MATH 350; Assignment 7; Jake R. Gameroff; ID: 261115879.

Question 1. Let G be a graph with $\chi(G) = k$ for some positive integer k. Show that G contains at least k vertices with degree at least $k - 1$.

Proof. We first note that G must be loopless, else $\chi(G) = \infty$. We colour G using the greedy colouring algorithm. Consider an ordering $(v_1, v_2, \ldots, v_k)$ of the vertices of G such that $\deg v_1 \geq \deg v_2 \geq \cdots \geq \deg v_k$. By lecture, the greedy colouring algorithm will colour G using m colours for some integer $m \geq k$ (since k is the chromatic number of G). Thus, when running the algorithm, there must be some vertex v_j with $1 \leq j \leq k$ such that v_j is given the k'th colour (otherwise $\chi(G) < k$). According to the algorithm, k is the smallest integer such that no neighbour of v_j has colour k. But this means that v_j has $k - 1$ neighbours which have already been coloured. Thus, $\deg v_j \geq k - 1$ and v_j has $k - 1$ neighbours that appear before which in the ordering of $V(G)$. Since the vertex degrees in the ordering are non-increasing, it follows that v_j has $k - 1$ neighbours with degree $\geq k - 1$. Taking these neighbours together with v_j, we conclude that there are at least k vertices with degree at least $k - 1$, thereby completing the proof. ∎

Question 2. Show that if G is a graph, $k \geq 1$ is an integer and $\chi(G) > k$ then G has a path with k edges.

Proof. We proceed by contraposition. Suppose G contains no path with k edges. Thus, if P is a path in G of maximal length m, then $m \leq k - 1$.

We claim that G is m-degenerate. To prove the claim, suppose for a contradiction that G is not m-degenerate. So there exists a subgraph H of G such that for each $v \in V(H)$, we have $\deg_H v \geq m+1$. Let P be a path of maximal length in H; since P is also a subgraph of G, it has length $\leq m$. Let u be one end of P. Then $\deg_H u \geq m + 1$. But then each neighbour of u in H must be in P, otherwise if there was a neighbour $w \in V(H)$ of u, then $P + w$ would be a path of greater length than P, contradicting its maximality. But then P contains the vertex u and all $m+1$ of its neighbours, so P has length at least $m+1$, a contradiction to the hypothesis that the maximum size path in G has length m. Thus, G is m-degenerate.

Since G is m-degenerate we have by lecture $\chi(G) \leq m+1 \leq k-1+1 = k$. By contraposition, the proof is complete. ∎

Question 3. Let G be a graph in which every two odd cycles share a vertex. Show that $\chi(G) \leq 5$.

Proof. Let $\mathcal{O}$ be the smallest odd cycle in G and let G' be the graph obtained from G by deleting all vertices in $V(\mathcal{O})$. It follows that G' does not contain an odd cycle. Indeed, if C is an odd cycle in G there exists a vertex $v \in V(C) \cap V(\mathcal{O})$ by the hypothesis that every two cycles share a vertex. But when v is deleted from G, C can no longer be a cycle in G'. Since G' contains no odd cycles, it is bipartite by lecture. Hence, G' is 2-colourable by lecture. Since $\mathcal{O}$ is an odd cycle, it can be coloured using 3 colours[1] (cf. Lemma 3.1). Thus, $G' = G \setminus V(\mathcal{O})$ is 2-colourable and $\mathcal{O}$ is 3-colourable. Thus, G is 5-colourable. It follows that $\chi(G) \leq 5$. ∎

Lemma 3.1. Let C be an odd cycle. Then C is 3-colourable. This proof is trivial. Since C is a cycle, $\Delta(G) = 2$. Thus, by lecture, $\chi(G) \leq \Delta(G) + 1 = 3$. Thus C is 3-colourable. ∎

[1] We note that $\mathcal{O}$ can not contain an edge that is not apart of the cycle in G. Indeed, if $v_1, v_2, \ldots, v_{2k+1}$ are the vertices of $\mathcal{O}$ written in order and $v_i v_j$ is an edge in $\mathcal{O}$ that is not apart of the cycle, we let P_1 and P_2 be the distinct paths in $\mathcal{O}$ from v_i to v_j along vertices in the cycle. Then since $|V(P_1)| + |V(P_2)| = 2k + 3$ is odd (we count v_i, v_j twice), we must have that $|V(P_1)|$ is odd. Thus, the path P_1 and the edge $v_i v_j$ is a cycle with an odd number of vertices, and since P_2 must have at least 3 vertices, $|V(P_1)| + 2 < |V(P_1)| + |V(P_2)| = 2k + 3 \implies |V(P_1)| < 2k + 1$. Thus, the existence of this smaller odd cycle is a contradiction; thus $\mathcal{O}$ has no such edge.