MATH 350; Assignment 3; Jake R. Gameroff; ID: 261115879.

Question 1. We say that $F \subseteq E(G)$ is **even-degree** if every vertex of G is incident to an even number of edges in F. Show that if T is a spanning tree of G, there is an even-degree set $F \subseteq E(G)$ with $F \cup E(T) = E(G)$. *Hint. First show that if F_1, F_2 are both even-degree, then so is $F_1 \triangle F_2 \coloneqq (F_1 \setminus F_2) \cup (F_2 \setminus F_1)$.*

Proof. Let G be a graph and T be a spanning tree of G. We construct an appropriate set $F \subseteq E(G)$ to satisfy the claim.

We will denote the set of edges $E(G) \setminus E(T) =: \{f_1, f_2, \dots, f_n\}$, where $n \in \mathbb{N}_{\geq 0}$. For each integer i with $1 \leq i \leq n$, we denote the fundamental cycle of f_i with respect to T by C_i. Then we claim that $E(C_i)$ is even-degree for each fixed $1 \leq i \leq n$. Indeed, if $v \in V(G)$ is arbitrary, either $v \in V(C_i)$ or $v \notin V(C_i)$. In the former case, since C_i is a cycle, v is incident to exactly 2 edges within which by definition. In the latter case, $v \notin V(C_i)$ means that v is not incident to any edge in $E(C_i)$, as otherwise, v would be in $V(C)$, since each edge has exactly two ends, pertaining to two vertices in the cycle. In either case, v is incident to an even number of edges in $E(C_i)$ (either 0 or 2), so $E(C_i)$ is even-degree. By Lemma 1.1 below, it follows that $F \coloneqq E(C_1) \triangle E(C_2) \triangle \cdots \triangle E(C_n)$ is even-degree, since i was arbitrary.

Obviously $F \cup E(T) \subseteq E(G)$, since any edge in $F \cup E(T)$ is in $E(G)$ by construction. Likewise, $E(G) \subseteq F \cup E(T)$. Indeed, if $e \in E(G)$ is arbitrary, then $e \in E(T)$ or $e \notin E(T)$. In the former case, we obtain $e \in F \cup E(T)$ as needed. In the latter case, this implies that there exists an integer j with $1 \leq j \leq n$ such that $e = f_j$, since $e \in E(G) \setminus E(T) = \{f_1, \dots, f_n\}$. Hence $e \in E(C_j)$. To show that $e \in F$, we must show that for fixed k such that $1 \leq k \leq n$ and $j \neq k$, $e \notin E(C_k)$ (as if $e \in E(C_k)$, e would not satisfy the "exclusive or" condition of the symmetric difference to be a member of F). But e can not be a member of $E(C_k)$, as otherwise, C_k would be a second fundamental cycle of e with respect to T, a contradiction to Lemma 4.3 (from lecture, which asserts the fundamental cycle's uniqueness). Hence, since e is in exactly one $E(C_j)$, $e \in F \implies e \in F \cup E(T)$. Thus, we conclude that $F \cup E(T) = E(G)$, since these sets are subsets of each other.

Thus, we have constructed a set $F \subseteq E(G)$ of even degree with $F \cup E(T) = E(G)$, as needed. ∎

Lemma 1.1. Let G be a graph. If $F_1, F_2 \subseteq E(G)$ are both even-degree, then so is $F_1 \triangle F_2 \coloneqq (F_1 \setminus F_2) \cup (F_2 \setminus F_1)$. This generalises to the following: if $F_1, \dots, F_n \subseteq E(G)$ are even degree, then $F_1 \triangle \cdots \triangle F_n$ is too.

Base case. Let G be a graph with two even-degree edge sets $F_1, F_2 \subseteq E(G)$. Let $v \in V(G)$ be an arbitrary fixed vertex. By hypothesis, there exist integers n, m with $n, m \geq 0$ such that exactly $2n$ edges in $E_1 \coloneqq \{e_1, e_2, \dots, e_{2n}\} \subseteq F_1$ are incident to v and exactly $2m$ edges in $E_2 \coloneqq \{f_1, f_2, \dots, f_{2m}\} \subseteq F_2$ are incident to v. Let j be the non-negative integer such that $|E_1 \cap E_2| = j$. If $j = 0$, i.e. E_1 and E_2 are disjoint, then v is incident to an even number $2n + 2m = 2(n+m)$ of edges in $F_1 \triangle F_2$. Hence, suppose $j \geq 1$. Denote the set of edges in both E_1 and E_2 by $E \coloneqq \{\ell_1, \ell_2, \dots, \ell_j\}$. Then v is incident to $2n - j$ edges which are in F_1 but not F_2 and $2m - j$ edges which are in F_2 but not F_1. Since the sets of these $2n - j$ and $2m - j$ edges are disjoint, we conclude that v is incident to $2n - j + 2m - j = 2(n + m - j)$ edges in $F_1 \triangle F_2$, an even number of edges. Since v was arbitrary, we conclude that $F_1 \triangle F_2$ is even-degree.[1]

Induction. Now suppose the claim holds for some $n \geq 1$, i.e. for even-degree $F_1, F_2, \dots, F_n$ as given in the claim, $F_1 \triangle \cdots \triangle F_n$ is even-degree. Then by the base case, if we let $F \coloneqq F_1 \triangle \cdots \triangle F_n$ and F_{n+1} be another even-degree set in $E(G)$, then we obtain that $F \triangle F_{n+1} = F_1 \triangle \cdots \triangle F_n \triangle F_{n+1}$ is even degree (via the base case), completing the induction and hence the proof. ∎

[1] Implicitly, this proof assumes the identity: $(F_1 \setminus F_2) \cup (F_2 \setminus F_1) = (F_1 \cup F_2) \setminus (F_1 \cap F_2)$. We prove this now via double inclusion. If $x \in (F_1 \setminus F_2) \cup (F_2 \setminus F_1) \iff x \in F_1, x \notin F_2$ or $x \in F_2, x \notin F_1 \iff x \in F_1$ or $x \in F_2, x \notin F_1$ or $x \notin F_2 \iff x \in (F_1 \cup F_2) \cap (F_1^c \cup F_2^c) = (F_1 \cup F_2) \setminus (F_1^c \cup F_2^c)^c = (F_1 \cup F_2) \setminus (F_1 \cap F_2)$, via DeMorgan's law.

Question 2. Let G be a connceted graph with m edges $e_1, e_2, \ldots, e_m$ and let $w : E(G) \to \mathbb{R}_+$ be such that $w(e_i) = 2^i$ for $i = 1, 2, \ldots, m$. Let T be the minimum cost spanning tree of (G, w). For $u, v \in V(G)$, let $P(u, v)$ be a path in G from u to v of minimum weight; that is, $P(u, v)$ is chosen among all paths from u to v so that $\sum_{e \in E(P(u,v))} w(e)$ is minimum. Show that $P(u, v) \subseteq T$.

Proof. Let $u, v \in V(G)$ be arbitrary vertices. Let T be the minimum cost spanning tree of (G, w) and $P(u, v)$ be the path in G from u to v of minimum weight. It suffices to prove that $E(P(u, v)) \subseteq E(T)$. Indeed, $V(P(u, v)) \subseteq V(G) = V(T)$ since T is a spanning tree.

Suppose for the sake of contradiction that there is an edge $f \in E(P(u, v))$ such that $f \notin E(T)$. Let C denote the fundamental cycle of f with respect to T. Corollary 4.5 from lecture then tells us that for any edge $e \in E(T)$ with $e \in E(C)$, $w(f) \geq w(e)$. Since, by hypothesis, there are integers i, j with $1 \leq i < j \leq m$ such that $w(f) = 2^j \neq 2^i = w(e) \implies w(f) > w(e)$. But notice that by the geometric series formula,

$$\sum_{k=1}^{j-1} 2^k = \sum_{k=0}^{j-1} 2^k - 2^0 = \frac{2^{j-1+1} - 1}{2 - 1} - 1 = 2^j - 2 < 2^j. \tag{2.1}$$

Suppose f has ends $x, y \in V(C)$. Then the path from x to y along f is a sub-path of $P(u, v)$ that is a sub-graph of C. Hence, let P be the longest of such paths with ends $s, t \in V(C)$, i.e. P is the path with the most vertices that is a sub-path of $P(u, v)$ and a sub-graph of C. Then P has length $1 \leq \ell < k$, where k denotes the number of edges in C (P can not have length k as otherwise it would contain a cycle and could not be a path). But C is a cycle, and hence there is a path P' from s to t in C with edges $E(C) \setminus E(P)$ and vertices $V(C) \setminus (V(P) \setminus \{s, t\})$.

We now note that there are at most $j - 1$ edges $e \in E(C) \setminus \{f\}$. Indeed, the path P contains at least the edge f with $w(f) = 2^j$; and we argued earlier that for every edge e with $e \in E(C)$ and $e \in E(T)$, there exists a unique i with $1 \leq i < j$ so that $w(e) = 2^i < 2^j = w(f)$. Hence, if there were more than $j - 1$ edges in $E(C) \setminus \{f\}$, then there would either be two edges with the same weight or an edge e_k of weight $2^k > 2^j$ for some $k > j$, a contradiction. Thus, using equation 2.1, we obtain

$$w(P) \geq w(f) > \sum_{k=1}^{j-1} 2^k \geq \sum_{e \in E(C) \setminus \{f\}} w(e) \geq \sum_{e \in E(C) \setminus E(P)} w(e) = w(P).$$

Thus, $w(P) > w(P')$, a contradiction to the minimality of $P(u, v)$. Indeed, if $P(u, v)$ were minimal, then it would have sub-path P' instead of P, since P' is of lesser weight (i.e. in either case we must travel from s to t, so we must take the sub-path of least weight).

Therefore, there exists no such edge $f \in E(P(u, v)) \setminus E(T)$. Thus, $f \in E(P(u, v)) \implies f \in E(T)$ so that $E(P(u, v)) \subseteq E(T)$ and thus $P(u, v) \subseteq T$ as was to be shown.

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Question 3.

a) Let e be an edge of the complete graph K_n with $n \geq 2$. Show that K_n has exactly $2n^{n-3}$ spanning trees containing e.

b) Let G_n be a simple graph obtained from the complete graph K_n by adding one extra vertex adjacent to exactly two vertices of K_n. Find the number of spanning trees of G_n.

Proof of a. Let $f \in E(K_n)$ be fixed for some $n \geq 2$. By lemma 3.1, a tree with n vertices has $n-1$ edges, thus each spanning tree T of K_n has $n-1$ edges. Meanwhile, K_n has $\binom{n}{2}$ total edges.[2] Thus, there are $(n-1)n^{n-2}$ edges in all spanning trees of K_n, since there are n^{n-2} total spanning trees of K_n by Cayley's theorem (obviously we may be counting edges more than once here).

Similarly, by the completeness of K_n, there is a positive integer k such that for each edge $e \in E(K_n)$, e appears in exactly k spanning trees. We must find this k. It follows that $k \cdot \binom{n}{2}$ also denotes the number of edges in all spanning trees, since each edge is in exactly k spanning trees and there are $\binom{n}{2}$ edges (we may count edges more than once here in the same way as above). Thus, we must have that

$$(n-1)n^{n-2} = k \cdot \binom{n}{2} \implies k = \frac{(n-1)n^{n-2}}{\binom{n}{2}} = \frac{(n-1)n^{n-2}}{\frac{n(n-1)}{2}} = \frac{2}{n} \cdot n^{n-2} = \boxed{2n^{n-3}}$$

spanning trees containing each edge $e \in E(K_n)$. Thus, for our fixed edge f, there are $2n^{n-3}$ spanning trees of K_n containing which. ■

Proof of b. Let G_n be a simple graph obtained from the complete graph K_n by adding one extra vertex adjacent to exactly two vertices of K_n. Denote this vertex by $v \in V(G_n)$, and suppose it is adjacent to vertices $k_1, k_2 \in K_n$. Let ℓ be the total number of spanning trees in G_n. Any of such spanning trees T fall into one of the following categories. Based on the number of spanning trees in each, we will calculate ℓ.

- T contains k_1v but not k_2v. There are, trivially, exactly n^{n-2} of such spanning trees. Indeed, Cayley's formula tells us that there are n^{n-2} spanning trees of K_n. Appending to each of these trees the vertex v and edge k_1v encompasses every possible spanning tree with the property in question, since appending a leaf to a spanning tree can not create a cycle.

- T contains k_2v but not k_1v. The number of such spanning trees is likewise n^{n-2} by the exact same reasoning as above.

- T contains k_1v and k_2v. In this case, we must note that if k_1k_2 is in T, the vertices k_1, k_2, v will form a cycle, so this can not happen. By problem 3a, there are $2n^{n-3}$ spanning trees of K_n containing the edge k_1k_2. Thus, for any spanning tree of K_n containing k_1k_2, deleting this edge and appending the edges k_1v and k_2v corresponds to a spanning tree of G_n. Indeed, if X is a spanning tree of K_n containing k_1k_2 and k_1v (or k_2v resp.) then by lemma 4.4, $(X + k_2v) \setminus k_1k_2$ is a spanning tree, as k_1k_2 would lie on the fundamental cycle of k_2v with respect to X (resp. $(X + k_1v) \setminus k_1k_2$) (since the edges k_1k_2, k_1v, k_2v form a cycle with vertices k_1, k_2, v and the fundamental cycle is unique for each edge). Thus, the number of spanning trees containing k_1v and k_2v of G_n equals the number of spanning trees of K_2 containing k_1k_2; that is: $2n^{n-3}$ by (a). If there were extra, unaccounted for spanning trees containing k_1v and k_2v but not k_1k_2, then deleting the vertex v would give a spanning tree (since doing this can

[2] Indeed, since all vertices within which are pairwise adjacent, we have must have $n \cdot n - 1$ edges forming these connections, but we count each twice (once for each of its ends), giving a total of $\frac{n(n-1)}{2} = \frac{n!}{2!(n-2)!} = \binom{n}{2}$ distinct edges.

not create a cycle, otherwise we would not have had a spanning tree to begin with) containing k_1k_2 in K_n, so we would have already counted this spanning tree.

Since these three cases encompass all possible spanning trees of G_n, we conclude that G_n has exactly $\ell = n^{n-2} + n^{n-2} + 2n^{n-3} = \boxed{2n^{n-2} + 2n^{n-3}}$ spanning trees. ■