100 Problems in Combinatorics

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ABSTRACT. I want to get better at combinatorics. Towards this end, I will solve 100 problems in combinatorics which I find illustrative or interesting.

Problem 1.

For an integer $k \geq 3$, let $N = R_3(k,k,k)$ be the minimum N such that in every edge-coloring of K_N in 3 colors there is a set X of k vertices so that all edges between vertices of X have the same color. Prove that

$$(*) \qquad \binom{N}{k}\left(\frac{1}{3}\right)^{\binom{k}{2}-1} \geq 1.$$

Proof. Suppose

$$\binom{N}{k}\left(\frac{1}{3}\right)^{\binom{k}{2}-1} < 1$$

for a contradiction. Let Ω be the space of all colorings of the edges of K_N in 3 colors. Let $R \subseteq V(K_N)$ be a k-element subset of vertices, and let A_R be the event where R is monochromatic. So

$$P(A_R) = 3 \prod_{1 \leq i \leq \binom{k}{2}} \frac{1}{3} = 3^{1-\binom{k}{2}}.$$

By sub-additivity,

$$P\left(\bigcup_{R \in [V(K_N)]^k} A_R\right) \leq \sum_{R \in [V(K_N)]^k} 3^{1-k} = \binom{N}{k} 3^{1-\binom{k}{2}} < 1.$$

Thus there is a non-zero probability that there is no monochromatic k-element subgraph of K_N, contradicting $N = R_3(k,k,k)$. Hence $(*)$ follows and we're done. $\square$

Problem 2.

Let F be a forest on n vertices with c connected components. (a) prove that F has $n-c$ edges; (b) find the average degree of G; (c) prove that the intersection of k connected subgraphs of F is either empty or a tree.

Proof. (a) We proceed by induction on c. If $c = 1$ then F is a tree so that $|E(F)| = |V(F)| - 1 = n - 1$ as needed. Now fix $c \geq 2$ and let C be any component of F. Let $F' = F \setminus C$. By induction, $|E(F')| = |V(F')| - (c - 1)$. Since C is maximally connected in F, $|E(C)| = |V(C)| + 1$ since it is a tree. Since there was no edge between F' and C (it was a component),

$$\begin{aligned} |E(F)| &= |E(F')| + |E(C)| \\ &= |V(F')| - (c - 1) + |V(C)| + 1 \\ &= |V(F)| - c = n - c, \end{aligned}$$

which completes the proof.

(b) Note that by handshaking and (a),

$$\sum_{v \in V(F)} \deg v = 2|E(F)| = 2(n - c)$$

so that the average degree of F is $\boxed{\frac{2}{n}(n - c).}$

(c) As in (a) we proceed by induction on k. The intersection of 1 connected subgraph of F must be a tree since F is a forest. Now let F_1 and F_2 be connected subgraphs of F. Assume for a contradiction that $F_1 \cap F_2$ is not a tree *and* non-empty. Since $F_1 \cap F_2$ is a subgraph of a forest, it is acyclic; thus $F_1 \cap F_2$ must not be connected, otherwise it is a tree. But then $F_1 \cap F_2$ is itself a forest. Using this and non-emptyness, there are vertices $u, v \in V(F_1 \cap F_2)$ which lie in different conneted components. But then u, v lie in different connected components of F, and $u, v \in V(F_1)$ contradicts its connectivity, as needed. If $k \geq 3$, write $F' = F_1 \cap F_2 \cdots \cap F_{k-1}$ and note that F' is either a tree or empty. In the first case, F' is connected so the induction hypothesis implies the result for $F' \cap F_k$; otherwise F' is null so that $F' \cap F_k$ is too. This completes the proof. $\square$