100 Problems in Combinatorics

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ABSTRACT. I want to get better at combinatorics. Here I solve 100 problems which I find illustrative, interesting, or fun. Aside from assessing the correctness of proofs, this document is not heavily revised, so typos are inevitable.

1. Graph Theory

1.1. **Question 1.** For an integer $k \geq 3$, let $N = R_3(k,k,k)$ be the minimum N such that in every edge-coloring of K_N in 3 colors there is a set X of k vertices so that all edges between vertices of X have the same color. Prove that

$$\binom{N}{k}\left(\frac{1}{3}\right)^{\binom{k}{2}-1} \geq 1. \tag{$*$}$$

Proof. Suppose

$$\binom{N}{k}\left(\frac{1}{3}\right)^{\binom{k}{2}-1} < 1$$

for a contradiction. Let Ω be the space of all colorings of the edges of K_N in 3 colors. Let $R \subseteq V(K_N)$ be a k-element subset of vertices, and let A_R be the event where R is monochromatic. So

$$P(A_R) = 3 \prod_{1 \leq i \leq \binom{k}{2}} \frac{1}{3} = 3^{1-\binom{k}{2}}.$$

By sub-additivity,

$$P\left(\bigcup_{R \in [V(K_N)]^k} A_R\right) \leq \sum_{R \in [V(K_N)]^k} 3^{1-\binom{k}{2}} = \binom{N}{k} 3^{1-\binom{k}{2}} < 1.$$

Thus there is a non-zero probability that there is no monochromatic k-element subgraph of K_N, contradicting $N = R_3(k,k,k)$. Hence $(*)$ follows and we're done. $\square$

1.2. **Question 2.** Let F be a forest on n vertices. Prove that the intersection of k connected subgraphs of F is either empty or a tree.

Proof. It suffices to prove the claim when $k = 2$, since given connected subgraphs $C_1, C_2, \dots, C_k$, we have by induction that $C = C_1 \cap \cdots \cap C_{k-1}$ is either a tree or empty. If C is a tree then it is connected and we may apply the case when $k = 2$ to $C \cap C_k$. Otherwise C is empty so that $C \cap C_k$ is too. Thus, we just need to prove the base case now.

Let C_1, C_2 be two connected subgraphs of F. If $C_1 \cap C_2$ is not empty, then there is a vertex v in their intersection. Thus $C_1 \cap C_2$ is connected. Since $C_1 \cap C_2 \subseteq C_1$ and C_1 is acyclic, so is $C_1 \cap C_2$. Hence $C_1 \cap C_2$ is a tree. If $C_1 \cap C_2$ is not a tree, then it is either not connected or contains a cycl. The latter case can not hold, since $C_1 \cap C_2 \subseteq F$ is a forest.

If $C_1 \cap C_2$ is non-null, then it must be connected. Indeed, as before, there is a vertex v in $C_1 \cap C_2$. Then there is a path between the two components so that $C_1 \cap C_2$ must be connected. Applying the contrapositive, we obtain that $C_1 \cap C_2 = \emptyset$. □

1.3. **Question 3.** Let G be a k-connected graph on n vertices.

(a) Prove that $|E(G)| \geq kn/2$.
(b) Let G' be obtained from G by adding a vertex v adjacent to every vertex in G. Show that G' is $(k+1)$-connected.
(c) Show that for every integer $k \geq 2$ and $n \geq k+1$ there is a k-connected graph with $|V(G)| = n$ and $|E(G)| \leq (k-1)n$.

Proof. For (a) note that since G is k-connected, every vertex has degree at least k. Otherwise, there is a vertex with degree at most $k-1$; deleting its neighbours disconnects the graph, contradicting k-connectivity. By handshaking,

$$|E(G)| = \frac{1}{2} \sum_{v \in V(G)} \deg v \leq nk/2.$$

For (b), let v be adjacent to every vertex in $V(G)$ in G'. Let X be a subset of $\leq k$ vertices in $V(G')$. If $v \in X$ then $G' \setminus X = G \setminus (X \setminus \{v\})$ is connected since G is k-connected and $|X \setminus \{v\}| \leq k-1$. Otherwise, $v \notin X$. Let $w \in X$ and note that $G' \setminus (X \setminus \{w\})$ is connected by k-connectivity. Then, we may remove w while preserving connectedness, since any two vertices x, y in $G' \setminus X$ are both adjacent to v. Hence G' is $(k+1)$-connected since X was arbitrary.

For (c) we may assume that $n > 2k-2$, otherwise $k+1 \leq n \leq 2k-2$ and

$$\binom{n}{2} = \frac{n(n-1)}{2} \leq \frac{2k-2}{2} \cdot n = (k-1)n$$

so we can just take $G = K_n$, which is $(n-1)$-connected so that it is k-connected since $k+1 \leq n$. Let G be d-regular, where $d = 2k-2$. Then by handshaking,

$$|E(G)| = \frac{1}{2} \sum_{v \in V(G)} d = \frac{n}{2}(2k-2) = (k-1)n.$$

It remains to prove that G is k-connected. Let X be a set of at most $k-1$ vertices.

We prove (c) by induction on k. For $k \geq 2$ and $n \geq k+1$, the cycle C_n on n vertices is the graph we need. Indeed,

$$|E(C_n)| = n = 1 \cdot n = (k-1)n.$$

Now fix $k \geq 3$. By the IH we obtain a graph G' on n' vertices and m' edges such that G' is $(k-1)$-connected, $n' \geq k$, and $m' \leq (k-2)n'$. Now let G be obtained by taking a vertex $v \in V(G')$ and connecting it to every vertex in G'. Then from (b) G is k-connected. This uses n' edges. Let $n = n'+1$ and $m = m'+n'$. Then $n = n'+1 \geq k+1$ and

$$\begin{aligned} m = m' + n' &\leq (k-2)n' + n' = (k-1)n' \\ &\leq (k-1)(n'+1) = (k-1)n \end{aligned}$$

which completes the proof. $\square$

1.4. **Question 4.** Prove that:

(a) If T is a tree then $|V(T)| = |E(T)| + 1$.
(b) If F is a forest and $c(F)$ is the number of components of F, then $c(F) = |V(F)| - |E(F)|$.

Proof. For (a) we proceed by induction on $|V(T)|$. If $|V(T)| = 1$ then T is edgeless so that $|V(T)| = 0 + 1 = |E(T)| + 1$. Now fix $|V(T)| \geq 2$. Then T has at least one leaf v. Let $T' = T \setminus v$. Then T' is a tree on $|V(T)| - 1$ vertices, so by the IH we have $|V(T')| = |E(T')| + 1$. Appending v to T' gives one new vertex and one new edge. Hence $|V(T)| = |V(T')| + 1 = |E(T')| + 1 + 1 = |E(T)| + 1$ and we're done.

For (b) we induct on $c(F)$. If $c(F) = 1$ then from (a)

$$|V(F)| = |E(F)| + 1 \Leftrightarrow 1 = |V(F)| - |E(F)|.$$

Then if $c(F) \geq 2$, let F' be obtained from F by deleting one entire component C. Then by the IH we have $c(F') = |V(F')| - |E(F')|$ and since C is a tree, (a) implies that $|V(C)| = |E(C)| + 1$. Putting these two together, we obtain that

$$\begin{aligned} c(F) = c(F') + 1 &= |V(F')| - |E(F')| + |V(C)| - |E(C)| \\ &= |V(F)| - |E(F)|, \end{aligned}$$

since $V(F') \cap V(C) = E(F') \cap E(C) = \emptyset$. This completes the proof. $\square$

1.5. **Question 5.** Let G be a k-connected graph. Show using the definitions that if G' is obtained from G by adding a new vertex v adjacent to at least k vertices of G, then G' is k-connected.

Proof. Let $N(v)$ be the set of neighbours of v in G'. Suppose for a contradiction that G' is not k-connected. Then there is a set X with $|X| \leq k-1$ such that $G' \setminus X$ is disconnected. Then, we have two cases:

(1) If $v \in X$ then $G' \setminus X = G \setminus (X \setminus v)$ can not be disconnected since $|X \setminus v| \leq k-2$ and G is k-connected, a contradiction.
(2) If $v \notin X$ then from k-connectivity, $H = G \setminus X = (G' \setminus X) \setminus v$ is connected. Then H contains a vertex $w \in N(v)$, since $|X| = k-1 < k \leq |N(v)|$ and $v \notin N(v)$. Thus, $G' \setminus X$ is still connected since v has positive degree within which, a contradiction.

This completes the proof. $\square$

1.6. **Question 6.** Let $\omega(G)$ be the size of the largest clique in G. Prove that $\omega(G) \leq \chi(G)$. Find a graph G with $\omega(G) < \chi(G)$.

Proof. Let $X = \{x_1, x_2, \ldots, x_n\}$ be a clique in G of size $n = \omega(G)$. Assign x_1 color 1; then $x_2, x_3, \ldots, x_n$ can not be colored using color 1. Then color x_2 using color 2; then $x_3, x_4, \ldots, x_n$ can not be colored using color 2. Continuing this way, we need n colors to color X and hence G. Thus, $\omega(G) \leq \chi(G)$.

Consider the cycle C_5. We can not color C_5 using two colors, but we can using three. But $\omega(C_5) = 2$. Thus, $\omega(G) = 2 < 3 = \chi(C_5)$. $\square$