100 Combinatorics Problems
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Q1. Let G be k-connected and suppose H is a graph obtained from G by appending a new vertex v adjacent to at least k vertices in G. Show that H is k-connected.

Proof. Consider a subset $X \subseteq V(H)$ of less than k vertices. If $v \notin X$, then $H \setminus X$ is connected since G is k-connected. Otherwise, $v \in X$, so by the k-connectivity of G and since v has a neighbour not in X (as $\deg v \geq k$), it follows that $F = H \setminus (X \setminus v)$ is connected. Then $F \setminus v = G \setminus (X \setminus v) = H \setminus X$ is connected by the choice of v. $\square$

Q2. Let $|V(G)| \geq 3$. Show that G is 2-connected if and only if for all vertices x, y, z there is a path between x and y containing z.

Proof. For "$\Rightarrow$", suppose G is 2-connected. Define $R = \{x, y\}$ and $Q = \{z\}$. Since G has no separation of order 2, Menger's theorem implies that there are two vertex disjoint paths (other than at z) with ends in R and Q. So $P_1 \cup P_2$ is a path between x and y containing z.

For "$\Leftarrow$", we prove the contrapositive. If G is not 2-connected then there is a cut vertex ℓ. So there are vertices s, t such that there is no path between s and t in $G \setminus \ell$. So there can be no path between s and ℓ containing t, else there is a path between s and t in $G \setminus \ell$. $\square$

Q3. Let $k \geq 2$. Show that if G is k-connected, then every k vertices are contained in a cycle.

Proof. For $k = 2$, the result immediately follows from Menger's theorem. Now fix $k \geq 3$ and let $X = \{x_1, x_2, \ldots, x_k\}$ be a set of k vertices in G. By induction, there is a cycle C containing $\{x_1, x_2, \ldots, x_{k-1}\}$. Let $x, y \in \{x_1, x_2, \ldots, x_{k-1}\}$ be such that no vertex in $\{x_1, x_2, \ldots, x_{k-1}\}$ lies between them in C. Then, Menger's theorem implies that there are k internally disjoint paths with ends in $\{x, y\}$ and $\{x_k\}$. So $C \cup P_x \cup P_y$ is the desired cycle, where P_x and P_y are paths from x to x_k and y to x_k respectively. $\square$

Q4. Let G be a connected, 3-regular graph. Show that if G has no cut-edge, then every pair of edges lie on a common cycle.

Proof. Let $e = (e_1, e_2)$ and $f = (f_1, f_2)$ be two edges in G. If there exist two vertex disjoint paths with ends in $\{e_1, e_2\}$ and $\{f_1, f_2\}$, then these paths and the two edges form the needed cycle. Otherwise, Menger's theorem implies that there is a separation (A, B) of order 1 with $\{e_1, e_2\} \subseteq A$ and $\{f_1, f_2\} \subseteq B$. Let $A \cap B = \{v\}$ and note that $\deg v = 3$. If v_1, v_2, v_3 are the neighbours of v, assume without loss of generality that $v_1, v_2 \in A$ and $v_3 \in B$. Then (v_2, v_3) is a cut-edge, so this case can not occur; thus e and f are contained in a cycle. $\square$

Q5. For any graph G, $\nu(G) \leq \tau(G) \leq 2\nu(G)$.

Proof. We first show that $\nu(G) \leq \tau(G)$. Given a maximal matching M, let X be a set consisting of one end from each edge in M. Then $|X| = |M|$ and X is a vertex cover by the maximality of M. Thus, $\tau(G) \leq |X| = \nu(G)$.

To show that $\tau(G) \leq 2\nu(G)$, let M be a maximal matching in G. Let X be the set of ends of edges in M. Then X is a vertex cover: if there was an edge $e \in E$ with no ends in X, then M is not maximal since $M \cup \{e\}$ is a matching. Hence $\tau(G) \leq |X| = 2|M| \leq 2\nu(G)$. □

Q6. Prove König's theorem using Menger's theorem: $\tau(G) = \nu(G)$ for any bipartite graph G.

Proof. We may assume that G has vertices no degree-zero vertices, as they don't change anything. It suffices to prove that $\tau(G) \leq \nu(G)$, given Q5. Suppose G is bipartite with bipartition (A, B). Let G' be obtained from G by appending two vertices u, v such that u is adjacent to every vertex in A and v is adjacent to every vertex in B. Let $k \geq 0$ be the greatest integer for which there exist k (internally) vertex disjoint paths between u and v. This k corresponds to a matching: each path contains an unique edge with an end in $X - Y$ and the other in $Y - X$; the set of such edges is a matching of size k.

Now observe that Menger's theorem implies that there is a separation (X, Y) of G' such that $\in X - Y$, $v \in Y - X$, and $|X \cap Y| \leq k$. Then $X \cap Y$ is a vertex cover as there can be no edge $e \in E(G')$ with no end in $X \cap Y$. Otherwise, one of the following must occur, a contradiction:

- e has both ends in X or both ends in Y, but this can not occur as G is bipartite.
- e has u or v as one of its ends, but this can not occur as otherwise the other end of e would need to have degree 0.
- e connects $X - Y$ to $Y - X$, but this would mean that (X, Y) is not a separation.

Thus, $\tau(G) \leq |X \cap Y| = k \leq \nu(G)$. □

Q7. Show that a matching M is maximal if and only if G contains no M-augmenting path.

Proof. If P is an M-augmenting with m edges, note that an odd number of edges in P are in M. In particular, $\frac{m-1}{2}$ edges are in M, and $\frac{m+1}{2}$ are not. Since the ends of P are unmatched, the edges in M not on P together with the edges on P not in M form a matching with more edges than $|M|$, a contradiction.

Now suppose M is a matching in G with no M-augmenting path. Assume for a contradiction that M' is a larger matching than M. We define a new graph G' by $V(G') = V(G)$ and $E(G') = M \cup M'$. Then since every vertex in G has degree 2, every component of G' is a path or a cycle. Since $|M'| > |M|$, there is a component C with more edges in M' than in M, i.e. $|E(C) \cap M'| > |E(C) \cap M|$.

- If C is a path, then one of its ends is matched by M (else C is M-augmenting). So the first edge of C is in M. From there, the edges alternate between M' and M, so $|E(C) \cap M'| \leq |E(C) \cap M|$.

- If C is a cycle, then it has an even number of edges (else two edges in M or M' will share a vertex as an end), so $|E(C) \cap M'| = |E(C) \cap M|$.

Thus, there can be no such component C, contradicting $|M'| > |M|$. Thus M is maximal. $\square$

Q8. For an integer $k \geq 3$, let $N = R_3(k, k, k)$ be the minimum N such that in every edge-coloring of K_N in 3 colors there is a set X of k vertices so that all edges between vertices of X have the same color. Prove that

$$\binom{N}{k}\left(\frac{1}{3}\right)^{\binom{k}{2}-1} \geq 1. \tag{$*$}$$

Proof. Suppose

$$\binom{N}{k}\left(\frac{1}{3}\right)^{\binom{k}{2}-1} < 1$$

for a contradiction. Let Ω be the space of all colorings of the edges of K_N in 3 colors. Let $R \subseteq V(K_N)$ be a k-element subset of vertices, and let A_R be the event where R is monochromatic. So

$$\mathbb{P}(A_R) = 3 \prod_{1 \leq i \leq \binom{k}{2}} \frac{1}{3} = 3^{1-\binom{k}{2}}.$$

By sub-additivity,

$$\mathbb{P}\left(\bigcup_{R \in [V(K_N)]^k} A_R\right) \leq \sum_{R \in [V(K_N)]^k} 3^{1-\binom{k}{2}} = \binom{N}{k} 3^{1-\binom{k}{2}} < 1.$$

Thus there is a non-zero probability that there is no monochromatic k-element subgraph of K_N, contradicting $N = R_3(k, k, k)$. Hence $(*)$ follows and we're done. $\square$

Q9. Let F be a forest on n vertices. Prove that the intersection of k connected subgraphs of F is either empty or a tree.

Proof. It suffices to prove the claim when $k = 2$, since given connected subgraphs $C_1, C_2, \ldots, C_k$, we have by induction that $C = C_1 \cap \cdots \cap C_{k-1}$ is either a tree or empty. If C is a tree then it is connected and we may apply the case when $k = 2$ to $C \cap C_k$. Otherwise C is empty so that $C \cap C_k$ is too. Thus, we just need to prove the base case now.

Let C_1 and C_2 be two connected subgraphs of F. Assume that $C_1 \cap C_2$ is not a tree and non-empty. Since $F \supseteq C_1 \cap C_2$ is acyclic, so is $C_1 \cap C_2$. Thus, $C_1 \cap C_2$ cannot be connected. Since $C_1 \cap C_2 \neq \emptyset$, there are vertices u, v in $C_1 \cap C_2$ which have no path between them. Let C be the connected component containing $\{u, v\}$. Since F is a forest, there is a unique path P between u and v in F. Since $u, v \in V(C_1) \cap V(C_2)$ and these are connected subgraphs, it follows that $C_1 \cap C_2$ contains P, a contradiction. $\square$

Q10. Let G be a k-connected graph on n vertices.

(a) Prove that $|E(G)| \geq kn/2$.

(b) Show that for every integer $k \geq 2$ and $n \geq k+1$ there is a k-connected graph with $|V(G)| = n$ and $|E(G)| \leq (k-1)n$.

Proof. For (a) note that since G is k-connected, every vertex has degree at least k. Otherwise, there is a vertex with degree at most $k-1$; deleting its neighbours disconnects the graph, contradicting k-connectivity. By handshaking,

$$|E(G)| = \frac{1}{2} \sum_{v \in V(G)} \deg v \leq nk/2.$$

We prove (b) by induction on k. For $k = 2$ and $n \geq k+1$, the cycle C_n on n vertices is the graph we need. Indeed,

$$|E(C_n)| = n = 1 \cdot n \leq (k-1) \cdot n.$$

Now fix $k \geq 3$. By the IH we obtain a graph G' on n' vertices and m' edges such that G' is $(k-1)$-connected, $n' \geq k$, and $m' \leq (k-2)n'$. Now let G be obtained by taking a vertex $v \in V(G')$ and connecting it to every vertex in G'. Then from Q1 G is k-connected. This uses n' edges. Let $n = n'+1$ and $m = m'+n'$. Then $n = n'+1 \geq k+1$ and

$$\begin{aligned} m = m' + n' &\leq (k-2)n' + n' = (k-1)n' \\ &\leq (k-1)(n'+1) = (k-1)n \end{aligned}$$

which completes the proof. □

Q11. Prove that:

(a) If T is a tree then $|V(T)| = |E(T)| + 1$.

(b) If F is a forest and $c(F)$ is the number of components of F, then $c(F) = |V(F)| - |E(F)|$.

Proof. For (a) we proceed by induction on $|V(T)|$. If $|V(T)| = 1$ then T is edgeless so that $|V(T)| = 0 + 1 = |E(T)| + 1$. Now fix $|V(T)| \geq 2$. Then T has at least one leaf v. Let $T' = T \setminus v$. Then T' is a tree on $|V(T)|-1$ vertices, so by the IH we have $|V(T')| = |E(T')| + 1$. Appending v to T' gives one new vertex and one new edge. Hence $|V(T)| = |V(T')| + 1 = |E(T')| + 1 + 1 = |E(T)| + 1$ and we're done.

For (b) we induct on $c(F)$. If $c(F) = 1$ then from (a)

$$|V(F)| = |E(F)| + 1 \Leftrightarrow 1 = |V(F)| - |E(F)|.$$

Then if $c(F) \geq 2$, let F' be obtained from F by deleting one entire component C. Then by the IH we have $c(F') = |V(F')| - |E(F')|$ and since C is a tree, (a) implies that $|V(C)| = |E(C)| + 1$. Putting these two together, we obtain that

$$\begin{aligned} c(F) = c(F') + 1 &= |V(F')| - |E(F')| + |V(C)| - |E(C)| \\ &= |V(F)| - |E(F)|, \end{aligned}$$

since $V(F') \cap V(C) = E(F') \cap E(C) = \emptyset$. This completes the proof. □

Q12. Prove Hall's theorem: Let G be bipartite with bipartition (A, B). Show that G has a matching which saturates A if and only if

$$|N(S)| \geq |S|$$

for every subset $S \subseteq A$.

Proof. The forward implication is easy: suppose M is a matching which saturates A. If $S \subseteq A$ were such that $|N(S)| < |S|$, then no matching could saturate S and hence A.

Conversely, first note that Hall's condition implies that $|A| \leq |B|$. Without loss of generality, we assume $|A| = |B|$. Thus, a matching M is perfect if and only if it saturates A. So let M be a matching in G; we will show that either G has an M-augmenting path or there is a set $S \subseteq A$ violating Hall's condition. Then, the contrapositive implies Hall's theorem if we take M to be maximum.

If M does not already saturate A, there is an unmatched vertex $a_0 \in A$ with a neighbour $b_1 \in B$. If b_1 is unmatched, then the path with the single edge a_0b_1 is augmenting. Otherwise, there is a vertex $a_1 \in A$ which is matched with b_1 by M. Note that we have constructed an alternating path with ends a_0 and a_1.

We continue this process inductively as follows. Let $k \geq 2$ and suppose we have defined $\{a_0, a_1, \ldots, a_{k-1}\}$ and $\{b_1, b_2, \ldots, b_{k-1}\}$ such that there is an alternating path with ends a_0 and a_j, for each $j \in \{0, 1, \ldots, k-1\}$. Then exactly one of the following holds:

- If $S = \{a_0, a_1, \ldots, a_{k-1}\}$ satisfies Hall's condition, then there is a vertex $b_k \notin \{b_1, b_2, \ldots, b_{k-1}\}$ adjacent to a vertex in S. If b_k is unmatched, then there is an M-augmenting path. Otherwise, b_k is matched with a vertex $a_k \in A$ and so there is an alternating path between a_0 and a_k.
- Else, S violates Hall's condition and the process terminates.

Since this process must eventually terminate, we either halt with an augmenting path or a subset S violating Hall's condition. $\square$

Q13. Prove König's theorem using Hall's theorem.

Proof. Again, as in Q6, we first notice that it suffices to show that $\tau(G) \leq \nu(G)$. Let G have bipartition (A, B) and let M be a maximal matching in G. If M saturates A, then $\nu(G) = |M| = |A|$ and A is a vertex cover so that $\tau(G) = \nu(G)$. Thus we assume that $L \subsetneq A$ is the largest subset of A saturated by M. Hence all vertices in $A - L$ are unmatched.

By induction on $|A - L|$, we show that a vertex cover of size $|L|$ exists. We note that if $|A - L| \geq 1$ then there is a subset $S \subseteq A - L$ with $|N(S)| < |S|$. Indeed, it follows from Hall's theorem that since A can not be saturated by a matching, there is a subset $S \subseteq A$ violating Hall's condition. Let's take $|S|$ minimum over all such sets S. Note then that $S \cap L = \emptyset$ since every vertex in L has at least one unique neighbour.

For the base case step, if $|A - L| = 1$ there is a unique unmatched vertex $a \in A - L$. So a has no neighbours since $|N(\{a\})| < 1$. Thus L is a vertex cover.

Furthermore, there is a maximal matching M in G such that every edge of M has exactly one end in L, and where every vertex in $A - L$ is still unmatched.

Now fix $|A - L| \geq 2$ and obtain a set $S \subseteq A - L$ which violates Hall's condition. Then the IH implies that $H = G \setminus (S \cup N(S))$ contains a vertex cover X of size $|L|$. So, in G, every edge with no end in $N(S)$ has an end in X. Also, there is a maximal matching M' in G such that each edge in M' has exactly one end in X and no vertex in $A - L$ is matched by M'.

Note that since every vertex in $S \subseteq A - L$ is unmatched by M', every vertex in $N(S)$ is matched, otherwise there is an augmenting path. Define Y to be the set of vertices in X which are not matched with a vertex in $N(S)$. Then we claim that $Y \cup N(S)$ is a vertex cover of G of size at most $|L|$.

- First, we prove that $Y \cup N(S)$ is a vertex cover. Let e be any edge in G. If $e \in M'$ then it has one end $x \in X$. If $x \notin Y$, then by definition $x \in N(S)$, as needed.

- Next, we show that $|Y \cup N(S)|$ has size at most $|L|$. By construction, every vertex in $X \setminus Y$ has a unique end in $N(S)$, so $|X \setminus Y| \leq |N(S)|$. Also, since every vertex $x \in N(S)$ is matched by M', either $x \in X \setminus Y$ (if $x \in X$) or x has a unique neighbour $y \in X \setminus Y$, so $|N(S)| \leq |X \setminus Y|$. Since $Y \cap N(S) = \emptyset$, we have $|Y \cup N(S)| = |Y| + |N(S)| = |Y| + |X \setminus Y| = |X| = |L|$.

Therefore, G has a vertex cover of size $|L|$. Therefore, $\tau(G) \leq |L| = \nu(G)$. □

Q14. Prove Tutte's matching theorem: A graph G has a perfect matching if and only if

$$c_o(G - X) \leq |X|$$

for every subset $X \subseteq V(G)$.

Proof. Let's start with the "only if" direction. Let M be a perfect matching in G, and suppose there is a set $X \subseteq V(G)$ with $c_o(G - X) > |X|$. Suppose $C_1, C_2, \ldots, C_k$ are the odd components of $G - X$. Since M is perfect and each component is odd, for each $j \in [k]$ there is a vertex $v_j \in V(C_j)$ which is matched with a vertex $x_j \notin C_j$; hence $x_j \in X$. But then two vertices v_{i_1}, v_{i_2} must receive the same match in X, a contradiction.

The "if" direction is much harder. We proceed by induction on $|V(G)|$. The theorem is trivial if $|V(G)| \leq 2$, so fix $|V(G)| \geq 3$ and suppose $c_o(G - X) \leq |X|$ for every $X \subseteq V(G)$. We prove a sequence of claims as follows:

- **Claim 1:** $|V(G)|$ *is even.* It suffices to check that G has only even components. Set $X = \emptyset$. Then $c_o(G - X) = c_o(G) \leq |\emptyset| = 0$, so the claim holds.
- **Claim 2:** $c_o(G - X) + |X|$ *is always even.* If $|X|$ is odd, then $|V(G)| - |X|$ is too by claim 1, so $G - X$ must have an odd number of components. The exact same reasoning shows that if $|X|$ is even then so is $c_o(G - X)$. Then, the claim holds since $a + b$ is even if and only if a and b have the same parity.

- **Claim 3:** *There is a subset $X \subseteq V(G)$ such that $c_o(G - X) = |X|$.* We will call such sets X *critical.* If $X = \emptyset$ then $c_o(G - X) = 0$ as shown above, so $\emptyset$ is critical.
- **Claim 4:** *Let $Z \subseteq V(G)$ be critical with $|Z|$ maximum. Then $G - Z$ has no even components.* Suppose by contradiction that C is an even component of $G - Z$, and fix $x \in V(C)$. Then $Z' = Z \cup \{x\}$ is critical so that Z is not maximal: $c_o(G - Z') = c_o(G - Z) + 1 = |Z| + 1 = |Z'|$, since deleting x from C will either give one odd component (if $C - x$ is connected) or one even and one odd component.
- **Claim 5:** *For each $j \in [k]$, fix $v_j \in V(C_j)$. Then $C_j^* = C_j - v_j$ has a perfect matching.* Suppose not. Then the IH implies that there is a set $X \subseteq V(C_j^*)$ with $c_o(C_j^* - X) > |X|$. But then $Z' = Z \cup X \cup \{v_j\}$ is critical:

$$\begin{aligned} c_o(G - Z') &= c_o(G - Z) - 1 + c_o(C_j^* - X) \\ &> c_o(G - Z) - 1 + |X| \\ &= |Z| - 1 + |X| = |Z'| - 2, \end{aligned}$$

  so $c_o(G - Z') \geq |Z'| - 1$, but $c_o(G - Z')$ and $|Z'|$ have the same parity by claim 2, so we must have $c_o(G - Z') \geq |Z'|$. By hypothesis, $c_o(G - Z') \leq |Z'|$, so Z' is critical.
- **Claim 6:** *G has a perfect matching.* Claim 5 shows that $C_1^*, C_2^*, \ldots, C_k^*$ have perfect matchings; so now we must match the points in Z with points in $Y = \{v_1, v_2, \ldots, v_k\}$. If there is no such (perfect) matching between Z and Y, then Hall's theorem implies that there is a subset $S \subseteq Y$ with $|N(S)| < |S|$ (note that the induced graph is bipartite, deleting any edges $z_i z_j$). Set $X = N(S)$ and then note that $c_o(G - X) \geq |S| > |N(S)| = |X|$ is a contradiction and the proof is done. □

Q15. Let G be a d-regular bipartite graph. Show that G has a perfect matching.

Proof. Suppose not. Let (A, B) be a bipartition of G. Then Hall's theorem says that there is a set $S \subseteq A$ with $|N(S)| < |S|$. Note that there are $d \cdot |S|$ edges leaving S, and there are $d \cdot |N(S)$ edges leaving $N(S)$. But every edge leaving S has another end in $N(S)$, so $d \cdot |N(S)| < d \cdot |S|$ is a contradiction. □

Q16. Given $n \in \mathbb{N}$, determine the minimum $\delta = \delta(n)$ such that every graph G on $2n$ vertices with minimum degree δ has a perfect matching.

Proof. First, we show that $\delta = n$ suffices to guarantee a perfect matching in G. We may assume that G is δ-regular, otherwise just delete any extra edges.

Given any graph G, let (A_0, B_0) be any partition of $V(G)$ such that $|A_0| = |B_0|$. For any partition (A, B), let $e(A)$ denote the number of edges with both ends in A, and define $e(B)$ analagously. We give an algorithm to obtain a bipartite subgraph of G which will contain a perfect matching.

Pick any $a \in A_0$ and $b \in B_0$. Define $A_1 = A_0 \setminus \{a\} \cup \{b\}$ and $B_1 = B_0 \setminus \{b\} \cup \{a\}$. If $e(A_0) + e(B_0) > e(A_1) + e(B_1)$, then repeat the algorithm again with the partition (A_1, B_1), otherwise try again with any other pair of vertices. Since G is finite, this process terminates with a partition (A, B) of $V(G)$ which minimises the number of edges with both ends in A or both ends in B. Then, it follows that each $a \in A$ has at least $\lceil n/2 \rceil$ neighbours in B and each $b \in B$ has at least $\lceil n/2 \rceil$ neighbours in A. Let H be obtained from G by deleting all edges with both ends in either A or B. Then H is $\lceil n/2 \rceil$-regular and bipartite, and so by Q15 it has a perfect matching. Thus, G does too.

We complete the proof by showing that $\delta = n-1$ does not suffice to guarantee a perfect matching. Consider the complete bipartite graph $K_{n-1,n+1}$. Deleting the smaller partite set of $n-1$ vertices results in $n+1$ isolated vertices, which violates Tutte's condition. Hence, $K_{n-1,n+1}$ has no perfect matching. Note that $\delta(K_{n-1,n+2}) = n-1$. $\square$

Q17. Show that for every bridgeless cubic graph G and every $e \in E(G)$ there is a perfect matching in G containing e.

Proof. Fix any edge $uv \in E(G)$ and let $e, f \in E(G)$ be the other two edges incident to v. Let G' be obtained from G by deleting the edges e, f. To show that G has a perfect matching containing uv, it suffices to prove that G' has a perfect matching. Hence, we use Tutte's theorem to show that $c_o(G'-X) \leq |X|$ for every subset $X \subseteq V(G')$.

Suppose for a contradiction that there is a set $X \subseteq V(G)$ such that $c_o(G' - X) > |X|$. Note that $c_o(G' - X)$ and $|X|$ have the same parity since $|V(G')|$ is even, hence $c_o(G'-X) \geq |X|+2$. Let $C_1, C_2, \ldots, C_n$ denote the odd components of $G' - X$. For each $j \in [n]$, let ℓ_j be the number of edges in G leaving C_j. Then since G is cubic,

$$\sum_{v \in V(C_j)} \deg_G v = 2|E(C_j)| + \ell_j = 3|V(C_j)|.$$

Observe that LHS is even (since $|V(C_j)|$ is odd), hence ℓ_j must be odd. Furthermore, $\ell_j \geq 2$ since G is bridgeless, and since ℓ_j is odd we have $\ell_j \geq 3$. Let r_j be the number of edges in G with one end in C_j and the other in another odd component C_i, and let $q_j = \ell_j - r_j$ be the remaining edges. Then $\sum_{j=1}^{n} r_j \leq 4$ since we only deleted two edges, and since G is 3-regular, $\sum_{j=1}^{n} q_j \leq 3|X|$.

Then, putting everything together,

$$\begin{aligned} c_o(G' - X) = n &\leq \frac{1}{3}\sum_{j=1}^{n} \ell_j = \frac{1}{3}\sum_{j=1}^{n} (q_j + r_j) \\ &\leq \frac{1}{3}\sum_{j=1}^{n} q_j + 4/3 \leq |X| + 4/3 \end{aligned}$$

But recall that $c_o(G' - X) = n \geq |X| + 2$, so we may rearrange the above to obtain $|X| + 2 - 2/3 \geq n$ and hence $n - 2/3 \geq n$ is a contradiction. Therefore, G' has a perfect matching. Since v has degree one in G', it follows that G has a perfect matching containing uv. $\square$

Q18. Prove the Tutte-Burge formula: For a graph G, its *deficiency* $\operatorname{def}(G)$ is the minimal number of vertices avoided by a matching. Clearly $\operatorname{def}(G) = |V(G)| - 2\nu(G)$. Show that

$$\operatorname{def}(G) = \max_{X \subseteq V(G)} (c_o(G - X) - |X|). \qquad (*)$$

Proof. First note that $(*)$ holds if G has a perfect matching: $c_o(G-X)-|X| \leq 0$ always by Tutte's theorem, and the inequality is tight taking $X = \emptyset$. Therefore,

$$\operatorname{def}(G) = |V(G)| - 2\nu(G) = 0 = \max_{X \subseteq V(G)} (c_o(G - X) - |X|).$$

Let $X \subseteq V(G)$ be such that $c_o(G - X) - |X| = k$ is maximum. Let M be any maximal matching in G and let $C_1, C_2, \ldots, C_n$ be the odd components of $G - X$. By deleting edges, we may assume without loss of generality that G has no edge with one end in $\cup_{j=1}^{k} V(C_j)$ and another in X, since at most $n-k$ edges in M can have an end in X. Then each component $C_1, C_2, \ldots, C_k$ contains an unmatched vertex, since $|V(C_j)|$ is odd for every $j \in [k]$. So k vertices can be avoided by a matching, hence

$$\operatorname{def}(G) = |V(G)| - 2\nu(G) \leq k = \max_{X \subseteq V(G)} (c_o(G - X) - |X|).$$

The reverse inequality is much harder. Let's set $k = |V(G)| - 2\nu(G)$. We construct an auxiliary graph H as follows. Let $Y = \{y_1, y_2, \ldots, y_k\}$ be a set of k new vertices. Let $H = G \cup Y$ be obtained from G by adding each vertex in Y, with each $y_j \in Y$ adjacent to every other vertex in H (including those in $Y - v_j$).

Then H has a perfect matching: take any maximal matching M in G and suppose $v_1, v_2, \ldots, v_k$ are unmatched; then $M \cup \{y_j v_j : j \in [k]\}$ is a perfect matching in H. So H satisfies Tutte's condition: we have $c_o(H - X) \leq |X|$ for every $X \subseteq V(H)$. So fix $X \subseteq V(G)$ and observe that

$$\begin{aligned} c_o(G - X) &= c_o(H - (X \cup Y)) \leq |X \cup Y| \\ &= |X| + |Y| = |X| + |V(G)| - 2\nu(G), \end{aligned}$$

consequently $c_o(G-X)-|X| \leq |V(G)|-2\nu(G) = \operatorname{def}(G)$. Since X was arbitrary, the proof is complete. ☐

Q19. Use LP duality to prove König's theorem.

Proof. Fix a bipartite graph $G = (V, E)$. First note that the following LP computes $\nu(G)$:

$$\begin{aligned} \max \quad & z(x) = \sum_{e \in E} x_e \\ \text{subject to} \quad & \sum_{e \in \delta(v)} x_e \leq 1, \ \forall v \in V \\ & x_e \geq 0, \ \forall e \in E. \end{aligned}$$

Since G is bipartite, its incidence matrix A is totally unimodular. Hence the polyhedron $P = \{x \geq 0 : Ax \leq 1\}$ has integral corners. By construction, $x \in P$ if and only if it is feasible, so any optimum x^* is integral.

Now fix a maximal matching M and define $x_e = 1$ if $e \in M$ and $x_e = 0$ otherwise. Then x is a feasible point and $z(x) = |M| = \nu(G)$. On the other hand, if x^* is an optimum of the LP, put $M = \{e \in E : x_e^* = 1\}$. Then $|M| = \sum_{e \in E} x_e^*$, and if M is not a matching there is a vertex $v \in V$ incident with two edges in M, violating the first constraint. So the primal has optimum value $\nu(G)$.

Consequently, LP duality implies that its dual program also has optimum value $\nu(G)$. The dual is as follows:

$$\begin{aligned} \min \quad & w(x) = \sum_{v \in V} y_v \\ \text{subject to} \quad & y_u + y_v \geq 1, \ \forall uv \in E \\ & y_v \geq 0, \ \forall v \in V. \end{aligned}$$

But the dual also computes $\tau(G)$. If X is a minimal vertex cover of G, set $y_v = 1$ if $v \in X$ and $y_v = 0$ otherwise. Then y is a feasible point and $w(y) = |X|$. Conversely, if y^* is an optimum of the LP, put $X = \{v \in V : y_v^* = 1\}$. If X is not a vertex cover, then there is an edge $uv \in E$ with no end in X. So $y_u^* = y_v^* = 0$, contradicting $y_u^* + y_v^* \geq 1$. It follows that $\nu(G) = \tau(G)$. □