot
bet_old = 0.8;  % old steady state
k_old=((1/bet_old-1+del)/(A*alp))^(1/(alp-1)); % old steady state k
bet_new = 0.95;  % new steady state
k_new=((1/bet_new-1+del)/(A*alp))^(1/(alp-1)); % new steady state k
x_axis = linspace(0.01,3,100); % It is for x-axis of Delta c=0 line
delta_c0 = A*x_axis.^alp-del*x_axis; % Delta c=0 line. Calcularate for each x_axis grid
%
figure
xline(k_old,'Color','#0072BD','LineWidth',3);  % vertical line, old Delta k=0
hold on  % Draw a new line on the same figure with keeping the old one.
xline(k_new,'--','Color','#7E2F8E','LineWidth',3); % vertical line, new Delta k=0, '--' means a dashed line
hold on  % Draw a new line on the same figure with keeping the old one.
plot(x_axis,delta_c0,'Color','#0072BD','LineWidth',3);
hold on
plot(k,c,'Color','#D95319','LineWidth',3);
% add explanation for each line
legend({'Delta k=0, old','Delta k=0, new','Delta c=0','Transition path'},'Location','southeast');
xlabel('k, capital'); % add x axis name
ylabel('c, consumption'); % add y axis name
```

■Extension

- You can specify the timing of the shock.
- Example: β will increase at period 30. Agents suddenly notice this shock at period 0. News shock.
- You can add Section 4.5 to the program.

```
// 4.5 shock /////////////////////
shocks;
var bet;   // exogeneous variable
// beta will change from 0.8 to 0.95 at period 30.
// At time 0, people happen to realize this shock.
// keep beta at 0.8 until period 30.
```

```
periods 1:30;   // periods of change
values 0.8;   // keep beta at 0.8
end;
```