MATH 350; Assignment 8; Jake R. Gameroff; ID: 261115879.

Problem 1.

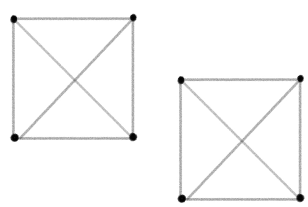
1. Let G be a 3-regular graph. Show that if G contains a Hamiltonian cycle then $\chi'(G) = 3$.
2. Does there exist a 3-regular graph G with $\chi'(G) = 3$ such that G does not contain a Hamiltonian cycle?

Proof of 1. We first note that since G is 3-regular, every vertex in G has degree 3 by definition. By the handshake lemma,

$$\sum_{v \in V(G)} \deg v = 3|V(G)| = 2|E(G)|.$$

Thus, $3|V(G)|$ is even so that $|V(G)|$ is even (otherwise, $3|V(G)|$ would be odd). Then if C is a Hamiltonian cycle in G, we note that C must be even since G has an even number of vertices. Since C is even, we can colour its edges by alternating between two colours. Furthermore, we must have that $E(G) \setminus E(C)$ is a perfect matching in G, since every vertex can be incident to at most one edge in $E(G) \setminus E(C)$ (i.e. every vertex has degree 3 and is already incident to two edges in C). Thus, we can colour the edges of $E(G) \setminus E(C)$ using the third colour. We must have that $\chi'(G) = 3$ as if $\chi'(G) = 2$ then G would have to be a cycle, a contradiction to 3-regularity. Therefore, $\chi'(G) = 3$ as needed. ■

Solution for 2. Yes! There does exist such a graph G. Consider the following:



This graph is 3-regular since it is two copies of K_4 (i.e. every vertex has degree 3). Furthermore, the edge-colouring number of this graph is 3 since $\chi'(K_4) = 3$ by lecture, so we need at least 3 colours to colour each connected component. Since this graph is disconnected, it can not contain a Hamiltonian cycle. Thus, we have found a 3-regular graph G with $\chi'(G) = 3$ that does not contain a Hamiltonian cycle. ■

Problem 2. What is the maximum possible number of edges in a graph with n vertices and no K_4 minor?

Proof. We commence by noting that we will consider only simple graphs in order to find this maximum number of edges, otherwise we could create graphs containing no K_4 minor with an arbitrary number of edges: that is, to any simple graph G with no K_4 minor, we can add as many loop or parallel edges as we want and G will still have no K_4 minor.

For $n = 1$ we can have no edges and for $n = 2$ we can have one (since G is simple). For $n \geq 3$, the maximum possible number of edges in a graph with n vertices and no K_4 minor is $\boxed{2n-3}$.[1]

[1]Here is an intuitive explanation: If G has n vertices (for $n \geq 3$) and the maximum number of edges so that it has no K_4 minor, then by lemma 13.2 G has a vertex of degree 2. Indeed, G has a clique $X \subseteq V(G)$ of size at least 2, otherwise we can add an edge between any two vertices which contradicts the maximality of G's edges. Then lemma 13.2 gives us a vertex v not in X of degree ≤ 2 ($X \neq V(G)$ as $|V(G)| \geq 3$). By maximality, this vertex must be of degree exactly 2, otherwise we could add another edge. By the transitivity of "being a minor of", deleting v still ensures that $G \setminus v$ has no K_4 minor. Continue this

We prove this claim by induction on $|V(G)|$. If G is *any* graph with $|V(G)| = 3$ then the most edges G can have is 3 (it would be K_3) so that $|E(G)| \leq 2|V(G)| - 3 = 2(3) - 3 = 3$ as needed (thus the claim holds for when G has no K_4 minor). Now suppose G is a graph with $|V(G)| \geq 4$ and with the maximum possible number of edges such that it has no K_4 minor. We claim that there exists a pair of adjacent vertices $X \coloneqq \{u, v\} \subseteq V(G)$, otherwise the number of edges in G is not maximal. Since $|V(G)| \geq 4$, $X \neq V(G)$. Thus, by lemma 13.2, there exists a vertex $v \in V(G) \setminus X$ such that $\deg v \leq 2$. Now let $G' \coloneqq G \setminus v$. Since G' is a minor of G, G' contains no K_4 minor as otherwise, by transitivity, G would as well. By the inductive hypothesis,

$$|E(G)| \leq |E(G')| + 2 \leq 2|V(G')| - 3 + 2 = 2(|V(G)| - 1) - 1 = 2|V(G)| - 2 - 1 = 2|V(G)| - 3.$$

Therefore, $|E(G)| \leq 2|V(G)| - 3$ completes the inductive step and hence the proof. ∎

Problem 3. Let G be a graph with n vertices and $\alpha(G) \leq 2$. Show that G contains $K_{\lceil n/3 \rceil}$ as a minor.

Proof. Without loss of generality, suppose G is simple (delete all loop and parallel edges and use the transitivity property of minors). We prove the claim with the additional assumption that $\alpha(G) = 2$. Indeed, if $\alpha(G) = 1$ then $G = K_n$, since any pair of vertices are adjacent. Thus $K_{\lceil n/3 \rceil}$ is a subgraph (and hence a minor) of G, since we can isolate $\lceil n/3 \rceil$ vertices in G and they will be pairwise adjacent, forming $K_{\lceil n/3 \rceil}$.

Since $\alpha(G) = 2$ $|V(G)| \geq 2$. For the base case, if $|V(G)| = 2$ then $K_{\lceil n/3 \rceil} = K_1$ which is a subgraph (and hence a minor) of G (just delete all but one vertex to obtain K_1). By the exact same reasoning, if $|V(G)| = 3$ then G contains a $K_{\lceil n/3 \rceil} = K_1$ minor. If $|V(G)| = 4$ then G has a $K_{\lceil 4/3 \rceil} = K_2$ minor since by lecture this is equivalent to G containing two adjacent vertices, which holds as $\alpha(G) = 2$ (G has a K_2 minor iff. G has K_2 as a subgraph by lecture).

For the induction step, suppose $|V(G)| \geq 5$ and $\alpha(G) = 2$. Thus there exists a pair $\{x, y\}$ of non-adjacent vertices G. Define the following sets:[2]

- $S_x \coloneqq \{v \in V(G) : x \leftrightarrow v,\ y \not\leftrightarrow v\}$
- $S_y \coloneqq \{v \in V(G) : x \not\leftrightarrow v,\ y \leftrightarrow v\}$
- $S_{xy} \coloneqq \{v \in V(G) : x \leftrightarrow v,\ y \leftrightarrow v\}$

and note that $S_x \cup S_y \cup S_{xy} = V(G) \setminus \{x, y\}$ is a disjoint union. Indeed, no vertex can be in two or more of these sets; by construction, $S_x \cup S_y \cup S_{xy} \subseteq V(G) \setminus \{x, y\}$ and if $\ell \in V(G) \setminus \{x, y\}$ then ℓ is adjacent to x or ℓ is adjacent to y, otherwise the set $\{x, y, \ell\}$ is an independent set of size 3 which contradicts the choice of G; thus $\ell \in S_x \cup S_y \cup S_{xy}$. Thus

$$S_x \cup S_y \cup S_{xy} \cup \{x, y\} = V(G). \tag{$*$}$$

Furthermore, we note that the vertices in S_x are pairwise adjacent; indeed, if there were two non-adjacent vertices $u, v \in S_x$ then $\{u, v, y\}$ is an independent set of size 3, a contradiction. By definition, x is adjacent to each vertex in S_x so that $S_x \cup \{x\}$ is a set of pairwise adjacent vertices. It follows that $S_y \cup \{y\}$ is also a set of pairwise adjacent vertices.

process for $n - 2$ vertices (we delete $2(n - 2) = 2n - 4$ edges). Then, there can only be one edge left between the remaining two vertices, giving $2n - 3$ edges. Thus, it follows that there is a graph on n vertices with $2n - 3$ edges and with no K_4 minor. Thus the proposed bound will be shown to be tight.

[2] We write $v \leftrightarrow u$ to mean that v and u are adjacent vertices and $v \not\leftrightarrow u$ to mean that v and u are non-adjacent vertices.

Using $(*)$, it follows by disjointness that $|S_x|+|S_y|+|S_{xy}|=n-2$. Hence at least one of $|S_x|,|S_y|,|S_{xy}| \geq \frac{n-2}{3}$ as otherwise, $|S_x|+|S_y|+|S_{xy}| < 3 \cdot \frac{n-2}{3} = n-2$, a contradiction. But then one of these three sets actually has order $\geq \lceil \frac{n-2}{3} \rceil$, since we require integer values for orders. If $|S_x| \geq \lceil \frac{n-2}{3} \rceil$ then G has a $K_{\lceil \frac{n-2}{3} \rceil + 1}$ minor since G contains $S_x \cup \{x\}$, a set of at least $\lceil \frac{n-2}{3} \rceil + 1$ pairwise adjacent vertices. Since $\lceil \frac{n-2}{3} \rceil + 1 \geq \lceil \frac{n-2}{3} + 1 \rceil = \lceil \frac{n+1}{3} \rceil \geq \lceil \frac{n}{3} \rceil$, G contains $K_{\lceil n/3 \rceil}$ as a subgraph and hence a minor as needed. By the exact same reasoning, the same holds if $|S_y| \geq \lceil \frac{n-2}{3} \rceil$.

The only other possibility is that $|S_{xy}| \geq \lceil \frac{n-2}{3} \rceil$. In this case, we consider a vertex $v \in S_{xy}$. Let H be the graph obtained from G by first contracting vertices v and x into a vertex v'' and then contracting y and v'' to a vertex v'. Then $\alpha(H) = 2$ as otherwise $\alpha(G) \neq 2$ likewise (i.e. contractions preserve independence numbers). It follows that v' is adjacent to every vertex in H; indeed, $V(H) = S_x \cup S_y \cup S_{xy} \cup \{v'\}$, and v' is adjacent to all vertices in each of these sets via the contractions with x and y. Define H' to be the subgraph obtained from H by deleting v'. Then $\alpha(H') = 2$; indeed, if $\alpha(H') > 2$ then there is a set X of pairwise non-adjacent vertices in H', but then this set is also independent in H, a contradiction to $\alpha(H) = 2$. Since $|V(H')| = n-3$, we can apply the induction hypothesis to find that H' has a $K_{\lceil \frac{n-3}{3} \rceil}$ minor. But then H has a $K_{\lceil \frac{n-3}{3} \rceil}$ minor that does not include the vertex v'. But v' is adjacent to every vertex in H' so that $K_{\lceil \frac{n-3}{3} \rceil + 1}$ is a minor in H and hence in G by transitivity (since H is a minor of G by construction). But $\lceil \frac{n-3}{3} \rceil + 1 = \lceil \frac{n}{3} - 1 \rceil + 1 = \lceil \frac{n}{3} \rceil$. Thus, G has a $K_{\lceil n/3 \rceil}$ minor as required. $\blacksquare$