2016 年度春学期 経済数学 I（解析学基礎）（月 2）

【第 14 回: 長期利潤の最適化問題】(担当: 瀧澤 武信) 提出期限: 7 月 25 日 (月) 17:00 (出題日：7 月 18 日 (月))

問題

労働投入量 $\ell(>0)$ と資本投入量 $k(>0)$ の関数 $y = f\begin{pmatrix}\ell\\ k\end{pmatrix}$ を生産関数とする. y は財の産出量である. いま, 労働 1 単位あたりの賃金を $w(>0)$, 資本 1 単位当たりのレンタル料を $r(>0)$ とする. また, 財は販売価格 $p(>0)$ ですべて売れるものとする. このとき, 次の各問に答えよ.

1. $f\begin{pmatrix}\ell\\ k\end{pmatrix} = \ell^{\frac{1}{3}}k^{\frac{1}{4}}$, $p = 1, w = 3, r = 2$ のとき, 利潤関数 $\Pi\begin{pmatrix}\ell\\ k\end{pmatrix}$ を求めよ. また, 最適労働・資本投入量 $\begin{pmatrix}\ell^*\\ k^*\end{pmatrix}$ とそのときの生産量 y^*(y の最大値) を求めよ. (3*1 点)
2. $f\begin{pmatrix}\ell\\ k\end{pmatrix} = \ell^{\frac{1}{3}}k^{\frac{1}{4}}$, $w = 3, r = 2$ のとき, 供給関数 $y^*(p)$ を求めよ (y^*を p の関数として表わせ). (1 点)

解答例

1. $\Pi\begin{pmatrix}\ell\\ k\end{pmatrix} = \ell^{\frac{1}{3}}k^{\frac{1}{4}} - 3\ell - 2k$

   $\begin{pmatrix}\ell^*\\ k^*\end{pmatrix} = \begin{pmatrix}2^{-\frac{9}{5}}\cdot 3^{-\frac{18}{5}}\\ 2^{-\frac{24}{5}}\cdot 3^{-\frac{8}{5}}\end{pmatrix}$

   $y^* = 2^{-\frac{9}{5}}3^{-\frac{8}{5}}$
2. $y^*(p) = 2^{-\frac{9}{5}}3^{-\frac{8}{5}}p^{\frac{7}{5}}$

解説

1. $\Pi\begin{pmatrix}\ell\\ k\end{pmatrix} = p\cdot f\begin{pmatrix}\ell\\ k\end{pmatrix} - w\ell - rk = 1\cdot \ell^{\frac{1}{3}}k^{\frac{1}{4}} - 3\ell - 2k$

   $\Pi_\ell\begin{pmatrix}\ell\\ k\end{pmatrix} = \frac{1}{3}\ell^{-\frac{2}{3}}k^{\frac{1}{4}} - 3$, $\Pi_k\begin{pmatrix}\ell\\ k\end{pmatrix} = \frac{1}{4}\ell^{\frac{1}{3}}k^{-\frac{3}{4}} - 2$,

   $\Pi_{\ell\ell}\begin{pmatrix}\ell\\ k\end{pmatrix} = -\frac{2}{9}\ell^{-\frac{5}{3}}k^{\frac{1}{4}}$, $\Pi_{\ell k}\begin{pmatrix}\ell\\ k\end{pmatrix} = \frac{1}{12}\ell^{-\frac{2}{3}}k^{-\frac{3}{4}}$, $\Pi_{k\ell}\begin{pmatrix}\ell\\ k\end{pmatrix} = \frac{1}{12}\ell^{-\frac{2}{3}}k^{-\frac{3}{4}}$, $\Pi_{kk}\begin{pmatrix}\ell\\ k\end{pmatrix} = -\frac{3}{16}\ell^{\frac{1}{3}}k^{-\frac{7}{4}}$

極値の候補 (必要条件を満たす点) を求める.

$\Pi_\ell \begin{pmatrix} \ell \\ k \end{pmatrix} = 0\ ,\ \Pi_k \begin{pmatrix} \ell \\ k \end{pmatrix} = 0$

$(1)\ell^{-\frac{2}{3}}k^{\frac{1}{4}} = 9 = 3^2\ ,\ (2)\ell^{\frac{1}{3}}k^{-\frac{3}{4}} = 8 = 2^3 \Rightarrow (3)\ell^{\frac{2}{3}}k^{-\frac{3}{2}} = (2^3)^2 = 2^6$

$\Longrightarrow (1)\times(3):\ k^{-\frac{5}{4}} = 2^6\cdot 3^2$

$k^{-5} = (2^6\cdot 3^2)^4 = 2^{24}\cdot 3^8 \Rightarrow k^5 = 2^{-24}\cdot 3^{-8} \Rightarrow k = 2^{-\frac{24}{5}}\cdot 3^{-\frac{8}{5}}$

一方, $(2)^3:\ \ell\cdot k^{-\frac{9}{4}} = 8^3 = (2^3)^3 = 2^9$ より

$\ell = k^{\frac{9}{4}}\cdot 2^9 = (2^{-\frac{24}{5}}\cdot 3^{-\frac{8}{5}})^{\frac{9}{4}}\cdot 2^9 = 2^{(-\frac{24}{5}\cdot\frac{9}{4}+9)}\cdot 3^{(-\frac{8}{5}\cdot\frac{9}{4})} = 2^{-\frac{9}{5}}\cdot 3^{-\frac{18}{5}}$

$\begin{pmatrix} \ell^* \\ k^* \end{pmatrix} = \begin{pmatrix} 2^{-\frac{9}{5}}\cdot 3^{-\frac{18}{5}} \\ 2^{-\frac{24}{5}}\cdot 3^{-\frac{8}{5}} \end{pmatrix}$

判別する.

$$\Delta_2 \begin{pmatrix} \ell \\ k \end{pmatrix} = \begin{vmatrix} \Pi_{\ell\ell}\begin{pmatrix} \ell \\ k \end{pmatrix} & \Pi_{\ell k}\begin{pmatrix} \ell \\ k \end{pmatrix} \\ \Pi_{k\ell}\begin{pmatrix} \ell \\ k \end{pmatrix} & \Pi_{kk}\begin{pmatrix} \ell \\ k \end{pmatrix} \end{vmatrix}$$

$$= \begin{vmatrix} -\frac{2}{9}\ell^{-\frac{5}{3}}k^{\frac{1}{4}} & \frac{1}{12}\ell^{-\frac{2}{3}}k^{-\frac{3}{4}} \\ \frac{1}{12}\ell^{-\frac{2}{3}}k^{-\frac{3}{4}} & -\frac{3}{16}\ell^{\frac{1}{3}}k^{-\frac{7}{4}} \end{vmatrix} = (-\frac{2}{9})\cdot(-\frac{3}{16})\ell^{-\frac{4}{3}}k^{-\frac{6}{4}} - (\frac{1}{12})^2\ell^{-\frac{4}{3}}k^{-\frac{6}{4}} = (\frac{1}{24}-\frac{1}{144})\ell^{-\frac{4}{3}}k^{-\frac{6}{4}} > 0,$$

$\Delta_1 \begin{pmatrix} \ell \\ k \end{pmatrix} = \Pi_{\ell\ell}\begin{pmatrix} \ell \\ k \end{pmatrix} = -\frac{2}{9}\ell^{-\frac{5}{3}}k^{\frac{1}{4}} < 0$ より $\Pi\begin{pmatrix} \ell \\ k \end{pmatrix}$ は上に凸だから, $\Pi\begin{pmatrix} 2^{-\frac{9}{5}}\cdot 3^{-\frac{18}{5}} \\ 2^{-\frac{24}{5}}\cdot 3^{-\frac{8}{5}} \end{pmatrix}$ は極大値.

$$y^* = f\begin{pmatrix} \ell^* \\ k^* \end{pmatrix} = (\ell^*)^{\frac{1}{3}}(k^*)^{\frac{1}{4}} = (2^{-\frac{9}{5}}\cdot 3^{-\frac{18}{5}})^{\frac{1}{3}}\cdot(2^{-\frac{24}{5}}\cdot 3^{-\frac{8}{5}})^{\frac{1}{4}}$$

$$= 2^{(-\frac{9}{5})\cdot\frac{1}{3}+(-\frac{24}{5})\cdot\frac{1}{4}}\cdot 3^{(-\frac{18}{5})\cdot\frac{1}{3}+(-\frac{8}{5})\cdot\frac{1}{4}} = 2^{-\frac{3}{5}-\frac{6}{5}}\cdot 3^{-\frac{6}{5}-\frac{2}{5}} = 2^{-\frac{9}{5}}3^{-\frac{8}{5}}$$

2. $\Pi\begin{pmatrix} \ell \\ k \end{pmatrix} = p\cdot f\begin{pmatrix} \ell \\ k \end{pmatrix} - w\ell - rk = p\cdot\ell^{\frac{1}{3}}k^{\frac{1}{4}} - 3\ell - 2k$

$\Pi_\ell\begin{pmatrix} \ell \\ k \end{pmatrix} = \frac{1}{3}p\ell^{-\frac{2}{3}}k^{\frac{1}{4}} - 3\ ,\ \Pi_k\begin{pmatrix} \ell \\ k \end{pmatrix} = \frac{1}{4}p\ell^{\frac{1}{3}}k^{-\frac{3}{4}} - 2\ ,$

$\Pi_{\ell\ell}\begin{pmatrix} \ell \\ k \end{pmatrix} = -\frac{2}{9}p\ell^{-\frac{5}{3}}k^{\frac{1}{4}}\ ,\ \Pi_{\ell k}\begin{pmatrix} \ell \\ k \end{pmatrix} = \frac{1}{12}p\ell^{-\frac{2}{3}}k^{-\frac{3}{4}}\ ,\ \Pi_{k\ell}\begin{pmatrix} \ell \\ k \end{pmatrix} = \frac{1}{12}p\ell^{-\frac{2}{3}}k^{-\frac{3}{4}}\ ,\ \Pi_{kk}\begin{pmatrix} \ell \\ k \end{pmatrix} = -\frac{3}{16}p\ell^{\frac{1}{3}}k^{-\frac{7}{4}}$

極値の候補 (必要条件を満たす点) を求める.

$\Pi_\ell \begin{pmatrix} \ell \\ k \end{pmatrix} = 0\ ,\ \Pi_k \begin{pmatrix} \ell \\ k \end{pmatrix} = 0$

$(1)\ell^{-\frac{2}{3}}k^{\frac{1}{4}}p = 9 = 3^2\ ,\ (2)\ell^{\frac{1}{3}}k^{-\frac{3}{4}}p = 8 = 2^3 \Rightarrow (3)\ell^{\frac{2}{3}}k^{-\frac{3}{2}}p^2 = (2^3)^2 = 2^6$

$\Longrightarrow (1)\times(3):\ k^{-\frac{5}{4}}p^3 = 2^6\cdot 3^2 \Longrightarrow k^{-\frac{5}{4}} = 2^6\cdot 3^2\cdot p^{-3}$

$k^{-5} = (2^6\cdot 3^2\cdot p^{-3})^4 = 2^{24}\cdot 3^8\cdot p^{-12} \Rightarrow k^5 = 2^{-24}\cdot 3^{-8}\cdot p^{12} \Rightarrow k = 2^{-\frac{24}{5}}3^{-\frac{8}{5}}p^{\frac{12}{5}}$

一方, $(2)^3:\ \ell \cdot k^{-\frac{9}{4}} \cdot p^3 = 8^3 = (2^3)^3 = 2^9$ より

$\ell = k^{\frac{9}{4}} \cdot 2^9 \cdot p^{-3} = (2^{-\frac{24}{5}} \cdot 3^{-\frac{8}{5}} \cdot p^{\frac{12}{5}})^{\frac{9}{4}} \cdot 2^9 \cdot p^{-3} = 2^{(-\frac{24}{5}\cdot\frac{9}{4}+9)} \cdot 3^{(-\frac{8}{5}\cdot\frac{9}{4})} \cdot p^{\frac{12}{5}\cdot\frac{9}{4}-3} = 2^{-\frac{9}{5}} \cdot 3^{-\frac{18}{5}} \cdot p^{\frac{12}{5}}$

$$\begin{pmatrix} \ell^* \\ k^* \end{pmatrix} = \begin{pmatrix} 2^{-\frac{9}{5}} \cdot 3^{-\frac{18}{5}} \cdot p^{\frac{12}{5}} \\ 2^{-\frac{24}{5}} \cdot 3^{-\frac{8}{5}} \cdot p^{\frac{12}{5}} \end{pmatrix}$$

判別する.

$$\Delta_2 \begin{pmatrix} \ell \\ k \end{pmatrix} = \begin{vmatrix} \Pi_{\ell\ell} \begin{pmatrix} \ell \\ k \end{pmatrix} & \Pi_{\ell k} \begin{pmatrix} \ell \\ k \end{pmatrix} \\ \Pi_{k\ell} \begin{pmatrix} \ell \\ k \end{pmatrix} & \Pi_{kk} \begin{pmatrix} \ell \\ k \end{pmatrix} \end{vmatrix}$$

$$= \begin{vmatrix} -\frac{2}{9}p\ell^{-\frac{5}{3}}k^{\frac{1}{4}} & \frac{1}{12}p\ell^{-\frac{2}{3}}k^{-\frac{3}{4}} \\ \frac{1}{12}p\ell^{-\frac{2}{3}}k^{-\frac{3}{4}} & -\frac{3}{16}p\ell^{\frac{1}{3}}k^{-\frac{7}{4}} \end{vmatrix} = (-\frac{2}{9}) \cdot (-\frac{3}{16})\ell^{-\frac{4}{3}}k^{-\frac{6}{4}}p^2 - (\frac{1}{12})^2\ell^{-\frac{4}{3}}k^{-\frac{6}{4}}p^2 = (\frac{1}{24} - \frac{1}{144})\ell^{-\frac{4}{3}}k^{-\frac{6}{4}}p^2 > 0,$$

$\Delta_1 \begin{pmatrix} \ell \\ k \end{pmatrix} = \Pi_{\ell\ell} \begin{pmatrix} \ell \\ k \end{pmatrix} = -\frac{2}{9}p\ell^{-\frac{5}{3}}k^{\frac{1}{4}} < 0$ より $\Pi \begin{pmatrix} \ell \\ k \end{pmatrix}$ は上に凸だから, $\Pi \begin{pmatrix} 2^{-\frac{9}{5}} \cdot 3^{-\frac{18}{5}} \cdot p^{\frac{12}{5}} \\ 2^{-\frac{24}{5}} \cdot 3^{-\frac{8}{5}} \cdot p^{\frac{12}{5}} \end{pmatrix}$ は極大値.

$$y^*(p) = f \begin{pmatrix} \ell^* \\ k^* \end{pmatrix} = (\ell^*)^{\frac{1}{3}}(k^*)^{\frac{1}{4}} = (2^{-\frac{9}{5}} \cdot 3^{-\frac{18}{5}} \cdot p^{\frac{12}{5}})^{\frac{1}{3}} \cdot (2^{-\frac{24}{5}} \cdot 3^{-\frac{8}{5}} \cdot p^{\frac{12}{5}})^{\frac{1}{4}}$$

$$= 2^{(-\frac{9}{5})\cdot\frac{1}{3}+(-\frac{24}{5})\cdot\frac{1}{4}} \cdot 3^{(-\frac{18}{5})\cdot\frac{1}{3}+(-\frac{8}{5})\cdot\frac{1}{4}} \cdot p^{\frac{12}{5}\cdot\frac{1}{3}+\frac{12}{5}\cdot\frac{1}{4}} = 2^{-\frac{3}{5}-\frac{6}{5}} \cdot 3^{-\frac{6}{5}-\frac{2}{5}} \cdot p^{\frac{4}{5}+\frac{3}{5}} = 2^{-\frac{9}{5}}3^{-\frac{8}{5}}p^{\frac{7}{5}}$$