<u>例題</u> $$f(x) = e^x$$

(解)

$$f \in C^\infty \text{すなわち} {}^\forall n \in \mathbf{N}, f \in C^n(\mathbf{R})$$
$$f^{(n)}(x) = e^x, f^{(n)}(0) = 1$$

よって， Maclaurin の定理から,

$$e^x = 1 + \frac{x}{1!}\cdot 1 + \frac{x^2}{2!}\cdot 1 + \cdots + \frac{x^{n-1}}{(n-1)!}\cdot 1 + \frac{x^n}{n!}\cdot e^{\theta x}, \ 0 < {}^\exists\theta < 1$$

$$\text{ここで}, R_n(x) \fallingdotseq \frac{x^n}{n!}\cdot e^{\theta x} \text{とおくと},$$

$$\lim_{n\to+\infty}|R_n(x)| = \lim_{n\to+\infty}\left|\frac{x^n}{n!}\cdot e^{\theta x}\right| = \lim_{n\to+\infty}\frac{|x^n|}{n!}e^{\theta x}$$

$$\text{また}, {}^\exists K \in \mathbf{N}, K+1 > 2|x| \text{とできる}.$$

$$\text{すると}, \frac{|x|}{K+1} < \frac{1}{2}$$

また, $0 < \theta < 1$ だから, $x > 0$ のとき, $e^{\theta x} < e^x$, $x < 0$ のとき $e^{\theta x} < 1$ よって, ($e^{\theta x}$ は有限の値だから)

$$\begin{aligned}\lim_{n\to+\infty}|R_n(x)| &= \lim_{n\to+\infty}\frac{|x|^K}{K!}\cdot\frac{|x|}{K+1}\cdot\frac{|x|}{K+2}\cdots\frac{|x|}{n}\cdot e^{\theta x} \\ &\leq \lim_{n\to+\infty}\frac{|x|^K}{K!}\left(\frac{1}{2}\right)^{n-K}e^{\theta x} \\ &= 0\end{aligned}$$

したがって,

$$\lim_{n\to+\infty} R_n = 0 \text{ をみたすので}, x \in \mathbf{R} \text{ で } e^x \text{は Maclaurin 展開が可能である}.$$

$$\begin{aligned}e^x &= 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots \\ &\left(= \sum_{n=0}^{+\infty}\frac{x^n}{n!}\right)\end{aligned}$$

$$\text{但し}, 0! = 1 \text{ とする}$$

<u>例題</u> $$f(x) = \log(1+x) \quad (-1 < x \leq 1)$$

(解)

$$f^{(n)}(x) = (-1)^{n-1}\frac{(n-1)!}{(1+x)^n}, \ f^{(n)}(0) = (-1)^{n-1}(n-1)!$$

$$\log(1+x) = 0 + \frac{x}{1!}\cdot 1 + \frac{x^2}{2!}\cdot(-1)\cdot 1! + \frac{x^3}{3!}\cdot(-1)^2\cdot 2! + \frac{x^4}{4!}\cdot(-1)^3\cdot 3! + \cdots$$

$$\Longrightarrow \log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots$$

$$\left(= \sum_{n=1}^{+\infty}(-1)^{n+1}\frac{x^n}{n}\right)$$

問 : 次の関数を級数展開 (Maclaurin 展開) せよ.

(1)
$$f(x) = \frac{1}{\sqrt{1-x}} \quad (-1 \leq x < 1)$$

(2)
$$f(x) = \log(1-x) \quad (-1 \leq x < 1)$$

問の解 : 次の関数を級数展開 (Maclaurin 展開) せよ.

(1)
$$f(x) = \tfrac{1}{\sqrt{1-x}} = (1-x)^{-\frac{1}{2}} \Rightarrow f'(x) = -\tfrac{1}{2}(1-x)^{-\frac{3}{2}} \cdot (-1) = \tfrac{1}{2}(1-x)^{-\frac{3}{2}}$$
$$f''(x) = \frac{1}{2} \cdot \frac{3}{2} \cdot (1+x)^{-\frac{5}{2}}$$
$$f'''(x) = \frac{1}{2} \cdot \frac{3}{2} \cdot \frac{5}{2} \cdot (1+x)^{-\frac{7}{2}}$$
$$f^{(4)}(x) = \frac{1}{2} \cdot \frac{3}{2} \cdot \frac{5}{2} \cdot \frac{7}{2} \cdot (1+x)^{-\frac{9}{2}}$$
$$f^{(n)}(x) = \tfrac{1}{2} \cdot \tfrac{3}{2} \cdots \tfrac{(2n-1)}{2} \cdot (1+x)^{-\frac{(2n+1)}{2}}$$
$$\begin{aligned} f^{(n)}(0) &= \frac{1}{2} \cdot \frac{3}{2} \cdots \frac{(2n-1)}{2} \\ &= \frac{1}{2^n}\{1 \cdot 3 \cdots (2n-1)\} \\ &= \frac{1}{2^n} \cdot \frac{(2n)!}{2^n \cdot n!} \\ &= \frac{(2n)!}{2^{2n} n!} \end{aligned}$$

$$\frac{1}{\sqrt{1-x}} = 1 + \frac{x}{1!}\frac{2!}{2^2 \cdot 1!} + \frac{x^2}{2!}\frac{4!}{2^4 \cdot 2!} + \frac{x^3}{3!}\frac{6!}{2^6 \cdot 3!} + \frac{x^4}{4!}\frac{8!}{2^8 \cdot 4!} + \cdots$$

$$\begin{aligned} &= 1 + \frac{2!x}{2^2(1!)^2} + \frac{4!x^2}{2^4(2!)^2} + \frac{6!x^3}{2^6(3!)^2} + \frac{8!x^4}{2^8(4!)^2} + \cdots \\ &= \sum_{k=0}^{+\infty} (-1)^k \frac{(2k)!x^k}{2^{2k}(k!)^2} \quad (\text{但し}, 0! = 1 \text{ とする}) \end{aligned}$$
$$\tfrac{1}{\sqrt{1+x}} = 1 + \tfrac{x}{2} + \tfrac{3x^2}{8} + \tfrac{5x^3}{16} + \tfrac{35x^4}{128} + \cdots \quad (4 \text{ 次の項まで})$$

(2)
$$f(x) = \log(1-x) \Rightarrow f'(x) = \tfrac{1}{1-x} \cdot (1-x)' = -(1-x)^{-1}$$
$$f''(x) = -(-1)(1-x)^{-2} \cdot (-1) = -1!(1-x)^{-2}$$
$$f'''(x) = -(-1)(-2)(1-x)^{-3} \cdot (-1)^2 = -2!(1-x)^{-3}$$
$$f^{(4)}(x) = -(-1)(-2)(-3)(1-x)^{-4} \cdot (-1)^3 = -3!(1-x)^{-4}$$
$$f^{(n)}(x) = -(n-1)!(1-x)^{-n}$$
$$f^{(n)}(0) = -(n-1)!$$
$$\begin{aligned} \log(1-x) &= 0 + \frac{x}{1!} \cdot (-1) + \frac{x^2}{2!} \cdot (-1!) + \frac{x^3}{3!} \cdot (-2!) + \frac{x^4}{4!} \cdot (-3!) + \cdots \\ &= -[\frac{x}{1} + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} + \cdots] \\ &= -\sum_{k=1}^{+\infty} \frac{x^k}{k} \end{aligned}$$
$$\log(1-x) = -(x + \tfrac{x^2}{2} + \tfrac{x^3}{3} + \tfrac{x^4}{4} + \cdots) \quad (4 \text{ 次の項まで})$$