MATH 350; Assignment 6; Jake R. Gameroff; ID: 261115879.

Problem 1. Let D be a directed graph and for each edge $e \in E(D)$ let $\varphi(e) \geq 0$ be an integer so that for every $v \in V(D)$

$$\sum_{e\in\delta^-(v)} \varphi(e) = \sum_{e\in\delta^+(v)} \varphi(e).$$

Show that there is a list $C_1, \ldots, C_n$ of directed cycles (possibly with repetition) so that every edge e of D belongs to exactly $\varphi(e)$ of these cycles.

Proof. Let D be a digraph as given above. We induct on $\sum_{e\in E(D)} \varphi(e)$.

Base case. If $\sum_{e\in E(G)} \varphi(e) = 0$, then $\varphi(e) = 0$ for each $e \in E(G)$. Thus, for every $v \in V(D)$, $\sum_{e\in\delta^-(v)} \varphi(e) = \sum_{e\in\delta^+(v)} \varphi(e) = 0$. Thus there are zero cycles such that every edge e of D belongs to $\varphi(e) = 0$ cycles (the claim is vaccuously true).

Induction. Let $k \geq 1$ be an integer so that $\sum_{e\in E(D)} \varphi(e) = k$. Thus, there is at least one edge $f \in E(D)$ for which $\varphi(f) > 0$. Let D' be the graph obtained from D by deleting f, i.e. $D' = D \setminus f$. Suppose f has ends $a, b \in V(D)$, where a is the head and b is the tail. Then $\psi \coloneqq \varphi|_{E(D')}$ is an (a, b)-flow. The value of ψ is at least 1, since $\varphi(f) \geq 1$.

Thus, by Lemma 9.3, there exists at least one directed path P from a to b in D' such that for every $e \in E(P)$, e belongs to at most $\psi(e)$ of these obtained paths (there is just one in this case). Thus, P has positive flow, i.e. $\psi(e) > 0$ for every $e \in E(P)$. Indeed, if P has an edge e such that $\psi(e) = 0$, then e can not belong to P, a contradiction (it would belong to at most 0 of the obtained paths). Thus, there is a directed cycle C in D given by $V(C) \coloneqq V(P)$ and $E(C) \coloneqq E(P) \cup \{f\}$ and such that for every edge $e \in E(C)$, $\varphi(e) \geq 1$.

Using this cycle C, we define

$$p(e) \coloneqq \begin{cases} \varphi(e) - 1 & e \in E(C) \\ \varphi(e) & e \notin E(C). \end{cases}$$

Thus, $p(e) \geq 0$ for $e \in E(D)$, and for $v \in V(D)$, it follows that $\sum_{e\in\delta^-(v)} p(e) = \sum_{e\in\delta^+(v)} p(e)$, since for every edge in C with head v there is an edge in C with tail v (since C is a directed cycle) and for any other edge e, $p(e) = \varphi(e)$. Thus, p satisfies the assumtpions of the problem statement and we have

$$\sum_{e\in E(D)} p(e) \leq \sum_{e\in E(D)} \varphi(e) = k,$$

thus we can apply the induction hypothesis! Hence there exists a list of directed cycles $C'_1, C'_2, \ldots, C'_m$ in D such that every edge e of D belongs to exactly $p(e)$ of these cycles.

Now, consider the list of cycles $C, C'_1, C'_2, \ldots, C'_m$. For an edge e not in C, it belongs to exactly $p(e) = \varphi(e)$ cycles in the list. For an edge e in C, it belongs to $p(e) + 1 = \varphi(e)$ cycles in the list, since we have added the cycle C to the list. Thus, every edge e of D belongs to exactly $\varphi(e)$ cycles in the list $C, C'_1, C'_2, \ldots, C'_m$, as desired.

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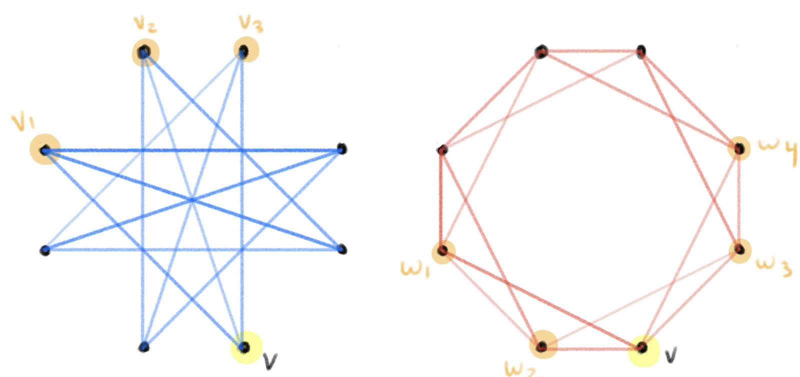
Problem 2. Show that $R(3,4) = 9$.

Proof. We commence by proving that $R(3,4) \leq 9$. Let G be an arbitrary graph on 9 vertices. We have two cases.

- ***Case 1.*** *Suppose there is a vertex $v \in V(G)$ incident to at least 4 edges $e_1, e_2, e_3, e_4 \in E(G)$.* Then, if f_1, f_2, f_3, f_4 are the respective ends of e_1, e_2, e_3, e_4, if there is an edge from f_i to f_j for $1 \leq i < j \leq 4$, then G has a clique of size 3 and we are done. Otherwise, the ends f_1, f_2, f_3, f_4 form an independent set in G since they are pairwise non-adjacent and we are also done. Thus, in case 1, we can always find either a clique of size 3 or an independent set of size 4 in G, as needed.

- ***Case 2.*** *There is a vertex of G is incident to at most 3 edges.* Thus, in $\overline{G}$, there is a vertex $v \in V(\overline{G})$ which is incident to at least 6 edges $va_1, va_2, \ldots, va_6 \in E(\overline{G})$. Now we consider the subgraph X of G given by $V(X) \coloneqq \{a_1, a_2, \ldots, a_6\}$ and whose edge set $E(X)$ contains all edges in G which contain a vertex a_i as an end $(1 \leq i \leq 6)$. By lecture, $R(3,3) = 6$, thus X either has a clique of size 3 or an independent set of size 3. If X has a clique of size 3, then so does G and we are done; and if X has an independent set Y of size 3, then $Y \cup \{v\}$ is an independent set in G of size 4 as needed, since $va_i \notin E(\overline{G})$ for every $i = 1, 2, \ldots, 6$.

Thus, in either case, G either has a clique of size 3 or an independent set of size 4; hence $R(3,4) \leq 9$. To show that $R(3,4) \geq 9$, we suppose otherwise for a contradiction. Thus, if $R(3,4) \leq 8$ then *every* graph on 8 vertices contains a clique of size 3 or an independent set of size 4. However, consider the following graph G (drawn in blue) along with its complement $\overline{G}$ (drawn in red):



By our supposition, G either has a clique of size 3 or an independent set of size 4. We show that either case produces a falsehood.

- If G contains a clique of size 3, then K_3 is a subgraph of which. Thus there is a vertex $w \in V(G)$ such that $w \in V(K_3)$. By symmetry, this implies that *every* vertex of G is in K_3. Thus $v \in V(G)$ (highlighted in yellow) is in K_3. But its three neighbours v_1, v_2, v_3 are pairwise non-adjacent, a contradiction since v must be in K_3.

- Thus, G must contain an independent set of size 4. By lecture, equivalently, $\overline{G}$ contains a clique of size 4. Thus $\overline{G}$ contains K_4. Thus there is a vertex $w \in V(\overline{G})$ that is in K_4. But then, by symmetry, *every* vertex of $\overline{G}$ is in K_4. Thus $v \in V(\overline{G})$ (highlighted in yellow) is in K_4. Thus there must be a 3 element subset of $\{w_1, w_2, w_3, w_4\}$ (neighbours of v) that contains K_3. However, looking at the graph of $\overline{G}$, its easy to see that this is impossible, i.e. no three vertices in $\{w_1, w_2, w_3, w_4\}$ are pairwise adjacent. Thus, we have derived a contradiction.

Since both cases provide a contradiction, we conclude that $R(3,4) \not\leq 8$ so that $R(3,4) \geq 9$. Since $R(3,4)$ is simultaneously ≤ 9 and ≥ 9, we conclude that $R(3,4) = 9$. ■

Problem 3. For an integer $k \geq 3$, let $N = R_3(k,k,k)$ (N is the minimum integer such that in every coloring of edges of K_N in 3 colors there exists a set X of k vertices so that all edges between the vertices of X have the same color). Show that

1. $$\binom{N}{k}\left(\frac{1}{3}\right)^{\binom{k}{2}-1} \geq 1$$

2. $N \geq 3^{k/2}$.

Proof of 1. Suppose by contradiction that $\binom{N}{k}(\frac{1}{3})^{\binom{k}{2}-1} = \binom{N}{k}3^{\binom{k}{2}-1} < 1$. We consider the graph K_N. There are $3^{\binom{N}{2}}$ possible colorings of K_N, since K_N has $\binom{N}{2}$ total edges which can each be coloured in one of three ways. For a given subset $S \subseteq V(K_N)$ with $|S| = k$, there are $3^{\binom{N}{2}-\binom{k}{2}}$ colorings of K_N in which the edges between vertices of S are monochromatic (since S has $\binom{k}{2}$ edges). Since there are 3 colours available, the total number of subsets $S \subseteq V(K_N)$ of size k such that the edges between vertices in S are monochromatic is

$$\underbrace{\binom{N}{k}}_{\text{ways to choose } S} \cdot \underbrace{3 \cdot 3^{\binom{N}{2}-\binom{k}{2}}}_{\text{graphs s.t. } S \text{ has monochromatic edges}},$$

i.e. this is the number of colorings of K_N in which there is such a subset S. However, by hypothesis we have

$$\binom{N}{k} \cdot 3^{\binom{N}{2}-\binom{k}{2}+1} = \binom{N}{k}3^{1-\binom{k}{2}}3^{\binom{N}{2}} = \binom{N}{k}\left(\frac{1}{3}\right)^{\binom{k}{2}-1} \cdot 3^{\binom{N}{2}} < 3^{\binom{N}{2}},$$

since we assumed that $\binom{N}{k}(\frac{1}{3})^{\binom{k}{2}-1} < 1$. Thus there exists a colouring of K_N in which there is no subset S with k elements and such that edges between its vertices are monochromatic, a contradiction to the hypothesis that $R_3(k,k,k) = N$. Thus, we conclude that $\binom{N}{k}\left(\frac{1}{3}\right)^{\binom{k}{2}-1} \geq 1$, thereby completing the proof. ■

Proof of 2. By (1), it suffices to show that if $N < 3^{k/2}$, then $\binom{N}{k}3^{1-\binom{k}{2}} < 1$ (as this would imply that $N \neq R_3(k,k,k)$), a contradiction. Notice that

$$\binom{N}{k}3^{1-\binom{k}{2}} = 3^{1-\binom{k}{2}} \cdot \frac{N(N-1)\cdots(N-k+1)}{k!} \leq 3^{1-\binom{k}{2}} \cdot \frac{N^k}{k!} < 3^{1-\binom{k}{2}} \cdot \frac{(3^{k/2})^k}{k!} \tag{2.1}$$

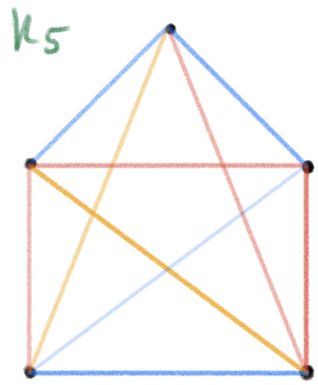
$$= \frac{3^{k^2/2} \cdot 3^{1-\binom{k}{2}}}{k!} \leq \frac{3^{k^2/2} \cdot 3^{1-\binom{k}{2}}}{3^{k-1}} \tag{2.2}$$

$$= \frac{3 \cdot 3^{k^2/2}}{3^{k-1}3^{k(k-1)/2}} = 3^{1+k^2/2-(k-1+k(k-1)/2)} = 3^{2-k/2} \leq 1,$$

as required (such holds for $k \geq 5$). (2.1) holds as $N < 3^{k/2}$ by hypothesis, and so does (2.2) since $k! \geq 3^{k-1}$ for $k \geq 5$ by induction.[1]

[1]For $k = 5$, $5! = 120 > 3^{5-1} = 3^4 = 81$. If $k \geq 5$, $(k+1)! = (k+1)k! \geq (k+1)3^{k-1} \geq 3 \cdot 3^{k-1} = 3^k$ by the inductive hypothesis and since $k+1 \geq 6$.

When $k = 4$, if $N = R_3(4,4,4) < 3^{4/2} = 9 \implies N \leq 8$. Thus, $\binom{N}{k}3^{1-\binom{k}{2}} \leq \binom{8}{4}3^{1-\binom{4}{2}} \approx 0.2 < 1$ as needed, thus $N \geq 9$ for $k = 4$ as needed. When $k = 3$, we need show that $R_3(3,3,3) = N \geq 3^{3/2} \approx 5.1$ which is equivalent to showing that $N \geq 6$ since N is an integer. Suppose towards a contradiction that $N \leq 5$. We show that 5 vertices will not suffice, from which it follows trivially that neither 4, 3, 2, nor 1 vertex will suffice either. We must find a coloring of K_5 such that for every collection of three vertices, the edges between them are not monochromatic. Consider the following coloring of K_5:



It is easy to see that for any collection of three vertices, the edges between them are not monochromatic. Thus, $N \geq 6 \geq 3^{k/2}$ when $k = 3$. Thus, the proof is complete. ■