MATH 350; Assignment 2; Jake R. Gameroff; ID: 261115879.

Question 1. Let $d_1, d_2, \ldots, d_n$ be positive integers with $n \geq 2$. Prove that there exists a tree T with vertex degrees $d_1, d_2, \ldots, d_n$ if and only if $\sum_{i=1}^{n} d_i = 2n - 2$.

Proof.
[$\implies$] Let $n \geq 2$ be fixed and $d_1, d_2, \ldots, d_n$ be positive integers. Suppose T is a tree with $V(T) \coloneqq \{v_1, v_2, \ldots, v_n\}$ such that for $1 \leq i \leq n$, $\deg v_i = d_i$. Then

$$\begin{aligned}
\sum_{i=1}^{n} d_i &= \sum_{v_i \in V(T)} \deg v_i \\
&= 2|E(T)| && \text{(By handshake lemma from lecture)} \\
&= 2(|V(T)| - 1) && (*) \\
&= 2(n-1) = 2n - 2.
\end{aligned}$$

We note that $(*)$ holds by lemma 3.1 (F non-null forest, then $\text{comp}(F) = |V(F)| - |E(F)|$) applied to a tree T which must have $\text{comp}(T) = 1$ since it is connected by definition: $1 = |V(T)| - |E(T)| \implies |E(T)| = |V(T)| - 1$. Thus, $\sum_{i=1}^{n} d_i = 2n - 2$ as required.

[$\impliedby$] We induct on n. For $n = 2$ (base case), $d_1 + d_2 = 2 \implies d_1 = d_2 = 1$, since $d_1, d_2 > 0$. Hence, we construct the complete graph on 2 vertices, which contains no cycles, is connected, and is non-null trivially. Hence, we have constructed a tree with the desired properties, thereby completing the base case step.

Suppose $d_1, d_2, \ldots, d_n, d_{n+1}$ is a sequence of positive integers for which $\sum_{i=1}^{n+1} d_i = 2(n+1) - 2 = 2n + 2 - 2 = 2n$ for some $n \geq 2$. As our inductive hypothesis, we suppose that if $d_1, d_2, \ldots, d_n > 0$ are such that $\sum_{i=1}^{n} d_i = 2n - 2$, then there exists a tree with precisely these vertex degrees.

Notice that $n + 1 < \sum_{i=1}^{n+1} d_i = 2n < 2(n+1)$ implies that there is at least one vertex degree $d_i \geq 2$, (otherwise the sum would equal $n + 1$) and at least one vertex degree $d_j = 1$ (otherwise the sum would be $\geq 2(n+1)$).

Without loss of generality, we can assign $d_j = d_{n+1} = 1$, and $d_i = d_n \geq 2$, since out of $d_1, d_2, \ldots, d_{n+1}$ there is at least one vertex for each of these requirements; hence we can re-label the vertices so that this holds. With this, define $c_n \coloneqq d_n - 1 > 0$ so that

$$\begin{aligned}
d_1 + d_2 + \cdots + d_n + d_{n+1} = 2n \implies d_1 + d_2 + \cdots + d_{n-1} + d_n - 1 &= \sum_{i=1}^{n-1} d_i + c_n \\
&= 2n - 1 - d_{n+1} = 2n - 2.
\end{aligned}$$

Thus, the inductive hypothesis implies that there exists a tree T_n with vertex degrees $d_1, d_2, \ldots, d_n - 1 = c_n$ for vertices $v_1, v_2, \ldots, v_n$. Suppose without loss of generality that the vertex $v_n \in V(T_n)$ is such that $\deg v_n = c_n$. Then we construct a tree T from T_n with vertex degrees $d_1, \ldots, d_{n+1}$. Extend the vertex set of T_n to $V(T) \coloneqq V(T_n) \cup \{v_{n+1}\} \coloneqq \{v_1, v_2, \ldots, v_{n+1}\}$ and we set the vertex v_{n+1} to be neighbours with the vertex v_n, giving it degree $\deg v_n = c_n + 1 = d_n$. Hence $E(T) \coloneqq E(T_n) \cup \{v_n v_{n+1}\}$.

To see that T is a tree, notice that $T \setminus v_{n+1}$ is a tree by the inductive hypothesis and v_{n+1} is a leaf; applying lemma 3.4 from lecture implies that T is hence a tree. Therefore, we have constructed a tree with vertex degrees $d_1, d_2, \ldots, d_{n+1}$, thereby completing the induction and hence the proof. ∎

Question 2. Let G be a non-null graph such that for every pair of vertices $u, v \in V(G)$ there exists a path in G from u to v of length at most k. Show that either G contains a cycle of length at most $2k+1$ or G is a tree.

Proof. Let G be a non-null graph such that for every pair of vertices $u, v \in V(G)$ there is a path in G from u to v of length at most k. We must prove two things:

1. Suppose G does not contain a cycle of length at most $2k+1$. Then we must show that G is a tree. We note that this does not immediately imply that G contains no cycles; to conclude that G has no cycles, we must prove that there can not exist a cycle in G of length greater than $2k+1$, given the assumption that there does not exist a cycle of length $\leq 2k+1$ in G. Hence, suppose there exists a cycle C in G of length $2k+j$, $j \geq 2$, with

$$V(C) \coloneqq \{v_1, v_2, \ldots, v_{2k+j}\} \text{ and } E(C) \coloneqq \{v_1v_2, v_2v_3, \ldots, v_{2k+j-1}v_{2k+j}, v_{2k+j}v_1\}.$$

Thus there is a path P_1 from v_1 to v_{k+2} in G (and along C) of length $k+1$. By hypothesis, there is a path from v_{k+2} to v_1 of length $\leq k$. Hence, we know that there is a path P_2 of lesser length from v_{k+2} to v_1 than that from v_1 to v_{k+2} along C. Then either $(V(P_2) \setminus \{v_1, v_{k+2}\}) \cap V(P_1) = \emptyset$, or there is some index $1 < i < k+2$ such that P_2 is a path from v_{k+2} to v_i (along P_1) to v_1 (along vertices not in $V(P_1)$); in either case, the length of P_2 is $\leq k$. So let $1 < j \leq k+2$ denote the least index for which there is a sub-path P_2' of P_2 from v_j to v_1 along vertices that are not in $V(P_1)$ (aside from v_1 and v_j), i.e. P_2' is the path with one end v_j in $V(P_1) \setminus \{v_1\}$ and another equaling v_1, where $1 < j \leq k+2$ is minimal and P_2' only intersects P_1 at v_1 and v_j.

Then the sub-path of P_1 from v_1 to v_j has length $\leq k+1$ and the path P_2' from v_j to v_1 has length $\leq k$; hence, we have constructed a cycle of length $\leq k+k+1 = 2k+1$, a contradiction to the choice of G. Therefore, we conclude that G has no cycles.

Since there exists a path between any two vertices $u, v \in V(G)$, G is connected by definition. Finally, since we assumed that G was non-null, we conclude that G is a non-null connected graph with no cycles, i.e. G is a tree by definition, as was to be shown.

2. Suppose G is not a tree. Then we must show that G contains a cycle of length at most $2k+1$. Since G is connected and non-null by hypothesis, we must have that G contains a cycle (otherwise G is a tree). Suppose this cycle has length $= 2k+j$ for $j \geq 2$. Then, we can apply the argument above to construct a cycle in G whose length is $\leq 2k+1$. If G does not have a cycle of length $\geq 2k+2$, then since G contains a cycle, its length must be $\leq 2k+1$, as needed. Therefore, if G has the hypothesized properties and is not a tree, then we can always construct a cycle in G of length at most $2k+1$, as needed.

Therefore, we have shown that either G contains a cycle of length at most $2k+1$ or G is a tree, thereby completing the proof. ■

__**Question 3.**__ Let T be a tree, and let $T_1, \dots, T_n$ be connected subgraphs of T so that $V(T_i \cap T_j) \neq \emptyset$ for all i, j with $1 \leq i < j \leq n$. Show that

$$V(T_1 \cap T_2 \cap \cdots \cap T_n) \neq \emptyset.$$

Hint. Delete a leaf and use induction on $|V(T)|$.

Proof. We induct on $|V(T)|$ and follow the hint.

Base case. If T is a tree with $|V(T)| = 1$ and connected subgraphs $T_1, \dots, T_n$ that pairwise intersect, then $V(T_1) = V(T_2) = \cdots = V(T_n) \implies \bigcap_{i=1}^n V(T_i) = V(T) \neq \emptyset$, since T only has one vertex and the T_i's can not have an empty vertex sets since they pairwise intersect; hence the base case step is complete.

Induction. Suppose that for any tree T with $|V(T)| = k-1$ for some $k \geq 2$, if $T_1, \dots, T_n$ are connected subgraphs of T such that $V(T_i) \cap V(T_j) \neq \emptyset$ for each $1 \leq i < j \leq n$, then $\bigcap_{i=1}^n V(T_i) \neq \emptyset$. So let T be a tree on k vertices with connected subgraphs $T_1, \dots, T_n$ where $V(T_i) \cap V(T_j) \neq \emptyset$ whenever $1 \leq i < j \leq n$. Let $\ell \in V(T)$ be a leaf, there must be at least one by lemma 3.2 (i.e. we may deduce from which that since T is a tree with ≥ 2 vertices, it has at least two leaves).

We can suppose that there does not exist a subgraph T_i for $1 \leq i \leq n$ such that $V(T_i) = \{\ell\}$, as otherwise, the fact that the connected subgraphs of T pairwise intersect implies that for each $j = 1, 2, \dots, n$, $V(T_i) \cap V(T_j) = \{\ell\} \implies \ell \in \bigcap_{i=1}^n V(T_i)$, which would complete the proof.

By lemma 3.4 (lecture), we know that $J := T \setminus \ell$ is a tree, since T is a tree and ℓ is a leaf. Likewise, since $T_1, T_2, \dots, T_n$ are (non-null) connected subgraphs of a tree T, they can not contain cycles, which implies that they are subtrees of T. Hence, for $i = 1, 2, \dots, n$, define the tree $J_i := T_i \setminus \ell$ (if T_i does not contain the vertex ℓ, then let $J_i = T_i$). We now show that the vertex sets of the J_i's pairwise intersect.

Suppose otherwise; then there exist i, j with $1 \leq i < j \leq n$ such that $V(J_i) \cap V(J_j) = \emptyset$. But recall that $V(T_i) \cap V(T_j) \neq \emptyset$ by supposition, hence $V(J_i) \cap V(J_j) = V(T_i \setminus \ell) \cap V(T_j \setminus \ell) = \emptyset$ implies that $\ell \in V(T_i) \cap V(T_j)$. But since $V(T_i), V(T_j) \neq \{\ell\}$ by supposition, they must both contain at least one other vertex. Thus, if we denote the neighbour of ℓ by $p \in V(T)$, we must have that $p \in V(T_i)$ and $p \in V(T_j)$ by connectedness (since there must be a path in T_i from its other vertex to ℓ, and since ℓ is a leaf, this path must include p; likewise for T_j). But this is a contradiction, as this implies that $p \in V(J_i) \cap V(J_j) \neq \emptyset$.

Therefore, $V(J_i) \cap V(J_j) \neq \emptyset$ whenever $1 \leq i < j \leq n$. Since $|V(J)| = |V(T)| - 1 = k - 1$, the inductive hypothesis implies that $\bigcap_{i=1}^n V(J_i) \neq \emptyset$. Thus, there is a vertex

$$w \in \bigcap_{i=1}^n V(J_i) = \bigcap_{i=1}^n V(T_i \setminus \ell) = \bigcap_{i=1}^n (V(T_i) \setminus \{\ell\}) \subseteq \bigcap_{i=1}^n V(T_i) \implies w \in \bigcap_{i=1}^n V(T_i).$$

Thus, we have shown that $V(T_1) \cap V(T_2) \cap \cdots V(T_n) \neq \emptyset$, thereby completing the inductive step.

Hence, we conclude that for a tree T with connected subgraphs $T_1, \dots, T_n$ whose vertex sets intersect pairwise, the global intersection of their vertex sets is non-empty.

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