5 SIR Macro model

5.1 Introduction

- Research Question: What is the tradeoff between COVID-19 infection and economy?
- Epidemiologists use SIR model to study new coronavirus transmission. Quantitatively calculate how much social activity should be limited. But what's the economics cost?
- SIR Macro model: SIR + DSGE. incentive of preventing behavior, market equilibrium effects, externality. Rapid development since Spring 2020.
- SIR Macro models are used to answer real-time problems.
  - What kind of policies are effective? lockdown? PCR test?
  - Alpha, Delta, Omicron variants?
  - How many vaccines?
- Lecture material
  - Eichenbaum & Rebelo & Trabandt "The Macroeconomics of Epidemics"
    `https://doi.org/10.1093/rfs/hhab040`
  - I wrote a note, but only in japanese.
    `https://www.ihep.jp/publications/journal/`
- Key concept: Pandemic possibility frontier

5.2 Basic SIR model

- Suppose $t =$ one week.
- 4 groups of people. Total population is 1.

$$\begin{aligned}
\text{Susceptible (not infected yet): } & S_{t+1} = S_t - T_t && (1)\\
\text{Infected (currently): } & I_{t+1} = I_t + T_t - (\pi_r + \pi_d) I_t && (2)\\
\text{Recovered (no more infection): } & R_{t+1} = R_t + \pi_r I_t && (3)\\
\text{Dead: } & D_{t+1} = D_t + \pi_d I_t && (4)
\end{aligned}$$

- Infected person recovers with probability π_r and dead with π_d each week.
- New infection

  $$T_t = \pi S_t I_t$$

  If each person randomly meets another one at time t, the total number meeting of S and I is $S_t I_t$. π is the transmission rate of virus.
- Reproduction Numbers

  $$I_{t+1} - I_t = \pi S_t I_t - (\pi_r + \pi_d) I_t$$

  $$\text{percent change: } \%\Delta I_{t+1} = \frac{I_{t+1} - I_t}{I_t} = \pi S_t - (\pi_r + \pi_d)$$
- Effective reproduction number:

  $$\mathcal{R}_t \equiv \frac{\pi S_t}{\pi_r + \pi_d} \begin{array}{c} > \\ = \\ < \end{array} 1 \quad \Leftrightarrow \quad \%\Delta I_{t+1} \begin{array}{c} > \\ = \\ < \end{array} 0$$
- Basic reproduction number

  $$\mathcal{R}_0 = \frac{\pi}{\pi_r + \pi_d},$$

  that is, $\mathcal{R}_t$ if $S = 1$.

5.3 SIR model with economy, theory

- Uniform lockdown: order L fraction of people to stay at home. Since the government does not identify disease, everyone is randomly locked down.

  $$T_t = \pi[S_t(1 - L_t)][I_t(1 - L_t)]$$
- Economic loss is

  $$\hat{Y}_t = L_t(S_t + I_t + R_t) + D_t$$
- Targetted lockdown. The government knows who is infected by the PCR test. Test-Tracing-Quarantine (TTQ) policy

  $$T_t = \pi S_t [I_t(1 - Q_t)]$$

  Economic loss is significantly small. The average length of disease is about two weeks, I is tiny.

  $$\hat{Y}_t = L_t I_t + D_t$$

5.4 SIR model with economy, simulation

Simulation is simple because there is no optimization. Assume $I_t = 0.001$ at time 0, and then, repeatedly calculate $t+1$ period variables from t.

```
%SIR model for DSGE lecture of Waseda, 2021, So Kubota
close all;clc;clear;

% parameters
pi_r    = (7/18)*(1-0.005);
pi_d    = (7/18)*0.005;
pi      = 0.5852;   %  60% will be infected eventually

% vectors of 100 periods
periods=100;
S=ones(periods,1);
I=zeros(periods,1);
R=zeros(periods,1);
D=zeros(periods,1);
T=zeros(periods,1);
Rt=zeros(periods,1);   % effective reproduction number
Y=zeros(periods,1);   % output loss

% uniform or targetted lockdowns
L=zeros(periods,1); % Fraction of lockdown
Q=zeros(periods,1); % Fraction of quarantine among infected
L(20:40) = 0.0;
Q(20:40) = 0.4;

% initial period
Rt(1) = pi*S(1)/(pi_r+pi_d);  % basic reproduction rate
I(1) = 0.001;  % 0.1% initial infection
Y(1) = L(1)*(S(1)+R(1)) + (L(1)*(1-Q(1))+Q(1))*I(1) + D(1);
T(1) = pi * (1-L(1))*S(1) * (1-L(1))*(1-Q(1))*I(1);

% calculation
for t=1:periods-1
    S(t+1) = S(t) - T(t);
    I(t+1) = I(t) + T(t) - pi_r*I(t) - pi_d*I(t);
    R(t+1) = R(t) + pi_r*I(t);
    D(t+1) = D(t) + pi_d*I(t);
    Y(t+1) = L(t+1)*(S(t+1)+R(t+1)) + (L(t+1)*(1-Q(t+1))+Q(t+1))*I(t+1) + D(t+1);
    Rt(t+1) = pi*S(t+1)/(pi_r+pi_d);
    T(t+1) = pi * (1-L(t+1))*S(t+1) * (1-L(t+1))*(1-Q(t+1))*I(t+1);
```

```
end

% plot
SIR_path = [I S R D T Y Rt];
titles = {'Infected', 'Susceptible', 'Recovered', 'Deceased',...
          'New Infection','Output loss','Effective Reproduction Rate'};
figure;
for x = 1:size(SIR_path,2)
    subplot(2,4,x);
    plot(SIR_path(1:periods, x), 'linewidth', 2);
    title(titles{x});
end
```

5.5 SIR Macro model, theory

- Based on Eichenbaum, Rebelo, and Trabandt (2020)
- We must consider a market equilibrium, instead of the social planner's problem. why?
- Assume infection is through economic activity (consumption) or non-economic activity.

  $$T_t = \pi \left(S_t C_t^s\right)\left(I_t C_t^i\right) \tag{5}$$

  C_t^s is susceptibles' total consumption, C_t^i is infecteds'. They are "Big K".
- U_t^s is a susceptible's value function. U_t^i is infected's. Suppose the infection probability as τ_t.
- Susceptible's Bellman equation

  $$U_t^s = u(c_t^s, n_t^s) + \beta\left[(1-\tau_t)U_{t+1}^s + \tau_t U_{t+1}^i\right] \tag{6}$$

  c_t^s and n_t^s are each susceptible's consumption and labor.
- Why time t is needed? Because the pandemic will eventually end. It is a finite period problem.
- For simplicity, assume

  $$u(c,n) = \ln c - \left(\frac{\theta}{2}\right) n^2$$

- Infection probability depends on susceptible's individual consumption c_t^s

  $$\tau_t(c_t^s) = \frac{T_t}{S_t} = \pi c_t^s \left(I_t C_t^i\right) \tag{7}$$

  little k, Big K trick. Each susceptible chooses c_t^s given macroeconomic level variables, including C_t^s. In the equilibrium, $c_t^s = C_t^s$.
- Budget constraint

  $$(1+\mu_t)c_t^s = A n_t^s \tag{8}$$

  Assume that the production function is AN. Then the marginal labor productivity is always A, which is equivalent to the wage. Then, An_t^s is the labor income.

- How to simulate lockdown? Add a consumption tax μ_t. The government pushes down the economic activity. The tax revenue is thrown away to the pacific ocean.
- No Saving! critical assumption.
- Susceptible's problem

$$\max_{c_t^s, n_t^s} U_t^s = \ln c_t^s - \left(\frac{\theta}{2}\right)(n_t^s)^2 + \beta\left\{[1-\tau_t(c_t^s)]U_{t+1}^s + \tau_t(c_t^s)U_{t+1}^i\right\}$$
$$\text{s.t. } (1+\mu_t)c_t^s = An_t^s,$$
$$\tau_t(c_t^s) = \pi c_t^s \left(I_t C_t^i\right)$$

- FOC

$$\frac{1}{c_t^s} = (1+\mu_t)\left(\frac{\theta}{A}\right)n_t^s + \beta(U_{t+1}^s - U_{t+1}^i)\pi\left(I_t C_t^i\right) \tag{9}$$

- Infected's problem

$$\max_{c_t^i, n_t^i} U_t^i = \ln c_t^i - \left(\frac{\theta}{2}\right)(n_t^i)^2 + \beta\left[(1-\gamma_r-\gamma_d)U_{t+1}^i + \gamma_r U_{t+1}^r + \gamma_d \times 0\right], \tag{10}$$
$$\text{s.t. } (1+\mu_t)c_t^i = An_t^i$$

- FOCs are static.

$$n_t^i = \frac{1}{(1+\mu_t)\sqrt{\theta}} \tag{11}$$
$$c_t^i = \frac{A}{(1+\mu_t)\sqrt{\theta}} \tag{12}$$

- Recovered problem

$$\max_{c_t^r, n_t^r} U_t^r = \ln c_t^r - \left(\frac{\theta}{2}\right)(n_t^r)^2 + \beta U_{t+1}^r, \tag{13}$$
$$\text{s.t. } (1+\mu_t)c_t^r = An_t^r$$

- FOCs are the same as (11) and (12). $n_t^r = \frac{1}{(1+\mu_t)\sqrt{\theta}}$ and $c_t^r = \frac{A}{(1+\mu_t)\sqrt{\theta}}$
- The system of equations?
- μ_t is exogeneous
- Initial shock is $I_t = 0.001$
- 11 variables

$$S_t, I_t, R_t, D_t, T_t, \tau_t, c_t^s, n_t^s, U_t^s, U_t^i, U_t^r$$

- 11 equations
  - Equations (1)-(4) about S, I, R, D
  - Budget constarint (8)
  - Substituting (12) to (7)

$$\tau_t = \frac{T_t}{S_t} = \pi c_t^s I_t \left(\frac{A}{\sqrt{\theta}}\right)$$

  - Rewrite new infection (5): $T_t = S_t \tau_t$
  - Substitute (9) to (12)

$$\frac{1}{c_t^s} = (1+\mu_t)\left(\frac{\theta}{A}\right)n_t^s + \beta(U_{t+1}^s - U_{t+1}^i)\pi I_t\left(\frac{A}{(1+\mu_t)\sqrt{\theta}}\right)$$

– Three Bellman equations. For infected and recovered, substitute (11) and (12)

$$U_t^s = \ln c_t^s - (\theta/2)(n_t^s)^2 + \beta\left[(1-\tau_t)U_{t+1}^s + \tau_t U_{t+1}^i\right]$$

$$U_t^i = \ln\left(\frac{A}{(1+\mu_t)\sqrt{\theta}}\right) - \frac{1}{2} + \beta\left[(1-\gamma_r-\gamma_d)U_{t+1}^i + \gamma_r U_{t+1}^r\right]$$

$$U_t^r = \ln\left(\frac{A}{(1+\mu_t)\sqrt{\theta}}\right) - \frac{1}{2} + \beta U_{t+1}^r$$

5.6 SIR Macro model, simulation

```
// Eichnbaum-Rebelo-Trandbont model
// for DSGE lecture of Waseda, 2021
// based on Krueger-Uhli-Xie
// https://github.com/tjxie/KUX_PandemicMacro
// addpath /Applications/Dynare/4.6.4/matlab
// addpath C:\dynare\4.6.4\matlab

// 0 preparation ////////////////////////////////////////////
close all;clc; // Do not "clear;"

// 1 variables //////////////////////////////////////////////

// This program will run the model many times from pi=0 to pi = target one.
// If Dynare runs with true pi from the beginning, it fails to find solution.
// But, if dynare starts from pi=0 and eventually increase pi,
// Dynare uses the last path as the initial guess for each calculation,
// then eventually find the solution under the true pi_target.

@#define pi_target  = 5e-7 // The true value of pi

// endogeneous variables
var         ns          (long_name = 'Labor supply, Susceptible')
            cs          (long_name = 'Consumption, Susceptible')
            tau         (long_name = 'Probability of infection')
            I           (long_name = 'Infected')
            T           (long_name = 'Newly infected')
            S           (long_name = 'Susceptible')
            R           (long_name = 'Recovered')
            D           (long_name = 'Deceased')
            Ui          (long_name = 'Lifetime utility of infected')
            Us          (long_name = 'Lifetime utility of susceptible')
            Ur          (long_name = 'Lifetime utility of recovered');

// exogeneous variables
```

```
varexo      eps         (long_name = 'Initial Infection')
            mu          (long_name = 'Consumption Tax');

// 2 Parameters ///////////////////////////////////////////

// definitions
parameters  pi      (long_name = 'Probability of becoming infected through consumption')
            pi_r        (long_name = 'Probability of recovery')
            pi_d        (long_name = 'Probability of death')
            theta       (long_name = 'Labor supply parameter')
            A           (long_name = 'Productivity')
            betta       (long_name = 'Discount factor');

// values
pi_r    = (7/18)*(1-0.005);
pi_d    = (7/18)*0.005;
betta   = 0.96^(1/52);
A       = 39.835;
theta   = 0.001275;
pi  = 0;  // pi in the leture note, it is zero, not true value

// 3 model ////////////////////////////////////////////////
model;

// SIR Equations
S - S(-1) + I - (1 - pi_r - pi_d) * I(-1) = 0;
I = T(-1) + (1 - pi_r - pi_d) * I(-1) + eps;
R = R(-1) + pi_r*I(-1);
D = D(-1) + pi_d*I(-1);

// First-order condition of susceptible
(1/cs) = (1+mu)*(theta/A)*ns + betta * (Us(+1) - Ui(+1)) * pi * (A/((1+mu)*sqrt(theta))) * I;
// Budget constraint of susceptible
(1+mu)*cs = A*ns;
// infection probability of sceptible
tau = pi * cs * (A/((1+mu)*sqrt(theta))) * I;
// total new infection;
T = S * tau;

// value functions
Us = log(cs) - (theta/2)*ns^2 + betta*((1 - tau)*Us(+1) + tau*Ui(+1));
Ui = log(A/((1+mu)*sqrt(theta))) - (1/2)*(1+mu)^(-2) + betta*((1 - pi_d - pi_r) * Ui(+1) + pi_r * Ur(+1));
Ur = log(A/((1+mu)*sqrt(theta))) - (1/2)*(1+mu)^(-2) + betta*Ur(+1);
```

```
end;

// 4 Steady state ///////////////////////////////////////////////

// only one steady state.
// I do not specify initial & terminal states

steady_state_model;
ns          = 1/sqrt(theta);
cs          = A*ns;
tau         = 0;
I           = 0;
T           = 0;
S           = 1;
R           = 0;
D           = 0;
Us          = (log(cs)-theta/2*ns^2)/(1-betta);
Ur          = Us;
Ui          = (log(cs)-theta/2*ns^2+betta*pi_r*Ur)/(1-betta*(1-pi_d-pi_r));
end;
steady;
check; //model_diagnostics

// 5 simulation /////////////////////////////////////////////////

shocks;
    var     eps;
    periods 1:1;
    values  0.001;
    var     mu;
    periods  10:30;
    values  0.3;
end;

perfect_foresight_setup(periods = 100);

// loop from pi=0 to pi=pi_target
// Dynare macro
@#for value in 0:2e-8:pi_target
    pi = @{value};  // substitute pi with each iteration value
    perfect_foresight_solver;  // Dynare uses the past result as initial guess of Newton method
@#endfor

%% 6 Plot ////////////////////////////////////////////////////
```

```
varmat = [I S R D cs ns];
titles = {'Infected', 'Susceptible', 'Recovered',...
          'Deceased', 'Susceptible Consumption', 'Susceptible Labor'};
figure;
plot_periods=100;
for x = 1:size(varmat,2)
    subplot(2,3,x);
    plot(varmat(2:(plot_periods+1), x), 'linewidth', 2);
    title(titles{x});
end
```