2016年度春学期 経済数学I（解析学基礎）（月2）

【第9回: Taylorの定理】(担当: 瀧澤 武信) 提出期限: 6月20日 (月) 17:00 (出題日：6月13日 (月))

問題

1. $x = 0$ の周りでTaylorの定理を用いて1次以下の式で近似せよ (2次の剰余項まで示せ).　(2点)

   $f(x) = e^{\sqrt{1-x}}$

2. Maclaurin級数展開せよ (3次の項まで示せ).　(2点)

   $f(x) = e^{\sqrt{1-x}}$

解答例

1. $$e^{\sqrt{1-x}} = e - \frac{e}{2}x + \frac{x^2}{8}e^{(1-\theta x)^{\frac{1}{2}}}\{(1-\theta x)^{-1} - (1-\theta x)^{-\frac{3}{2}}\}$$

2. $$f(x) = e - \frac{ex}{2} - \frac{ex^3}{48} + \cdots \quad \text{(3次の項まで)}$$

$\boxed{\text{解説}}$

1.

$$
\begin{aligned}
f(x) = f(x) = e^{\sqrt{1-x}} = e^{(1-x)^{\frac{1}{2}}} \text{のとき} \\
f'(x) = e^{(1-x)^{\frac{1}{2}}} \cdot \tfrac{1}{2}(1-x)^{-\frac{1}{2}} \cdot (-1) \\
f''(x) = \{e^{(1-x)^{\frac{1}{2}}}\}' \cdot \tfrac{1}{2}(1-x)^{-\frac{1}{2}} \cdot (-1) + e^{(1-x)^{\frac{1}{2}}} \cdot \{\tfrac{1}{2}(1-x)^{-\frac{1}{2}} \cdot (-1)\}' \\
= e^{(1-x)^{\frac{1}{2}}} \cdot \{\tfrac{1}{2}(1-x)^{-\frac{1}{2}} \cdot (-1)\}^2 + e^{(1-x)^{\frac{1}{2}}} \cdot \tfrac{1}{2}(-\tfrac{1}{2})(1-x)^{-\frac{3}{2}} \cdot (-1)^2 \\
= e^{(1-x)^{\frac{1}{2}}} \cdot \left[\tfrac{1}{4}(1-x)^{-1} - \tfrac{1}{4}(1-x)^{-\frac{3}{2}}\right] \text{であるから} \\
f(0) = e, f'(0) = -\tfrac{1}{2}e, f''(\theta x) = e^{(1-\theta x)^{\frac{1}{2}}} \cdot \left[\tfrac{1}{4}(1-\theta x)^{-1} - \tfrac{1}{4}(1-\theta x)^{-\frac{3}{2}}\right] \\
\Longrightarrow f(x) = f(0) + xf'(0) + \tfrac{x^2}{2}f''(\theta x) \\
= e + x \cdot (-\tfrac{1}{2}e) + \tfrac{x^2}{2} \cdot e^{(1-\theta x)^{\frac{1}{2}}} \cdot \left[\tfrac{1}{4}(1-\theta x)^{-1} - \tfrac{1}{4}(1-\theta x)^{-\frac{3}{2}}\right] \\
e^{\sqrt{1-x}} = e - \tfrac{e}{2}x + \tfrac{x^2}{8}e^{(1-\theta x)^{\frac{1}{2}}}\{(1-\theta x)^{-1} - (1-\theta x)^{-\frac{3}{2}}\}
\end{aligned}
$$

2.

$$
\begin{aligned}
f(x) = e^{\sqrt{1-x}} = e^{(1-x)^{\frac{1}{2}}} \\
f'(x) = e^{(1-x)^{\frac{1}{2}}} \cdot \tfrac{1}{2}(1-x)^{-\frac{1}{2}} \cdot (-1) \\
f''(x) = e^{(1-x)^{\frac{1}{2}}} \cdot \{\tfrac{1}{2}(1-x)^{-\frac{1}{2}} \cdot (-1)\}^2 + e^{(1-x)^{\frac{1}{2}}} \cdot \tfrac{1}{2}(-\tfrac{1}{2})(1-x)^{-\frac{3}{2}} \cdot (-1)^2 \\
= e^{(1-x)^{\frac{1}{2}}} \cdot [\tfrac{1}{4}(1-x)^{-1} - \tfrac{1}{4}(1-x)^{-\frac{3}{2}}] \\
f'''(x) = e^{(1-x)^{\frac{1}{2}}}\{\tfrac{1}{2}(1-x)^{-\frac{1}{2}} \cdot (-1)\}[\tfrac{1}{4}(1-x)^{-1} - \tfrac{1}{4}(1-x)^{-\frac{3}{2}}] \\
+e^{(1-x)^{\frac{1}{2}}} \cdot [\tfrac{1}{4}(-1)(1-x)^{-2} \cdot (-1) - \tfrac{1}{4}(-\tfrac{3}{2})(1-x)^{-\frac{5}{2}} \cdot (-1)] \\
\text{よって，} f(0) = e,\ f'(0) = -\tfrac{1}{2}e,\ f''(0) = 0,\ f'''(0) = -\tfrac{1}{8}e
\end{aligned}
$$

$$\Longrightarrow f(x) = f(0) + \frac{x}{1!}f'(0) + \frac{x^2}{2!}f''(0) + \frac{x^3}{3!}f'''(0) + \cdots$$

$$= e - \frac{x}{1!}\frac{e}{2} - \frac{x^3}{3!}\frac{e}{8} + \cdots$$

$$= e - \frac{ex}{2} - \frac{ex^3}{48} + \cdots \quad (3\text{ 次の項まで})$$