Colorful Ultrametric Spaces and Ramsey Ultrafilters

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1. PRELIMINARIES

1.1. **Ultrametric Analysis.** Recall that a *metric* on a set X is a function $d : X \times X \to [0, +\infty)$ such that for every $x, y, z \in X$ the following hold:

(1) $d(x, y) = 0 \Leftrightarrow x = y$;
(2) $d(x, y) = d(y, x)$; and
(3) $d(x, y) \leq d(x, z) + d(z, y)$.

The pair (X, d) is called a *metric space*. A stronger version of (3) is called the *strong triangle inequality*:

$$d(x, y) \leq \max\{d(x, z), d(z, y)\}.$$

If the strong triangle inequality holds, then we call d an *ultrametric* and (X, d) an *ultrametric space*. In what follows, unless otherwise specified, (X, d) (or simply X) is an ultrametric space.

Lemma 1.1. *If x, y, z are distinct points in an ultrametric space X and $d(x, y) < d(y, z)$ then $d(x, z) = d(x, y)$. (That is, every triangle in an ultrametric space is isosceles.*

Proof. □

An *open ball* (or simply a *ball*) of radius $\varepsilon > 0$ centered about a point $x \in X$ is defined as the set $B_\varepsilon(x) \coloneqq \{y \in X : d(x, y) < \varepsilon\}$. If left unspecified, note that x denotes a point in X and ε a positive real number.

A subset $\mathcal{O}$ of X is called *open* if it can be written as a union of open balls. A set is *closed* if its complement is open. Open balls (and hence open sets) in ultrametric spaces exhibit properties that are quite counterintuitive, as delineated by the following simple results.

Lemma 1.2. *Let $B_\varepsilon(x)$ be an open ball in X. Then $B_\varepsilon(x) = B_\varepsilon(y)$ for every point $y \in B_\varepsilon(x)$. (That is, every point in a ball is its center.)*

Proof. Fix $y \in B_\varepsilon(x)$ so that $d(x,y) < \varepsilon$. If $t \in B_\varepsilon(y)$ then $d(y,t) < \varepsilon$. Then $t \in B_\varepsilon(x)$ since

$$d(x,t) \leq \max\{d(x,y), d(y,t)\} < \varepsilon.$$

The reverse inclusion follows symmetrically as $x \in B_\varepsilon(y)$. $\square$

Corollary 1.1. *Fix $\varepsilon_1, \varepsilon_2 > 0$ and $x, y \in X$. If $B_{\varepsilon_1}(x) \cap B_{\varepsilon_2}(y) \neq \emptyset$ then $B_{\varepsilon_1}(x) \subseteq B_{\varepsilon_2}(y)$ or $B_{\varepsilon_2}(y) \subseteq B_{\varepsilon_1}(x)$. (That is, two balls are either disjoint or one of them contains the other.)*

Proof. Assume without loss of generality that $\varepsilon_1 \leq \varepsilon_2$. If $B_{\varepsilon_1}(x) \cap B_{\varepsilon_2}(y) \neq \emptyset$, there is a point $t \in X$ with $t \in B_{\varepsilon_1}(x)$ and $t \in B_{\varepsilon_2}(y)$. By (lemma) and since $\varepsilon_1 \leq \varepsilon_2$, we obtain $B_{\varepsilon_1}(x) = B_{\varepsilon_1}(t) \subseteq B_{\varepsilon_2}(t) = B_{\varepsilon_2}(y)$ as required. $\square$

Corollary 1.2. *If $B_\varepsilon(x)$ is an open ball in X, then $X \setminus B_\varepsilon(x)$ is a union of open balls. (That is, $B_\varepsilon(x)$ is closed in X).*

Proof. Suppose for a contradiction that there is a point $t \in X \setminus B_\varepsilon(x)$ such that for every $r > 0$ the ball $B_r(t)$ is not contained in $X \setminus B_\varepsilon(x)$, so $B_r(t) \cap B_\varepsilon(x) \neq \emptyset$. Setting $r = \varepsilon$ and using (lemma), we have $B_\varepsilon(t) \subseteq B_\varepsilon(x)$ or $B_\varepsilon(t) \subseteq B_\varepsilon(x)$. In either case, (lemma) implies equality. But then $t \in X \setminus B_\varepsilon(x) = X \setminus B_\varepsilon(t)$ is a contradiction. Hence $X \setminus B_\varepsilon(x)$ is a union of open balls so that $B_\varepsilon(x)$ is closed in X. $\square$