MATH 350 Results.

Connectivity

Handshaking lemma. For any graph G, $\sum_{v\in V(G)} \deg v = 2|E(G)|$.
Lemma 2.1. If there exists a walk in G with ends u, v then there exists a path in G with these ends.
Lemma 2.2. A graph G is not connected iff there exists a partition (X, Y) of $V(G)$ such that no edge of G has one end in X and the other in Y.
Lemma 2.3. If H_1, H_2 are connected subgraphs of G and $V(H_1) \cap V(H_2) \neq \emptyset$, then $H_1 \cup H_2$ is connected.
Lemma 2.4. Every vertex of G belongs to a unique connected component.
Lemma 2.5. A subgraph $H \subseteq G$ is a connected component of G iff H is connected, and if $e \in E(G)$ has an end in $V(H)$ then $e \in E(H)$.
Lemma 2.6. Let $e \in E(G)$ with ends u, v. Then exactly one of the following holds.

1. e is a cut edge, u and v belong to different conneected components of $G \setminus e$, and $\mathrm{comp}(G \setminus e) = \mathrm{comp}(G) + 1$.
2. e is not a cut edge, u and v belong to the same connected component of $G \setminus e$, and $\mathrm{comp}(G \setminus e) = \mathrm{comp}(G)$.

Trees & Forests

Lemma 3.1. Let F be a non-null forest. Then $\mathrm{comp}(F) = |V(F)| - |E(F)|$. If F is a tree, then $|V(F)| = |E(F)| + 1$.
Lemma 3.2. Let T tree with $|V(T)| \geq 2$. Let X be the set of leaves of T and Y be the set of vertices of degree ≥ 3. Then $|X| \geq |Y| + 2$. In particular, T has at least 2 leaves.
Lemma 3.3. If a tree T has exactly two leaves u and v, then T is a path with ends u and v.
Lemma 3.4. Let v be a leaf in a tree T. Then $T \setminus v$ is a tree.
Lemma 3.5. Let v be a leaf in a graph G. If $G \setminus v$ is a tree then G is a tree.
Lemma 3.6. Let T be a tree, $u, v \in V(T)$. Then there exists a unique path in T with ends u, v.

Spanning Trees

Lemma 4.1. Let G be a connected, non-null graph. Let $H \subseteq G$ be chosen minimal such that $V(H) = V(G)$ and H is connected. Then H is a spanning tree of G.
Lemma 4.2. Let G be a connected, non-null graph. Let $H \subseteq G$ be chosen maximal such that H has no cycles. Then H is a spanning tree of G.
Lemma 4.3. Let T be a spanning tree of G. Let $f \in E(G) \setminus E(T)$. Then there exists a unique fundamental cycle of f with respect to T.
Lemma 4.4. Let T be a spanning tree of G, $f \in E(G) \setminus E(T)$, and C be the fundamental cycle of f with respect to T. Let $T' = (T + f) \setminus e$ be the graph obtained from T by adding f and deleting some $e \in E(C)$. Then T' is a spanning tree of G.
Corollary 4.5. Let G, T, f, C, e be as in Lemma 4.4. Let $w : E(G) \to \mathbb{R}_+$. If T is $\mathrm{mst}(G, w)$, then $w(f) \geq w(e)$.
Kruskal's Algorithm: Input a connected non-null graph G and $w : E(G) \to \mathbb{R}_+$. For $i = 1, 2, \ldots, |V(G)| - 1$, let $e_i \in E(G)$ be chosen with $w(e_i)$ minimum such that $e_i \notin \{e_1, e_2, \ldots, e_{i-1}\}$ and $\{e_1, e_2, \ldots, e_i\}$ does not contain the edge set of a cycle. Output is a tree T with $V(T) = V(G)$ and $E(T) = \{e_1, \ldots, e_{|V(G)|-1}\}$.
Theorem 4.6. Let G be a graph, $w : E(G) \to \mathbb{R}_+$ be such that $w(e) \neq w(f)$ for any $e, f \in E(G)$ with $e \neq f$. Let T be $\mathrm{mst}(G, w)$ and $E(T) = \{e_1, e_2, \ldots, e_k\}$ be such that $w(e_1) < \cdots < w(e_k)$. Then for every i with $1 \leq i \leq k$, e_i is the edge of minimum weight subject to $e_i \notin \{e_1, \ldots, e_{i-1}\}$ and $\{e_1, \ldots, e_i\}$ does not contain the edge set of a cycle.
Theorem 4.7. Kruskal's algorithm outputs $\mathrm{mst}(G, w)$.
Cayley's formula. The complete graph on n vertices has n^{n-2} spanning trees.

Euler Tours & Hamiltonian Cycles

Facts. A walk uses two edges incident to a vertex each time this vertex occurs in the walk (except for ends); thus if an Eulerian trail exists, then at most two vertices odd degree; if an Euler tour exists, then all vertices must have even degree.
Lemma 5.1. Let $E(G) \neq \emptyset$ and suppose G has no leaves. Then G contains a cycle.
Lemma 5.2. Let G be a graph such that every vertex of G has even degree. Then there exist cycles $C_1, \ldots, C_k$ in G such that $(E(C_1), \ldots, E(C_k))$ is a partition of $E(G)$.
Euler's theorem. If G is a connected graph such that the degree of every vertex of G is even, then G has an Euler tour.
Corollary 5.4. If G is a connected graph such that G contains at most two vertices of odd degree, then G has an Eulerian trail.
Lemma 5.5. Let G be a graph. If there exists $X \subseteq V(G)$ with $X \neq \emptyset$ such that $G \setminus X$ has more than $|X|$ components, then G has no Hamiltonian cycle.
Dirac. Let G be a simple graph on $n \geq 3$ vertices. Suppose that for every pair of non-adjacent vertices, $u, v \in V(G)$, $\deg u + \deg v \geq n$. Then G has a Hamiltonian cycle.
Corollary 5.7. Let G be a simple graph with $n \geq 3$ vertices. Suppose that either

- $\deg v \geq n/2$ for all $v \in V(G)$; or
- $|E(G)| \geq \binom{n}{2} - n + 3$.

Then G has a Hamiltonian cycle.

Bipartite Graphs

Lemma 6.1. Trees are bipartite.
Theorem 6.2. Let G be a graph. TFAE:

1. G is bipartite;
2. G contains no closed walk of odd length;
3. G contains no odd cycle.

Matchings

Fact. $\nu(G) \leq \frac{1}{2}|V(G)|$.
Lemma 7.1. Let G be a loopless graph. Then $\nu(G) \leq \tau(G) \leq 2\nu(G)$.
Lemma 7.2. A matching M in G has maximum size iff there does not exist an M-augmenting path in G.
Konig. If G is bipartite, then $\nu(G) = \tau(G)$.
Konig rephrased. Let G bipartite. G has a matching M with $|M| \geq k \iff$ There does not exist $X \subseteq V(G)$ with $|X| < k$ such that every edge of G has an end in X, i.e. there is no vertex cover with $< k$ vertices.
Theorem 7.4. Let $d \geq 1$ be an integer, let G bipartite such that $\deg v = d$ for every $v \in V(G)$. Then G has a perfect matching.
Hall. Let G bipartite with bipartition (A, B). Then G has a matching M covering A iff $|N(S)| \geq |S|$ for every $S \subseteq A$.
Menger. Let $s, t \in V(G)$ be a pair of distinct non-adjacent vertices of G, and let $k \geq 1$ be an integer. Then exactly one of the following holds.

1. There exist pairwise internally disjoint paths $P_1, \ldots, P_k$ in G with ends s, t.
2. There exists a separation (A, B) of G such that $s \in A \setminus B$, $t \in B \setminus A$ of order $< k$.

Theorem 8.2. Let $Q, R \subseteq V(G)$, $k \geq 1$ an integer. Then exactly one of the following holds. (1) There exist pairwise disjoint paths $P_1, \ldots, P_k$ in G each with one end in Q and another in R. (2) There exists a separation (A, B) of G of order $< k$ such that $Q \subseteq A$ and $R \subseteq B$.
Corollary 8.3. Let G be a k-connected graph, $s, t \in V(G)$ be distinct. Then there exist paths $P_1, \ldots, P_k$ in G from s to t that are internally disjoint.
Menger 2. Let $s, t \in V(G)$ be distinct and $k \geq 1$. Then exactly one of the following holds. (1) There exist paths $P_1, \ldots, P_k$ in G with ends s, t and such that for $1 \leq i < j \leq n$, $E(P_i) \cap E(P_j) = \emptyset$ (pairwise edge disjoint). (2) There exists $X \subseteq V(G)$ such that $s \in X$, $t \in V(G) \setminus X$, and $|\delta(X)| < k$.

Directed Graphs & Network Flows

Lemma 9.1. Let G be a digraph. Let $s, t \in V(G)$. Then there does not exist a directed path in G from s to t iff there exists $X \subseteq V(G)$ such that $s \in X$, $t \in V(G) \setminus X$, and $\delta^+(X) = \emptyset$.
Lemma 9.2. Let φ be an (s, t)-flow on a digraph G with value k. Then for any $X \subseteq V(G)$ such that $s \in X$, $t \in V(G) \setminus X$, we have

$$\sum_{e\in\delta^+(X)} \varphi(e) - \sum_{e\in\delta^-(X)} \varphi(e) = k.$$

Lemma 9.3. Let φ be an integral (s, t)-flow on a digraph G withvalue $k \geq 0$. Then there exist directed paths $P_1, \ldots, P_k$ from s to t such that every edge of G belongs to at most $\varphi(e)$ of these paths (paths may not be unique).
Lemma 9.4. Let φ be an integral c-admissible (s, t)-flow on G with value k. If there exists a φ-augmenting path P from s to

t then there exists an integral c-admissible (s,t)-flow on G of value $k+1$.
Theorem 9.5 (Max flow min cut). Let $k \in \mathbb{N}$. Then exactly one of the following holds:

1. There exists an integral c-admissible (s,t)-flow of value $\geq k$.
2. There exists $X \subseteq V(G)$, $s \in X$, $t \in V(G) \setminus X$ such that $\sum_{e \in \delta^+(X)} c(e) < k$.

Independence & Cliques

Properties: For P_n: $\nu = \lfloor n/2 \rfloor$, $\tau = \lfloor n/2 \rfloor$, $\alpha = \lceil n/2 \rceil$, $\rho = \lceil n/2 \rceil$; for C_n: $\nu = \lfloor n/2 \rfloor$, $\tau = \lceil n/2 \rceil$, $\alpha = \lfloor n/2 \rfloor$, $\rho = \lceil n/2 \rceil$; for K_n: $\nu = \lfloor n/2 \rfloor$, $\tau = n-1$, $\alpha = 1$, $\rho = \lceil n/2 \rceil$.
ALSO: $\rho(G) \geq \alpha(G)$ and $\rho(G) \geq |V(G)|/2$.
Lemma 10.1. For any graph G, $\alpha(G) + \tau(G) = |V(G)|$.
Lemma 10.2. Let G be simple and such that every vertex of G is incident to an edge, then $\nu(G) + \rho(G) = |V(G)|$.
Corollary 10.3. Let G be a simple bipartite graph such that every vertex is incident to an edge, then $\alpha(G) = \rho(G)$.

Ramsey Numbers

Properties: $R(s,t) = R(t,s)$, $R(1,t) = 1$, $R(2,t) = t$, $R(3,3) = 6$, $R(3,4) = 9$.
Ramsey Theorem. $R(s,t)$ exists for every $s,t \geq 1$ and $R(s,t) \leq R(s-1,t) + R(s,t-1)$.
Corollary 10.5. For $s,t \geq 1$, $R(s,t) \leq \binom{s+t-2}{s-1}$. **Lemma 10.6.** If $\binom{N}{s} 2^{1-\binom{s}{2}} < 1$ then there exists a simple graph G with $|V(G)| = N$ and no clique or independent set of size s (so $R(s,s) > N$).
Theorem 10.7. For $s \geq 2$, $R(s,s) \geq 2^{s/2} = (\sqrt{2})^s$.
Theoreom 10.8. $R_k(s_1, \ldots, s_k)$ exists for all $k, s_1, \ldots, s_k \geq 1$. Furthermore, $R_k(s_1, \ldots, s_k) \leq R_{k-1}(R_2(s_1,s_2), s_3, \ldots, s_k)$.
Schur. For every $k \geq 1$ there exists $N \geq 1$ such that in every coloring of $\{1, 2, \ldots, N\}$ in k colors there exist x, y, z of the same color and not necessarily distinct such that $x + y = z$.

Vertex Coloring

Properties. G is 1-colorable iff G is edgeless; G is 2-colorable iff G is bipartite. G is 1-degenerate iff forest. $\chi(K_n) = n$
Lemma 11.1. Let G be loopless, then

1. $\chi(G) \geq \omega(G)$
2. $\chi(G) \geq \lceil \frac{|V(G)|}{\alpha(G)} \rceil$.

Lemma 11.2. If G is loopless and k-degenerate then $\chi(G) \leq k+1$. In particular, $\chi(G) \leq \Delta(G) + 1$.
Greedy coloring algorithm. Input a loopless graph G and an ordering $(v_1, v_2, \ldots, v_n)$ of $V(G)$. Outputs a k-coloring of G for some k.

1. Colour v_1 using color 1
2. once $\{v_1, \ldots, v_{i-1}\}$ receive colors, color v_i with the smallest integer $l \in \mathbb{N}$ that is not a color of any of the neighbours of v_i so far.

Brooks. Let G be connected loopless s.t. G is not complete and not an odd cycle. Then $\chi(G) \leq \Delta(G)$.

Edge Coloring

Properties. $\chi'(K_4) = 3$, $\chi'(C_{2k+1}) = 3$. 1-factor iff perfect matching.
Lemma 12.1. $\Delta(G) \leq \chi'(G) \leq 2\Delta(G) - 1$, for any loopless graph G with $\Delta(G) \geq 1$.
Lemma 12.2. Let G be a graph with $\Delta(G) \leq k$. Then there exists a k-regular graph H such that G is a subgraph of H. Moreover, if G is loopless (resp. bipartite) then H can be chosen to be loopless (resp. bipartite). (If G is simple then H can be chosen to be simple).
Theorem 12.3. If G is bipartite then $\chi'(G) = \Delta(G)$.
Lemma 12.4. Let G be a loopless $2k$-regular graph. Then $E(G)$ can be partitioned into k 2-factors.
Shannon. Let G be a loopless graph then $\chi'(G) \leq 3\lceil \frac{\Delta(G)}{2} \rceil$.
Vizing. If G simple then $\chi'(G) \leq \Delta(G) + 1$.

Minors

Facts. No loop minor iff forest; no K_2 minor iff no K_2 subgraph; no K_3 minor iff no cycles of length ≥ 3 iff forest with added loops and parallel edges.
Hadwiger: For $0 \leq t \geq 5$ if G is a loopless graph with no K_{t+1} minor then $\chi(G) \leq t$. Also, K_{t+1} subgraph implies $\chi(G) \geq t+1$.
Subdivision. If G is a subdivision of H then H is a minor of G (converse does not hold).
Lemma 13.1. If G is 3-connected then G has a K_4 minor.
Lemma 13.2. Let G be a simple graph with no K_4 minoir. Let X be a clique in G with $|X| \leq 2$ (can $= 0$) and $X \neq V(G)$. Then there is a vertex $v \in V(G) - X$ such that $\deg_G v \leq 2$.
Theorem 13.3. If G is a loopless graph with no K_4 minor then $\chi(G) \leq 3$.

Planar Graphs

Jordan Curve. Any closed simple curve separates the plane into two regions.
Lemma 14.1. Let G be a graph drawn in the plane, $e \in E(G)$. Then the regions on different sides of e are the same if and only if e is a cut-edge of G.
Euler's Formula. Let G be a planar non-null graph. Then

$$|V(G)| - |E(G)| + \mathrm{Reg}(G) = 1 + \mathrm{comp}(G)$$

Lemma 14.3. Let G be a connected simple graph drawn in the plane with $|E(G)| \geq 2$. Then the length of every region of G is at least 3, and if it is 3 then the boundary is a cycle of length 3.
Lemma 14.4. If G is a simple planar graph, $|E(G)| \geq 2$, then

$$|E(G)| \leq 3|V(G)| - 6$$

and if G has no K_3 subgraph then

$$|E(G)| \leq 2|V(G)| - 4.$$

Fact: K_5, $K_{3,3}$, and their subdivisions are the only minimal non-planar graphs.
Corollary 14.5. Let G be a simple planar graph, $|E(G)| \geq 2$, then $\sum_{v \in V(G)} (6 - \deg v) \geq 12$.
Corollary 14.6. If G is a simple non-null planar grapg then for some $v \in V(G)$ $\deg_G v \leq 5$. Thus, planar graphs are 5-degenerate so that $\chi(G) \leq 6$.

Kuratowski

Lemma 15.1. Let G be a 2-connected loopless graph drawn in the plane. Then every region is bounded by a cycle.
Lemma 15.2. Let C be a cycle, $X, Y \subseteq V(G)$, $|V(C)| \geq 2$, then one of the following holds:

1. There exist $z_1, z_2 \in V(C)$ distinct, two paths P, Q with ends z_1 and z_2 such that $P \cup Q = C$, $X \subseteq V(P)$ and $Y \subseteq V(Q)$.
2. There exist distinct $x_1, x_2 \in X$, $y_1, y_2 \in Y$ such that x_1, y_1, x_2, y_2 appear on C in this order.
3. $X = Y$ and $|X| = |Y| = 3$.

Kuratowski-Wagner. A graph G is non-planar if and only if either K_5 or $K_{3,3}$ is a minor of G.
EQUIVALENTLY: A graph G is non-planar if and only if it contains a subdivision of K_5 or $K_{3,3}$ as a **subgraph**.
Four color. If G is planar and loopless then $\chi(G) \leq 4$.
Tait. Let G be a planar triangulation and let G^* be its dual. THen $\chi(G) \leq 4 \iff \chi'(G^*) = 3$.
Kaufman. For any pair of bracketings of the product $u_1 \times \cdots \times u_m$ there is a choice of $u_n \in \{i, j, k\}$ for every $1 \leq n \leq m$ such that the corresponding products are the same and non-zero.

Definitions

1. **Matching.** A matching $M \subseteq E(G)$ in a graph G is a collection of non-loop edges so that every vertex of G is incident to at most one edge of M.
2. **M-alternating path.** An M-alternating path P is a path whose edges alternate between belonging to M and belonging to $E(G) \setminus M$. Equivalently, every internal vertex of the path is incident to one edge in $M \cap E(P)$ and one edge of $(E(G) \setminus M) \cap E(P)$.
3. **M-augmenting path.** A path P is M-augmenting if $|P| \geq 1$, it is M-alternating, and the ends of P belong to no edge of M (whether in P or not).
4. **Vertex cover.** A vertex cover is a subset $X \subseteq V(G)$ if every edge of G has an end in X.
5. $\nu(G)$ is the **matching number** of G, i.e. the maximum size of a matching in G. $\tau(G)$ is the minimum size of a vertex cover in G.
6. **Cover.** A matching M covers $Y \subseteq V(G)$ is every vertex in Y is incident to an edge of M.
7. **Perfect matching.** M is a perfect matching if it covers $V(G)$.
8. **Separation.** A pair (A, B), $A, B \subseteq V(G)$ is called a separation if $A \cup B = V(G)$ and no edge of G has one end in $A \setminus B$ and another in $B \setminus A$, i.e. any edge from A to B has an end in $A \cap B$.
9. **k-Connected.** A graph if k-connected if $|V(G)| \geq k+1$ and $G \setminus X$ is connected for every $X \subseteq V(G)$ with $|X| \leq k+1$.

10. For $X \subseteq V(G)$, let $\delta(X)$ denote the set of all edges of G with one end in X and another in $V(G) \setminus X$. Let $N(X)$ the set of all neighbours of vertices in X.

11. **Line graph.** The line graph $L(G)$ of G has $V(L(G)) = E(G)$, and two vertices of $L(G)$ are adjacent iff the edges they represent share an end in G.

12. **Directed graph.** A directed graph (digraph) is a graph where for every edge, one of its ends is chosen as a head and the other as a tail. An edge is said to be directed from its tail to its head.

13. **Directed path.** A directed path from s to t in a digraph is a path in which every edge is traversed from its tail to its head as we follow the path from s to t.

14. For a set $X \subseteq V(G)$, we let $\delta^+(X)$ denote the set of all edges with tail in X and head in $V(G) \setminus X$; we let $\delta^-(X) = \delta^+(V(G) \setminus X)$.

15. **(s,t)-Flow.** Let G be a digraph, $s, t \in V(G)$. A function $\varphi : E(G) \to \mathbb{R}^+$ is an (s,t)-flow if for every $v \in V(G) \setminus \{s,t\}$, $\sum_{e \in \delta^+(v)} \varphi(e) = \sum_{e \in \delta^-(v)} \varphi(e)$. The **value** of φ is $\sum_{e \in \delta^+(s)} \varphi(e) - \sum_{e \in \delta^-(s)} \varphi(e)$.

Combinatorics

1. Max number of edges in simple graph on $n \geq 1$ vertices (i.e. $|E(K_n)|$) is $\frac{n(n-1)}{2}$.
2. No formula for number of graphs up to isomorphism on n vertices; number of graphs up to isomorphism with n edges is $2^{\binom{n}{2}}$.
3. Number of spanning trees on n labelled vertices (i.e. in K_n) is n^{n-2}.
4. Number of ways to label a path on n vertices: $n!/2$ and a star on n vertices: n.
5. $\nu(K_n) = \lfloor n/2 \rfloor$, $\tau(K_n) = n - 1$, $\nu(C_n) = \lfloor n/2 \rfloor$, $\tau(C_n) = \lceil n/2 \rceil$.

Strategy

1. **Is G connected**: look at connected components and derive a contradiction.
2. **Is G bipartite**: look at possible sizes of A, B in a bipartition, does this contradict the vertex degrees?
3. **Is G 2-connected**:
4. **Does G have an Euler tour**: yes if every vertex degree even (and must be connected)?
5. **Does G have an Euler trail**: yes if G connected and has at most two vertices of odd degree.
6. **Does G have Hamiltonian cycle**:

   (a) No if there is non-null $X \subseteq V(G)$ such that $G \setminus X$ has more than $|X|$ components;

   (b) Yes if $n \geq 3$ vertices, and if for every pair of non-adjacent vertices $\deg u + \deg v \geq |V(G)| = n$.

   (c) Yes if $|V(G)| \geq 3$ and either (1) $\deg v \geq n/2$ (2) $|E(G)| \geq \binom{n}{2} - n + 3$.

7. **Does G have a perfect matching**:

   (a) Yes if bipartite and all vertices have degree $d \geq 1$.

   (b) Yes iff $\nu(G) = \frac{|V(G)|}{2}$. If bipartite, this is equivalent to every vertex cover X is such that $|X| \geq \frac{|V(G)|}{2}$.

Useful inequalities:

1. If every vertex in G has degree at most d, then $d|X| \geq \sum_{v \in V(G)} \deg v \geq |E(G)|$