Diametric Colorings in Ultrametric Spaces

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Abstract. Let Γ_X denote the family of compact subsets of (X, d). A coloring $\chi : \Gamma_X \to [k]$ is *diametric* if every pair of compact subsets with equal diameters receive the same color. A free ultrafilter $\mathcal{F}$ is called *diametrically Ramsey* if every diametric coloring admits a set $A \in \mathcal{F}$ whose compact subsets are monochrome. We show that every infinite ultrametric space contains a sequence (x_n) so that every free ultrafilter containing (x_n) is diametrically Ramsey, extending a result of Prostatov and Prostatova [1].

1. Introduction

Ramsey Theory explores the underlying structure emerging in "large enough" complex systems. For example, Frank Ramsey [1] proved that for each $k \in \mathbb{N}$ there is a sufficiently large $n \in \mathbb{N}$ such that in any red-blue coloring of the edges of the complete graph K_n there is a set of k vertices joined by edges of the same color.

Another seminal result is Van der Waerden's theorem, which states that for all positive integers $r, k \in \mathbb{N}$, there is a large enough $n \in \mathbb{N}$ such that if we color the integers in $[n] := \{1, 2, \ldots, n\}$ using k colors, one can always find a set of r monochromatic integers in arithmetic progression [2].

Motivated by these kinds of classical results, we study the structural properties of infinite spaces using a Ramsey-theoretic lens. In particular, we will color the compact subsets of the space and search for an infinite set whose compact subsets receive the same color. We formalize this shortly, but we need to introduce some terminology first.

Fix an infinite metric space (X, d) and let $k \in \mathbb{N}$ be a postive integer.

This raises a natural question: given $n, k \in \mathbb{N}$, is there an integer $N \in \mathbb{N}$ such that in any edge-coloring of K_N in k colors there is a monochromatic clique of size n? Its positive answer is due to (authors) [2]. Analogously, in what follows, we generalize and strengthen a Ramsey-type coloring theorem of Protasov and Protasova [3] to hold for k colors, where $k \in \mathbb{N}$.

Fix an infinite metric space (X, d) and let $k \in \mathbb{N}$. A *k-coloring* on X is a map $\chi : [X]^k \to [k]$ which assigns one of k colors to each k-element subset of X. For a given k-coloring χ, we would like to find a set $A \subseteq X$ such that $\chi([A]^k) = \{c\}$ for some $c \in [k]$; in this case, we call $[A]^k$ *χ-monochrome.*

In this context, the "large" objects containing underlying structure in (X, d) are free ultrafilters [4]. A *filter* $\mathcal{F}$ on X is a collection of subsets of X satisfying the following for all sets $A, B \subseteq X$:

(1) $\emptyset \notin \mathcal{F}$ and $X \in \mathcal{F}$;
(2) If $A \in \mathcal{F}$ and $A \subseteq B$ then $B \in \mathcal{F}$;
(3) If $A, B \in \mathcal{F}$ then $A \cap B \in \mathcal{F}$.

A filter $\mathcal{F}$ is called an *ultrafilter* if it is not properly contained in a filter on X. A filter $\mathcal{F}$ is called *free* if $\bigcap \mathcal{F} = \emptyset$. Free filters are "spread out" throughout the space and ultrafilters are maximal filters, so we consider *free ultrafilters* as "large" objects in X.

Given this, one naturally asks if there is a free ultrafilter $\mathcal{F}$ such that for every coloring χ there is a set $A \in \mathcal{F}$ so that $[A]^k$ is χ-monochrome. It turns out that this question is undecidable in ZFC even with $X = \mathbb{N}$ and $d(x, y) = |x - y|$, though the statement is true if we declare the continuum hypothesis as axiomatic [5]. Hence we must define more restrictive classes of colorings.

To this end, we say that a k-coloring $\chi : [X]^k \to [k]$ is an *isometric k-coloring* if $\chi(A_1) = \chi(A_2)$ whenever A_1, A_2 is a pair of isodiametric k-element subsets of X. A free ultrafilter $\mathcal{F}$ is called *metrically Ramsey* if for every isometric k-coloring χ there is a set $A \in \mathcal{F}$ such that $[A]^k$ is χ-monochrome.

Recall that an *ultrametric* d on a set X is a metric satisfying the *strong triangle inequality:* for all $x, y, z \in X$

$$d(x, y) \leq \max\{d(x, z), d(z, y)\}.$$

It turns out that in the particular case of ultrametric spaces, metrically Ramsey free ultrafilters are not too hard to construct. Indeed, Protasov and Protasova [6] prove that for $k = 2$ every infinite ultrametric space X contains a sequence (x_n) such that every free ultrafilter containing (x_n) is metrically Ramsey. The authors leverage the properties of the ultrametric coupled with (lemma) (to construct (x_n)) to prove the main result when $k = 2$. We expand on this approach, strengthening their result to hold for all k-colorings.

In the following two sections, we review some elementary properties of ultrametrics and filters. We then prove the main result in (section 4).

2. Ultrametric Analysis

2.1. **The space $\mathbb{N}^{\mathbb{N}}$ and ε-chains.** In this short section, we provide some examples of ultrametric spaces. In particular, we construct ultrametrics on the Baire space $\mathbb{N}^{\mathbb{N}}$ and general uniformly disconnected metric spaces.

The simplest example of an ultrametric on a set X is the discrete metric d, where $d(x, y)$ is 1 if $x \neq y$ and 0 otherwise. Another simple example is $(\mathbb{N}, d)$, where $d(n, m) = \max\{1 + 1/n, 1 + 1/m\}$ if $n \neq m$ and $d(n, m) = 0$ otherwise.

The *Baire space* $\mathbb{N}^{\mathbb{N}}$ is the space of all sequences of natural numbers.

Example 1. For two distinct sequences $x = (x_n)$, $y = (y_n)$ of naturals, we define $m(x, y) = \min\{k \in \mathbb{N} : x_k \neq y_k\}$, the first index at which x and y do not coincide. For distinct sequences $x, y \in \mathbb{N}^{\mathbb{N}}$, set $d(x, y) = m(x, y)^{-1}$ with $d(x, x) = 0$. Then d is an ultrametric on $\mathbb{N}^{\mathbb{N}}$.

Proof. The symmetry and non-negativity of d is immediate. Furthermore, $d(x, x) = 0$ by definition, and if $d(x, y) = 0$ then we must have $x = y$ since $m(x, y) \geq 1$. To prove the strong triangle inequality for d, fix $x, y, z \in \mathbb{N}^{\mathbb{N}}$. We assume x, y, z are all distinct sequences as the result is immediate otherwise. Observe that

$$\begin{aligned}
&d(x, y) \leq \max\{d(x, z), d(z, y)\} \\
&\Leftrightarrow m(x, y)^{-1} \leq \max\{m(x, z)^{-1}, m(z, y)^{-1}\} \\
&\Leftrightarrow m(x, y)^{-1} \leq \min\{m(x, z), m(z, y)\}^{-1} \\
&\Leftrightarrow m(x, y) \geq \min\{m(x, z), m(z, y)\}.
\end{aligned}$$

Clearly $m(x, y) \geq \min\{m(x, z), m(z, y)\}$: if $\ell = m(x, y)$ then for each $k \in \{1, 2, \ldots, \ell - 1\}$ the terms $x_k = y_k$, so if $m(x, z) \geq \ell + 1$ and $m(z, y) \geq \ell + 1$ then $x_\ell = z_\ell = y_\ell$ so that $m(x, y) \geq \ell + 1$ is a contradiction. Therefore, d is an ultrametric. $\square$

Let (X, d) be any metric space and $\varepsilon > 0$ be given. An *ε-chain* between the pair $x, y \in X$ is a finite sequence $x = x_0, x_1, \ldots, x_n = y$ such that

$$\max_{1 \leq i \leq n} d(x_{i-1}, x_i) \leq \varepsilon.$$

In this case, we say that x, y are ε-connected. Observe that if x, y are ε-connected then they are ε'-connected for every $\varepsilon' \geq \varepsilon$; and if x, y can not be ε-connected then they can not be ε'-connected for every $\varepsilon' \leq \varepsilon$.

The space X is called *uniformly disconnected* if there is an $\varepsilon > 0$ such that no two points in X can be ε-connected.

Example 2. Let (X, d) be a uniformly disconnected metric space. For $x, y \in X$, let $c(x, y)$ be the infimum over all $\varepsilon > 0$ such that x and y are ε-connected. Then c is an ultrametric on X.

Proof. Clearly c is symmetric and non-negative. If $x = y$ then clearly x, y are ε-connected for every $\varepsilon > 0$ so that $c(x, y) = 0$. On the other hand, if $c(x, y) = 0$ then $x = y$, otherwise for each $\varepsilon > 0$ the points x, y are ε-connected, so X is not uniformly disconnected, a contradiction.

The last thing to prove is the strong triangle inequality for c. To this end, fix $x, y, z \in X$ and let $\varepsilon > 0$ be given. We may assume the points x, y, z are distinct, otherwise the claim is immediate. By definition of the infimum, there exist $\gamma_1, \gamma_2 > 0$ with $\gamma_1 \leq c(x, z) + \varepsilon$ and $\gamma_2 \leq c(z, y) + \varepsilon$ such that x, z are γ_1-connected and z, y are γ_2-connected. Set $\gamma = \max\{\gamma_1, \gamma_2\}$ and note that by combining the x, z and z, y chains it follows that x, y are γ-connected. Thus

$$c(x, y) \leq \gamma \leq \max\{c(x, z), c(z, y)\} + \varepsilon.$$

Since ε was arbitrary, it follows that $c(x, y) \leq \max\{c(x, z), c(z, y)\}$. □

2.2. **Properties of ultrametric spaces.**

Lemma 2.1. *If x, y, z are distinct points in an ultrametric space X and $d(x, y) < d(y, z)$ then $d(y, z) = d(x, z)$. (That is, every triangle in an ultrametric space is isosceles.)*

Proof. Note that d is an ultrametric and $d(x, y) < d(y, z)$ so that

$$d(x, z) \leq \max\{d(x, y), d(y, z)\} = d(y, z),$$

and since $d(y, z) > d(x, y)$, we have $d(x, y) < d(x, z)$ as otherwise

$$d(y, z) \leq \max\{d(y, x), d(x, z)\} = \max\{d(x, y), d(x, z)\} = d(x, y)$$

is a contradiction. Thus $d(y, z) \leq \max\{d(x, y), d(x, z)\} = d(x, z)$. Combining everything together, we obtain $d(y, z) = d(x, z)$. □

An *open ball* (or simply a *ball*) of radius $\varepsilon > 0$ centered about a point $x \in X$ is defined as the set $B_\varepsilon(x) := \{y \in X : d(x, y) < \varepsilon\}$. If left unspecified, note that x denotes a point in X and ε a positive real number.

A subset $\mathcal{O}$ of X is called *open* if it can be written as a union of open balls. A set is *closed* if its complement is open. Open balls and general sets in ultrametric spaces exhibit properties that are quite counterintuitive, as delineated by the following simple lemmas.

Lemma 2.2. *Let $B_\varepsilon(x)$ be an open ball in X. Then $B_\varepsilon(x) = B_\varepsilon(y)$ for every point $y \in B_\varepsilon(x)$. (That is, every point in a ball is its center.)*

Proof. Fix $y \in B_\varepsilon(x)$ so that $d(x, y) < \varepsilon$. If $t \in B_\varepsilon(y)$ then $d(y, t) < \varepsilon$. Then $t \in B_\varepsilon(x)$ since

$$d(x, t) \leq \max\{d(x, y), d(y, t)\} < \varepsilon.$$

The reverse inclusion follows symmetrically as $x \in B_\varepsilon(y)$. □

Lemma 2.3. *Let $A \subseteq X$ and consider any ball $B_\varepsilon(x)$ in X. If $B_\varepsilon(x)$ meets A then $A \cap B_\varepsilon(x)$ is a ball in the space (A, d).*

Proof. Since $A \cap B_\varepsilon(x) \neq \emptyset$, there is a point $a \in A \cap B_\varepsilon(x)$. Let $B_A = \{t \in A : d(t, a) < \varepsilon\}$ be a ball in A. We show that $A \cap B_\varepsilon(x) = B_A$.

Note from (lemma) that a is the center of $B_\varepsilon(x)$ so that $A \cap B_\varepsilon(x) = A \cap B_\varepsilon(a)$. Then if $t \in A \cap B_\varepsilon(x)$ we have $t \in A \cap B_\varepsilon(a)$ so that $d(t, a) < \varepsilon$ and hence $t \in B_A$. On the other hand, if $t \in B_A$ then from the strong triangle inequality we obtain

$$d(x, t) \leq \max\{d(x, a), d(a, t)\} < \varepsilon,$$

since $a \in B_\varepsilon(x)$. Hence, $t \in A \cap B_\varepsilon(x)$ completes the proof. □

Lemma 2.4. *Fix $\varepsilon_1, \varepsilon_2 > 0$ and $x, y \in X$. If $B_{\varepsilon_1}(x) \cap B_{\varepsilon_2}(y) \neq \emptyset$ then $B_{\varepsilon_1}(x) \subseteq B_{\varepsilon_2}(y)$ or $B_{\varepsilon_2}(y) \subseteq B_{\varepsilon_1}(x)$. (That is, two balls are either disjoint or one of them contains the other.)*

Proof. This is a weaker version of (obs), but we give the statement its own proof for clarity. Assume without loss of generality that $\varepsilon_1 \leq \varepsilon_2$. If $B_{\varepsilon_1}(x) \cap B_{\varepsilon_2}(y) \neq \emptyset$, there is a point $t \in X$ with $t \in B_{\varepsilon_1}(x)$ and $t \in B_{\varepsilon_2}(y)$. By (lemma) and since $\varepsilon_1 \leq \varepsilon_2$, we obtain $B_{\varepsilon_1}(x) = B_{\varepsilon_1}(t) \subseteq B_{\varepsilon_2}(t) = B_{\varepsilon_2}(y)$ as required. □

Lemma 2.5. *If $B_\varepsilon(x)$ is an open ball in X, then $X \setminus B_\varepsilon(x)$ is a union of open balls. (That is, open balls are closed in X.)*

Proof. Suppose for a contradiction that there is a point $t \in X \setminus B_\varepsilon(x)$ such that for every $r > 0$ the ball $B_r(t)$ is not contained in $X \setminus B_\varepsilon(x)$, so $B_r(t) \cap B_\varepsilon(x) \neq \emptyset$. Setting $r = \varepsilon$ and using (lemma), we have $B_\varepsilon(t) \subseteq B_\varepsilon(x)$ or $B_\varepsilon(t) \subseteq B_\varepsilon(x)$. In either case, (lemma) implies equality. But then $t \in X \setminus B_\varepsilon(x) = X \setminus B_\varepsilon(t)$ is a contradiction. Hence $X \setminus B_\varepsilon(x)$ is a union of open balls so that $B_\varepsilon(x)$ is closed in X. □

Lemma 2.6. *Let $A \subseteq X$ be non-empty with $a \in A$. Then* $\operatorname{diam} A = \sup\{d(a, x) : x \in A\}$.

Proof. Set $u = \sup\{d(a,x) : x \in A\}$ and fix $x, y \in A$. Then

$$d(x,y) \leq \max\{d(x,a), d(a,y)\} = \max\{d(a,x), d(a,y)\} \leq u$$

so that u is an upper bound of $D = \{d(x,y) : x, y \in A\}$. Since $u = \sup\{d(a,x) : x \in A\}$, for any given $\varepsilon > 0$, there exists a point $x_\varepsilon \in A$ with $u \leq d(a, x_\varepsilon) + \varepsilon$. But $a, x_\varepsilon \in A$ so that $u = \sup D = \operatorname{diam} A$ which completes the proof. $\square$

3. Filters

Lemma 3.1. *If $\mathcal{K}$ is a family of subsets satisfying the FIP, then there is a filter $\mathcal{F}$ containing each element of $\mathcal{K}$.*

Proof. First let $\mathcal{F}' = \mathcal{K} \cup \mathcal{I}$, where $\mathcal{I}$ is the set of all finite intersections of elements of $\mathcal{K}$. Hence $\mathcal{F}'$ is closed under finite intersections. Then, let $\mathcal{F} = \mathcal{F}' \cup \mathcal{S}$, where $A \in \mathcal{S}$ if and only if A contains a set in $\mathcal{F}'$.

Clearly $\mathcal{F}$ is closed when taking supersets. If $A, B \in \mathcal{F}$, the only non-trivial needing consideration is, without loss of generality, when $A \in \mathcal{S}$. So, if $B \in \mathcal{S}$ or $B \in \mathcal{F}'$, then there exist sets $A', B' \in \mathcal{F}'$ such that $A' \subseteq A$ and $B' \subseteq B$ (if $B \in \mathcal{F}'$ then $B' = B$). Since $\mathcal{F}'$ is closed under finite intersections, $A' \cap B' \in \mathcal{F}'$. Then, from $A' \cap B' \subseteq A \cap B$ it follows that $A \cap B \in \mathcal{S} \subseteq \mathcal{F}$. Hence $\mathcal{F}$ is closed under finite intersections. $\square$

Lemma 3.2. *A family $\mathcal{F}$ is an ultrafilter if and only if for every subset $A \subseteq \mathbb{N}$ either $A \in \mathcal{F}$ or $A^c \in \mathcal{F}$.*

Proof. Let $\mathcal{F}$ be an ultrafilter and suppose for a contradiction that there is a subset $A \subseteq \mathbb{N}$ with $A, A^c \notin \mathcal{F}$. So every set in $\mathcal{F}$ intersects both A and A^c. It follows from (lemma) that the filter extending $\mathcal{F} \cup \{A, A^c\}$ properly contains $\mathcal{F}$, which is a contradiction. Now suppose $\mathcal{F}$ is a filter such that for every subset $A \subseteq \mathbb{N}$, either $A \in \mathcal{F}$ or $A^c \in \mathcal{F}$. Assume for a contradiction that $\mathcal{F}'$ is a filter which properly contains $\mathcal{F}$. So there is a subset $E \subseteq \mathbb{N}$ with $E \in \mathcal{F}'$ and $E \notin \mathcal{F}$. Thus $E^c \in \mathcal{F}$ and hence $E^c \in \mathcal{F}'$. But then $\mathcal{F}'$ contains the emptyset since it is closed under finite intersections and $\emptyset = E \cap E^c \in \mathcal{F}'$, a contradiction. $\square$

Lemma 3.3. *A family $\mathcal{F}$ is an ultrafilter if and only if $\mathcal{F}$ has the Ramsey property.*

Proof. Let $\mathcal{F}$ be an ultrafilter and suppose for a contradiction that $A = A_1 \cup A_2$ is in $\mathcal{F}$ but $A_1, A_2 \notin \mathcal{F}$. Hence (lemma) we have $A_1^c, A_2^c \in \mathcal{F}$ so that $A^c = A_1^c \cap A_2^c \in \mathcal{F}$. Consequently, $\emptyset = A \cap A^c \in \mathcal{F}$ is a contradiction. The case where $A = A_1 \cup \cdots \cup A_n$ follows via elementary induction. Conversely, assume $\mathcal{F}$ has the Ramsey property. If $\mathcal{F}$ is not an ultrafilter, then there is a subset $A \subseteq \mathbb{N}$ with $A, A^c \notin \mathcal{F}$. But $\mathbb{N} \in \mathcal{F}$ and $\mathbb{N} = A \cup A^c$, so we must have $A \in \mathcal{F}$ or $A^c \in \mathcal{F}$, a contradiction. □

Proposition 3.1. *A family $\mathcal{F}$ is an ultrafilter if and only if $\mathcal{F}^*$ is a filter.*

Proof. Let us first assume that $\mathcal{F}$ is an ultrafilter, and fix $A, B \in \mathcal{F}^*$. To prove that $A \cap B \in \mathcal{F}^*$, it suffices to fix a set $E \in \mathcal{F}$ and prove that E intersects $A \cap B$. Noting that since $A \in \mathcal{F}^*$ we have $E \cap A \neq \emptyset$, write

$$E = (E \cap A) \cup (E \setminus A).$$

Certainly $E \setminus A \notin \mathcal{F}$, as otherwise $A \in \mathcal{F}^*$ implies $(E \setminus A) \cap A \neq \emptyset$. (lemma) implies that $\mathcal{F}$ satisfies the Ramsey property so that the set $E \cap A \in \mathcal{F}$. Since $B \in \mathcal{F}^*$, it follows as needed that $E \cap A \cap B \neq \emptyset$.

Conversely, we suppose $\mathcal{F}^*$ is a filter. Suppose towards a contradiction that $\mathcal{F}$ is not an ultrafilter. Hence by (lemma) there is a subset $A \subseteq \mathbb{N}$ with $A, A^c \notin \mathcal{F}$. So no set in $\mathcal{F}$ is contained in A and likewise for A^c. That is, for every set $B \in \mathcal{F}$, we have $A \cap B \neq \emptyset$ and $A^c \cap B \neq \emptyset$. By definition, then, $A, A^c \in \mathcal{F}^*$. But $\mathcal{F}^*$ is a filter, so $\emptyset = A \cap A^c \in \mathcal{F}^*$ is a contradiction, and the proof is complete. □

4. Main Result

4.1. **Preliminaries.** In the present work, we extend a result due to (Authors) [3]. A coloring $\chi : [X]^2 \to [k]$ of the two-element subsets of X is called *isometric* if $\chi(\{x_1, y_1\}) = \chi(\{x_2, y_2\})$ whenever $d(x_1, y_1) = d(x_2, y_2)$. A free ultrafilter $\mathcal{F}$ on X is called *metrically Ramsey* if every isometric coloring χ admits a set $A \in \mathcal{F}$ such that every two-element subset of A receives the same color.

With $k = 2$, (Authors) present the following theorem.

Theorem 4.1. *Fix an infinite ultrametric space X. There is a sequence (x_n) in X such that every free ultrafilter $\mathcal{F}$ containing (x_n) is metrically Ramsey.*

It is natural to question whether similar structure exists when coloring a larger class of subsets of X. The positive answer to this question is the keynote of this paper. In this connection, we will analyze the family Γ_X of all compact subsets of X.

Before proceeding to the main result, we first introduce some terminology. The map $\chi : \Gamma_X \to [k]$ is called a *diametric coloring* if $\chi(A_1) = \chi(A_2)$ for every pair A_1, A_2 of compact subsets of X with $\operatorname{diam} A_1 = \operatorname{diam} A_2$. Since finite sets are compact, clearly every diametric coloring is isometric.

A subset A of X is called *monochrome* if its compact subsets receive the same color; that is, there is a color $\varphi \in [k]$ such that $\chi(\Gamma_A) = \{\varphi\}$. A free ultrafilter $\mathcal{F}$ on X is called *diametrically Ramsey* if for every diametric coloring χ there is a monochrome set $A \in \mathcal{F}$.

4.2. **A Technical Lemma.** We note that (authors) construct (x_n) using the following lemma, whose proof is provided for the sake of completeness [2].

Lemma 4.1. *Let (X, d) be an infinite metric space. Then there is a sequence $\{x_n\}_{n=1}^{\infty}$ of distinct points in X such that either*

(1) The sequence $\{d(x_1, x_n)\}_{n=1}^{\infty}$ is strictly monotone; or
(2) For every $n \in \mathbb{N}$ and $i, j \geq n$ the distances $d(x_n, x_i) = d(x_n, x_j)$.

Proof. We first assume that there is a point $x_0 \in X$ such that $d(x_0, X) \coloneqq \{d(x_1, x_n) : x_n \in X\}$ is not finite. Hence, there is a countably infinite subset $E \subseteq X$ with $x_0 \notin E$ and $d(x_0, x) \neq d(x_0, y)$ for every $x, y \in E$. We obtain from E the sequence $\xi = \{d(x_0, x) : x \in E\}$. Since ξ is a sequence of reals, it has a monotone subsequence $\{d(x_0, x_n)\}_{n=1}^{\infty}$ whose points are distinct by construction of E. Since d is a metric, $x_i \neq x_j$ for every $i, j \in \mathbb{N}$ and so $\{x_n\}_{n=1}^{\infty}$ is the desired sequence.

Otherwise, $d(x, X)$ is finite for every $x \in X$. Fix $x_0 \in X$ and assume without loss of generality that $\ell_1 \in d(x_0, X)$ is non-zero. Let E_1 be a countable subset of X with $E_1 \subseteq \{x \in X : d(x_0, x) = \ell_1\}$. Choose $x_1 \in E_1$, and note that $d(x_0, E_1) = \{\ell_1\}$ is a singleton set and $x_0 \notin E_1$.

For $n \geq 2$, we choose x_n and define E_n inductively as follows. As above, let $\ell_n \in d(x_{n-1}, X)$ be non-zero and let E_n be a countable subset of X with $E_n \subseteq \{x \in X : d(x_{n-1}, x) = \ell_n\}$. Again, we choose $x_n \in E_n$ and observe that $d(x_{n-1}, E_n) = \{\ell_n\}$. Continuing this way, we obtain the sequence from (2). $\square$

4.3. **Main Result.** We are now ready to state the main result.

Theorem 4.2. *Fix an infinite ultrametric space X. There is a sequence (x_n) in X such that every free ultrafilter $\mathcal{F}$ containing (x_n) is diametrically Ramsey.*

Proof. Let χ be any diametric coloring on X and fix a free ultrafilter $\mathcal{F}$ containing the sequence (x_n) as obtained in (lemma).

Let $h : (x_n) \to \mathbb{R}^+$ be a fixed map. How we define h depends on (x_n), so we proceed in this regard later on. Moreover, suppose $f : \mathbb{R}^+ \to [k]$ is any mapping satisfying $f(h(x_n)) = \chi(A)$ whenever $A \in \Gamma_X$ is a compact subset of X with $\operatorname{diam} A = h(x_n)$. Finally, we set $c = f \circ h$.

Write $(x_n) = c^{-1}([k]) = \cup_{j=1}^{k} c^{-1}(\{j\})$ and observe that since $E \in \mathcal{F}$, (lemma) implies that there is a color $\varphi \in [k]$ with $c^{-1}(\{\varphi\}) \in \mathcal{F}$.

We set $A = c^{-1}(\{\varphi\})$ and complete the proof by showing that A is monochrome. Specifically, we show that if K is a compact subset of A then $\chi(K) = \varphi$. So fix such a set K.

Since K is compact, there are points $x_i, x_j \in K$ with $i < j$ and $d(x_i, x_j) = \operatorname{diam} K$, applying (lemma). We now consider the conditions on (x_n) as described in (lemma), and define h accordingly to complete the proof.

Case 1. We will first assume that case (1) of (lemma) holds, namely that (x_n) is a sequence of distinct points in X where $\{d(x_0, x_n)\}_{n=1}^{\infty}$ is strictly monotone for some point $x_0 \in X$. In this case, h will indicate the distance to x_0 from a term $x_n \in E$, given by $h(x_n) = d(x_0, x_n)$.

If h is strictly increasing, then we have $d(x_0, x_i) < d(x_0, x_j)$. Hence (lemma) implies that $d(x_i, x_j) = d(x_0, x_j)$, since d is an ultrametric. Otherwise $d(x_0, x_i) > d(x_0, x_j)$ so that $d(x_i, x_j) = d(x_0, x_i)$ using (lemma) once more. Possibly swapping the symbols i, j, we assume the latter case holds. Since χ is diametric, $x_j \in A$, and $\operatorname{diam} K = d(x_i, x_j) = d(x_0, x_j)$, we have

$$c(x_j) = f(d(x_0, x_j)) = \chi(K) = \varphi,$$

as needed.

Case 2. We now assume that case (2) of (lemma) applies to (x_n). Namely, for each $n \in \mathbb{N}$ and $i, j \geq n$ we have $d(x_n, x_i) = d(x_n, x_j)$. Define $h : E \to \mathbb{R}^+$ by $h(x_n) = d(x_n, x_{n+1})$, and note that $h(x_n) = d(x_n, x_j)$ for every $j > n$. Again, since χ is diametric and $i < j$, we have $h(x_i) = d(x_i, x_j)$ and, as desired,

$$c(x_i) = f(d(x_i, x_j)) = \chi(K) = \varphi.$$

This completes the proof, since A is χ-monochrome in both cases. □

Rewrite abstract, add text throughout, more results, conclusion, checking things, clean up, acknowledgements, email, formatting, etc. A lot more to do!