MATH 350; Midterm Review.

Connectivity

Handshaking lemma. For any graph G, $\sum_{v\in V(G)} \deg v = 2|E(G)|$.
Lemma 2.1. If there exists a walk in G with ends u, v then there exists a path in G with these ends.
Lemma 2.2. A graph G is not connected iff there exists a partition (X, Y) of $V(G)$ such that no edge of G has one end in X and the other in Y.
Lemma 2.3. If H_1, H_2 are connected subgraphs of G and $V(H_1) \cap V(H_2) \neq \emptyset$, then $H_1 \cup H_2$ is connected.
Lemma 2.4. Every vertex of G belongs to a unique connected component.
Lemma 2.5. A subgraph $H \subseteq G$ is a connected component of G iff H is connected, and if $e \in E(G)$ has an end in $V(H)$ then $e \in E(H)$.
Lemma 2.6. Let $e \in E(G)$ with ends u, v. Then exactly one of the following holds.

1. e is a cut edge, u and v belong to different conneected components of $G \setminus e$, and $\mathrm{comp}(G \setminus e) = \mathrm{comp}(G) + 1$.
2. e is not a cut edge, u and v belong to the same connected component of $G \setminus e$, and $\mathrm{comp}(G \setminus e) = \mathrm{comp}(G)$.

Trees & Forests

Lemma 3.1. Let F be a non-null forest. Then $\mathrm{comp}(F) = |V(F)| - |E(F)|$. If F is a tree, then $|V(F)| = |E(F)| + 1$.
Lemma 3.2. Let T tree with $|V(T)| \geq 2$. Let X be the set of leaves of T and Y be the set of vertices of degree ≥ 3. Then $|X| \geq |Y| + 2$. In particular, T has at least 2 leaves.
Lemma 3.3. If a tree T has exactly two leaves u and v, then T is a path with ends u and v.
Lemma 3.4. Let v be a leaf in a tree T. Then $T \setminus v$ is a tree.
Lemma 3.5. Let v be a leaf in a graph G. If $G \setminus v$ is a tree then G is a tree.
Lemma 3.6. Let T be a tree, $u, v \in V(T)$. Then there exists a unique path in T with ends u, v.

Spanning Trees

Lemma 4.1. Let G be a connected, non-null graph. Let $H \subseteq G$ be chosen minimal such that $V(H) = V(G)$ and H is connected. Then H is a spanning tree of G.
Lemma 4.2. Let G be a connected, non-null graph. Let $H \subseteq G$ be chosen maximal such that H has no cycles. Then H is a spanning tree of G.
Lemma 4.3. Let T be a spanning tree of G. Let $f \in E(G) \setminus E(T)$. Then there exists a unique fundamental cycle of f with respect to T.
Lemma 4.4. Let T be a spanning tree of G, $f \in E(G) \setminus E(T)$, and C be the fundamental cycle of f with respect to T. Let $T' = (T + f) \setminus e$ be the graph obtained from T by adding f and deleting some $e \in E(C)$. Then T' is a spanning tree of G.
Corollary 4.5. Let G, T, f, C, e be as in Lemma 4.4. Let $w : E(G) \to \mathbb{R}_+$. If T is $\mathrm{mst}(G, w)$, then $w(f) \geq w(e)$.
Theorem 4.6. Let G be a graph, $w : E(G) \to \mathbb{R}_+$ be such that $w(e) \neq w(f)$ for any $e, f \in E(G)$ with $e \neq f$. Let T be $\mathrm{mst}(G, w)$ and $E(T) = \{e_1, e_2, \ldots, e_k\}$ be such that $w(e_1) < \cdots < w(e_k)$. Then for every i with $1 \leq i \leq k$, e_i is the edge of minimum weight subject to $e_i \notin \{e_1, \ldots, e_{i-1}\}$ and $\{e_1, \ldots, e_i\}$ does not contain the edge set of a cycle.
Theorem 4.7. Kruskal's algorithm outputs $\mathrm{mst}(G, w)$.
Cayley's formula. The complete graph on n vertices has n^{n-2} spanning trees.

Euler Tours & Hamiltonian Cycles

Facts. A walk uses two edges incident to a vertex each time this vertex occurs in the walk (except for ends); thus if an Eulerian trail exists, then at most two vertices odd degree; if an Euler tour exists, then all vertices must have even degree.
Lemma 5.1. Let $E(G) \neq \emptyset$ and suppose G has no leaves. Then G contains a cycle.
Lemma 5.2. Let G be a graph such that every vertex of G has even degree. Then there exist cycles $C_1, \ldots, C_k$ in G such that $(E(C_1), \ldots, E(C_k))$ is a partition of $E(G)$.
Euler's theorem. If G is a connected graph such that the degree of every vertex of G is even, then G has an Euler tour.
Corollary 5.4. If G is a connected graph such that G contains at most two vertices of odd degree, then G has an Eulerian trail.
Lemma 5.5. Let G be a graph. If there exists $X \subseteq V(G)$ with $X \neq \emptyset$ such that $G \setminus X$ has more than $|X|$ components, then G has no Hamiltonian cycle.
Dirac. Let G be a simple graph on $n \geq 3$ vertices. Suppose that for every pair of non-adjacent vertices, $u, v \in V(G)$, $\deg u + \deg v \geq n$. Then G has a Hamiltonian cycle.
Corollary 5.7. Let G be a simple graph with $n \geq 3$ vertices. Suppose that either

- $\deg v \geq n/2$ for all $v \in V(G)$; or
- $|E(G)| \geq \binom{n}{2} - n + 3$.

Then G has a Hamiltonian cycle.

Bipartite Graphs

Lemma 6.1. Trees are bipartite.
Theorem 6.2. Let G be a graph. TFAE:

1. G is bipartite;
2. G contains no closed walk of odd length;
3. G contains no odd cycle.

Matchings

Fact. $\nu(G) \leq \frac{1}{2}|V(G)|$.
Lemma 7.1. Let G be a loopless graph. Then $\nu(G) \leq \tau(G) \leq 2\nu(G)$.
Lemma 7.2. A matching M in G has maximum size iff there does not exist an M-augmenting path in G.
Konig. If G is bipartite, then $\nu(G) = \tau(G)$.
Theorem 7.4. Let $d \geq 1$ be an integer, let G bipartite such that $\deg v = d$ for every $v \in V(G)$. Then G has a perfect matching.
Hall. Let G bipartite with bipartition (A, B). Then G has a matching M covering A iff $|N(S)| \geq |S|$ for every $S \subseteq A$.
Menger. Let $s, t \in V(G)$ be a pair of distinct non-adjacent vertices of G, and let $k \geq 1$ be an integer. Then exactly one of the following holds.

1. There exist pairwise internally disjoint paths $P_1, \ldots, P_k$ in G with ends s, t.
2. There exists a separation (A, B) of G such that $s \in A \setminus B$, $t \in B \setminus A$ of order $< k$.

Theorem 8.2. Let $Q, R \subseteq V(G)$, $k \geq 1$ an integer. Then exactly one of the following holds. (1) There exist pairwise disjoint paths $P_1, \ldots, P_k$ in G each with one end in Q and another in R. (2) There exists a separation (A, B) of G of order $< k$ such that $Q \subseteq A$ and $R \subseteq B$.
Corollary 8.3. Let G be a k-connected graph, $s, t \in V(G)$ be distinct. Then there exist paths $P_1, \ldots, P_k$ in G from s to t that are internally disjoint.
Menger 2. Let $s, t \in V(G)$ be distinct and $k \geq 1$. Then exactly one of the following holds. (1) There exist paths $P_1, \ldots, P_k$ in G with ends s, t and such that for $1 \leq i < j \leq n$, $E(P_i) \cap E(P_j) = \emptyset$ (pairwise edge disjoint). (2) There exists $X \subseteq V(G)$ such that $s \in X$, $t \in V(G) \setminus X$, and $|\delta(X)| < k$.

Directed Graphs & Network Flows

Lemma 9.1. Let G be a digraph. Let $s, t \in V(G)$. Then there does not exist a directed path in G from s to t iff there exists $X \subseteq V(G)$ such that $s \in X$, $t \in V(G) \setminus X$, and $\delta^+(X) = \emptyset$.
Lemma 9.2. Let φ be an (s, t)-flow on a digraph G with value k. Then for any $X \subseteq V(G)$ such that $s \in X$, $t \in V(G) \setminus X$, we have

$$\sum_{e\in\delta^+(X)} \varphi(e) - \sum_{e\in\delta^-(X)} \varphi(e) = k.$$

Definitions

1. **Matching.** A matching $M \subseteq E(G)$ in a graph G is a collection of non-loop edges so that every vertex of G is incident to at most one edge of M.
2. **M-alternating path.** An M-alternating path P is a path whose edges alternate between belonging to M and belonging to $E(G) \setminus M$. Equivalently, every internal vertex of the path is incident to one edge in $M \cap E(P)$ and one edge of $(E(G) \setminus M) \cap E(P)$.
3. **M-augmenting path.** A path P is M-augmenting if $|P| \geq 1$, it is M-alternating, and the ends of P belong to no edge of M (whether in P or not).
4. **Vertex cover.** A vertex cover is a subset $X \subseteq V(G)$ if every edge of G has an end in X.

5. $\nu(G)$ is the **matching number** of G, i.e. the maximum size of a matching in G. $\tau(G)$ is the minimum size of a vertex cover in G.
6. **Cover.** A matching M covers $Y \subseteq V(G)$ is every vertex in Y is incident to an edge of M.
7. **Perfect matching.** M is a perfect matching if it covers $V(G)$.
8. **Separation.** A pair (A, B), $A, B \subseteq V(G)$ is called a separation if $A \cup B = V(G)$ and no edge of G has one end in $A \setminus B$ and another in $B \setminus A$, i.e. any edge from A to B has an end in $A \cap B$.
9. **k-Connected.** A graph if k-connected if $|V(G)| \geq k+1$ and $G \setminus X$ is connected for every $X \subseteq V(G)$ with $|X| \leq k+1$.
10. For $X \subseteq V(G)$, let $\delta(X)$ denote the set of all edges of G with one end in X and another in $V(G) \setminus X$. Let $N(X)$ the set of all neighbours of vertices in X.
11. **Line graph.** The line graph $L(G)$ of G has $V(L(G)) = E(G)$, and two vertices of $L(G)$ are adjacent iff the edges they represent share an end in G.
12. **Directed graph.** A directed graph (digraph) is a graph where for every edge, one of its ends is chosen as a head and the other as a tail. An edge is said to be directed from its tail to its head.
13. **Directed path.** A directed path from s to t in a digraph is a path in which every edge is traversed from its tail to its head as we follow the path from s to t.
14. For a set $X \subseteq V(G)$, we let $\delta^+(X)$ denote the set of all edges with tail in X and head in $V(G) \setminus X$; we let $\delta^-(X) = \delta^+(V(G) \setminus X)$.
15. **(s,t)-Flow.** Let G be a digraph, $s, t \in V(G)$. A function $\varphi : E(G) \to \mathbb{R}^+$ is an (s,t)-flow if for every $v \in V(G) \setminus \{s, t\}$, $\sum_{e \in \delta^{-1}(v)} \varphi(e) = \sum_{e \in \delta^-(v)} \varphi(e)$. The **value** of φ is $\sum_{e \in \delta^+(s)} \varphi(e) - \sum_{e \in \delta^-(s)} \varphi(e)$.