7 RBC model and Stochastic Simulation

7.1 What we will learn?

- How to incorporate stochastic shocks to DSGE models
- How to conduct quantitative model. Parameters? Simulation? Data?
- See Miao, Chapter 14.2

7.2 The Real Business Cycle (RBC) model (実物的景気循環理論)

Model

- It is a dynamic general equilibrium model with stochastic shock. (DSGE)
- An extension of Ramsey model to have labor-leisure choice and stochastic TFP shocks. (全要素生産性)
- No friction in this economy. Let's consider the representative agent's problem (代表的期個人).

$$
\begin{aligned}
\max_{\{C_t, N_t, K_{t+1}\}_t^\infty} \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t u(C_t, N_t) \\
\text{s.t.} K_{t+1} + C_t = z_t F(K_t, N_t) + (1-\delta) K_t \\
\ln(z_t) = \rho \ln(z_{t-1}) + \sigma \epsilon_t \\
\epsilon_t \sim N(0,1)
\end{aligned}
$$

- Utility function $u(C_t, N_t)$
  - consumption C_t and hours of work N_t.
  - Specify as follows.

$$u(C_t, N_t) = \ln(C_t) + \chi \ln(1 - N_t)$$

  - natural log? Constant Relative Risk Aversion (CRRA) utility $\frac{c^{1-\sigma}}{1-\sigma}$ with $\sigma = 1$. In this case, if wage w increases, the substitution and income effects offset each other. It is consistent with stable N for the long-run.
  - χ is a relative weight on labor.
- Stochastic shock z_t
  - z_t is productivity. (TFP)
  - Take $\ln(z_t)$ so that z_t is always positive.
  - Follows AR(1) process with $0 < \rho < 1$. A white noise shock ε_t is persistent.
- production function $F(K_t, N_t)$
  - Assume Cobb-Douglas. $K_t^\alpha N_t^{1-\alpha}$
- Expectation $\mathbb{E}_0$
  - The social planner maximizes the Von Neumann – Morgenstern expected utility.
  - At time 0? The plan should be changed at time 1 given new shock ε_t?

- – It's a kind of cheating. We will derive the optimality condition at time 0. Then, the result can be also used as a decision at period $1, 2, \cdots$.
  – Dynamic Programming (DP) has no problem about it. Expectation is introduced for each period.

$$\begin{aligned}
&V(K,z) = u(C,N) + \beta \mathbb{E}\big[V(K',z')\big|z\big] \\
&\text{ s.t. } K' + C = zF(K,N) + (1-\delta)K \\
&\ln(z') = \rho\ln(z) + \sigma\varepsilon'
\end{aligned}$$

  where

$$\mathbb{E}\big[V(K',z')\big|z\big] = \int V\big(K', \exp(\rho\ln(z) + \sigma\varepsilon')\big) f(\varepsilon')d\varepsilon'$$

  But macroeconomists usually use sequential problems for convenience.

- Lagrangean

$$\mathcal{L} = \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t \big\{\ln(C_t) + \chi\ln(1-N_t) + \lambda_t[z_t K_t^\alpha N_t^{1-\alpha} + (1-\delta)K_t - K_{t+1} - C_t]\big\}$$

- FOCs

$$\begin{aligned}
&C_t: \quad \beta^t\left(\frac{1}{C_t} - \lambda_t\right) = 0 \\
&N_t: \quad \beta^t\left(-\frac{\chi}{1-N_t} + \lambda_t z_t(1-\alpha)K_t^\alpha N_t^{-\alpha}\right) = 0 \\
&K_{t+1}: \quad -\beta^t\lambda_t + \mathbb{E}_t\lambda_{t+1}\big[z_{t+1}\alpha K_{t+1}^{\alpha-1}N_{t+1}^{1-\alpha} + 1 - \delta\big] = 0
\end{aligned}$$

- What does $\mathbb{E}_t$ mean? It is the expected value about $t+1$ variables given the current information (K_t, z_t).
- intertemporal condition (Euler)

$$\frac{1}{C_t} = \beta\mathbb{E}_t\left[\big(\alpha z_{t+1}K_{t+1}^{\alpha-1}N_{t+1}^{1-\alpha} + 1 - \delta\big)\frac{1}{C_{t+1}}\right]$$

- intratemporal condition

$$\frac{\chi C_t}{1-N_t} = (1-\alpha)z_t K_t^\alpha N_t^{-\alpha}$$

Calibration

- The process to set the model's parameter is called calibration. The RBC model decides parameters from the micro-estimation and long-run values in the steady state. Originally, researchers dislike the estimation of the model's parameters because of identification problems in econometric methods. But nowadays many macro economists estimate the DSGE model's parameters by Bayesian time-series methods.
- Let's use Miao's values, although I slightly disagree.
- Parameters from other sources
  – log utility function ($\sigma = 1$). It is from empirical findings that income & substitution effects seem to offset.

  - $\alpha = 0.33$. Macro-level firm's problem

$$\max K^\alpha N^{1-\alpha} - rK - wN$$

    implies that $\alpha = rK/Y$. It is called capital share. Empirically rK/Y is about 1/3 in many countries and unchanged overtime
  - $\beta = 0.99$. Assume the model's frequency is quarterly. In the household model, $\beta(1+R_t) = 1$ is the steady state condition. Let's say, R_t is 4% annual, and 1% quarterly.
  - $\delta = 0.025$. The annual capital depreciation rate is 10%. It is from the national accounting estimation.
  - $\rho = 0.99$ and $\sigma = 0.0089$. z_t can be measured as the Solow residual.

$$\ln(z_t) = \ln(Y_t) - \alpha \ln K_t - (1-\alpha)\ln N_t$$

    By the sequence of z_t, we can directly estimate the coefficient ρ by OLS of AR(1) process. The standard deviation σ is also estimated as the standard error.
  - N^{ss}. It's not a parameter, but we usually calculate for other parameters' calibrations. In simulation, N_t is calculated as average annual hours of work $\times$ employment with normalization to 1. For example, employment rate is 60%, hours of work is 40 per week for 50 weeks. Suppose 16 is total hours per day excluding sleep and some others. Then,

$$N^{ss} = \frac{0.6 \times 40 \times 50}{16 \times 7 \times 52} \approx 0.2$$

    But Miao uses $N^{ss} = 0.33$.
- Parameters internally decided.
  - Usually, the steady state values are used. For business cycle analysis, those are interpreted as external conditions.
  - Normalization: $z^{ss} = 1$.
  - By Euler,

$$\beta\left[\alpha(K^{ss})^{\alpha-1}(N^{ss})^{1-\alpha} + 1 - \delta\right]$$

$$\iff K^{ss} = \left(\frac{\alpha}{\beta^{-1} - (1-\delta)}\right)^{\frac{1}{1-\alpha}} N^{ss}$$

  - Given the parameter values, we get $K^{ss} = 9.355$.
  - By the resource constraint,

$$C^{ss} = (K^{ss})^\alpha (N^{ss})^{1-\alpha} - \delta K^{ss} = 0.7612.$$

  - By the intratemporal condition,

$$\chi = \frac{(1-\alpha)(K^{ss})^\alpha(1-N^{ss})}{(N^{ss})^\alpha C^{ss}} = 1.7783$$

Dynare

- Dynare automatically calculrat the log-linearized system as

$$A\begin{pmatrix}\hat{C}_{t+1}\\ \hat{N}_{t+1}\\ \hat{K}_{t+1}\\ z_{t+1}\end{pmatrix} = B\begin{pmatrix}\hat{C}_{t}\\ \hat{N}_{t}\\ \hat{K}_{t}\\ z_{t}\end{pmatrix} + \begin{pmatrix}0\\ 0\\ 0\\ \sigma\end{pmatrix}\varepsilon_{t+1}$$

  z_t is already log.
- A and B are coefficients evaluated at the steady state.
- Dynare calculates the eigenvalues of $A^{-1}B$. In this case, C and N are jump variables, so two eigenvalues should be larger than 1.
- Two types of simulations: stochastic simulations and Impulse response function (IRF)

7.3 stochastic simulation

- This method simulates the economy with random shock for thousands of periods and derive business cycle moments. They Include the endogenous variables' standard deviations, auto correlations, and correlation between each one. Then, these moments are compared with the moments obtained from data. To isolate the business cycle from the trend component, simple decomposition such as Hodrick-Prescott (HP) filter is used.
- Take a look at `rbc1.mod`
  - ε_t is a stochastic exogeneous variable. It is defined as `varexo e;`
  - K_t is a pre-determined variable. It must be written as `K(-1)`
  - z_t represents a stochastic shock. It is also a pre-determined variable, but no need to push back. z_t is `z`
  - ε_t follows the standard normal distribution. It is introduced as `var e; stderr 1;`
  - Stochastic simulation for 100,000 periods. We use linear approximation. To compare with data, HP-filter is applied.
    `stoch_simul(order = 1, hp_filter = 1600, periods =100000);`
  - Option: You can add GDP Y_t, investment I_t, real interest rate r_t, and wage w_t.
- To replicate Miao's result, we need to use natural log of each variable. Take a look at `rbc2.mod`
- Overall, RBC model matches data well!
- A major problem is too much fluctuation of N. Adjustment of hours & employment are not so smooth.

7.4 Impulse response function (IRF)

- If ε_{t+1} suddenly increases by one standard deviation, then what happens?
- Dynare automatically plots one standard deviation shock of ε_t.
- It is also compared with data. IRF can be estimated by Vector Auto Re-

Table 14.1
Business cycle statistics

	Standard deviation (%)	Relative standard deviation	First order auto correlation	Contemporaneous correlation with Y
Y	1.52 (1.85)	1.00 (1.00)	0.72 (0.85)	1 (1)
C	0.74 (1.17)	0.45 (0.63)	0.76 (0.86)	0.97 (0.80)
I	4.19 (4.41)	2.75 (2.38)	0.71 (0.84)	0.99 (0.62)
N	0.55 (1.95)	0.36 (1.05)	0.71 (0.90)	0.98 (0.82)
Y/N	0.99 (1.12)	0.65 (0.60)	0.74 (0.72)	0.99 (0.26)
w	0.99 (0.87)	0.65 (0.47)	0.74 (0.72)	0.99 (−0.06)
TFP	0.89 (1.10)	0.54 (0.60)	0.99 (0.80)	1.00 (0.68)
Price	NA (0.94)	NA (0.51)	NA (0.91)	NA (0.00)
$\ln R^f$	0.05 (0.70)	0.04 (0.38)	0.71 (0.71)	0.95 (0.02)
$\ln R^s$	0.05 (8.40)	0.03 (4.51)	0.71 (0.09)	0.96 (−0.23)

Note: All variables are in logarithms. Numbers in the brackets are computed from the US quarterly data from 1948Q1–2010Q4. Other numbers are computed from the basic RBC model.

gression (VAR) model. It is a direct estimation of the linearized system. Endogeneity problem is possible, but it can be adjusted by Structural VAR approach. It imposes identification assumption of the system of equations, for example, TFP's effects on labor must vanish in the long-run.

- Interpretation of IRF.
  - To get better intuition, assume $\rho = 0.1$ and think about very temporary effects.
  - In general, an increase in z causes both income and substitution effects. They may offset each other and do not change N. But, the shock is very temporary in this case. Now is the chance to produce more! The substitution effect dominates and N increases.
  - More labor input, then Y increases.
  - By the consumption smoothing, C does not change so much.
  - Given More Y and unchanged C, the resource goes to investment. I jumps up, and K increases.
  - If $\rho = 0.99$, the long-run (income) effects are also included. Implications are ambiguous.

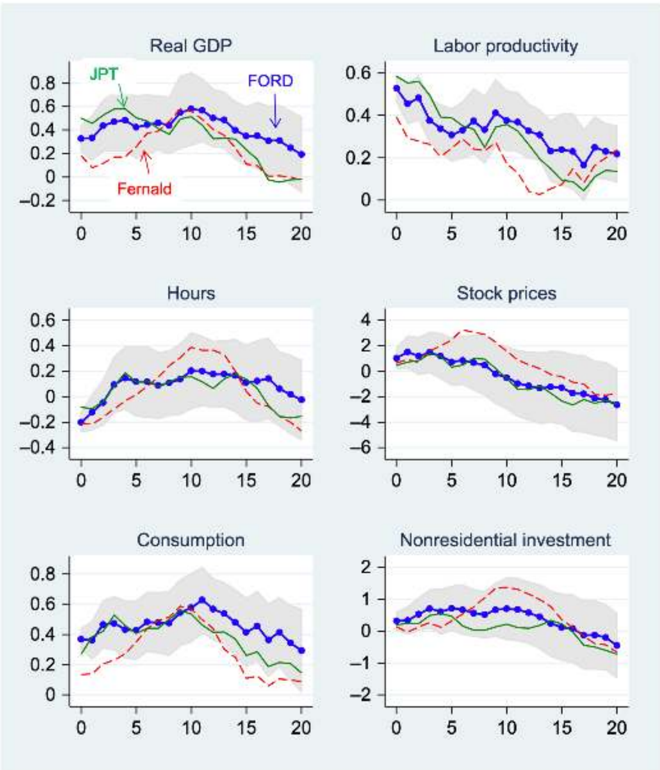

Fig. 9 Effects of TFP shock, Jordà local projection, various samples. Francis, Owyang, Roush, DiCecio (FORD): *blue lines with circles* (*black* in the print version); Fernald utilization-adj TFP: *dashed red* (*gray* in the print version) *lines*; Justiniano, Primiceri, Tambalotti (JPT) DSGE TFP: *solid green* (*gray* in the print version) *lines*. Light gray bands are 90% confidence bands.

Ramey, V. A. (2016). Macroeconomic shocks and their propagation. Handbook of macroeconomics, 2, 71-162.

7.5 Appendix: Hodrick-Prescott (HP) filter

- It is a method to separate time-series data to trend and cycle.
- x_t is original data from period 1 to T. TR_t is trend.
- TR_t is obtained by minimizing the equation.

$$\sum_{t=1}^{T}(x_t - TR_t)^2 + \lambda \sum_{t=2}^{T-1}\left[(TR_{t+1} - TR_t)^2 - (TR_t - TR_{t-1})^2\right]$$

- λ controls the smoothness. If $\lambda = 0$, $x_t - TR_t$, so the trend also flucutuates. If $\lambda \to \infty$, The growth of TR_t, that is $TR_{t+1} - TR_t$, becomes a constant. Linear trend.

- Practically, $\lambda = 100$ for annual data, $\lambda = 1600$ for quarterly data, and $\lambda = 14400$ for monthly data.

```
% HP filter, example
clear;clc;close all;
load Data_GDP; % US Quarterly GDP sample data provided by Matlab
quarters = datetime(dates,'ConvertFrom','datenum','Format',"yyyy-MM-dd");
% Builtin function to separate log GDP into trend and cycle
[Trend,Cycle] = hpfilter(log(Data),1600);
% convert dates to time-series data
quarters = datetime(dates,'ConvertFrom','datenum','Format',"yyyy-MM-dd");

% plot
figure;
subplot(2,1,1);
plot(quarters, log(Data));
hold on;
plot(quarters, Trend);
hold off;
subplot(2,1,2);
plot(quarters, Cycle);
```