6 Log-linearization

6.1 Overview

- The modern business cycle models study the economic fluctuations around the steady state. That is, the steady state is a long-run equilibrium, and the deviation from the steady state is a short-run.
- Then, we can focus on small movement around the steady state. As the spirit of Taylor expansion, the linearized system may be a good enough approximation. It eliminates a difficutly due to non-linearity, like complicated divergent economic paths on the phase diagram of Ramsey model.
- Three topics
  1. The linear dynamic system
  2. Log linearization
  3. The solution of the log-linearized Ramsey model.
- useful lecture slides
  `http://www.chrisedmond.net/hons2019.html`
- Miao, chapter 1,2,3,15. But too much detail

6.2 Linear dynamic system

- A dynamic linear system of two variables

$$x_{t+1} = a_{11}x_t + a_{12}y_t$$
$$y_{t+1} = a_{21}x_t + a_{22}y_t$$

- matrix representation

$$\begin{pmatrix} x_{t+1} \\ y_{t+1} \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} x_t \\ y_t \end{pmatrix}$$

- The matrix $A \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$ can be diagnolazied (対角化)

$$A = V^{-1} \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} V$$

- λ_1 and λ_2 are called eigenvalues (固有値) and V is called eigenvectors (固有ベクトル).
- How to derive? (Honestly speaking, I forgot) Calculate the determinant (行列式)

$$\left| \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} - \lambda I \right| = 0$$

$$\Leftrightarrow \ \lambda^2 - (a_{11} + a_{22})\lambda + a_{11}a_{22} - a_{12}a_{21} = 0$$

- Rewrite the system

$$\begin{pmatrix} x_{t+1} \\ y_{t+1} \end{pmatrix} = V^{-1} \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} V \begin{pmatrix} x_t \\ y_t \end{pmatrix}$$

$$\Leftrightarrow V\begin{pmatrix} x_{t+1} \\ y_{t+1} \end{pmatrix} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} V \begin{pmatrix} x_t \\ y_t \end{pmatrix}$$

- Let $V = \begin{pmatrix} v_{11} & v_{12} \\ v21 & v_{22} \end{pmatrix}$.
- Coordiante change. Define

$$z_{1,t} = v_{11}x_t + v_{12}y_t \qquad z_{2,t} = v_{21}x_t + v_{22}y_t$$

  Then,

$$\begin{pmatrix} z_{1,t+1} \\ z_{2,t+1} \end{pmatrix} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} V \begin{pmatrix} z_{1,t} \\ z_{2,t} \end{pmatrix}$$

- Two variables are separated!

$$z_{1,t} = \lambda_1^t z_{1,0} \qquad z_{2,t} = \lambda_1^t z_{2,0}$$

- Converge to zero or diverge to ∞. Depending on λ.
  1. Case1 (Sink) Both λ_1 and λ_2 are less than 1. In this case, the system eventually converges to the steady state wherever the initial point is.
  2. Case2 (Saddle) A λ is larger than 1, a λ is less than 1. In this case, the system diverges except the knife-edge path.
  3. Case3 (Source) Both λ_1 and λ_2 are larger than 1. In this case, the system diverges regardless of the initial point.

6.3 Log linearization

- Linear Approximation (線形近似)

$$f(x) \approx f(x^\star) + f'(x^\star)(x - x^\star)$$

- Log Linear Approximation (対数線形近似)

$$\begin{aligned} \ln f(x) &\approx \ln f(x^\star) + \frac{d \ln f(x^\star)}{dx}(x - x^\star) \\ &= \ln f(x^\star) + \frac{f'(x^\star)}{f(x^\star)}(x - x^\star) \\ &= \ln f(x^\star) + \frac{f'(x^\star)x^\star}{f(x^\star)} \underbrace{\left(\frac{(x - x^\star)}{x^\star}\right)}_{\hat{x}} \end{aligned}$$

  where ln denotes natural logarithm, $\approx$ denotes approximation, $\hat{x}$ denotes the percent deviation from the steady state value.
- If $y = f(x)$,

$$\hat{y} \approx \ln f(x) - \ln f(x^\star) = \frac{f'(x^\star)x^\star}{f(x^\star)}\hat{x}$$

- Similarly,

$$\ln f(x, y) \approx \ln f(x^\star, y^\star) + \frac{(\partial f/\partial x^\star)x^\star}{f(x^\star, y^\star)}\hat{x} + \frac{(\partial f/\partial y^\star)y^\star}{f(x^\star, y^\star)}\hat{y}$$

- In the business cycle analysis, we do not care the level of economic variables. To consider percent changes instead, log - linearization is better.

- Examples

$$y = x^a z^b \;\Rightarrow\; \hat{y} = a\hat{x} + b\hat{z}$$

$$y = ax + bz \;\Rightarrow\; \hat{y} = \left(\frac{ax^*}{ax^* + bz^*}\right)\hat{x} + \left(\frac{bz^*}{ax^* + bz^*}\right)\hat{z}$$

$$y = (ax+b)^c \;\Rightarrow\; \hat{y} = c\left(\frac{ax^*}{ax^* + b}\right)\hat{x}$$

$$y = f(x) + g(z) \;\Rightarrow\; \hat{y} = \left(\frac{f'(x^*)x^*}{f(x^*) + g(z^*)}\right)\hat{x} + \left(\frac{g'(z^*)}{f(x^*) + g(z^*)}\right)\hat{z}$$

6.4 Log linearization of Ramsey model

$$\begin{aligned} \text{Euler: } & u'(c_t) = \beta(1 + f'(k_{t+1}) - \delta)u'(c_{t+1}) \\ \text{Resource: } & k_{t+1} + c_t = (1-\delta)k_t + f(k_t) \end{aligned}$$

- Log linearization of Resource Constraint

$$k_{t+1} + c_t = f(k_t) + (1-\delta)k_t$$

  - Approximation around the steady state (k^{ss}, c^{ss})
  - LHS

$$\left(\frac{k^{ss}}{k^{ss} + c^{ss}}\right)\hat{k}_{t+1} + \left(\frac{c^{ss}}{k^{ss} + c^{ss}}\right)\hat{c}_t$$

  - RHS

$$\frac{[f'(k^{ss}) + 1 - \delta]\, k^{ss}}{f(k^{ss}) + (1-\delta)k^{ss}}\hat{k}_t$$

  - By the steady state condition, $k^{ss} + c^{ss} = f(k^{ss}) + (1-\delta)k^{ss}$. Then, by LHS=RHS,

$$k^{ss}\hat{k}_{t+1} + c^{ss}\hat{c}_t = [f'(k^{ss}) + 1 - \delta]k^{ss}\hat{k}_t$$

  - By the steady state condition of $\Delta c = 0$ line, we know that

$$1 = \beta[f'(k^{ss}) + 1 - \delta]$$

  - Finally, we get

$$k^{ss}\hat{k}_{t+1} + c^{ss}\hat{c}_t = \frac{1}{\beta}k^{ss}\hat{k}_t$$

- Log-linearization of Euler
  - Euler

$$\frac{u'(c_t)}{u'(c_{t+1})} = \beta[f'(k_{t+1}) + 1 - \delta]$$

  - LHS

$$\frac{u''(c^{ss})c^{ss}}{u'(c^{ss})}(\hat{c}_t - \hat{c}_{t+1})$$

– Notation: the Arrow/Pratt measure of relative risk aversion

$$\sigma(c) = -\frac{u''(c)c}{u'(c)} > 0$$

– LHS

$$\sigma(c^{ss})(\hat{c}_{t+1} - \hat{c}_t)$$

– RHS

$$\frac{\beta f''(k^{ss})k^{ss}}{\beta[f'(k^{ss}) + 1 - \delta]}\hat{k}_{t+1}$$

In the stedy state, $\beta[f'(k^{ss}) + 1 - \delta] = 1$. Then,

$$\beta f''(k^{ss})k^{ss}\hat{k}_{t+1}$$

– Then, by LHS=RHS,

$$\hat{c}_{t+1} - \hat{c}_t = \frac{1}{\sigma(c^{ss})}\beta f''(k^{ss})k^{ss}\hat{k}_{t+1}$$

- The linearized system of equations
  - Summary

$$\begin{aligned}\hat{k}_{t+1} &= \frac{1}{\beta}\hat{k}_t - \frac{c^{ss}}{k^{ss}}\hat{c}_t \\ \hat{c}_{t+1} - \frac{\beta f''(k^{ss})k^{ss}}{\sigma(c^{ss})}\hat{k}_{t+1} &= \hat{c}_t\end{aligned}$$

  - Substitute the first one to the second

$$\hat{c}_{t+1} = \frac{f''(k^{ss})k^{ss}}{\sigma(c^{ss})}\hat{k}_t + \left(1 - \frac{\beta f''(k^{ss})c^{ss}}{\sigma(c^{ss})}\right)\hat{c}_t$$

  - Matrix representation

$$\begin{pmatrix}\hat{k}_{t+1} \\ \hat{c}_{t+1}\end{pmatrix} = \begin{pmatrix}\frac{1}{\beta} & \frac{c^{ss}}{k^{ss}} \\ \frac{f''(k^{ss})k^{ss}}{\sigma(c^{ss})} & 1 - \frac{\beta f''(k^{ss})c^{ss}}{\sigma(c^{ss})}\end{pmatrix}\begin{pmatrix}\hat{k}_t \\ \hat{c}_t\end{pmatrix}$$

- Dynamic property
  - Determinants

$$\left|\begin{pmatrix}\frac{1}{\beta} & \frac{c^{ss}}{k^{ss}} \\ \frac{f''(k^{ss})k^{ss}}{\sigma(c^{ss})} & 1 - \frac{\beta f''(k^{ss})c^{ss}}{\sigma(c^{ss})}\end{pmatrix} - \lambda I\right| = 0$$

$$\begin{aligned}\Leftrightarrow p(\lambda) \equiv \lambda^2 - &\left(1 + \frac{1}{\beta} - \frac{\beta f''(k^{ss})c^{ss}}{\sigma(c^{ss})}\right)\lambda + \frac{1}{\beta} - \frac{f''(k^{ss})c^{ss}}{\sigma(c^{ss})} + \frac{f''(k^{ss})c^{ss}}{\sigma(c^{ss})} \\ &= \lambda^2 - \left(1 + \frac{1}{\beta} - \frac{\beta f''(k^{ss})c^{ss}}{\sigma(c^{ss})}\right)\lambda + \frac{1}{\beta} = 0\end{aligned}$$

  - Eigenvalues

$$p(0) = \frac{1}{\beta} > 0$$

$$p(1) = 1 - 1 - \frac{1}{\beta} + \frac{\beta f''(k^{ss})c^{ss}}{\sigma(c^{ss})} + \frac{1}{\beta} = \frac{\beta f''(k^{ss})c^{ss}}{\sigma(c^{ss})} < 0$$

  - One λ must be between 0 and 1, and the other must be larger than 1.
  - The linearized Ramsey model is saddle!

6.5 Rational Expectation

- Summary

$$
\begin{aligned}
& V \begin{bmatrix} \hat{k}_{t+1} \\ \hat{c}_{t+1} \end{bmatrix} = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} V \begin{bmatrix} \hat{k}_t \\ \hat{c}_t \end{bmatrix} \\
\Leftrightarrow\ & V \begin{bmatrix} \hat{k}_t \\ \hat{c}_t \end{bmatrix} = \begin{bmatrix} \lambda_1^t & 0 \\ 0 & \lambda_2^t \end{bmatrix} V \begin{bmatrix} \hat{k}_0 \\ \hat{c}_0 \end{bmatrix}
\end{aligned}
$$

- Define $V^{-1} = \begin{pmatrix} v_{11} & v_{12} \\ v21 & v_{22} \end{pmatrix}$. (It is different from the previous V)

$$
\begin{aligned}
\begin{bmatrix} \hat{k}_T \\ \hat{c}_T \end{bmatrix} &= V^{-1} \begin{bmatrix} \lambda_1^T & 0 \\ 0 & \lambda_2^T \end{bmatrix} V \begin{bmatrix} \hat{k}_0 \\ \hat{c}_0 \end{bmatrix} \\
&= \begin{bmatrix} v_{11} & v_{12} \\ v_{21} & v_{22} \end{bmatrix} \begin{bmatrix} \lambda_1^T & 0 \\ 0 & \lambda_2^T \end{bmatrix} \frac{1}{\det(V^{-1})} \begin{bmatrix} v_{22} & -v_{12} \\ -v_{21} & v_{11} \end{bmatrix} \begin{bmatrix} \hat{k}_0 \\ \hat{c}_0 \end{bmatrix} \\
&= \frac{1}{\det(V^{-1})} \begin{bmatrix} v_{11}\lambda_1^T & v_{12}\lambda_2^T \\ v_{21}\lambda_1^T & v_{22}\lambda_2^T \end{bmatrix} \begin{bmatrix} v_{22}\lambda_1^T & -v_{12}\lambda_2^T \\ -v_{21}\lambda_1^T & v_{11}\lambda_2^T \end{bmatrix}
\end{aligned}
$$

then

$$
\begin{bmatrix} \hat{k}_T \\ \hat{c}_T \end{bmatrix} = \frac{1}{\det(V^{-1})} \begin{bmatrix} v_{11}\lambda_1^T(v_{22}\hat{k}_0 - v_{12}\hat{c}_0) & v_{12}\lambda_2^T(-v_{21}\hat{k}_0 + v_{11}\hat{c}_0) \\ v_{21}\lambda_1^T(v_{22}\hat{k}_0 - v_{12}\hat{c}_0) & v_{22}\lambda_2^T(-v_{21}\hat{k}_0 + v_{11}\hat{c}_0) \end{bmatrix}
$$

- Suppose $\lambda_1 > 0$ and $\lambda_2 < 0$.
- Remember that the economy or the representative agent chooses c_0 so that the dynamic path converges to the steady state.
  - k is called a pre-determined variable (先決変数). It is usually the state variable (状態変数). Given k_t, the economy decide k_{t+1} and c_t.
  - c is called a jump variable. It can be flexibly chosen.
- The convergence requires that both $\hat{k}_0$ and $\hat{c}_0$ converge to zero. However, $\lambda_1 > 0$ tend to diverge the system.
- The only way to avoid the divergence: choose appropriate $\hat{c}_0$ so that

$$
v_{22}\hat{k}_0 = v_{12}\hat{c}_0
$$

- The solution.

$$
\begin{bmatrix} \hat{k}_T \\ \hat{c}_T \end{bmatrix} = \begin{bmatrix} v_{12}\lambda_2^T(-v_{21}\hat{k}_0 + v_{11}\hat{c}_0) \\ v_{22}\lambda_2^T(-v_{21}\hat{k}_0 + v_{11}\hat{c}_0) \end{bmatrix} = \begin{bmatrix} v_{12}\lambda_2^T(\frac{v_{11}v_{22}}{v_{12}})\hat{k}_0 \\ v_{22}\lambda_2^T(\frac{v_{11}v_{22}}{v_{12}})\hat{k}_0 \end{bmatrix}
$$

Therefore,

$$
\begin{aligned}
\hat{k}_T &= \lambda_2^T v_{11} v_{22} \hat{k}_0 \\
\hat{c}_T &= \lambda_2^T \left(\frac{v_{11}v_{22}^2}{v_{12}} \right) \hat{k}_0
\end{aligned}
$$

- Numerically, q and λ are obtained from parameters and steady state values of endogenous variables. Given initial $\tilde{k}_0$, the path is calculated.
- For three or more variable case, the rational expectation requires that
  - The number of the eigenvalues larger than 1
    = The number of jump variables