MATH 350; Assignment 4; Jake R. Gameroff; ID: 261115879.

1. Prove that every graph G has a bipartite subgraph H such that $V(H) = V(G)$ and $\deg_H v \geq \frac{1}{2} \deg_G v$ for each $v \in V(G)$.

Proof. Let G be any graph. For each partition (A, B) of $V(G)$, denote by $E_{A,B}$ the set of edges in $E(G)$ with an end in A and another end in B. Let (X_1, X_2) be a partition of $V(G)$ such that $|E_{X_1,X_2}|$ is maximal, and let $E \coloneqq E_{X_1,X_2}$. Define the subgraph $H \subseteq G$ by $V(H) = V(G)$ and $E(H) = E$. Then H is bipartite since by construction (X_1, X_2) is a bipartition of H.

Suppose for the sake of contradiction that there is a vertex $v \in V(G)$ with $2\deg_H v < \deg_G v$. Without loss of generality, suppose $v \in X_1$. This means that there are at least $\deg_H v + 1$ edges in $E(G)$ incident to v and with both ends in X_1. So, there are $\deg_H v$ edges which contain v as an end and whose other end is in X_2; moving v from X_1 to X_2 would admit at least $\deg_H v + 1$ such edges containing v as an end and whose other end is in $X_1 \setminus \{v\}$. That is, instead of $\deg_H v + 1$ edges with ends both in X_1, we obtain $\deg_H v$ edges with both ends in X_2 instead.

Concretely, if $X \coloneqq X_1 \setminus \{v\}$ and $Y \coloneqq X_2 \cup \{v\}$, the partition (X, Y) of $V(G)$ would admit a set $E_{X,Y}$ of edges as defined above such that $|E_{X,Y}| \geq |E| + 1$, contradicting the maximality of E under the partition (X_1, X_2). Therefore, $H \subseteq G$ is a bipartite subgraph of G with $V(H) = V(G)$ and $\deg_H v \geq \frac{1}{2} \deg_G v$ for each $v \in V(G)$. ∎

2. Let $k \geq 4$ be an integer. Let G be a bipartite graph such that for every $v \in V(G)$, $3 \leq \deg v \leq k$. Show that there exists a matching in G of size at least $\frac{3|V(G)|}{2k}$.

Proof. Let $k \geq 4$ and G be a bipartite graph such that for each vertex $v \in V(G)$, $3 \leq \deg v \leq k$. Since G is bipartite, König's theorem implies that $\nu(G) = \tau(G)$. We will show that $\tau(G) \geq \frac{3|V(G)|}{2k}$ to prove the claim.

We first note that $|E(G)| \geq \frac{3|V(G)|}{2}$. Indeed, for each vertex $v \in V(G)$, there exist at least 3 edges in $E(G)$ which are incident to v. But since an edge has two ends, we count each edge twice (once for each end), which is why we divide $3|V(G)|$ by 2 to obtain the minimum number of edges in G.

Let $X \subseteq V(G)$ be a minimal vertex cover; that is, $|X| = \tau(G)$. It follows that $|E(G)| \leq k|X|$, as each edge $e \in E(G)$ has an end in X by definition and each $v \in X$ has degree at most k.

Taking these two deductions together, we obtain

$$\tau(G) = |X| \geq \frac{|E(G)|}{k} \geq \frac{\frac{3|V(G)|}{2}}{k} = \frac{3|V(G)|}{2k}.$$

Therefore, we conclude that $\tau(G) = \nu(G) \geq \frac{3|V(G)|}{2k}$. Thus, G must have a matching of size at least $\frac{3|V(G)|}{2k}$, as required. ∎

3. Let G be a bipartite graph with bipartition (A, B) in which every vertex has degree at least one. Assume that for every edge of G with ends $a \in A$ and $b \in B$ we have $\deg a \geq \deg b$. Show that there exists a matching in G covering A.

Proof. We show that there exists a matching $\mathcal{M} \subseteq E(G)$ such that each vertex $a \in A$ is incident to an edge of $\mathcal{M}$. By Hall's theorem, it suffices to show that for each subset $S \subseteq A$, $|N(S)| \geq |S|$. Suppose for the sake of contradiction that there is no matching in G covering A. Thus, there is a set $S \subseteq A$ such that $|S| > |N(S)|$. Therefore, let S be chosen such that $|S| > |N(S)|$ and $|S|$ is minimal.

Since $|S| > |N(S)|$, there exists $b \in N(S)$ such that b is adjacent to two vertices $v_1, v_2 \in S$, as otherwise, $|S| = |N(S)|$. Indeed, in this case, there is a one-to-one correspondence between vertices in S and $N(S)$ since every vertex $b \in N(S)$ would be adjacent to exactly one $v \in S$; and since there are no vertices of degree < 1 in G, this means that each $v \in S$ is adjacent to exactly one $b \in N(S)$.

Let $u \in S$ be fixed and define $S' \coloneqq S \setminus \{u\}$. By the minimality of $|S|$, we have $|S'| \leq |N(S')|$. In fact, $|S'| = |N(S')|$, as otherwise $|N(S')| > |S'|$ implies $|N(S')| \geq |S'| + 1$ so that

$$|N(S)| \geq |N(S')| \geq |S'| + 1 = |S|$$

contradicts the choice of S.

Let X denote the set of edges with one end in S' and another in $N(S')$. Clearly, $\sum_{a \in S'} \deg a \leq |X|$, as for each $a \in S'$, $\deg a$ corresponds to the number of its neighbours, which are each in $N(S')$ and not in A since G is bipartite and by definition of $N(S')$; thus, for each neighbour of a there is a unique edge in X. Similarly, $|X| \leq \sum_{b \in N(S')} \deg b$, since each $b \in N(S')$ admits at most $\deg b$ edges in X, i.e. this holds as b can also be adjacent to other vertices not in S', so there may exist edges with an end in $N(S')$ and another not in S'.

We now combine these two observations to complete the proof. Since $|N(S')| = |S'|$ and each pair of adjacent vertices $a \in A,\ b \in B$ satisfy $\deg a \geq \deg b$, we deduce that for each $b \in N(S')$ (resp. $a \in S'$) there exists an $a \in S'$ (resp. $b \in N(S')$) adjacent to b (resp. a) so that $\deg b \leq \deg a$. Summing over all such vertices, we obtain

$$\sum_{b \in N(S')} \deg b \leq \sum_{a \in S'} \deg a \leq |X| \leq \sum_{b \in N(S')} \deg b$$

so that $\sum_{b \in N(S')} \deg b = |X|$.

Without loss of generality, suppose $N(\{u\}) \subseteq N(S')$, as if there was a vertex $\ell \in N(S) \setminus N(S')$ adjacent to u, then $|S| = |S'| + 1 \leq |N(S')| + 1 \leq |N(S)|$ and there is nothing to prove.[1] Thus, we have must have $N(S) = N(S') \cup N(\{u\}) = N(S')$. Since $\sum_{b \in N(S')} \deg b = |X|$, this means that each neighbour of every vertex in $N(S') = N(S)$ is in S', i.e. $N(N(S)) \subseteq S'$. This follows since each $b \in N(S')$ has a neighbour that is a vertex in S', corresponding to a unique edge in X, so if $b \in N(S')$ has a neighbour in $A \setminus S'$, then $\sum_{b \in N(S')} \deg b \geq |X| + 1$.

But note that since $\deg u \geq 1$, it has a neighbour $b \in N(S)$ and all neighbours of b (in particular, u) must be in S' since $N(N(S)) \subseteq S'$, thus $u \in S' = S \setminus \{u\}$, a contradiction. Therefore, we conclude that there exists a matching in G which covers A, thereby completing the proof. ■

[1] The last inequality holds as there is at least one more element in $N(S)$ than in $N(S')$, namely ℓ.