HW4

```r
periodogram <- function(x){
  xPer <- (1/length(x))*abs(fft(x)^2)
  f <- seq(0,1.0-1/length(x),by=1/length(x))
  return(list(f,xPer))
}
```

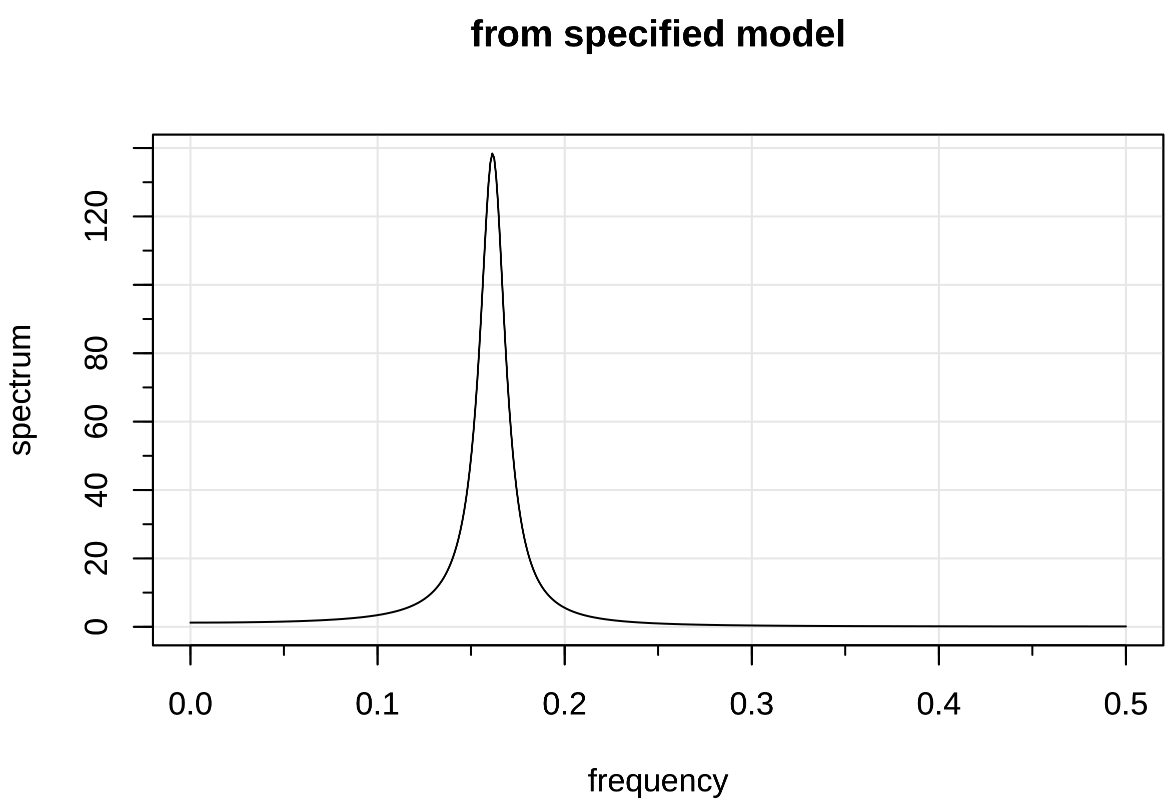
1

$$Z_t \sim WN(1)$$

a

$$X_t = X_{t-1} - 0.9X_{t-2} + Z_t$$

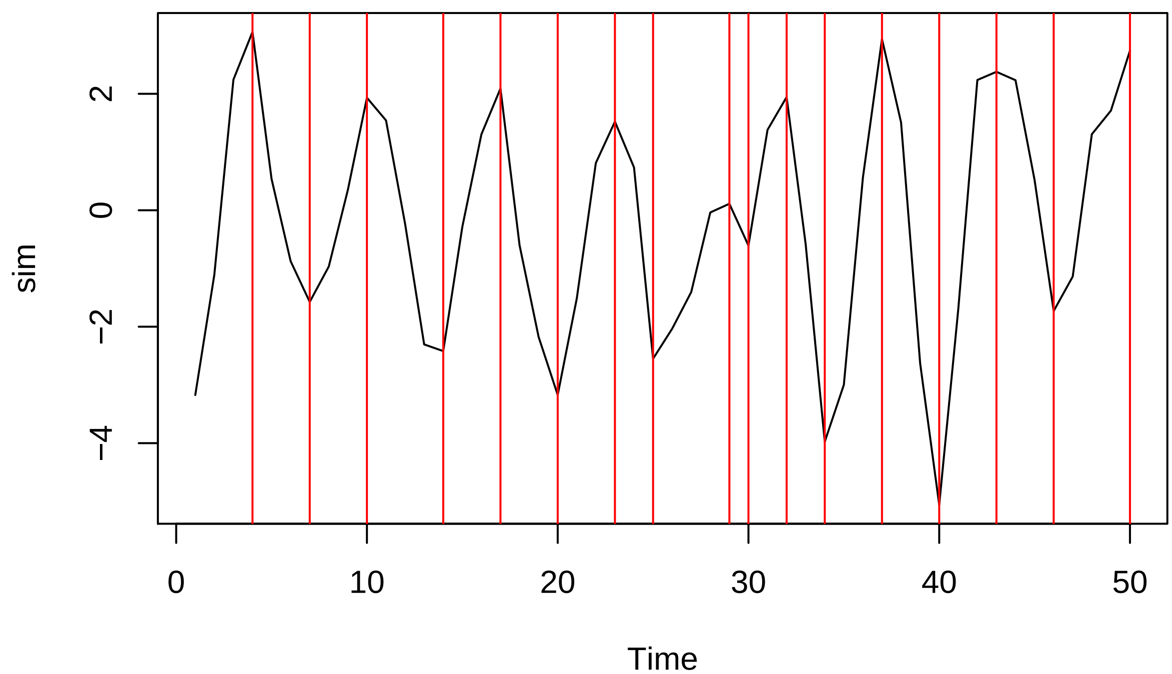
```r
library(astsa)
freq = arma.spec(ar = c(1,-0.9), var.noise = 1, type = "line")
```



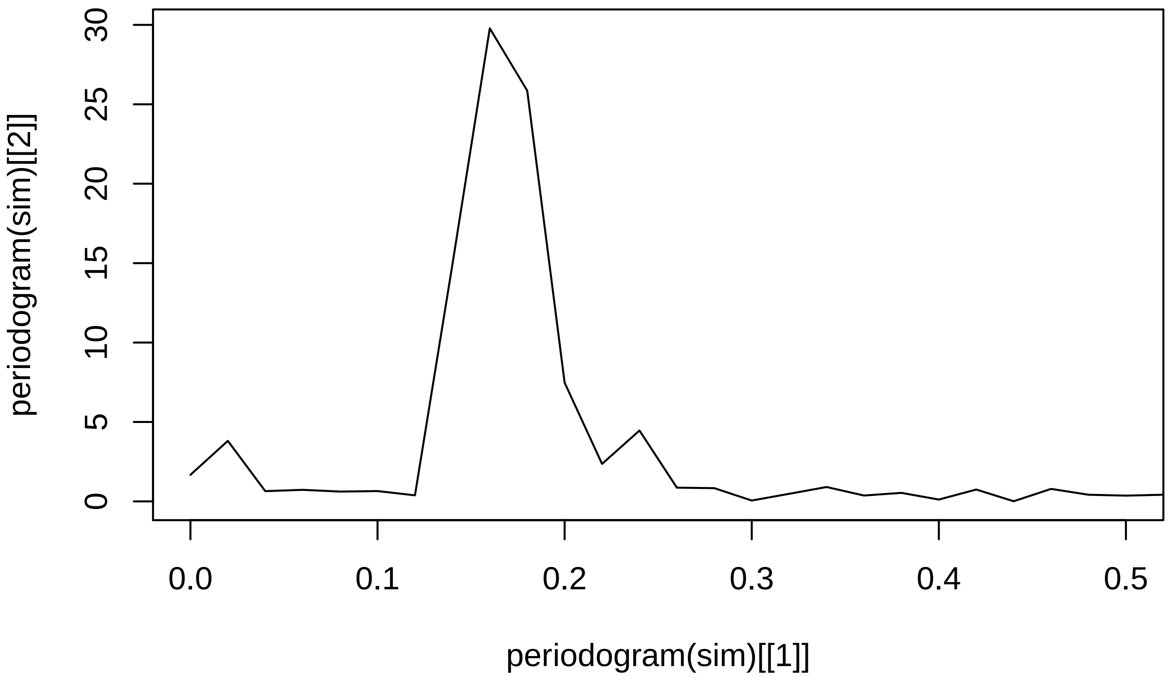
```
best = freq$freq[which.max(freq$spec)]
```

The frequencies that appear to be dominant are $f \in [0.125, 0.175]$. The peak occured at 0.1613226

```
set.seed(1)
sim = arima.sim(n = 50, list(ar = c(1,-0.9)))
period = cumsum(rle(as.vector(sign(diff(sim))))$lengths) + 1
plot(sim)
abline(v = period, col = 'red')
```




```
plot(periodogram(sim)[[1]],
     periodogram(sim)[[2]], type = 'l',
     xlim = c(0,0.5))
```




```r
mean(rle(as.vector(sign(diff(sim))))$lengths) * 2
```

```
## [1] 5.764706
```

```r
1/best
```

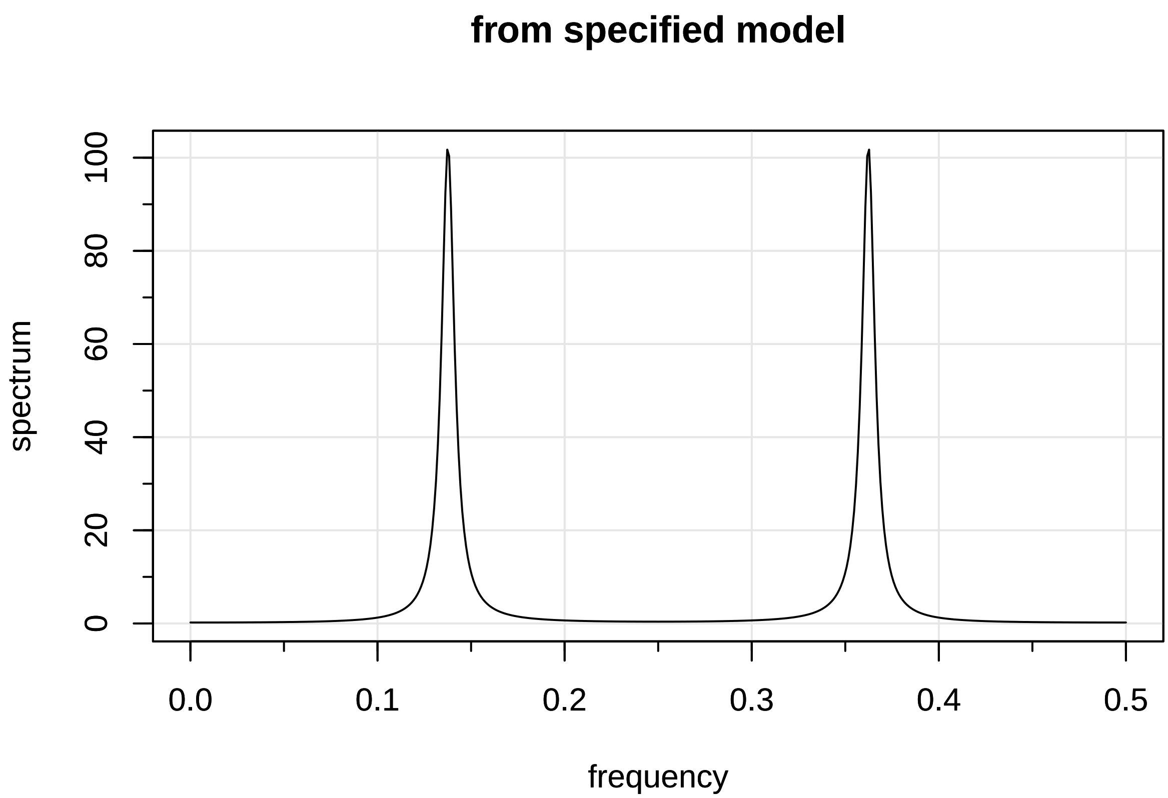
```
## [1] 6.198758
```

It appears that the observed period matches reasonably well with $\frac{1}{f}$

b

$$X'_t = -0.3X'_{t-2} - 0.9X'_{t-4} + Z_t$$

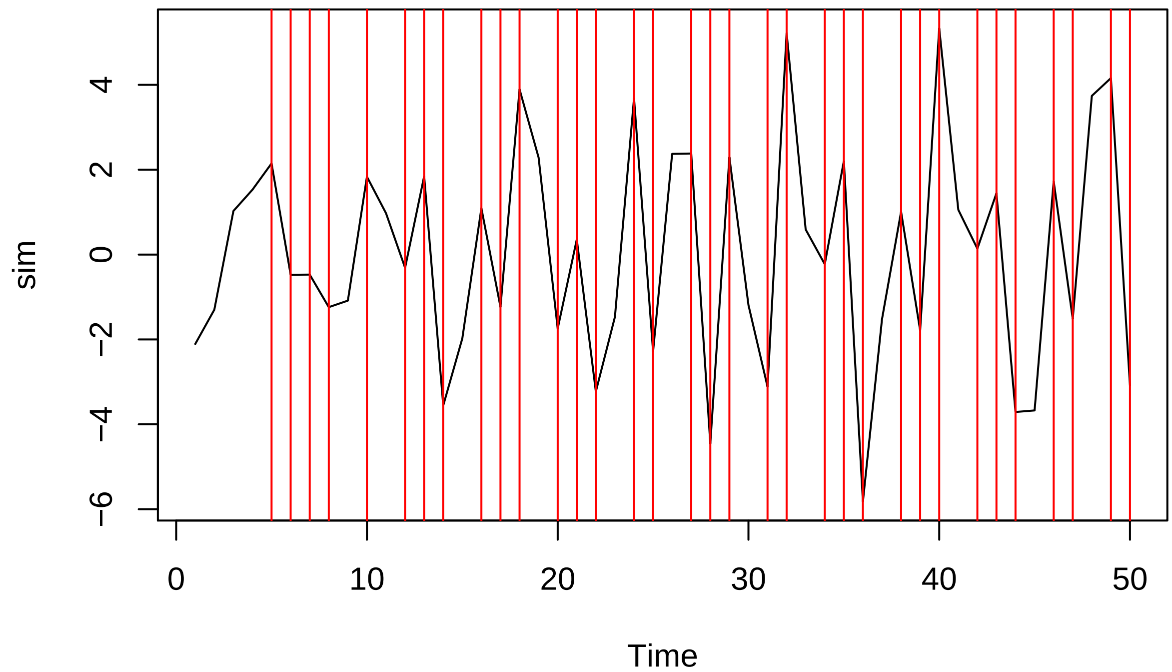
```
library(astsa)
freq = arma.spec(ar = c(0,-0.3,0,-0.9), var.noise = 1, type = "line")
```



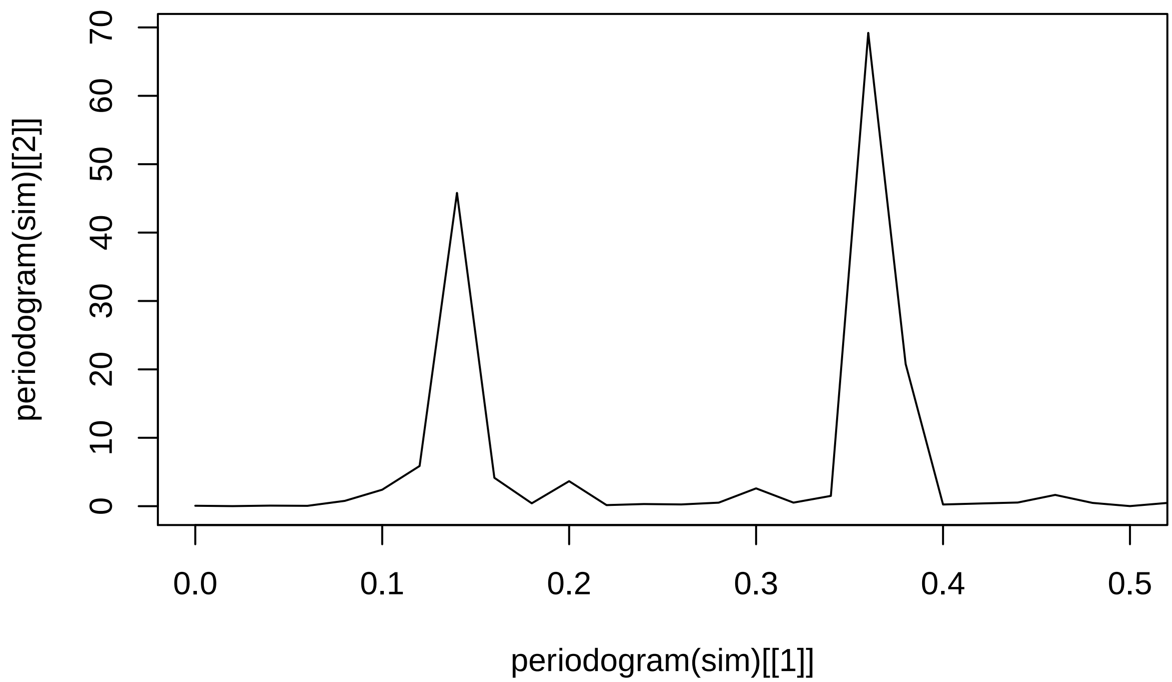

```
best1 = freq$freq[which.max(freq$spec[1:300])]
best2 = freq$freq[which.max(freq$spec)]
```

The frequencies that appear to be dominant are $f_1 \in [0.125, 0.15]$ and $f_1 \in [0.35, 0.375]$. The peaks occured at 0.1372745 and 0.3627255

```
set.seed(1)
sim = arima.sim(n = 50, list(ar = c(0,-0.3,0,-0.9)))
period = cumsum(rle(as.vector(sign(diff(sim))))$lengths) + 1
plot(sim)
abline(v = period, col = 'red')
```




```r
plot(periodogram(sim)[[1]],
     periodogram(sim)[[2]], type = 'l',
     xlim = c(0,0.5))
```




```
mean(rle(as.vector(sign(diff(sim))))$lengths) * 2
```

```
## [1] 2.882353
```

```
1/best1
```

```
## [1] 7.284672
```

```
1/best2
```

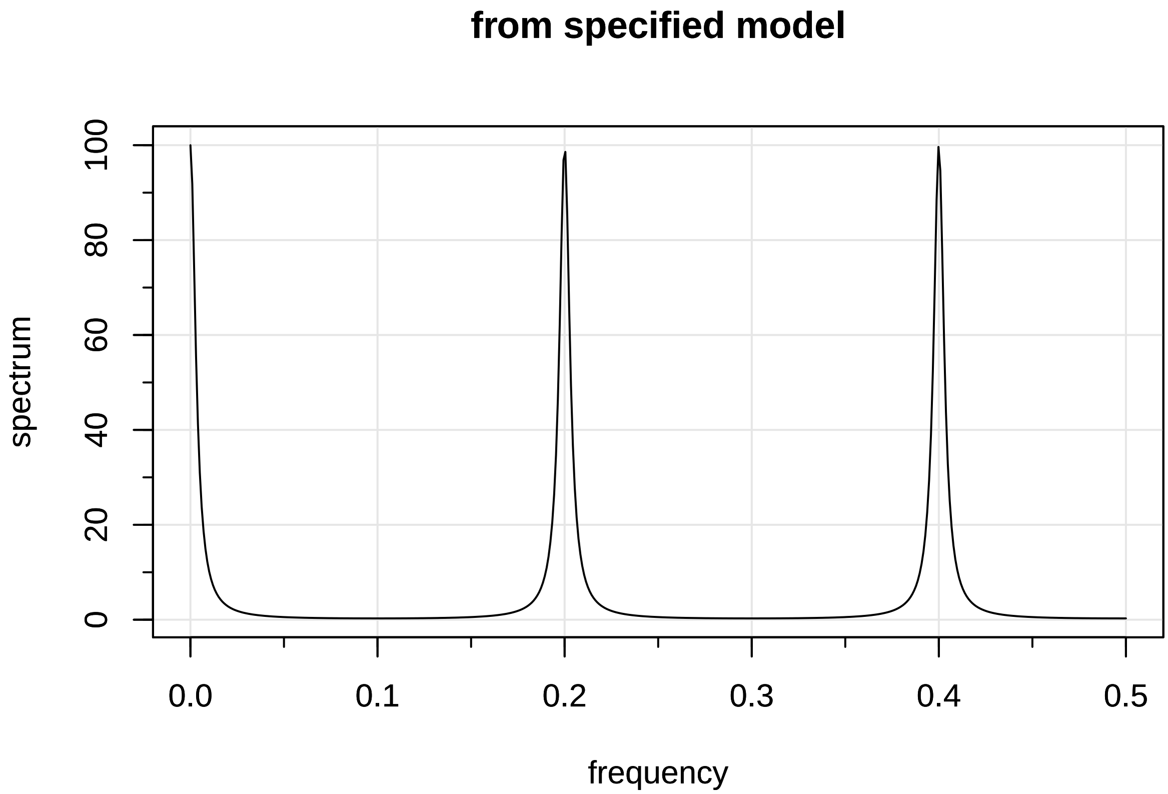
```
## [1] 2.756906
```

It appears that the observed period matches reasonably well with $\frac{1}{f_2}$ and not with $\frac{1}{f_1}$

c

$$X''_t = 0.9X''_{t-5} + Z_t$$

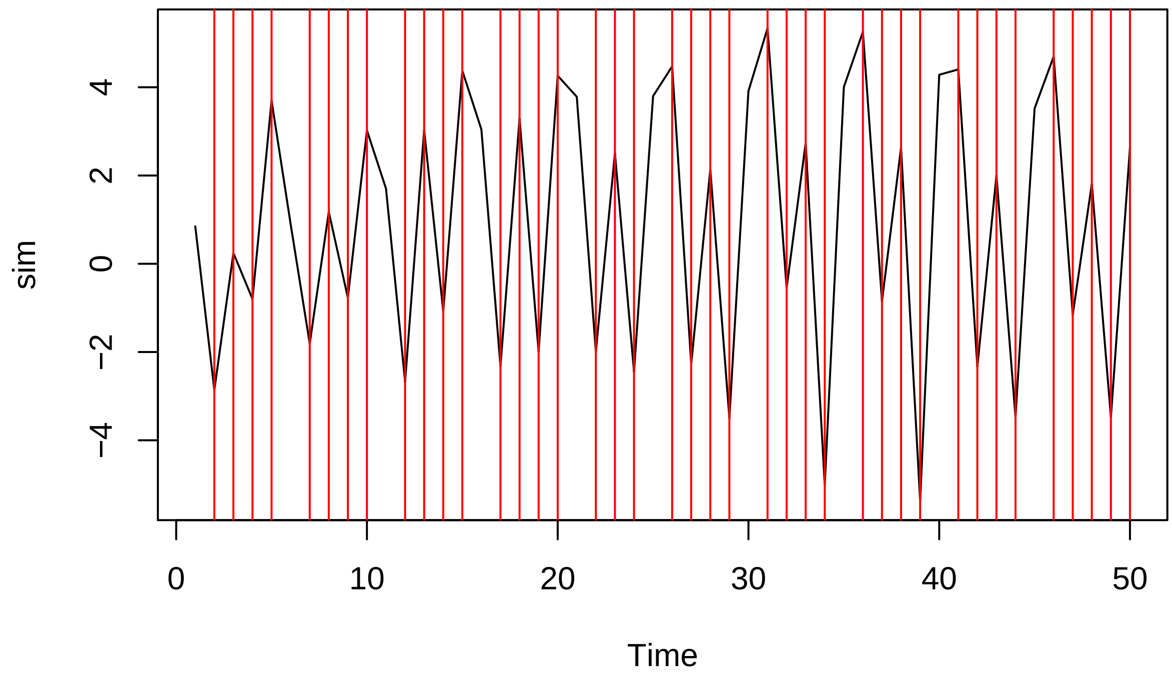
```
library(astsa)
freq = arma.spec(ar = c(0,0,0,0,0.9), var.noise = 1, type = "line")
```



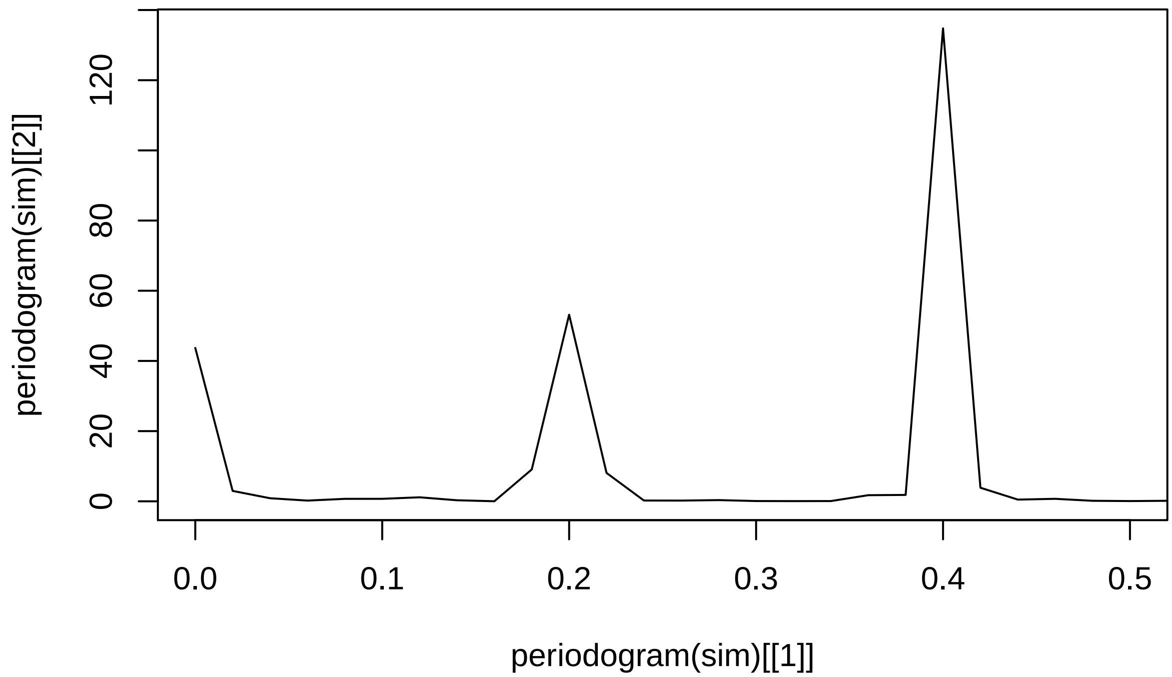

```
best1 = 0
best2 = 0.2
best3 = 0.4
```

The frequencies that appear to be dominant are $f_1 \in [0, 0.05]$, $f_2 \in [0.19, 0.21]$, and $f_3 \in [0.39, 0.41]$. The peak occured at 0, 0.2, and 0.4.

```
set.seed(1)
sim = arima.sim(n = 50, list(ar = c(0,0,0,0,0.9)))
period = cumsum(rle(as.vector(sign(diff(sim))))$lengths) + 1
plot(sim)
abline(v = period, col = 'red')
```




```r
plot(periodogram(sim)[[1]],
     periodogram(sim)[[2]], type = 'l',
     xlim = c(0,0.5))
```




```r
mean(rle(as.vector(sign(diff(sim))))$lengths) * 2
```

```
## [1] 2.45
```

```r
1/best1
```

```
## [1] Inf
```

```r
1/best2
```

```
## [1] 5
```

```r
1/best3
```

```
## [1] 2.5
```

It appears that the observed period matches reasonably well with $\frac{1}{f_3}$ and not with $\frac{1}{f_2}$ or $\frac{1}{f_1}$.

2

$$(1 - 0.9B^3)X_t = Z_t, Z_t \sim WN(1)$$

a

Compute transfer, power transfer functions, and spectral density $f_X(\lambda)$ associated with AR polynomial $(1 - 0.9B^3)$.

$$A(\lambda) = \sum_j a_j e^{-2\pi ij\lambda}, -1/2 \leq \lambda \leq 1/2$$

Where $|A(\lambda)|^2$ is the power transfer function. Thus with $a_0 = 1$ and $a_3 = -0.9$ then,

$$= 1 - 0.9e^{-6\pi i\lambda} = \phi(e^{-2\pi i\lambda})$$

Now, the spectral density is given by,

$$f_X(\lambda) = \sigma_Z^2 \frac{|\theta(e^{-2\pi i\lambda})|^2}{|\phi(e^{-2\pi i\lambda})|^2}, -1/2 \leq \lambda \leq 1/2$$

With $\theta(z) = 1, \phi(z) = 1 - 0.9z^3$,

$$= \frac{\sigma_Z^2}{|1 - 0.9 * (e^{-2\pi i\lambda})^3|^2}$$

$$= \frac{\sigma_Z^2}{|1 - 0.9 * (e^{-6\pi i\lambda})|^2}$$

$$= \frac{\sigma_Z^2}{|1 - 0.9\cos(6\pi\lambda) - 0.9i\sin(6\pi\lambda)|^2}$$

$$= \frac{\sigma_Z^2}{(1 - 0.9\cos(6\pi\lambda))^2 - 0.9^2 i^2 \sin^2(6\pi\lambda)}$$

$$= \frac{\sigma_Z^2}{1 - 2 * 0.9\cos(6\pi\lambda) + 0.9^2\cos^2(6\pi\lambda) + 0.9^2\sin^2(6\pi\lambda)}$$

$$= \frac{\sigma_Z^2}{1 - 2 * 0.9\cos(6\pi\lambda) + 0.9^2(\cos^2(6\pi\lambda) + \sin^2(6\pi\lambda))}$$

Using $\cos^2(\theta) + \sin^2(\theta) = 1$ then,

$$= \frac{\sigma_Z^2}{1 + 0.9^2 - 2 * 0.9\cos(6\pi\lambda)}$$

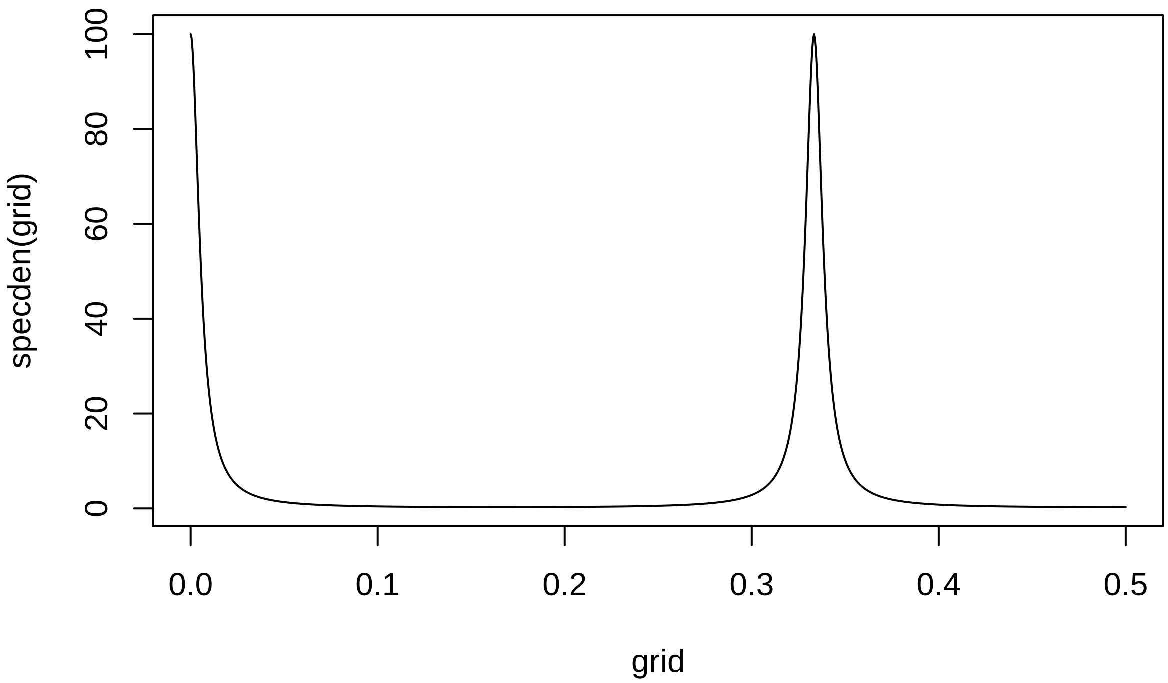
Thus the power transfer function is,

$$|A(\lambda)|^2 = 1 + 0.9^2 - 2 * 0.9\cos(6\pi\lambda)$$

b

Plotting the theoretical spectral density gives,

```r
specden <- function(x){1/(1 + 0.9^2 - 2*0.9*cos(6*pi*x))}
grid = seq(from = 0, to = 1/2, length.out = 1000)
grid2 = seq(from = 0.2, to = 1/2, length.out = 1000)
plot(grid, specden(grid), type = 'l')
```

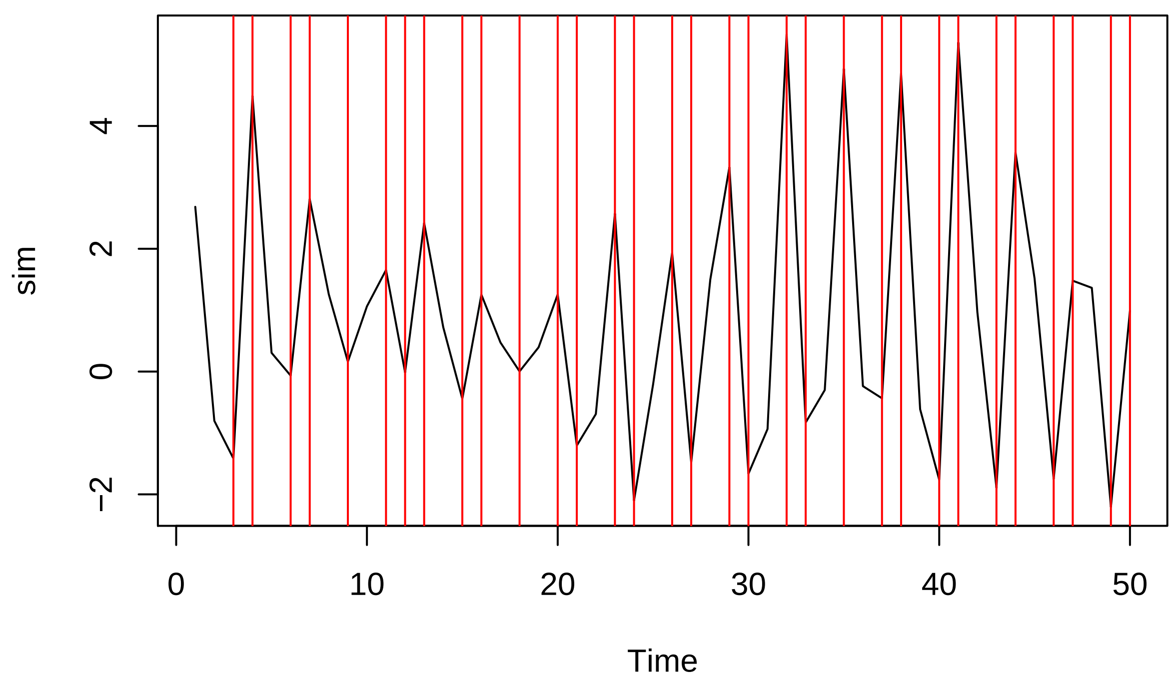



```r
best1 = grid[which.max(specden(grid))]
best2 = grid2[which.max(specden(grid2))]
```

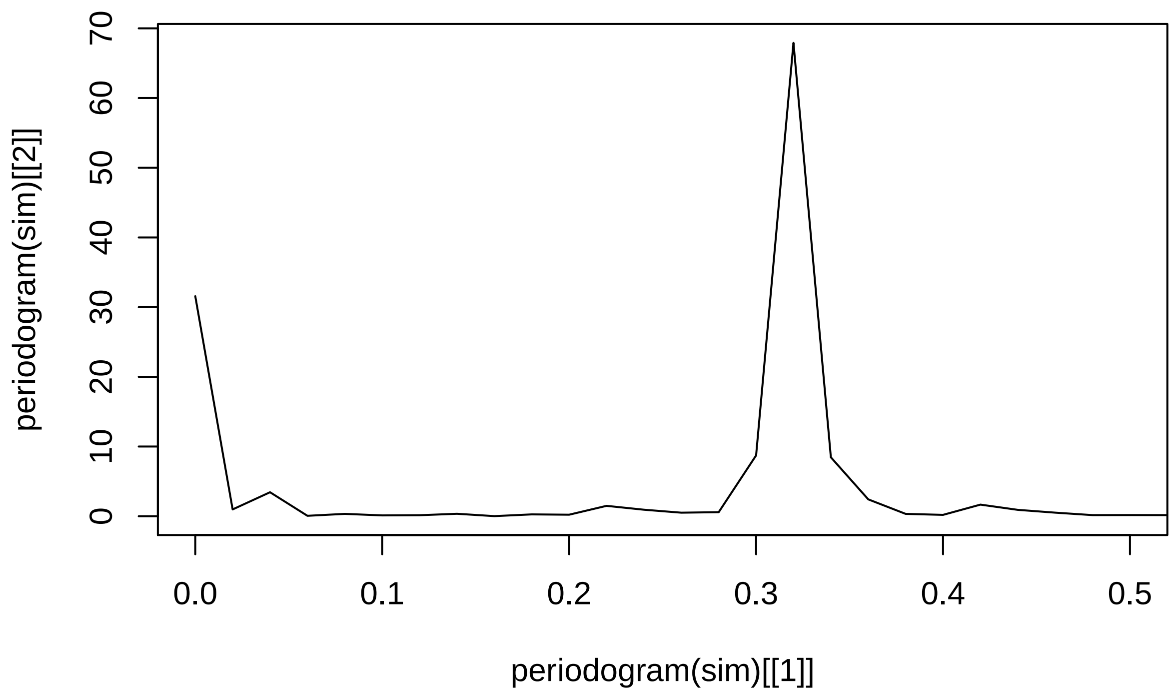
Based off this plot, I believe the series will oscillate. Since the frequency that contributes the most is 0.3333333, I believe it will have period 3.

c

```
set.seed(1)
sim = arima.sim(n = 50, list(ar = c(0,0,0.9)))
period = cumsum(rle(as.vector(sign(diff(sim))))$lengths) + 1
plot(sim)
abline(v = period, col = 'red')
```




```
plot(periodogram(sim)[[1]],
     periodogram(sim)[[2]], type = 'l',
     xlim = c(0,0.5))
```




```
mean(rle(as.vector(sign(diff(sim))))$lengths) * 2
```

```
## [1] 3.0625
```

```
1/best1
```

```
## [1] Inf
```

```
1/best2
```

```
## [1] 3
```

Since the period roughly matches the theoretical prediction, the two simulations are consistent.

d

Now consider the linear filter with weights $a_1 = a_0 = a_1 = \frac{1}{3}; a_j = 0$ otherwise.

$$A(\lambda) = \frac{1}{2q+1}(2\frac{\sin(\pi(q+1)\lambda)}{\sin(\pi\lambda)}\cos(\pi q\lambda) - 1)$$

and with $q = 1$,

$$= \frac{1}{3}(2\frac{\sin(2\pi\lambda)}{\sin(\pi\lambda)}\cos(\pi\lambda) - 1)$$

Then, the power transfer function would be,

$$|A(\lambda)|^2 = \frac{1}{9}(2\frac{\sin(2\pi\lambda)}{\sin(\pi\lambda)}\cos(\pi\lambda) - 1)^2$$

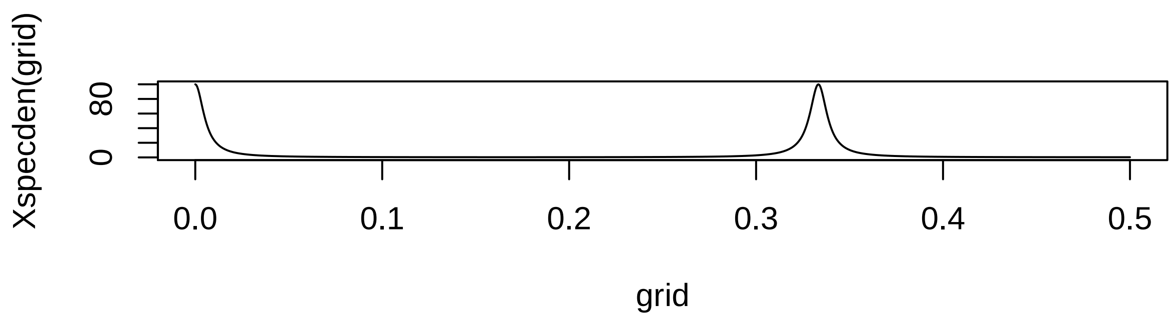
Thus, the spectral density for Y would be,

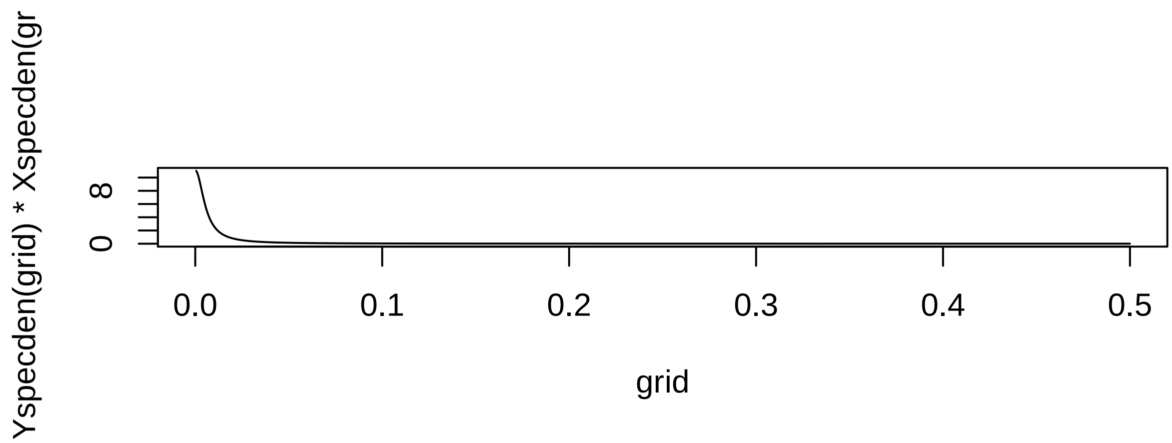
$$f_Y(\lambda) = |A(\lambda)|^2 f_X(\lambda)$$

$$= \frac{1}{9}(2\frac{\sin(2\pi\lambda)}{\sin(\pi\lambda)}\cos(\pi\lambda) - 1)^2 * \frac{\sigma_Z^2}{1 + 0.9^2 - 2 * 0.9\cos(6\pi\lambda)}$$

e

```r
Xspecden <- function(x){1/(1 + 0.9^2 - 2*0.9*cos(6*pi*x))}
Yspecden <- function(y){abs((1/9)*(2*(sin(2*pi*y)/sin(pi*y))*cos(pi*y)-1))^2}
grid = seq(from = 0, to = 1/2, length.out = 1000)
par(mfrow = c(2,1))
plot(grid, Xspecden(grid), type = 'l')
plot(grid, Yspecden(grid)*Xspecden(grid), type = 'l')
```

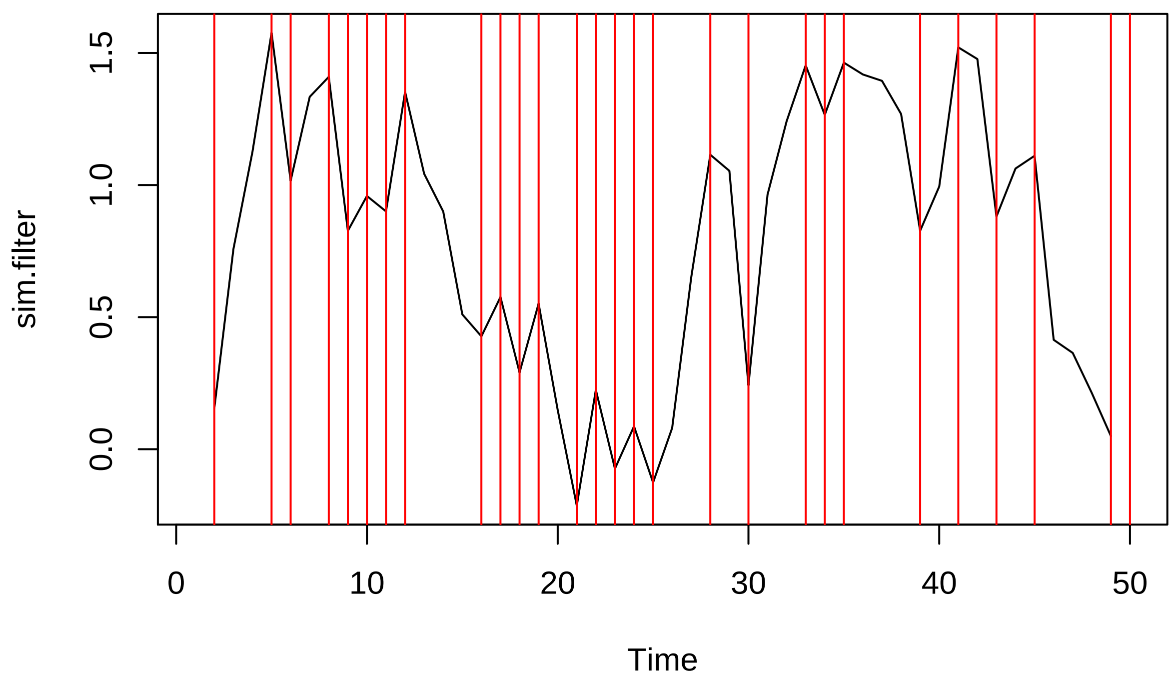




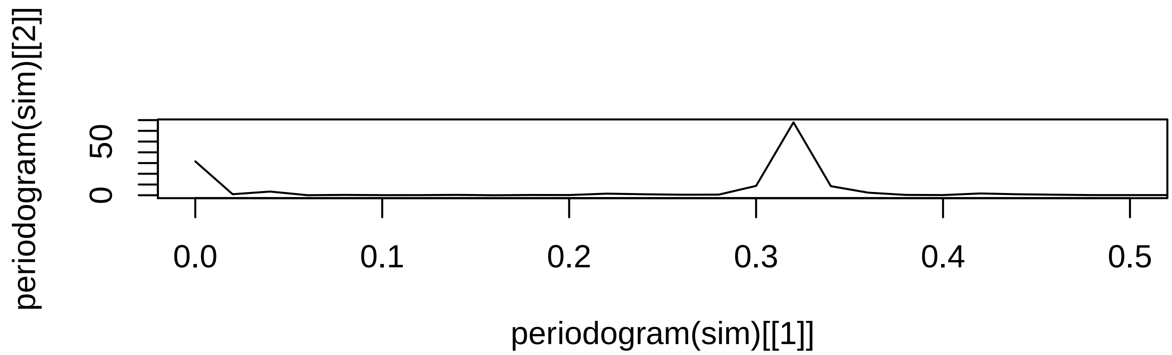
Looking at the plot, the frequency at 1/3 has been smoothed. Thus, I would expect the series to oscillate with a period larger than 3 after the smoothing.

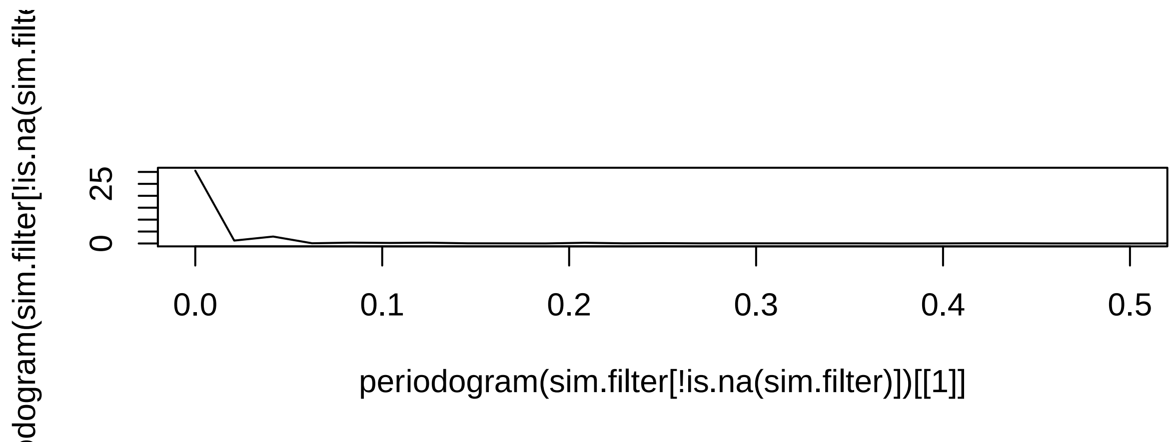
f

```r
set.seed(1)
sim = arima.sim(n = 50, list(ar = c(0,0,0.9)))
sim.filter = filter(sim, sides=2, filter=rep(1/3,3))
period = cumsum(rle(as.vector(sign(diff(sim.filter))))$lengths) + 1
plot(sim.filter)
abline(v = period, col = 'red')
```




```r
par(mfrow = c(2,1))
plot(periodogram(sim)[[1]],
     periodogram(sim)[[2]], type = 'l',
     xlim = c(0,0.5))
plot(periodogram(sim.filter[!is.na(sim.filter)])[[1]],
     periodogram(sim.filter[!is.na(sim.filter)])[[2]], type = 'l',
     xlim = c(0,0.5))
```






```
mean(rle(as.vector(sign(diff(sim.filter))))$lengths) * 2
```

```
## [1] 3.5
```

The simulation confirms the theory as the average period is now larger than before, thus indicating less weight from the frequency of 1/3.

3

For $x_0, ..., x_{n-1}$ and $y_0, ..., y_{n-1}$

$$z_t = \sum_{k=0}^{n-1} x_{t-k} y_k$$

With $x_{-m} = x_{n-m}$. The jth coefficient of a DFT is given by,

$$b_j = \sum_{t=0}^{n-1} x_t e^{-\frac{2\pi ijt}{n}}, j = 0, ..., n-1$$

Thus,

$$b_j^X = \sum_{t=0}^{n-1} x_t e^{-\frac{2\pi ijt}{n}}, j = 0, ..., n-1$$

and,

$$b_j^Y = \sum_{t=0}^{n-1} y_t e^{-\frac{2\pi ijt}{n}}, j = 0, ..., n-1$$

Now consider the product,

$$b_j^X b_j^Y = \sum_{t=0}^{n-1} x_t e^{-\frac{2\pi ijt}{n}} \cdot \sum_{t=0}^{n-1} y_t e^{-\frac{2\pi ijt}{n}}, j = 0, ..., n-1$$

$$= \sum_{k=0}^{n-1} \sum_{t=0}^{n-1} x_t e^{-\frac{2\pi ijt}{n}} y_k e^{-\frac{2\pi ijk}{n}}$$

$$= \sum_{k=0}^{n-1} \sum_{t=0}^{n-1} x_t y_k e^{\frac{-2\pi ij}{n}(k+t)}$$

Using the IFT, $x_t = \frac{1}{n} \sum_{j=0}^{n-1} b_j e^{\frac{2\pi ij(t)}{n}}$

$$= \frac{1}{n} \sum_{k=0}^{n-1} \sum_{t=0}^{n-1} \sum_{j=0}^{n-1} b_j y_k e^{\frac{-2\pi ij}{n}(k+t)} e^{\frac{2\pi ijt}{n}}$$

$$= \frac{1}{n} \sum_{k=0}^{n-1} \sum_{t=0}^{n-1} \sum_{j=0}^{n-1} b_j y_k e^{\frac{-2\pi ij}{n}(k)}$$

$$= \frac{1}{n} \sum_{t=0}^{n-1} \sum_{k=0}^{n-1} \sum_{j=0}^{n-1} b_j y_k e^{\frac{2\pi ij(-k)}{n}}$$

$$= \frac{1}{n} \sum_{t=0}^{n-1} \sum_{k=0}^{n-1} \sum_{j=0}^{n-1} b_j y_k e^{-\frac{2\pi ijt}{n}} e^{\frac{2\pi ij(t-k)}{n}}$$

$x_{t-k} = \frac{1}{n} \sum_{j=0}^{n-1} b_j e^{\frac{2\pi ij(t-k)}{n}}$,

$$= \sum_{t=0}^{n-1} \Big(\sum_{k=0}^{n-1} x_{t-k} y_k \Big) e^{-\frac{2\pi ijt}{n}}$$

$$= \sum_{t=0}^{n-1} z_t e^{-\frac{2\pi ijt}{n}} = b_j^Z, j = 0, ..., n-1$$

$$\therefore b_j^X b_j^Y = b_j^Z, j = 0, ..., n-1$$

4

Suppose x_t is h-cyclic for some h. Also that the DFT of $x_0, ..., x_{h-1}$ is $\beta_0, ..., \beta_{h-1}$. That is,

$$\{\beta_j\}_{j=0}^{j=h-1} = \{\sum_{t=0}^{h-1} x_t e^{-\frac{2\pi ijt}{n}}\}_{j=0}^{j=h-1}$$

For $n = kh$ data points instead of h, note that because $x_{t+h} = x_t \forall t$, then for $\forall j$,

$$b_j = \sum_{t=0}^{n-1} x_t e^{-\frac{2\pi ijt}{n}} = x_1 * e^{-\frac{2\pi ij(1)}{n}} + ... + x_{n-1} * e^{-\frac{2\pi ij(n-1)}{n}}$$

$$= x_1(e^{-\frac{2\pi ij(1)}{n}} + ... + e^{-\frac{2\pi ij(n-h)}{n}}) + ... + x_{h-1}(e^{-\frac{2\pi ij(h-1)}{n}} + ... + e^{-\frac{2\pi ij(n-1)}{n}})$$

$$= x_1(e^{-\frac{2\pi ij(1)}{n}} + e^{-\frac{2\pi ij(h)}{n}} + ... + e^{-\frac{2\pi ij(n-h)}{n}}) + ... + x_{h-1}(e^{-\frac{2\pi ij(h-1)}{n}} + ... + e^{-\frac{2\pi ij(n-1)}{n}})$$

Now note that the data can be written in k groups of h observations each,

$$\{t\} = \{0, ..., h-1\} \bigcup \{h, ..., 2h-1\} \bigcup ... \bigcup \{(k-1)h = n-h, ..., kh-1 = n-1\}$$

So we can rewrite the sums as,

$$= \sum_{t=0}^{h-1} x_t \sum_{c=0}^{k-1} e^{(-2\pi ij\frac{t+c\cdot h}{n})}$$

Factoring out the part that does not depend on c,

$$= \sum_{t=0}^{h-1} x_t \cdot e^{-2\pi ij\frac{t}{n}} \sum_{c=0}^{k-1} (e^{(-2\pi ij\frac{h}{n})})^c$$

Now recognize that for $j = mk$ for some $m \in \mathbb{N}^+$ and using that $\frac{h}{n} = \frac{1}{k}$,

$$(e^{(-2\pi ij\frac{h}{n})})^c = (e^{(\frac{-2\pi imk}{k})})^c = (e^{(-2\pi im)})^c = 1$$

And since there will be k of these terms of 1, we can break the sum into two cases,

$$= \sum_{t=0}^{h-1} x_t \cdot e^{-2\pi ij\frac{t}{n}} (k + \sum_{c=0, j\neq mk}^{k-1} (e^{(\frac{-2\pi ij}{k})})^c)$$

Now using the geometric formula, $\sum_{i=0}^{k-1} z^i = \frac{1-z^k}{1-z}$, on the second case,

$$= \sum_{t=0}^{h-1} x_t \cdot e^{-2\pi ij\frac{t}{n}} (k + \frac{1-(e^{(\frac{-2\pi ij}{k})})^k}{1-e^{(\frac{-2\pi ij}{k})}})$$

$$= \sum_{t=0}^{h-1} x_t \cdot e^{-2\pi ij\frac{t}{n}} (k + \frac{1-e^{(-2\pi ij)}}{1-e^{(-2\pi ij\frac{1}{k})}})$$

But $\frac{1-e^{(-2\pi ij)}}{1-e^{(-2\pi ij\frac{1}{k})}} = 0$ because the numerator is 0 and the denominator is nonzero. Thus,

$$= k\sum_{t=0}^{h-1} x_t \cdot e^{-2\pi ij\frac{t}{n}}$$

$$= k\beta_j, \forall j$$

Thus, if we consider $n = kh$ then $b_j = b_{mk} = k\beta_m$ for $0 \leq m \leq h-1$ and $b_j = 0$ otherwise.