

Ramsey Properties of Isometric k -Colorings

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ABSTRACT. A mapping $\chi : [X]^k \rightarrow [k]$ on a metric space X is an isometric k -coloring if $\chi(A_1) = \chi(A_2)$ for every pair A_1, A_2 of isodiametric k -element subsets of X . A free ultrafilter \mathcal{F} is called metrically Ramsey if for every isometric coloring χ there is a set $A \in \mathcal{F}$ such that $[A]^k$ is χ -monochrome. Protasov and Protasova [1] prove that every infinite ultrametric space X contains a sequence (x_n) such that every free ultrafilter containing (x_n) is metrically Ramsey when $k = 2$. We strengthen this result to hold for every $k \in \mathbb{N}$.

1. INTRODUCTION

Ramsey Theory explores the underlying structure emerging in “large enough” complex systems. For example, Frank Ramsey [1] proved that for each $n \in \mathbb{N}$ there is a sufficiently large $N \in \mathbb{N}$ such that in any red-blue coloring of the edges of the complete graph K_N there is a set of n vertices joined by pairwise monochromatic edges (i.e., a monochromatic clique). This raises a natural question: given $n, k \in \mathbb{N}$, is there an integer $N \in \mathbb{N}$ such that in any edge-coloring of K_N in k colors there is a monochromatic clique of size n ? Its positive answer is due to (authors) [2]. Analogously, in what follows, we generalize and strengthen a Ramsey-type coloring theorem of Protasov and Protasova [3] to hold for k colors, where $k \in \mathbb{N}$.

Fix a metric space (X, d) and let $k \in \mathbb{N}$. A mapping $\chi : [X]^k \rightarrow [k]$ is called an *isometric coloring* if $\chi(A_1) = \chi(A_2)$ whenever A_1, A_2 is a pair of isodiametric k -element subsets of X .

2. ULTRAMETRIC ANALYSIS

Recall that a *metric* on a set X is a function $d : X \times X \rightarrow [0, +\infty)$ such that for every $x, y, z \in X$ the following hold:

- (1) $d(x, y) = 0 \Leftrightarrow x = y$;
- (2) $d(x, y) = d(y, x)$; and
- (3) $d(x, y) \leq d(x, z) + d(z, y)$.

The pair (X, d) is called a *metric space*. A stronger version of (3) is called the *strong triangle inequality*:

$$d(x, y) \leq \max\{d(x, z), d(z, y)\}.$$

If the strong triangle inequality holds, then we call d an *ultrametric* and (X, d) an *ultrametric space*. In what follows, unless otherwise specified, (X, d) (or simply X) is an ultrametric space.

2.1. The space $\mathbb{N}^{\mathbb{N}}$ and ε -chains. In this short section, we provide some examples of ultrametric spaces. In particular, we construct ultrametries on the Baire space $\mathbb{N}^{\mathbb{N}}$ and general uniformly disconnected metric spaces.

The simplest example of an ultrametric on a set X is the discrete metric d , where $d(x, y)$ is 1 if $x \neq y$ and 0 otherwise. Another simple example is (\mathbb{N}, d) , where $d(n, m) = \max\{1 + 1/n, 1 + 1/m\}$ if $n \neq m$ and $d(n, m) = 0$ otherwise.

The *Baire space* $\mathbb{N}^{\mathbb{N}}$ is the space of all sequences of natural numbers.

Example 1. For two distinct sequences $x = (x_n), y = (y_n)$ of naturals, we define $m(x, y) = \min\{k \in \mathbb{N} : x_k \neq y_k\}$, the first index at which x and y do not coincide. For distinct sequences $x, y \in \mathbb{N}^{\mathbb{N}}$, set $d(x, y) = m(x, y)^{-1}$ with $d(x, x) = 0$. Then d is an ultrametric on $\mathbb{N}^{\mathbb{N}}$.

Proof. The symmetry and non-negativity of d is immediate. Furthermore, $d(x, x) = 0$ by definition, and if $d(x, y) = 0$ then we must have $x = y$ since $m(x, y) \geq 1$. To prove the strong triangle inequality for d , fix $x, y, z \in \mathbb{N}^{\mathbb{N}}$. We assume x, y, z are all distinct sequences as the result is immediate otherwise. Observe that

$$\begin{aligned} d(x, y) &\leq \max\{d(x, z), d(z, y)\} \\ &\Leftrightarrow m(x, y)^{-1} \leq \max\{m(x, z)^{-1}, m(z, y)^{-1}\} \\ &\Leftrightarrow m(x, y)^{-1} \leq \min\{m(x, z), m(z, y)\}^{-1} \\ &\Leftrightarrow m(x, y) \geq \min\{m(x, z), m(z, y)\}. \end{aligned}$$

Clearly $m(x, y) \geq \min\{m(x, z), m(z, y)\}$: if $\ell = m(x, y)$ then for each $k \in \{1, 2, \dots, \ell - 1\}$ the terms $x_k = y_k$, so if $m(x, z) \geq \ell + 1$ and $m(z, y) \geq \ell + 1$ then $x_\ell = z_\ell = y_\ell$ so that $m(x, y) \geq \ell + 1$ is a contradiction. Therefore, d is an ultrametric. \square

Let (X, d) be any metric space and $\varepsilon > 0$ be given. An ε -chain between the pair $x, y \in X$ is a finite sequence $x = x_0, x_1, \dots, x_n = y$ such that

$$\max_{1 \leq i \leq n} d(x_{i-1}, x_i) \leq \varepsilon.$$

In this case, we say that x, y are ε -connected. Observe that if x, y are ε -connected then they are ε' -connected for every $\varepsilon' \geq \varepsilon$; and if x, y can not be ε -connected then they can not be ε' -connected for every $\varepsilon' \leq \varepsilon$.

The space X is called *uniformly disconnected* if there is an $\varepsilon > 0$ such that no two points in X can be ε -connected.

Example 2. Let (X, d) be a uniformly disconnected metric space. For $x, y \in X$, let $c(x, y)$ be the infimum over all $\varepsilon > 0$ such that x and y are ε -connected. Then c is an ultrametric on X .

Proof. Clearly c is symmetric and non-negative. If $x = y$ then clearly x, y are ε -connected for every $\varepsilon > 0$ so that $c(x, y) = 0$. On the other hand, if $c(x, y) = 0$ then $x = y$, otherwise for each $\varepsilon > 0$ the points x, y are ε -connected, so X is not uniformly disconnected, a contradiction.

The last thing to prove is the strong triangle inequality for c . To this end, fix $x, y, z \in X$ and let $\varepsilon > 0$ be given. We may assume the points x, y, z are distinct, otherwise the claim is immediate. By definition of the infimum, there exist $\gamma_1, \gamma_2 > 0$ with $\gamma_1 \leq c(x, z) + \varepsilon$ and $\gamma_2 \leq c(z, y) + \varepsilon$ such that x, z are γ_1 -connected and z, y are γ_2 -connected. Set $\gamma = \max\{\gamma_1, \gamma_2\}$ and note that by combining the x, z and z, y chains it follows that x, y are γ -connected. Thus

$$c(x, y) \leq \gamma \leq \max\{c(x, z), c(z, y)\} + \varepsilon.$$

Since ε was arbitrary, it follows that $c(x, y) \leq \max\{c(x, z), c(z, y)\}$. \square

2.2. Properties of ultrametric spaces.

Lemma 2.1. *If x, y, z are distinct points in an ultrametric space X and $d(x, y) < d(y, z)$ then $d(y, z) = d(x, z)$. (That is, every triangle in an ultrametric space is isosceles.)*

Proof. Note that d is an ultrametric and $d(x, y) < d(y, z)$ so that

$$d(x, z) \leq \max\{d(x, y), d(y, z)\} = d(y, z),$$

and since $d(y, z) > d(x, y)$, we have $d(x, y) < d(x, z)$ as otherwise

$$d(y, z) \leq \max\{d(y, x), d(x, z)\} = \max\{d(x, y), d(x, z)\} = d(x, y)$$

is a contradiction. Thus $d(y, z) \leq \max\{d(x, y), d(x, z)\} = d(x, z)$. Combining everything together, we obtain $d(y, z) = d(x, z)$. \square

An *open ball* (or simply a *ball*) of radius $\varepsilon > 0$ centered about a point $x \in X$ is defined as the set $B_\varepsilon(x) := \{y \in X : d(x, y) < \varepsilon\}$. If left unspecified, note that x denotes a point in X and ε a positive real number.

A subset \mathcal{O} of X is called *open* if it can be written as a union of open balls. A set is *closed* if its complement is open. Open balls

and general sets in ultrametric spaces exhibit properties that are quite counterintuitive, as delineated by the following simple lemmas.

Lemma 2.2. *Let $B_\varepsilon(x)$ be an open ball in X . Then $B_\varepsilon(x) = B_\varepsilon(y)$ for every point $y \in B_\varepsilon(x)$. (That is, every point in a ball is its center.)*

Proof. Fix $y \in B_\varepsilon(x)$ so that $d(x, y) < \varepsilon$. If $t \in B_\varepsilon(y)$ then $d(y, t) < \varepsilon$. Then $t \in B_\varepsilon(x)$ since

$$d(x, t) \leq \max\{d(x, y), d(y, t)\} < \varepsilon.$$

The reverse inclusion follows symmetrically as $x \in B_\varepsilon(y)$. \square

Lemma 2.3. *Let $A \subseteq X$ be non-empty with $a \in A$. Then $\text{diam } A = \sup\{d(a, x) : x \in A\}$.*

Proof. Set $u = \sup\{d(a, x) : x \in A\}$ and fix $x, y \in A$. Then

$$d(x, y) \leq \max\{d(x, a), d(a, y)\} = \max\{d(a, x), d(a, y)\} \leq u$$

so that u is an upper bound of $D = \{d(x, y) : x, y \in A\}$. Since $u = \sup\{d(a, x) : x \in A\}$, for any given $\varepsilon > 0$, there exists a point $x_\varepsilon \in A$ with $u \leq d(a, x_\varepsilon) + \varepsilon$. But $a, x_\varepsilon \in A$ so that $u = \sup D = \text{diam } A$ which completes the proof. \square

Lemma 2.4. *Let $A \subseteq X$ and consider any ball $B_\varepsilon(x)$ in X . If $B = A \cap B_\varepsilon(x) \neq \emptyset$ and $|B| \geq 2$ then B is a ball.*

Proof. Since $A \cap B_\varepsilon(x) \neq \emptyset$, there is a point $a \in A \cap B_\varepsilon(x)$. Set $r = \text{diam}(A \cap B_\varepsilon(x))$. Then from (lemma),

$$r = \text{diam}(A \cap B_\varepsilon(x)) = \sup\{d(a, t) : t \in A \cap B_\varepsilon(x)\}.$$

We claim that $B_r(a) = A \cap B_\varepsilon(x)$. To this end, first fix $t \in B_r(a)$ so that $d(a, t) < \text{diam}(A \cap B_\varepsilon(x))$ \square

Lemma 2.5. *Fix $\varepsilon_1, \varepsilon_2 > 0$ and $x, y \in X$. If $B_{\varepsilon_1}(x) \cap B_{\varepsilon_2}(y) \neq \emptyset$ then $B_{\varepsilon_1}(x) \subseteq B_{\varepsilon_2}(y)$ or $B_{\varepsilon_2}(y) \subseteq B_{\varepsilon_1}(x)$. (That is, two balls are either disjoint or one of them contains the other.)*

Proof. This is a weaker version of (obs), but we give the statement its own proof for clarity. Assume without loss of generality that $\varepsilon_1 \leq \varepsilon_2$. If $B_{\varepsilon_1}(x) \cap B_{\varepsilon_2}(y) \neq \emptyset$, there is a point $t \in X$ with $t \in B_{\varepsilon_1}(x)$ and $t \in B_{\varepsilon_2}(y)$. By (lemma) and since $\varepsilon_1 \leq \varepsilon_2$, we obtain $B_{\varepsilon_1}(x) = B_{\varepsilon_1}(t) \subseteq B_{\varepsilon_2}(t) = B_{\varepsilon_2}(y)$ as required. \square

Lemma 2.6. *If $B_\varepsilon(x)$ is an open ball in X , then $X \setminus B_\varepsilon(x)$ is a union of open balls. (That is, open balls are closed in X .)*

Proof. Suppose for a contradiction that there is a point $t \in X \setminus B_\varepsilon(x)$ such that for every $r > 0$ the ball $B_r(t)$ is not contained in $X \setminus B_\varepsilon(x)$, so $B_r(t) \cap B_\varepsilon(x) \neq \emptyset$. Setting $r = \varepsilon$ and using (lemma), we have $B_\varepsilon(t) \subseteq B_\varepsilon(x)$ or $B_\varepsilon(t) \subseteq B_\varepsilon(x)$. In either case, (lemma) implies equality. But then $t \in X \setminus B_\varepsilon(x) = X \setminus B_\varepsilon(t)$ is a contradiction. Hence $X \setminus B_\varepsilon(x)$ is a union of open balls so that $B_\varepsilon(x)$ is closed in X . \square

3. FILTERS

Lemma 3.1. *If \mathcal{K} is a family of subsets satisfying the FIP, then there is a filter \mathcal{F} containing each element of \mathcal{K} .*

Proof. First let $\mathcal{F}' = \mathcal{K} \cup \mathcal{I}$, where \mathcal{I} is the set of all finite intersections of elements of \mathcal{K} . Hence \mathcal{F}' is closed under finite intersections. Then, let $\mathcal{F} = \mathcal{F}' \cup \mathcal{S}$, where $A \in \mathcal{S}$ if and only if A contains a set in \mathcal{F}' .

Clearly \mathcal{F} is closed when taking supersets. If $A, B \in \mathcal{F}$, the only non-trivial needing consideration is, without loss of generality, when $A \in \mathcal{S}$. So, if $B \in \mathcal{S}$ or $B \in \mathcal{F}'$, then there exist sets $A', B' \in \mathcal{F}'$ such that $A' \subseteq A$ and $B' \subseteq B$ (if $B \in \mathcal{F}'$ then $B' = B$). Since \mathcal{F}' is closed under finite intersections, $A' \cap B' \in \mathcal{F}'$. Then, from $A' \cap B' \subseteq A \cap B$ it follows that $A \cap B \in \mathcal{S} \subseteq \mathcal{F}$. Hence \mathcal{F} is closed under finite intersections. \square

Lemma 3.2. *A family \mathcal{F} is an ultrafilter if and only if for every subset $A \subseteq \mathbb{N}$ either $A \in \mathcal{F}$ or $A^c \in \mathcal{F}$.*

Proof. Let \mathcal{F} be an ultrafilter and suppose for a contradiction that there is a subset $A \subseteq \mathbb{N}$ with $A, A^c \notin \mathcal{F}$. So every set in \mathcal{F} intersects both A and A^c . It follows from (lemma) that the filter extending $\mathcal{F} \cup \{A, A^c\}$ properly contains \mathcal{F} , which is a contradiction. Now suppose \mathcal{F} is a filter such that for every subset $A \subseteq \mathbb{N}$, either $A \in \mathcal{F}$ or $A^c \in \mathcal{F}$. Assume for a contradiction that \mathcal{F}' is a filter which properly contains \mathcal{F} . So there is a subset $E \subseteq \mathbb{N}$ with $E \in \mathcal{F}'$ and $E \notin \mathcal{F}$. Thus $E^c \in \mathcal{F}$ and hence $E^c \in \mathcal{F}'$. But then \mathcal{F}' contains the emptyset since it is closed under finite intersections and $\emptyset = E \cap E^c \in \mathcal{F}'$, a contradiction. \square

Lemma 3.3. *A family \mathcal{F} is an ultrafilter if and only if \mathcal{F} has the Ramsey property.*

Proof. Let \mathcal{F} be an ultrafilter and suppose for a contradiction that $A = A_1 \cup A_2$ is in \mathcal{F} but $A_1, A_2 \notin \mathcal{F}$. Hence (lemma) we have $A_1^c, A_2^c \in \mathcal{F}$ so that $A^c = A_1^c \cap A_2^c \in \mathcal{F}$. Consequently, $\emptyset = A \cap A^c \in \mathcal{F}$ is a contradiction. The case where $A = A_1 \cup \dots \cup A_n$ follows via elementary induction. Conversely, assume \mathcal{F} has the Ramsey property. If \mathcal{F} is not an ultrafilter, then there is a subset $A \subseteq \mathbb{N}$ with $A, A^c \notin \mathcal{F}$. But $\mathbb{N} \in \mathcal{F}$ and $\mathbb{N} = A \cup A^c$, so we must have $A \in \mathcal{F}$ or $A^c \in \mathcal{F}$, a contradiction. \square

Proposition 3.1. *A family \mathcal{F} is an ultrafilter if and only if \mathcal{F}^* is a filter.*

Proof. Let us first assume that \mathcal{F} is an ultrafilter, and fix $A, B \in \mathcal{F}^*$. To prove that $A \cap B \in \mathcal{F}^*$, it suffices to fix a set $E \in \mathcal{F}$ and prove that E intersects $A \cap B$. Noting that since $A \in \mathcal{F}^*$ we have $E \cap A \neq \emptyset$, write

$$E = (E \cap A) \cup (E \setminus A).$$

Certainly $E \setminus A \notin \mathcal{F}$, as otherwise $A \in \mathcal{F}^*$ implies $(E \setminus A) \cap A \neq \emptyset$. (lemma) implies that \mathcal{F} satisfies the Ramsey property so that the set $E \cap A \in \mathcal{F}$. Since $B \in \mathcal{F}^*$, it follows as needed that $E \cap A \cap B \neq \emptyset$.

Conversely, we suppose \mathcal{F}^* is a filter. Suppose towards a contradiction that \mathcal{F} is not an ultrafilter. Hence by (lemma) there is a subset $A \subseteq \mathbb{N}$ with $A, A^c \notin \mathcal{F}$. So no set in \mathcal{F} is contained in A and likewise for A^c . That is, for every set $B \in \mathcal{F}$, we have $A \cap B \neq \emptyset$ and $A^c \cap B \neq \emptyset$. By definition, then, $A, A^c \in \mathcal{F}^*$. But \mathcal{F}^* is a filter, so $\emptyset = A \cap A^c \in \mathcal{F}^*$ is a contradiction, and the proof is complete. \square

4. MAIN RESULT

4.1. Some Lemmas and Preliminaries.

Lemma 4.1. *Let (X, d) be an infinite metric space. Then there is a sequence $\{x_n\}_{n=1}^{\infty}$ of distinct points in X such that either*

- (1) *The sequence $\{d(x_1, x_n)\}_{n=1}^{\infty}$ is strictly monotone; or*
- (2) *For every $n \in \mathbb{N}$ and $i, j \geq n$ the distances $d(x_n, x_i) = d(x_n, x_j)$.*

Proof. We first assume that there is a point $x_0 \in X$ such that $d(x_0, X) := \{d(x_0, x_n) : x_n \in X\}$ is not finite. Hence, there is a countably infinite subset $E \subseteq X$ with $x_0 \notin E$ and $d(x_0, x) \neq d(x_0, y)$ for every $x, y \in E$. We obtain from E the sequence $\xi = \{d(x_0, x) : x \in E\}$. Since ξ is a sequence of reals, it has a monotone subsequence $\{d(x_0, x_n)\}_{n=1}^{\infty}$ whose points are distinct by construction of E . Since d is a metric, $x_i \neq x_j$ for every $i, j \in \mathbb{N}$ and so $\{x_n\}_{n=1}^{\infty}$ is the desired sequence.

Otherwise, $d(x, X)$ is finite for every $x \in X$. Fix $x_0 \in X$ and assume without loss of generality that $\ell_1 \in d(x_0, X)$ is non-zero. Let E_1 be a countable subset of X with $E_1 \subseteq \{x \in X : d(x_0, x) = \ell_1\}$. Choose $x_1 \in E_1$, and note that $d(x_0, E_1) = \{\ell_1\}$ is a singleton set and $x_0 \notin E_1$.

For $n \geq 2$, we choose x_n and define E_n inductively as follows. As above, let $\ell_n \in d(x_{n-1}, X)$ be non-zero and let E_n be a countable subset of X with $E_n \subseteq \{x \in X : d(x_{n-1}, x) = \ell_n\}$. Again, we choose $x_n \in E_n$ and observe that $d(x_{n-1}, E_n) = \{\ell_n\}$. Continuing this way, we obtain the sequence from (2). \square

Set $k \in \mathbb{N}$ and let X be a metric space. A **k -coloring** on X is a function $\chi : [X]^k \rightarrow [k]$, where $[X]^k = \{\{x_1, x_2, \dots, x_k\} : x_i \in X, \forall i \in [k]\}$ is the set of all k element subsets of X . A subset $A \subseteq X$ is called **χ -monochrome** if χ is constant on $[A]^k$. The coloring χ is called **k -isometric** if $\chi(A_1) = \chi(A_2)$ whenever the pair $A_1, A_2 \in [X]^k$ of k element subsets satisfy $\text{diam } A_1 = \text{diam } A_2$.

A free ultrafilter \mathcal{F} on X is called **k -Ramsey** with respect to a collection \mathcal{C} of colorings on X if for every coloring $\chi \in \mathcal{C}$ there is a set $A \in \mathcal{F}$ such that $[A]^k$ is χ -monochrome.

4.2. Main Result. (Authors) in [3] prove that every free ultrafilter on an infinite ultrametric space X is 2-Ramsey with respect to the class of 2-isometric colorings on X . In this particular case, a coloring χ is 2-isometric if and only if all points $x_1, x_2, y_1, y_2 \in X$ with $d(x_1, y_1) = d(x_2, y_2)$ satisfy $\chi(\{x_1, y_1\}) = \chi(\{x_2, y_2\})$. (Authors) leverage the properties of the ultrametric coupled with this observation and (lemma) to prove the main result when $k = 2$. We expand on this approach, strengthening their result to hold for all k -colorings in the particular case of ultrametric spaces.

Theorem 4.1. *Let (X, d) be an infinite ultrametric space and let k be a positive integer. Let $E = \{c_n\}_{n=1}^\infty$ be a sequence of points obtained as in (lemma). Then every free ultrafilter \mathcal{F} in X containing E is k -Ramsey with respect to the collection \mathcal{C} of k -isometric colorings on X .*

Proof. Let $\chi \in \mathcal{C}$ be a k -isometric coloring on X and fix a free ultrafilter \mathcal{F} which contains E .

Let $h : E \rightarrow \mathbb{R}^+$ be a fixed map. How we define h will depend on the sequence E attained from (lemma), so we will proceed in this regard later on. Moreover, suppose $f : h(E) \rightarrow [k]$ is any mapping satisfying $f(h(c_\ell)) = \chi(A)$ whenever $A \in [X]^k$ is a k element subset of X with $\text{diam } A = h(c_\ell)$. Finally, we set $c = f \circ h$.

Write $E = c^{-1}([k]) = \cup_{j=1}^k c^{-1}(\{j\})$ and observe that since $E \in \mathcal{F}$, (lemma) implies that there is a color $\varphi \in [k]$ with $c^{-1}(\{\varphi\}) \in \mathcal{F}$. We set $A = c^{-1}(\{\varphi\})$ and complete the proof by showing that A is χ -monochrome. In particular, we will show that each k element set in $[A]^k$ has color φ .

Let $n_1 < n_2 < \dots < n_k$ be fixed positive integers and consider the k element subsequence $C_k = \{c_{n_1}, c_{n_2}, \dots, c_{n_k}\} \in [A]^k$ of E . Assume integers $n_i < n_j$ are such that

$$\{c_{n_i}, c_{n_j}\} = \arg \max_{\{x, y\} \subseteq C_k} d(x, y);$$

that is, $d(c_{n_i}, c_{n_j}) = \text{diam } C_k$. We now consider the conditions on E as described in (lemma), and define h accordingly to complete the proof.

Case 1. We will first assume that case (1) of (lemma) holds, namely that $\{c_n\}_{n=1}^\infty$ is a sequence of distinct points in X where $\{d(c_0, c_n)\}_{n=1}^\infty$ is strictly monotone. In this case, h will indicate the distance to c_0 from a term $c_\ell \in E$, given by $h(c_\ell) = d(c_0, c_\ell)$.

If h is strictly increasing, then we have $d(c_0, c_{n_i}) < d(c_0, c_{n_j})$. Hence (lemma) implies that $d(c_{n_i}, c_{n_j}) = d(c_0, c_{n_j})$, since d is an ultrametric. Otherwise $d(c_0, c_{n_i}) > d(c_0, c_{n_j})$ so that $d(c_{n_i}, c_{n_j}) = d(c_0, c_{n_i})$ using (lemma) once more. Possibly swapping the symbols i, j , we assume the latter case holds. Since $c_{n_j} \in A$ and $\text{diam } C_k = d(c_{n_i}, c_{n_j}) = d(c_0, c_{n_j})$, we have $f(d(c_0, c_{n_j})) = \chi(C_k) = \varphi$, as needed.

Case 2. We now assume that case (2) of (lemma) applies to $\{c_n\}_{n=1}^\infty$. Namely, for each $n \in \mathbb{N}$ and $i, j \geq n$ we have $d(c_n, c_i) = d(c_n, c_j)$. Define $h : E \rightarrow \mathbb{R}^+$ by $h(c_\ell) = d(c_\ell, c_{\ell+1})$, and note that $h(c_\ell) = d(c_\ell, c_j)$ for every $j > \ell$. Since $n_i < n_j$, we have $h(c_{n_i}) = d(c_{n_i}, c_{n_j})$ and, as desired,

$$f(h(c_{n_i})) = f(d(c_{n_i}, c_{n_j})) = \chi(C_k) = \varphi.$$

This completes the proof, since A is χ -monochrome in both cases. \square