Diametric Colorings in Ultrametric Spaces

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Abstract. Let Γ_X denote the family of compact subsets of (X, d). A coloring $\chi : \Gamma_X \to [k]$ is *diametric* if every pair of compact subsets with equal diameters receive the same color. A free ultrafilter $\mathcal{F}$ is called *diametrically Ramsey* if every diametric coloring admits a set $A \in \mathcal{F}$ whose compact subsets are monochrome. We show that every infinite ultrametric space contains a sequence (x_n) so that every free ultrafilter containing (x_n) is diametrically Ramsey, extending a result of Protasov and Protasova [1].

1. Introduction

Ramsey Theory explores the underlying structure emerging in "large enough" complex systems. For example, Frank Ramsey proved in [2] that for each $k \in \mathbb{N}$ there is a sufficiently large $n \in \mathbb{N}$ such that in any red-blue coloring of the edges of the complete graph K_n there is a set of k vertices joined by edges of the same color.

Another seminal result is Van der Waerden's theorem, which states that for all positive integers $r, k \in \mathbb{N}$ there is a large enough $n \in \mathbb{N}$ such that if we color the integers in $[n] := \{1, 2, \ldots, n\}$ using k colors, one can always find a set of r monochromatic integers in arithmetic progression [3].

Motivated by these classical results, we study the structural properties of infinite spaces using a Ramsey-theoretic lens. In particular, we will color a class of subsets of the space and search for a set whose subsets in this class are all the same color. We formalize this as follows.

Fix an infinite metric space (X, d) and let $k \in \mathbb{N}$ be a postive integer. For a family $\mathcal{A}$ of subsets of X, a *coloring* of $\mathcal{A}$ is any mapping $\chi : \mathcal{A} \to [k]$. We would like to find a set $M \subseteq X$ and a color $\varphi \in [k]$ such that $\chi(N) = \varphi$ for every subset $N \subseteq M$ with $N \in \mathcal{A}$. We then say that M is *monochrome* with respect to $\mathcal{A}$. A natural first choice is $\mathcal{A} = [X]^2$, the class of two-element subsets of X.

In this context, the "large" objects in X which we will work with are free ultrafilters. A *filter* $\mathcal{F}$ on X is a collection of subsets of X satisfying the following for all subsets $A, B \subseteq X$:

(1) If $A \in \mathcal{F}$ and $A \subseteq B$ then $B \in \mathcal{F}$;

(2) If $A, B \in \mathcal{F}$ then $A \cap B \in \mathcal{F}$;
(3) $\emptyset \notin \mathcal{F}$ and $X \in \mathcal{F}$.

A filter $\mathcal{F}$ is called an *ultrafilter* if it is not properly contained in another filter. We call $\mathcal{F}$ *free* if $\bigcap \mathcal{F} = \emptyset$. Intuitively, ultrafilters are maximal filters and free filters are "spread out" throughout the space, so we view *free ultrafilters* as "large" objects in X.

Given that these are the large objects in focus, it is natural to ask whether there is a free ultrafilter $\mathcal{F}$ such that for every coloring of $[X]^2$ there is a monochrome set $M \in \mathcal{F}$. It turns out that this question is undecidable in ZFC even with $X = \mathbb{N}$, though the statement is true if we accept the continuum hypothesis [1]. Consequently, we must define more restrictive classes of colorings.

Towards this end, Protasov and Protasova introduce the following theory in [1]. A coloring $\chi : [X]^2 \to [k]$ of the two-element subsets of X is called *isometric* if $\chi(\{x_1, y_1\}) = \chi(\{x_2, y_2\})$ whenever $d(x_1, y_1) = d(x_2, y_2)$. A free ultrafilter $\mathcal{F}$ is called *metrically Ramsey* if for every isometric coloring of $[X]^2$ there is a monochrome set $M \in \mathcal{F}$.

It turns out that in the particular case of ultrametric spaces, metrically Ramsey free ultrafilters are not too hard to construct. Recall that an *ultrametric* d on a set X is a metric satisfying the *ultrametric inequality*, which states that for all $x, y, z \in X$

$$d(x, y) \leq \max\{d(x, z), d(z, y)\}.$$

Protasov and Protasova leverage the properties of the ultrametric to prove the following theorem [1].

Theorem 1.1. *Every infinite ultrametric space X contains a sequence (x_n) such that any free ultrafilter $\mathcal{F}$ with $(x_n) \in \mathcal{F}$ is metrically Ramsey.*

As a follow up, one may ask if similar structure exists when coloring a larger class of subsets of X. The positive answer to this question is the keynote of this paper. In this connection, we will analyze the family Γ_X of all compact subsets of X.

We generalize isometric colorings along these lines. The map $\chi : \Gamma_X \to [k]$ is called a *diametric coloring* if $\chi(A_1) = \chi(A_2)$ for every pair A_1, A_2 of compact subsets of X with $\operatorname{diam} A_1 = \operatorname{diam} A_2$. Accordingly, a set $M \subseteq X$ is monochrome if its compact subsets are the same color.

Given this, we say that a free ultrafilter $\mathcal{F}$ on X is *diametrically Ramsey* if for every diametric coloring χ there is a monochrome set $M \in \mathcal{F}$. Since finite sets are compact, Γ_X contains $[X]^2$ so that every diametric coloring is isometric.

In this context, our main result is the following.

Theorem 1.2. *Every infinite ultrametric space X contains a sequence (x_n) such that every free ultrafilter $\mathcal{F}$ with $(x_n) \in \mathcal{F}$ is diametrically Ramsey.*

Building towards the main result, the following two sections review some elementary properties of ultrametric spaces and filters. We then prove Theorem 1.2 in Section 4.

2. Ultrametric Analysis

In this brief section, we introduce some notable examples of ultrametric spaces and survey some of their fundamental properties.

2.1. **Examples: The space $\mathbb{N}^\mathbb{N}$, graphs, and ε-chains.** The simplest example of an ultrametric on a set X is the discrete metric d, where $d(x, y)$ is 1 if $x \neq y$ and 0 otherwise. A slightly more complicated ultrametric space is $(\mathbb{N}, d)$, where $d(n, m) = \max\{1 + 1/n, 1 + 1/m\}$ if $n \neq m$ and $d(n, m) = 0$ otherwise.

However, there are much more interesting constructions. These include ultrametrics on the Baire space $\mathbb{N}^\mathbb{N}$, connected graphs, and uniformly disconnected metric spaces, which we will now construct.

2.1.1. *The Baire space.* We will first discuss the *Baire space* $\mathbb{N}^\mathbb{N}$, which is the space of all sequences of natural numbers.

For two distinct sequences $x = (x_n)$, $y = (y_n)$ in $\mathbb{N}^\mathbb{N}$, we define $m(x, y) = \min\{k \in \mathbb{N} : x_k \neq y_k\}$ to be the first index at which x and y do not coincide. Set $d(x, y) = m(x, y)^{-1}$ with $d(x, x) = 0$. We claim that d is an ultrametric on the Baire space.

Proof. The symmetry and non-negativity of d are immediate. Furthermore, $d(x, x) = 0$ by definition, and if $d(x, y) = 0$ then $x = y$ since $m(x, y)^{-1} > 0$ for distinct x, y. To prove the ultrametric inequality for d, fix $x, y, z \in \mathbb{N}^\mathbb{N}$. Assume x, y, z are distinct sequences, otherwise the inequality is clear. Observe that

$$\begin{aligned}
&d(x, y) \leq \max\{d(x, z), d(z, y)\} \\
&\Leftrightarrow m(x, y)^{-1} \leq \max\{m(x, z)^{-1}, m(z, y)^{-1}\} \\
&\Leftrightarrow m(x, y)^{-1} \leq \min\{m(x, z), m(z, y)\}^{-1} \\
&\Leftrightarrow m(x, y) \geq \min\{m(x, z), m(z, y)\}.
\end{aligned}$$

Clearly $m(x, y) \geq \min\{m(x, z), m(z, y)\}$. Otherwise, $m(x, y) < m(x, z)$ and $m(x, y) < m(z, y)$. Letting $\ell = m(x, y)$, we see that $x_\ell \neq y_\ell$, but since both $m(x, z), m(z, y) \geq \ell + 1$, we have $x_\ell = z_\ell = y_\ell$, a contradiction. Therefore, d is an ultrametric. $\square$

2.1.2. *Graphs.* The following construction is inspired by Leclerc's elegant work in [4]. Let G be a connected graph with positive edge-weights. For an edge e in G, let $w(e)$ denote its weight.

A *walk* in G is a finite sequence of adjacent vertices. Given a walk x, we will denote by e_x an edge in the walk with maximum weight. We say that a walk x between two vertices u, v is a *minimax walk* if there is no other walk x' between u, v whose max-weight edge is lighter than e_x. Equivalently, $w(e_x) \leq w(e_{x'})$ for every walk x' between u, v.

If we think of an edge's weight as the difficulty level of traveling from one of its ends to the other, the minimax walk from u to v minimizes the most challenging part of the journey.

We can use this notion to define an ultrametric on $V(G)$. Specifically, given two distinct vertices $u, v \in V(G)$ with minimax walk x, we set $d(u, v) = w(e_x)$, with $d(u, u) = 0$. The proof that d is an ultrametric is as follows.

Proof. That d is symmetric and non-negative is clear. By definition, $d(u, u) = 0$ and if $d(u, v) = 0$ then $u = v$, as otherwise there is an edge in G with weight 0 even though its edges have only positive weights.

To prove the ultrametric inequality, fix $u, v, w \in V(G)$. We may assume that u, v, w are distinct, else the inequality immediately follows. Let $x_{u,w} = (v_1, v_2, \ldots, v_m)$ and $x_{w,v} = (v_m, v_{m+1}, \ldots, v_n)$ denote minimax walks between u, w and w, v respectively, and let $x = (v_1, v_2, \ldots, v_n)$ be their union. Then the max-weight edge in x has weight $\max\{d(u, w), d(w, v)\}$. Since x is a walk between u, v, we have $d(u, v) \leq \max\{d(u, w), d(w, v)\}$, as claimed. $\square$

2.1.3. *Uniformly disconnected spaces.* The theory below is based on Guy and Semmes' work in [5] and Heinonen's construction in [6].

Let (X, d) be any metric space and $\varepsilon > 0$ be given. An *ε-chain* between the pair $x, y \in X$ is a finite sequence $x = x_0, x_1, \ldots, x_n = y$ with

$$\max_{1 \leq i \leq n} d(x_{i-1}, x_i) \leq \varepsilon \cdot d(x, y).$$

In this case, we say that x, y are *ε-connected*. Observe that if x, y are ε-connected then they are ε'-connected for every $\varepsilon' \geq \varepsilon$; and if x, y can not be ε-connected then they can not be ε'-connected for every $\varepsilon' \leq \varepsilon$.

The space X is called *uniformly disconnected* if there is an $\varepsilon > 0$ such that no two points in X can be ε-connected. It is not hard to prove that uniform disconnectivity is stronger than total disconnectivity. To give some intuition, we note that the middle thirds Cantor set is uniformly disconnected and the set $\{1/n : n \in \mathbb{N}\}$ is not [6].

Let (X,d) be a uniformly disconnected metric space. For $x,y\in X$, let $c(x,y)$ be the infimum over all $\varepsilon>0$ such that x and y are $d(x,y)^{-1}\cdot\varepsilon$-connected. Then c is an ultrametric on X.

Proof. Clearly c is symmetric and non-negative. If $x=y$ then x,y are ε-connected for every $\varepsilon>0$ so that $c(x,y)=0$. On the other hand, if $c(x,y)=0$ then $x=y$, otherwise x,y are ε-connected for every $\varepsilon>0$, contradicting the uniform disconnectivity of X.

The last thing to prove is the ultrametric inequality for c. To this end, fix $x,y,z\in X$ and let $\varepsilon>0$ be given. We may assume the points x,y,z are distinct, otherwise the claim is immediate. By definition of the infimum, there exist $\gamma_1,\gamma_2>0$ with $\gamma_1\le c(x,z)+\varepsilon$ and $\gamma_2\le c(z,y)+\varepsilon$ such that x,z are $d(x,z)^{-1}\cdot\gamma_1$-connected and z,y are $d(z,y)^{-1}\cdot\gamma_2$-connected. Hence we obtain sequences

- $x=x_0,x_1,\dots x_m=z$ with $\max_{1\le i\le m} d(x_{i-1},x_i)\le\gamma_1$; and
- $z=x_m,x_{m+1},\dots,x_n=y$ with $\max_{m+1\le i\le n} d(x_{i-1},x_i)\le\gamma_2$.

Now set $\gamma=\max\{\gamma_1,\gamma_2\}$ and observe that x,y are γ-connected since

$$\max_{1\le i\le n} d(x_{i-1},x_i)\le\gamma=\frac{\gamma}{d(x,y)}\cdot d(x,y).$$

Then

$$\begin{aligned}c(x,y)\le\gamma&=\max\{\gamma_1,\gamma_2\}\\&\le\max\{c(x,z)+\varepsilon,c(z,y)+\varepsilon\}\\&=\max\{c(x,z),c(z,y)\}+\varepsilon,\end{aligned}$$

and sending $\varepsilon\to0$ yields $c(x,y)\le\max\{c(x,z),c(z,y)\}$. □

2.2. **Properties of ultrametric spaces.** The ultrametric inequality is much stronger than the usual triangle inequality. Because of this, ultrametric spaces have some interesting properties which we explore now (see [7] for a comprehensive overview).

First, it turns out that every triangle is isosceles.

Lemma 2.1. *If x,y,z are distinct points in an ultrametric space X and $d(x,z)<d(z,y)$ then $d(x,y)=d(z,y)$.*

Proof. Since $d(x,z)<d(z,y)$, the ultrametric inequality implies

$$d(x,y)\le\max\{d(x,z),d(z,y)\}=d(z,y).$$

Furthermore, $d(x,z)<d(x,y)$ else

$$d(z,y)\le\max\{d(x,z),d(x,y)\}=d(x,z)$$

is a contradiction. Thus, $d(z,y)\le\max\{d(z,x),d(x,y)\}=d(x,y)$. Combining everything together, we conclude that $d(x,y)=d(z,y)$. □

We now examine open balls in ultrametric spaces. An *open ball* (or simply a ball) of radius $r > 0$ centered about $x \in X$ is the set

$$B_r(x) = \{y \in X : d(x, y) < r\}.$$

A subset $\mathcal{O}$ of X is called *open* if it can be written as a union of open balls. A set is *closed* if its complement is open.

Open balls in ultrametric spaces have vastly unintuitive properties. For example, every point in a ball is its center.

Lemma 2.2. *Let $B_r(x)$ be an open ball in X. Then $B_r(x) = B_r(y)$ for every point $y \in B_r(x)$.*

Proof. Let y be any point in $B_r(x)$. So $d(x, y) < r$, and if $t \in B_r(y)$ then $d(y, t) < r$. Then $t \in B_r(x)$ since $d(x, t) \leq \max\{d(x, y), d(y, t)\} < r$. Hence $B_r(y) \subseteq B_r(x)$, and the reverse inclusion follows symmetrically as $x \in B_r(y)$. □

Another important result is that if two balls intersect then one of them contains the other. More generally, we have the following lemma.

Lemma 2.3. *Let $B_r(x)$ and $B_q(y)$ be a pair of intersecting balls in X with $r \leq q$. Then $B_r(x) \subseteq B_q(y)$.*

Proof. Since the two balls intersect, there is a point $t \in B_r(x) \cap B_q(y)$. Thus, Lemma 2.2 implies that $B_r(t) = B_r(x)$ and $B_q(t) = B_q(y)$, and since $r \leq q$ we obtain $B_r(x) = B_r(t) \subseteq B_q(t) = B_q(y)$, as required. □

A generalization of Lemma 2.3 is as follows.

Lemma 2.4. *Let $A \subseteq X$ and consider any ball $B_r(x)$ in X. If $B_r(x)$ intersects A then $A \cap B_r(x)$ is a ball in the space (A, d).*

Proof. Since $A \cap B_r(x) \neq \emptyset$, there is a point $t \in A \cap B_r(x)$. Let $B_A = \{a \in A : d(t, a) < r\}$ be a ball in A. We show that $A \cap B_r(x) = B_A$.

Note from Lemma 2.2 that t is the center of $B_r(x)$ so that $A \cap B_r(x) = A \cap B_r(t)$. Then if $a \in A \cap B_r(x)$ we have $a \in A \cap B_r(t)$ so that $d(t, a) < r$ and hence $a \in B_A$. On the other hand, if $a \in B_A$ then from the ultrametric inequality we obtain

$$d(x, a) \leq \max\{d(x, t), d(t, a)\} < r,$$

since $t \in B_r(x)$. Hence $a \in A \cap B_r(x)$ and we're done. □

Furthermore, it turns out that all open balls are also closed in X.

Lemma 2.5. *If $B_r(x)$ is an open ball in X then $X \setminus B_r(x)$ is open. In particular, $B_r(x)$ is closed.*

Proof. Assume by contradiction that $X \setminus B_r(x)$ is not open. So there is a point $t \in X \setminus B_r(x)$ which is not an interior point of $X \setminus B_r(x)$. That is, for every $q > 0$ the ball $B_q(t)$ is not contained in $X \setminus B_r(x)$, meaning $B_q(t) \cap B_r(x) \neq \emptyset$. In particular, if $q = r$ one deduces from Lemma 2.3 that $B_r(x) = B_q(t)$. But then $t \in X \setminus B_r(x) = X \setminus B_q(t)$ is a contradiction. Therefore, $X \setminus B_r(x)$ is a open, so $B_r(x)$ is closed. $\square$

Our final application of the ultrametric inequality is to the diameter of subsets of the space.

Lemma 2.6. *Let $A \subseteq X$ be non-empty with $a \in A$. Then* $\operatorname{diam} A = \sup\{d(a,x) : x \in A\}$.

Proof. Set $u = \sup\{d(a,x) : x \in A\}$ and fix $x, y \in A$. Then

$$d(x,y) \leq \max\{d(a,x), d(a,y)\} \leq u$$

so that u is an upper bound of $D = \{d(x,y) : x, y \in A\}$. Since $u = \sup\{d(a,x) : x \in A\}$, for any given $\varepsilon > 0$ there is a point $x_\varepsilon \in A$ with $u \leq d(a, x_\varepsilon) + \varepsilon$. But $a, x_\varepsilon \in A$ so that $u = \sup D = \operatorname{diam} A$ which completes the proof. $\square$

3. Filters and Ultrafilters

In this short section, we introduce a sequence of lemmas building up to Lemma 3.3, which is fundamental to the proof of the main result. We then conclude the section with Lemma 3.4, which is yet another elegant application of Lemma 3.3. The development of these lemmas was informed by concepts discussed in the notes of Koppelberg [8] and the work of Brian [9].

Recall from the introduction that a *filter* $\mathcal{F}$ on X is a family of subsets of X with $\emptyset \notin \mathcal{F}$, $X \in \mathcal{F}$, and which is closed under the superset inclusion and finite intersections. A filter $\mathcal{F}$ is an *ultrafilter* if no filter properly contains it, and we call $\mathcal{F}$ *free* if $\bigcap \mathcal{F} = \emptyset$.

The first lemma is the following.

Lemma 3.1. *If $\mathcal{A}$ is a non-empty family of subsets of X such that any intersection of finitely many sets in $\mathcal{A}$ is non-empty, then there is a filter $\mathcal{F}$ which contains $\mathcal{A}$.*

Proof. First let $\mathcal{F}' = \mathcal{A} \cup \mathcal{I}$, where $\mathcal{I}$ is the set of all finite intersections of elements of $\mathcal{A}$. Then $\mathcal{F}'$ is closed under finite intersections. Then let $\mathcal{F} = \mathcal{F}' \cup \mathcal{S}$, where $A \in \mathcal{S}$ if and only if A contains a set in $\mathcal{A} \cup \mathcal{I}$.

We first note that $\emptyset \notin \mathcal{F}$ since the intersection of any finite subcollection of $\mathcal{A}$ is non-empty. Likewise, $X \in \mathcal{F}$ by the non-emptyness of $\mathcal{A}$ and the construction of $\mathcal{S}$. Clearly $\mathcal{F}$ is closed under the superset inclusion. It remains to show that if $A, B \in \mathcal{F}$ then $A \cap B \in \mathcal{F}$.

If $A, B \in \mathcal{F}$, the only non-trivial case needing consideration is, without loss of generality, when $A \in \mathcal{S}$. Then there exist sets $A', B' \in \mathcal{F}'$ with $A' \subseteq A$ and $B' \subseteq B$ (if $B \in \mathcal{F}'$ then $B' = B$). Since $\mathcal{F}'$ is closed under finite intersections, $A' \cap B' \in \mathcal{F}'$. Then, from $A' \cap B' \subseteq A \cap B$ it follows that $A \cap B \in \mathcal{S} \subseteq \mathcal{F}$. Hence $\mathcal{F}$ is closed under finite intersections, as needed. $\square$

Lemma 3.2. *A filter $\mathcal{F}$ is an ultrafilter if and only if for every subset $A \subseteq X$ either $A \in \mathcal{F}$ or $A^c \in \mathcal{F}$.*

Proof. For "$\Rightarrow$", let $\mathcal{F}$ be an ultrafilter and suppose for a contradiction that there is a subset $A \subseteq X$ with $A \notin \mathcal{F}$ and $A^c \notin \mathcal{F}$. So every set in $\mathcal{F}$ intersects both A and A^c. Using this and the fact that $\mathcal{F}$ is a filter, it follows that any finite sub-collection of $\mathcal{F} \cup \{A\}$ has a non-empty intersection. Then, Lemma 3.1 implies that there is a filter containing $\mathcal{F} \cup \{A\}$ and hence $\mathcal{F}$, which contradicts our choice of $\mathcal{F}$.

Now suppose for "$\Leftarrow$" that $\mathcal{F}$ is a filter such that for every subset $A \subseteq X$ either $A \in \mathcal{F}$ or $A^c \in \mathcal{F}$. Assume for a contradiction that $\mathcal{F}'$ is a filter which properly contains $\mathcal{F}$. So there is a subset $E \subseteq X$ with $E \in \mathcal{F}'$ and $E \notin \mathcal{F}$. Thus $E^c \in \mathcal{F}$ and hence $E^c \in \mathcal{F}'$. But then $\mathcal{F}'$ contains the empty set since it is closed under finite intersections and $\emptyset = E \cap E^c \in \mathcal{F}'$, which is a contradiction. $\square$

We will say that a filter $\mathcal{F}$ has the *Ramsey property* if whenever $\cup_{j=1}^{n} A_j \in \mathcal{F}$ there is a $j \in [n]$ with $A_j \in \mathcal{F}$. In this connection, we have the following surprising result.

Lemma 3.3. *A filter $\mathcal{F}$ is an ultrafilter if and only if $\mathcal{F}$ has the Ramsey property.*

Proof. Let $\mathcal{F}$ be an ultrafilter and suppose for a contradiction that $A = A_1 \cup A_2$ is in $\mathcal{F}$ but $A_1, A_2 \notin \mathcal{F}$. From Lemma 3.2, we have $A_1^c, A_2^c \in \mathcal{F}$ so that $A^c = A_1^c \cap A_2^c \in \mathcal{F}$. Consequently, $\emptyset = A \cap A^c \in \mathcal{F}$ is a contradiction. The case where $A = A_1 \cup \cdots \cup A_n$ follows via elementary induction. Conversely, assume $\mathcal{F}$ has the Ramsey property. If $\mathcal{F}$ is not an ultrafilter, then there is a subset $A \subseteq X$ with $A \notin \mathcal{F}$ and $A^c \notin \mathcal{F}$. But $X \in \mathcal{F}$ and $X = A \cup A^c$, so we must have $A \in \mathcal{F}$ or $A^c \in \mathcal{F}$, contradicting our choice of A. $\square$

For a family $\mathcal{F}$ of subsets of X, we define its *dual* $\mathcal{F}^*$ to be the collection of all subsets of X which intersect every set in $\mathcal{F}$.

A simple example of families and their duals is as follows [9]. We call a subset $A \subseteq \mathbb{N}$ *thick* if it contains intervals of arbitrary lengths; and we call A *syndetic* if the space between its intervals is bounded. That is, A is syndetic if there is an integer $N \in \mathbb{N}$ such that every interval of length N contains a point in A. Then, the dual of the family of syndetic subsets of $\mathbb{N}$ is the class of thick sets in $\mathbb{N}$.

The following result, due to Glasner, relates ultrafilters to their duals [10]. We provide its proof here for clarity.

Lemma 3.4. *A filter $\mathcal{F}$ is an ultrafilter if and only if $\mathcal{F}^*$ is a filter.*

Proof. For "$\Rightarrow$", assume that $\mathcal{F}$ is an ultrafilter, and fix $A, B \in \mathcal{F}^*$. To prove that $A \cap B \in \mathcal{F}^*$, it suffices to fix a set $E \in \mathcal{F}$ and prove that E intersects $A \cap B$. Note that since $A \in \mathcal{F}^*$ we have $E \cap A \neq \emptyset$. Hence we may write

$$E = (E \cap A) \cup (E \setminus A).$$

Certainly $E \setminus A \notin \mathcal{F}$, as otherwise $A \in \mathcal{F}^*$ implies $(E \setminus A) \cap A \neq \emptyset$. Since $\mathcal{F}$ is an ultrafilter, Lemma 3.3 implies that $\mathcal{F}$ has the Ramsey property so that $E \cap A \in \mathcal{F}$. Since $B \in \mathcal{F}^*$, it follows as needed that $E \cap A \cap B \neq \emptyset$.

Conversely, for "$\Leftarrow$", let $\mathcal{F}$ and $\mathcal{F}^*$ be filters. Suppose towards a contradiction that $\mathcal{F}$ is not an ultrafilter. Then Lemma 3.2 implies that there is a subset $A \subseteq X$ with $A \notin \mathcal{F}$ and $A^c \notin \mathcal{F}$. Thus, every set in $\mathcal{F}$ intersects both A and A^c. That is, for every $B \in \mathcal{F}$ we have $A \cap B \neq \emptyset$ and $A^c \cap B \neq \emptyset$. By definition, then, $A \in \mathcal{F}^*$ and $A^c \in \mathcal{F}^*$. But $\mathcal{F}^*$ is a filter, so $\emptyset = A \cap A^c \in \mathcal{F}^*$ is a contradiction, and the proof is complete. $\square$

4. Main Result

4.1. **A technical lemma.** Protasov and Protasova introduce the following lemma, contructing a sequence (x_n) with a unique property [1]. We use the existence of this sequence to prove the main result. Hence, for the sake of completeness, we start this section with its statement and proof. We conclude the section with a proof of Theorem 1.2.

Lemma 4.1. *Let (X, d) be an infinite metric space. Then there is a sequence $(x_n)_{n=0}^{\infty}$ of distinct points in X such that either*

(1) The sequence $(d(x_0, x_n))_{n=1}^{\infty}$ is strictly monotone; or
(2) For every $n \geq 0$ and $i, j \geq n$ the distances $d(x_n, x_i) = d(x_n, x_j)$.

Proof. For a point $x \in X$ and a subset $A \subseteq X$, define

$$D(x, A) = \{d(x, y) : y \in A\}.$$

We first assume that there is a point $x_0 \in X$ such that $D(x_0, X)$ is infinite. Hence, there is a countably infinite subset $A \subseteq X$ with $x_0 \notin A$ and $d(x_0, x) \neq d(x_0, y)$ for every $x, y \in A$. Define the sequence $\xi = \{d(x_0, x) : x \in A\}$, and note that since ξ is a sequence of reals with distinct points, it has a strictly monotone subsequence $(d(x_0, x_n))_{n=1}^{\infty}$ as required.

Otherwise, $D(x, X)$ is finite for every $x \in X$. Write $A_0 = X$ and fix a point $x_0 \in A_0$. Then there is a distance $\ell_1 \in D(x_0, A_0)$ such that $A_1 = \{y \in A_0 : d(x_0, y) = \ell_1\}$ is not finite. Indeed, otherwise one may write $D(x_0, A_0) = \{\ell_1, \ell_2, \ldots, \ell_m\}$ and note that

$$X = A_0 = \bigcup_{j=1}^{m} \{y \in X : d(x_0, y) = \ell_j\}$$

is finite, which contradicts our choice of X. Observe that $D(x_0, A_1) = \{\ell_1\}$ and $x_0 \notin A_1$ since otherwise $\ell_1 = 0$ so that $|A_1| = 1 < \infty$.

For each $n \geq 2$, we choose x_n and define A_n inductively as follows. By the same reasoning as above, there is a distance $\ell_n \in D(x_{n-1}, A_{n-1})$ such that $A_n = \{y \in A_{n-1} : d(x_{n-1}, y) = \ell_n\}$ is not finite (otherwise $|A_{n-1}| < \infty$ contradicts the inductive hypothesis), and we have $x_{n-1} \notin A_n$ since $\ell_n \neq 0$. Note also that $D(x_{n-1}, A_n) = \{\ell_n\}$. By induction, $A_{n-1} \subsetneq A_n$ for every $n \in \mathbb{N}$, and so (x_n) satisfies (2). □

4.2. **Main result.** Recall that the map $\chi : \Gamma_X \to [k]$ is called a *diametric coloring* if $\chi(A_1) = \chi(A_2)$ for every pair A_1, A_2 of compact subsets of X with $\operatorname{diam} A_1 = \operatorname{diam} A_2$. A subset M of X is called *monochrome* if its compact subsets receive the same color; that is, there is a color $\varphi \in [k]$ such that $\chi(\Gamma_M) = \{\varphi\}$. A free ultrafilter $\mathcal{F}$ on X is called *diametrically Ramsey* if for every diametric coloring χ there is a monochrome set $M \in \mathcal{F}$.

We are ready to prove Theorem 1.2. Its statement is the following: *Every infinite ultrametric space X contains a sequence (x_n) such that every free ultrafilter $\mathcal{F}$ with $(x_n) \in \mathcal{F}$ is diametrically Ramsey.*

Proof of Theorem 1.2. Let χ be any diametric coloring on X and consider any free ultrafilter $\mathcal{F}$ containing the sequence $(x_n)_{n=0}^{\infty}$ as obtained in Lemma 4.1.

Let $h : (x_n) \to \mathbb{R}^+$ be a fixed map. How we define h depends on (x_n), so we do this later on. Moreover, let $f : \mathbb{R}^+ \to [k]$ be a mapping satisfying $f(h(x_n)) = \chi(A)$ whenever $A \in \Gamma_X$ is a compact subset of X with $\operatorname{diam} A = h(x_n)$. Note that f is well-defined since χ is diametric. Finally, we set $c = f \circ h$.

Write $(x_n) = c^{-1}([k]) = \bigcup_{j=1}^{k} c^{-1}(\{j\})$ and note that since $(x_n) \in \mathcal{F}$, there is a color $\varphi \in [k]$ such that $c^{-1}(\{\varphi\}) \in \mathcal{F}$. Indeed, $\mathcal{F}$ is an ultrafilter so Lemma 3.3 implies that $\mathcal{F}$ has the Ramsey property.

We set $M = c^{-1}(\{\varphi\})$ and complete the proof by showing that M is monochrome. Specifically, we show that $\chi(\Gamma_M) = \{\varphi\}$, meaning that $\chi(N) = \varphi$ for every compact subset N of M.

Fix $N \in \Gamma_M$ and note that since N is a compact subset of M there is a pair of points $x_i, x_j \in N$ with $i < j$ and $d(x_i, x_j) = \operatorname{diam} N$. We now consider the conditions on (x_n) as described in the cases of Lemma 4.1 and define h accordingly to complete the proof.

Case 1. We will first assume that case (1) of Lemma 4.1 holds, namely that (x_n) is a sequence of distinct points in X where the sequence $(d(x_0, x_n))_{n=1}^{\infty}$ is strictly monotone. In this case, h will indicate the distance between x_0 and a term x_n, given by $h(x_n) = d(x_0, x_n)$.

If h is strictly increasing, then we have $d(x_0, x_i) < d(x_0, x_j)$. Hence Lemma 2.1 implies that $d(x_i, x_j) = d(x_0, x_j)$, since d is an ultrametric. Otherwise $d(x_0, x_i) > d(x_0, x_j)$ so that $d(x_i, x_j) = d(x_0, x_i)$ using Lemma 2.1 once more. Possibly swapping the symbols i, j, we may assume that the former case holds. Since $x_j \in M$, $\operatorname{diam} N = d(x_i, x_j)$, and using the construction of f, we have

$$\varphi = c(x_j) = f(d(x_0, x_j)) = f(d(x_i, x_j)) = \chi(N),$$

so N has color φ as needed.

Case 2. We now assume that case (2) of Lemma 4.1 applies to (x_n), so for each $n \in \mathbb{N}$ and $i, j \geq n$ we have $d(x_n, x_i) = d(x_n, x_j)$. Define $h : (x_n) \to \mathbb{R}^+$ by $h(x_n) = d(x_n, x_{n+1})$, and note that $h(x_n) = d(x_n, x_j)$ for every $j > n$. Using the same reasoning as above, N receives color φ since

$$\varphi = c(x_i) = f(d(x_i, x_{i+1})) = f(d(x_i, x_j)) = \chi(N).$$

This completes the proof, since M is monochrome in both cases. $\square$

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