Ramsey Properties of Isometric k-Colorings

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Abstract. A map $\chi : [X]^k \to [k]$ on a metric space X is an isometric k-coloring if $\chi(A_1) = \chi(A_2)$ for every pair A_1, A_2 of isodiametric k-element subsets of X. A free ultrafilter $\mathcal{F}$ is called metrically Ramsey if for every isometric coloring χ there is a set $A \in \mathcal{F}$ such that $[A]^k$ is χ-monochrome. Protasov and Protasova [1] prove that every infinite ultrametric space X contains a sequence (x_n) such that every free ultrafilter containing (x_n) is metrically Ramsey when $k = 2$. We strengthen this result to hold for every $k \in \mathbb{N}$.

1. Introduction

Ramsey Theory explores the underlying structure emerging in "large enough" complex systems. For example, Frank Ramsey [1] proved that for each $n \in \mathbb{N}$ there is a sufficiently large $N \in \mathbb{N}$ such that in any red-blue coloring of the edges of the complete graph K_N there is a set of n vertices joined by pairwise monochromatic edges (i.e., a monochromatic clique). This raises a natural question: given $n, k \in \mathbb{N}$, is there an integer $N \in \mathbb{N}$ such that in any edge-coloring of K_N in k colors there is a monochromatic clique of size n? Its positive answer is due to (authors) [2]. Analogously, in what follows, we generalize and strengthen a Ramsey-type coloring theorem of Protasov and Protasova [3] to hold for k colors, where $k \in \mathbb{N}$.

Fix an infinite metric space (X, d) and let $k \in \mathbb{N}$. A *k-coloring* on X is a map $\chi : [X]^k \to [k]$ which assigns one of k colors to each k-element subset of X. For a given k-coloring χ, we would like to find a set $A \subseteq X$ such that $\chi([A]^k) = \{c\}$ for some $c \in [k]$; in this case, we call $[A]^k$ *χ-monochrome.*

In this context, the "large" objects containing underlying structure in (X, d) are free ultrafilters [4]. A *filter* $\mathcal{F}$ on X is a collection of subsets of X satisfying the following for all sets $A, B \subseteq X$:

(1) $\emptyset \notin \mathcal{F}$ and $X \in \mathcal{F}$;
(2) If $A \in \mathcal{F}$ and $A \subseteq B$ then $B \in \mathcal{F}$;
(3) If $A, B \in \mathcal{F}$ then $A \cap B \in \mathcal{F}$.

A filter $\mathcal{F}$ is called an *ultrafilter* if it is not properly contained in a filter on X. A filter $\mathcal{F}$ is called *free* if $\bigcap \mathcal{F} = \emptyset$. Free filters are "spread out" throughout the space and ultrafilters are maximal filters, so we consider *free ultrafilters* as "large" objects in X.

Given this, one naturally asks if there is a free ultrafilter $\mathcal{F}$ such that for every coloring χ there is a set $A \in \mathcal{F}$ so that $[A]^k$ is χ-monochrome. It turns out that this question is undecidable in ZFC even with $X = \mathbb{N}$ and $d(x, y) = |x - y|$, though the statement is true if we declare the continuum hypothesis as axiomatic [5]. Hence we must define more restrictive classes of colorings.

To this end, we say that a k-coloring $\chi : [X]^k \to [k]$ is an *isometric k-coloring* if $\chi(A_1) = \chi(A_2)$ whenever A_1, A_2 is a pair of isodiametric k-element subsets of X. A free ultrafilter $\mathcal{F}$ is called *metrically Ramsey* if for every isometric k-coloring χ there is a set $A \in \mathcal{F}$ such that $[A]^k$ is χ-monochrome.

Recall that an *ultrametric* d on a set X is a metric satisfying the *strong triangle inequality:* for all $x, y, z \in X$

$$d(x, y) \leq \max\{d(x, z), d(z, y)\}.$$

It turns out that in the particular case of ultrametric spaces, metrically Ramsey free ultrafilters are not too hard to construct. Indeed, Protasov and Protasova [6] prove that for $k = 2$ every infinite ultrametric space X contains a sequence (x_n) such that every free ultrafilter containing (x_n) is metrically Ramsey. The authors leverage the properties of the ultrametric coupled with (lemma) (to construct (x_n)) to prove the main result when $k = 2$. We expand on this approach, strengthening their result to hold for all k-colorings.

In the following two sections, we review some elementary properties of ultrametrics and filters. We then prove the main result in (section 4).

2. Ultrametric Analysis

2.1. **The space $\mathbb{N}^\mathbb{N}$ and ε-chains.** In this short section, we provide some examples of ultrametric spaces. In particular, we construct ultrametrics on the Baire space $\mathbb{N}^\mathbb{N}$ and general uniformly disconnected metric spaces.

The simplest example of an ultrametric on a set X is the discrete metric d, where $d(x, y)$ is 1 if $x \neq y$ and 0 otherwise. Another simple example is $(\mathbb{N}, d)$, where $d(n, m) = \max\{1 + 1/n, 1 + 1/m\}$ if $n \neq m$ and $d(n, m) = 0$ otherwise.

The *Baire space* $\mathbb{N}^\mathbb{N}$ is the space of all sequences of natural numbers.

Example 1. For two distinct sequences $x = (x_n)$, $y = (y_n)$ of naturals, we define $m(x, y) = \min\{k \in \mathbb{N} : x_k \neq y_k\}$, the first index at which x and y do not coincide. For distinct sequences $x, y \in \mathbb{N}^{\mathbb{N}}$, set $d(x, y) = m(x, y)^{-1}$ with $d(x, x) = 0$. Then d is an ultrametric on $\mathbb{N}^{\mathbb{N}}$.

Proof. The symmetry and non-negativity of d is immediate. Furthermore, $d(x, x) = 0$ by definition, and if $d(x, y) = 0$ then we must have $x = y$ since $m(x, y) \geq 1$. To prove the strong triangle inequality for d, fix $x, y, z \in \mathbb{N}^{\mathbb{N}}$. We assume x, y, z are all distinct sequences as the result is immediate otherwise. Observe that

$$\begin{aligned}
&d(x, y) \leq \max\{d(x, z), d(z, y)\} \\
&\Leftrightarrow m(x, y)^{-1} \leq \max\{m(x, z)^{-1}, m(z, y)^{-1}\} \\
&\Leftrightarrow m(x, y)^{-1} \leq \min\{m(x, z), m(z, y)\}^{-1} \\
&\Leftrightarrow m(x, y) \geq \min\{m(x, z), m(z, y)\}.
\end{aligned}$$

Clearly $m(x, y) \geq \min\{m(x, z), m(z, y)\}$: if $\ell = m(x, y)$ then for each $k \in \{1, 2, \ldots, \ell - 1\}$ the terms $x_k = y_k$, so if $m(x, z) \geq \ell + 1$ and $m(z, y) \geq \ell + 1$ then $x_\ell = z_\ell = y_\ell$ so that $m(x, y) \geq \ell + 1$ is a contradiction. Therefore, d is an ultrametric. $\square$

Let (X, d) be any metric space and $\varepsilon > 0$ be given. An *ε-chain* between the pair $x, y \in X$ is a finite sequence $x = x_0, x_1, \ldots, x_n = y$ such that

$$\max_{1 \leq i \leq n} d(x_{i-1}, x_i) \leq \varepsilon.$$

In this case, we say that x, y are ε-connected. Observe that if x, y are ε-connected then they are ε'-connected for every $\varepsilon' \geq \varepsilon$; and if x, y can not be ε-connected then they can not be ε'-connected for every $\varepsilon' \leq \varepsilon$.

The space X is called *uniformly disconnected* if there is an $\varepsilon > 0$ such that no two points in X can be ε-connected.

Example 2. Let (X, d) be a uniformly disconnected metric space. For $x, y \in X$, let $c(x, y)$ be the infimum over all $\varepsilon > 0$ such that x and y are ε-connected. Then c is an ultrametric on X.

Proof. Clearly c is symmetric and non-negative. If $x = y$ then clearly x, y are ε-connected for every $\varepsilon > 0$ so that $c(x, y) = 0$. On the other hand, if $c(x, y) = 0$ then $x = y$, otherwise for each $\varepsilon > 0$ the points x, y are ε-connected, so X is not uniformly disconnected, a contradiction.

The last thing to prove is the strong triangle inequality for c. To this end, fix $x, y, z \in X$ and let $\varepsilon > 0$ be given. We may assume the points x, y, z are distinct, otherwise the claim is immediate. By definition of the infimum, there exist $\gamma_1, \gamma_2 > 0$ with $\gamma_1 \leq c(x, z) + \varepsilon$

and $\gamma_2 \le c(z,y)+\varepsilon$ such that x,z are γ_1-connected and z,y are γ_2-connected. Set $\gamma = \max\{\gamma_1,\gamma_2\}$ and note that by combining the x,z and z,y chains it follows that x,y are γ-connected. Thus

$$c(x,y) \le \gamma \le \max\{c(x,z), c(z,y)\} + \varepsilon.$$

Since ε was arbitrary, it follows that $c(x,y) \le \max\{c(x,z), c(z,y)\}$. $\square$

2.2. **Properties of ultrametric spaces.**

Lemma 2.1. *If x,y,z are distinct points in an ultrametric space X and $d(x,y) < d(y,z)$ then $d(y,z) = d(x,z)$. (That is, every triangle in an ultrametric space is isosceles.)*

Proof. Note that d is an ultrametric and $d(x,y) < d(y,z)$ so that

$$d(x,z) \le \max\{d(x,y), d(y,z)\} = d(y,z),$$

and since $d(y,z) > d(x,y)$, we have $d(x,y) < d(x,z)$ as otherwise

$$d(y,z) \le \max\{d(y,x), d(x,z)\} = \max\{d(x,y), d(x,z)\} = d(x,y)$$

is a contradiction. Thus $d(y,z) \le \max\{d(x,y), d(x,z)\} = d(x,z)$. Combining everything together, we obtain $d(y,z) = d(x,z)$. $\square$

An *open ball* (or simply a *ball*) of radius $\varepsilon > 0$ centered about a point $x \in X$ is defined as the set $B_\varepsilon(x) \coloneqq \{y \in X : d(x,y) < \varepsilon\}$. If left unspecified, note that x denotes a point in X and ε a positive real number.

A subset $\mathcal{O}$ of X is called *open* if it can be written as a union of open balls. A set is *closed* if its complement is open. Open balls and general sets in ultrametric spaces exhibit properties that are quite counterintuitive, as delineated by the following simple lemmas.

Lemma 2.2. *Let $B_\varepsilon(x)$ be an open ball in X. Then $B_\varepsilon(x) = B_\varepsilon(y)$ for every point $y \in B_\varepsilon(x)$. (That is, every point in a ball is its center.)*

Proof. Fix $y \in B_\varepsilon(x)$ so that $d(x,y) < \varepsilon$. If $t \in B_\varepsilon(y)$ then $d(y,t) < \varepsilon$. Then $t \in B_\varepsilon(x)$ since

$$d(x,t) \le \max\{d(x,y), d(y,t)\} < \varepsilon.$$

The reverse inclusion follows symmetrically as $x \in B_\varepsilon(y)$. $\square$

Lemma 2.3. *Let $A \subseteq X$ be non-empty with $a \in A$. Then* $\operatorname{diam} A = \sup\{d(a,x) : x \in A\}$.

Proof. Set $u = \sup\{d(a,x) : x \in A\}$ and fix $x,y \in A$. Then

$$d(x,y) \le \max\{d(x,a), d(a,y)\} = \max\{d(a,x), d(a,y)\} \le u$$

so that u is an upper bound of $D = \{d(x,y) : x,y \in A\}$. Since $u = \sup\{d(a,x) : x \in A\}$, for any given $\varepsilon > 0$, there exists a point

$x_\varepsilon \in A$ with $u \leq d(a, x_\varepsilon)+\varepsilon$. But $a, x_\varepsilon \in A$ so that $u = \sup D = \operatorname{diam} A$ which completes the proof. $\square$

Lemma 2.4. *Let $A \subseteq X$ and consider any ball B in X. If $A \cap B \neq \emptyset$ then B is a ball in the ultrametric space (A, d).*

Proof. Since $A \cap B \neq \emptyset$ there is a point $a \in A \cap B$. Write $B = B_\varepsilon(x)$ for some $x \in X$ and $\varepsilon > 0$. To show that $A \cap B$ is a ball in A, we must find a radius $r > 0$ such that for every $t \in A \cap B$, $d(a, t) < r$. Set $r = \varepsilon$ and fix $t \in A \cap B$. Then $a, t \in B$ so that $d(a, x) < \varepsilon$ and $d(t, x) < \varepsilon$. Using the strong triangle inequality,

$$d(a, t) \leq \max\{d(a, x), d(t, x)\} < \varepsilon = r.$$

Therefore, $A \cap B$ is a ball in A. $\square$

Lemma 2.5. *Fix $\varepsilon_1, \varepsilon_2 > 0$ and $x, y \in X$. If $B_{\varepsilon_1}(x) \cap B_{\varepsilon_2}(y) \neq \emptyset$ then $B_{\varepsilon_1}(x) \subseteq B_{\varepsilon_2}(y)$ or $B_{\varepsilon_2}(y) \subseteq B_{\varepsilon_1}(x)$. (That is, two balls are either disjoint or one of them contains the other.)*

Proof. This is a weaker version of (obs), but we give the statement its own proof for clarity. Assume without loss of generality that $\varepsilon_1 \leq \varepsilon_2$. If $B_{\varepsilon_1}(x) \cap B_{\varepsilon_2}(y) \neq \emptyset$, there is a point $t \in X$ with $t \in B_{\varepsilon_1}(x)$ and $t \in B_{\varepsilon_2}(y)$. By (lemma) and since $\varepsilon_1 \leq \varepsilon_2$, we obtain $B_{\varepsilon_1}(x) = B_{\varepsilon_1}(t) \subseteq B_{\varepsilon_2}(t) = B_{\varepsilon_2}(y)$ as required. $\square$

Lemma 2.6. *If $B_\varepsilon(x)$ is an open ball in X, then $X \setminus B_\varepsilon(x)$ is a union of open balls. (That is, open balls are closed in X.)*

Proof. Suppose for a contradiction that there is a point $t \in X \setminus B_\varepsilon(x)$ such that for every $r > 0$ the ball $B_r(t)$ is not contained in $X \setminus B_\varepsilon(x)$, so $B_r(t) \cap B_\varepsilon(x) \neq \emptyset$. Setting $r = \varepsilon$ and using (lemma), we have $B_\varepsilon(t) \subseteq B_\varepsilon(x)$ or $B_\varepsilon(t) \subseteq B_\varepsilon(x)$. In either case, (lemma) implies equality. But then $t \in X \setminus B_\varepsilon(x) = X \setminus B_\varepsilon(t)$ is a contradiction. Hence $X \setminus B_\varepsilon(x)$ is a union of open balls so that $B_\varepsilon(x)$ is closed in X. $\square$

3. Filters

Lemma 3.1. *If $\mathcal{K}$ is a family of subsets satisfying the FIP, then there is a filter $\mathcal{F}$ containing each element of $\mathcal{K}$.*

Proof. First let $\mathcal{F}' = \mathcal{K} \cup \mathcal{I}$, where $\mathcal{I}$ is the set of all finite intersections of elements of $\mathcal{K}$. Hence $\mathcal{F}'$ is closed under finite intersections. Then, let $\mathcal{F} = \mathcal{F}' \cup \mathcal{S}$, where $A \in \mathcal{S}$ if and only if A contains a set in $\mathcal{F}'$.

Clearly $\mathcal{F}$ is closed when taking supersets. If $A, B \in \mathcal{F}$, the only non-trivial needing consideration is, without loss of generality, when $A \in \mathcal{S}$. So, if $B \in \mathcal{S}$ or $B \in \mathcal{F}'$, then there exist sets $A', B' \in \mathcal{F}'$ such that $A' \subseteq A$ and $B' \subseteq B$ (if $B \in \mathcal{F}'$ then $B' = B$). Since $\mathcal{F}'$ is closed under finite intersections, $A' \cap B' \in \mathcal{F}'$. Then, from $A' \cap B' \subseteq A \cap B$ it follows that $A \cap B \in \mathcal{S} \subseteq \mathcal{F}$. Hence $\mathcal{F}$ is closed under finite intersections. $\square$

Lemma 3.2. *A family $\mathcal{F}$ is an ultrafilter if and only if for every subset $A \subseteq \mathbb{N}$ either $A \in \mathcal{F}$ or $A^c \in \mathcal{F}$.*

Proof. Let $\mathcal{F}$ be an ultrafilter and suppose for a contradiction that there is a subset $A \subseteq \mathbb{N}$ with $A, A^c \notin \mathcal{F}$. So every set in $\mathcal{F}$ intersects both A and A^c. It follows from (lemma) that the filter extending $\mathcal{F} \cup \{A, A^c\}$ properly contains $\mathcal{F}$, which is a contradiction. Now suppose $\mathcal{F}$ is a filter such that for every subset $A \subseteq \mathbb{N}$, either $A \in \mathcal{F}$ or $A^c \in \mathcal{F}$. Assume for a contradiction that $\mathcal{F}'$ is a filter which properly contains $\mathcal{F}$. So there is a subset $E \subseteq \mathbb{N}$ with $E \in \mathcal{F}'$ and $E \notin \mathcal{F}$. Thus $E^c \in \mathcal{F}$ and hence $E^c \in \mathcal{F}'$. But then $\mathcal{F}'$ contains the emptyset since it is closed under finite intersections and $\emptyset = E \cap E^c \in \mathcal{F}'$, a contradiction. $\square$

Lemma 3.3. *A family $\mathcal{F}$ is an ultrafilter if and only if $\mathcal{F}$ has the Ramsey property.*

Proof. Let $\mathcal{F}$ be an ultrafilter and suppose for a contradiction that $A = A_1 \cup A_2$ is in $\mathcal{F}$ but $A_1, A_2 \notin \mathcal{F}$. Hence (lemma) we have $A_1^c, A_2^c \in \mathcal{F}$ so that $A^c = A_1^c \cap A_2^c \in \mathcal{F}$. Consequently, $\emptyset = A \cap A^c \in \mathcal{F}$ is a contradiction. The case where $A = A_1 \cup \cdots \cup A_n$ follows via elementary induction. Conversely, assume $\mathcal{F}$ has the Ramsey property. If $\mathcal{F}$ is not an ultrafilter, then there is a subset $A \subseteq \mathbb{N}$ with $A, A^c \notin \mathcal{F}$. But $\mathbb{N} \in \mathcal{F}$ and $\mathbb{N} = A \cup A^c$, so we must have $A \in \mathcal{F}$ or $A^c \in \mathcal{F}$, a contradiction. $\square$

Proposition 3.1. *A family $\mathcal{F}$ is an ultrafilter if and only if $\mathcal{F}^*$ is a filter.*

Proof. Let us first assume that $\mathcal{F}$ is an ultrafilter, and fix $A, B \in \mathcal{F}^*$. To prove that $A \cap B \in \mathcal{F}^*$, it suffices to fix a set $E \in \mathcal{F}$ and prove that E intersects $A \cap B$. Noting that since $A \in \mathcal{F}^*$ we have $E \cap A \neq \emptyset$, write

$$E = (E \cap A) \cup (E \setminus A).$$

Certainly $E \setminus A \notin \mathcal{F}$, as otherwise $A \in \mathcal{F}^*$ implies $(E \setminus A) \cap A \neq \emptyset$. (lemma) implies that $\mathcal{F}$ satisfies the Ramsey property so that the set $E \cap A \in \mathcal{F}$. Since $B \in \mathcal{F}^*$, it follows as needed that $E \cap A \cap B \neq \emptyset$.

Conversely, we suppose $\mathcal{F}^*$ is a filter. Suppose towards a contradiction that $\mathcal{F}$ is not an ultrafilter. Hence by (lemma) there is a subset $A \subseteq \mathbb{N}$ with $A, A^c \notin \mathcal{F}$. So no set in $\mathcal{F}$ is contained in A and likewise for A^c. That is, for every set $B \in \mathcal{F}$, we have $A \cap B \neq \emptyset$ and $A^c \cap B \neq \emptyset$. By definition, then, $A, A^c \in \mathcal{F}^*$. But $\mathcal{F}^*$ is a filter, so $\emptyset = A \cap A^c \in \mathcal{F}^*$ is a contradiction, and the proof is complete. $\square$

4. Main Result

4.1. **Some Lemmas and Preliminaries.**

Lemma 4.1. *Let (X, d) be an infinite metric space. Then there is a sequence $\{x_n\}_{n=1}^{\infty}$ of distinct points in X such that either*

(1) The sequence $\{d(x_1, x_n)\}_{n=1}^{\infty}$ is strictly monotone; or
(2) For every $n \in \mathbb{N}$ and $i, j \geq n$ the distances $d(x_n, x_i) = d(x_n, x_j)$.

Proof. We first assume that there is a point $x_0 \in X$ such that $d(x_0, X) :=\{d(x_1, x_n) : x_n \in X\}$ is not finite. Hence, there is a countably infinite subset $E \subseteq X$ with $x_0 \notin E$ and $d(x_0, x) \neq d(x_0, y)$ for every $x, y \in E$. We obtain from E the sequence $\xi = \{d(x_0, x) : x \in E\}$. Since ξ is a sequence of reals, it has a monotone subsequence $\{d(x_0, x_n)\}_{n=1}^{\infty}$ whose points are distinct by construction of E. Since d is a metric, $x_i \neq x_j$ for every $i, j \in \mathbb{N}$ and so $\{x_n\}_{n=1}^{\infty}$ is the desired sequence.

Otherwise, $d(x, X)$ is finite for every $x \in X$. Fix $x_0 \in X$ and assume without loss of generality that $\ell_1 \in d(x_0, X)$ is non-zero. Let E_1 be a countable subset of X with $E_1 \subseteq \{x \in X : d(x_0, x) = \ell_1\}$. Choose $x_1 \in E_1$, and note that $d(x_0, E_1) = \{\ell_1\}$ is a singleton set and $x_0 \notin E_1$.

For $n \geq 2$, we choose x_n and define E_n inductively as follows. As above, let $\ell_n \in d(x_{n-1}, X)$ be non-zero and let E_n be a countable subset of X with $E_n \subseteq \{x \in X : d(x_{n-1}, x) = \ell_n\}$. Again, we choose $x_n \in E_n$ and observe that $d(x_{n-1}, E_n) = \{\ell_n\}$. Continuing this way, we obtain the sequence from (2). $\square$

Set $k \in \mathbb{N}$ and let X be a metric space. A k-**coloring** on X is a function $\chi : [X]^k \to [k]$, where $[X]^k = \{\{x_1, x_2, \ldots, x_k\} : x_i \in X,\ \forall i \in [k]\}$ is the set of all k element subsets of X. A subset $A \subseteq X$ is called χ-**monochrome** if χ is constant on $[A]^k$. The coloring χ is called k-**isometric** if $\chi(A_1) = \chi(A_2)$ whenever the pair $A_1, A_2 \in [X]^k$ of k element subsets satisfy $\operatorname{diam} A_1 = \operatorname{diam} A_2$.

A free ultrafilter $\mathcal{F}$ on X is called k-**Ramsey** with respect to a collection $\mathcal{C}$ of colorings on X if for every coloring $\chi \in \mathcal{C}$ there is a set $A \in \mathcal{F}$ such that $[A]^k$ is χ-monochrome.

4.2. **Main Result.** (Authors) in [3] prove that every free ultrafilter on an infinite ultrametric space X is 2-Ramsey with respect to the class of 2-isometric colorings on X. In this particular case, a coloring χ is 2-isometric if and only if all points $x_1, x_2, y_1, y_2 \in X$ with $d(x_1, y_1) = d(x_2, y_2)$ satisfy $\chi(\{x_1, y_1\}) = \chi(\{x_2, y_2\})$. (Authors) leverage the properties of the ultrametric coupled with this observation and (lemma) to prove the main result when $k = 2$. We expand on this approach, strengthening their result to hold for all k-colorings in the particular case of ultrametric spaces.

Theorem 4.1. *Let (X, d) be an infinite ultrametric space and let k be a positive integer. Let $E = \{c_n\}_{n=1}^{\infty}$ be a sequence of points obtained as in (lemma). Then every free ultrafilter $\mathcal{F}$ in X containing E is k-Ramsey with respect to the collection $\mathcal{C}$ of k-isometric colorings on X.*

Proof. Let $\chi \in \mathcal{C}$ be a k-isometric coloring on X and fix a free ultrafilter $\mathcal{F}$ which contains E.

Let $h : E \to \mathbb{R}^+$ be a fixed map. How we define h will depend on the sequence E attained from (lemma), so we will proceed in this regard later on. Moreover, suppose $f : h(E) \to [k]$ is any mapping satisfying $f(h(c_\ell)) = \chi(A)$ whenever $A \in [X]^k$ is a k element subset of X with $\operatorname{diam} A = h(c_\ell)$. Finally, we set $c = f \circ h$.

Write $E = c^{-1}([k]) = \cup_{j=1}^{k} c^{-1}(\{j\})$ and observe that since $E \in \mathcal{F}$, (lemma) implies that there is a color $\varphi \in [k]$ with $c^{-1}(\{\varphi\}) \in \mathcal{F}$. We set $A = c^{-1}(\{\varphi\})$ and complete the proof by showing that A is χ-monochrome. In particular, we will show that each k element set in $[A]^k$ has color φ.

Let $n_1 < n_2 < \cdots < n_k$ be fixed positive integers and consider the k element subsequence $C_k = \{c_{n_1}, c_{n_2}, \ldots, c_{n_k}\} \in [A]^k$ of E. Assume integers $n_i < n_j$ are such that

$$\{c_{n_i}, c_{n_j}\} = \underset{\{x,y\} \subseteq C_k}{\arg\max}\, d(x, y);$$

that is, $d(c_{n_i}, c_{n_j}) = \operatorname{diam} C_k$. We now consider the conditions on E as described in (lemma), and define h accordingly to complete the proof.

Case 1. We will first assume that case (1) of (lemma) holds, namely that $\{c_n\}_{n=1}^{\infty}$ is a sequence of distinct points in X where $\{d(c_0, c_n)\}_{n=1}^{\infty}$ is strictly monotone. In this case, h will indicate the distance to c_0 from a term $c_\ell \in E$, given by $h(c_\ell) = d(c_0, c_\ell)$.

If h is strictly increasing, then we have $d(c_0, c_{n_i}) < d(c_0, c_{n_j})$. Hence (lemma) implies that $d(c_{n_i}, c_{n_j}) = d(c_0, c_{n_j})$, since d is an ultrametric. Otherwise $d(c_0, c_{n_i}) > d(c_0, c_{n_j})$ so that $d(c_{n_i}, c_{n_j}) = d(c_0, c_{n_i})$ using (lemma) once more. Possibly swapping the symbols i, j, we assume the latter case holds. Since $c_{n_j} \in A$ and $\operatorname{diam} C_k = d(c_{n_i}, c_{n_j}) = d(c_0, c_{n_j})$, we have $f(d(c_0, c_{n_j})) = \chi(C_k) = \varphi$, as needed.

Case 2. We now assume that case (2) of (lemma) applies to $\{c_n\}_{n=1}^{\infty}$. Namely, for each $n \in \mathbb{N}$ and $i, j \geq n$ we have $d(c_n, c_i) = d(c_n, c_j)$. Define $h : E \to \mathbb{R}^+$ by $h(c_\ell) = d(c_\ell, c_{\ell+1})$, and note that $h(c_\ell) = d(c_\ell, c_j)$ for every $j > \ell$. Since $n_i < n_j$, we have $h(c_{n_i}) = d(c_{n_i}, c_{n_j})$ and, as desired,

$$f(h(c_{n_i})) = f(d(c_{n_i}, c_{n_j})) = \chi(C_k) = \varphi.$$

This completes the proof, since A is χ-monochrome in both cases. $\square$