100 Problems in Combinatorics

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ABSTRACT. I want to get better at combinatorics. Here I solve 100 problems which I find illustrative, interesting, or fun. Aside from assessing the correctness of proofs, this document is not heavily revised, so typos are inevitable.

1. Graph Theory

1.1. **Question 1.** For an integer $k \geq 3$, let $N = R_3(k,k,k)$ be the minimum N such that in every edge-coloring of K_N in 3 colors there is a set X of k vertices so that all edges between vertices of X have the same color. Prove that

$$\binom{N}{k}\left(\frac{1}{3}\right)^{\binom{k}{2}-1} \geq 1. \tag{$*$}$$

Proof. Suppose

$$\binom{N}{k}\left(\frac{1}{3}\right)^{\binom{k}{2}-1} < 1$$

for a contradiction. Let Ω be the space of all colorings of the edges of K_N in 3 colors. Let $R \subseteq V(K_N)$ be a k-element subset of vertices, and let A_R be the event where R is monochromatic. So

$$P(A_R) = 3 \prod_{1 \leq i \leq \binom{k}{2}} \frac{1}{3} = 3^{1-\binom{k}{2}}.$$

By sub-additivity,

$$P\left(\bigcup_{R \in [V(K_N)]^k} A_R\right) \leq \sum_{R \in [V(K_N)]^k} 3^{1-\binom{k}{2}} = \binom{N}{k} 3^{1-\binom{k}{2}} < 1.$$

Thus there is a non-zero probability that there is no monochromatic k-element subgraph of K_N, contradicting $N = R_3(k,k,k)$. Hence $(*)$ follows and we're done. $\square$

1.2. **Question 2.** Let F be a forest on n vertices with c connected components.

(a) Prove that F has $n - c$ edges.
(b) Find the average degree of G.
(c) Prove that the intersection of k connected subgraphs of F is either empty or a tree.

Proof. For (a) we proceed by induction on c. If $c = 1$ then F is a tree so that $|E(F)| = |V(F)| - 1 = n - 1$ as needed. Now fix $c \geq 2$ and let C be any component of F. Let $F' = F \setminus C$. By induction, $|E(F')| = |V(F')| - (c-1)$. Since C is maximally connected in F, $|E(C)| = |V(C)| + 1$ since it is a tree. Since there was no edge between F' and C (it was a component),

$$\begin{aligned} |E(F)| &= |E(F')| + |E(C)| \\ &= |V(F')| - (c-1) + |V(C)| + 1 \\ &= |V(F)| - c = n - c, \end{aligned}$$

which completes the proof.

For (b) note that by handshaking and (a),

$$\sum_{v \in V(F)} \deg v = 2|E(F)| = 2(n-c)$$

so that the average degree of F is $\frac{2}{n}(n-c)$.

Finally, for (c) we proceed as in (a) by induction on k. The intersection of 1 connected subgraph of F must be a tree since F is a forest. Now let F_1 and F_2 be connected subgraphs of F. Assume for a contradiction that $F_1 \cap F_2$ is not a tree *and* non-empty. Since $F_1 \cap F_2$ is a subgraph of a forest, it is acyclic; thus $F_1 \cap F_2$ must not be connected, otherwise it is a tree. But then $F_1 \cap F_2$ is itself a forest. Using this and non-emptyness, there are vertices $u, v \in V(F_1 \cap F_2)$ which lie in different conneted components. But then u, v lie in different connected components of F, and $u, v \in V(F_1)$ contradicts its connectivity, as needed. If $k \geq 3$, write $F' = F_1 \cap F_2 \cap \cdots \cap F_{k-1}$ and note that F' is either a tree or empty. In the first case, F' is connected so the induction hypothesis implies the result for $F' \cap F_k$; otherwise F' is null so that $F' \cap F_k$ is too. This completes the proof. $\square$

1.3. **Question 3.** Let G be a k-connected graph on n vertices.

(a) Prove that $|E(G)| \geq kn/2$.
(b) Let G' be obtained from G by adding a vertex v adjacent to every vertex in G. Show that G' is $(k+1)$-connected.

(c) Show that for every integer $k \geq 2$ and $n \geq k+1$ there is a k-connected graph with $|V(G)| = n$ and $|E(G)| \leq (k-1)n$.

Proof. For (a) note that since G is k-connected, every vertex has degree at least k. Otherwise, there is a vertex with degree at most $k-1$; deleting its neighbours disconnects the graph, contradicting k-connectivity. By handshaking,

$$|E(G)| = \frac{1}{2} \sum_{v \in V(G)} \deg v \leq nk/2.$$

For (b), let v be adjacent to every vertex in $V(G)$ in G'. Let X be a subset of $\leq k$ vertices in $V(G')$. If $v \in X$ then $G' \setminus X = G \setminus (X \setminus \{v\})$ is connected since G is k-connected and $|X \setminus \{v\}| \leq k-1$. Otherwise, $v \notin X$. Let $w \in X$ and note that $G' \setminus (X \setminus \{w\})$ is connected by k-connectivity. Then, we may remove w while preserving connectedness, since any two vertices x, y in $G' \setminus X$ are both adjacent to v. Hence G' is $(k+1)$-connected since X was arbitrary.

For (c) we may assume that $n > 2k-2$, otherwise $k+1 \leq n \leq 2k-2$ and

$$\binom{n}{2} = \frac{n(n-1)}{2} \leq \frac{2k-2}{2} \cdot n = (k-1)n$$

so we can just take $G = K_n$, which is $(n-1)$-connected so that it is k-connected since $k+1 \leq n$. Let G be d-regular, where $d = 2k-2$. Then by handshaking,

$$|E(G)| = \frac{1}{2} \sum_{v \in V(G)} d = \frac{n}{2}(2k-2) = (k-1)n.$$

It remains to prove that G is k-connected. Let X be a set of at most $k-1$ vertices.

We prove (c) by induction on k. For $k \geq 2$ and $n \geq k+1$, the cycle C_n on n vertices is the graph we need. Indeed,

$$|E(C_n)| = n = 1 \cdot n = (k-1)n.$$

Now fix $k \geq 3$. By the IH we obtain a graph G' on n' vertices and m' edges such that G' is $(k-1)$-connected, $n' \geq k$, and $m' \leq (k-2)n'$. Now let G be obtained by taking a vertex $v \in V(G')$ and connecting it to every vertex in G'. Then from (b) G is k-connected. This uses n' edges. Let $n = n'+1$ and $m = m'+n'$. Then $n = n'+1 \geq k+1$ and

$$\begin{aligned} m = m' + n' &\leq (k-2)n' + n' = (k-1)n' \\ &\leq (k-1)(n'+1) = (k-1)n \end{aligned}$$

which completes the proof. $\square$

1.4. **Question 4.** Prove that:

(a) If T is a tree then $|V(T)| = |E(T)| + 1$.

(b) If F is a forest and $c(F)$ is the number of components of F, show that $c(F) = |V(F)| - |E(F)|$.

Proof. For (a) we proceed by induction on $|V(T)|$. If $|V(T)| = 1$ then T is edgeless so that $|V(T)| = 0+1 = |E(T)|+1$. Now fix $|V(T)| \geq 2$. Then T has at least one leaf v. Let $T' = T \setminus v$. Then T' is a tree on $|V(T)| - 1$ vertices, so by the IH we have $|V(T')| = |E(T')| + 1$. Appending v to T' gives one new vertex and one new edge. Hence $|V(T)| = |V(T')| + 1 = |E(T')| + 1 + 1 = |E(T)| + 1$ and we're done.

For (b) we induct on $c(F)$. If $c(F) = 1$ then from (a)

$$|V(F)| = |E(F)| + 1 \Leftrightarrow 1 = |V(F)| - |E(F)|.$$

Then if $c(F) \geq 2$, let F' be obtained from F by deleting one entire component C. Then by the IH we have $c(F') = |V(F')| - |E(F')|$ and since C is a tree, (a) implies that $|V(C)| = |E(C)| + 1$. Putting these two together, we obtain that

$$\begin{aligned} c(F) = c(F') + 1 &= |V(F')| - |E(F')| + |V(C)| - |E(C)| \\ &= |V(F)| - |E(F)|, \end{aligned}$$

since $V(F') \cap V(C) = E(F') \cap E(C) = \emptyset$. This completes the proof. $\square$