MATH 350; Assignment 5; Jake R. Gameroff; ID: 261115879.

1. Let G be a connected graph in which every vertex has degree three. Show that if G has no cut-edge then every two edges of G lie on a common cycle.

Proof. Suppose $e, f \in E(G)$ are two edges in G with ends e_1, e_2 and f_1, f_2 respectively. Let $Q := \{e_1, e_2\}$ and $R := \{f_1, f_2\}$. We have two cases:

Firstly, if there exist two vertex disjoint paths P_1, P_2 each with one end in Q and the other in R, then the cycle C given by $V(C) := V(P_1) \cup V(P_2)$ and $E(C) := E(P_1) \cup E(P_2) \cup \{e, f\}$ is a common cycle between e and f; hence, there is nothing to prove. Otherwise, there do not exist such paths. Hence by theorem 8.2 there exists a separation (A, B) of G of order < 2 and such that $Q \subseteq A$ and $R \subseteq B$.

However, $|A \cap B| > 0$, since $|A \cap B| = 0$ implies that there does not exist a path in G with an end in Q and another in R, a contradiction to the connectivity of G. Thus $|A \cap B| = 1$. Hence there is a vertex $w \in V(G)$ such that $\{w\} = A \cap B$. By hypothesis, w has neighbours $w_1, w_2, w_3 \in V(G)$ as it has degree three. Without loss of generality, take $w_1, w_2 \in A$ and $w_3 \in B$ (all three neighbours can not be in either A or B as G would not be connected). Then the edge $ww_3 \in E(G)$ must be a cut-edge, as otherwise there is another path with an end in B and another in A that does not include w, a contradiction to the order of (A, B). Thus, ww_3 is a cut-edge which contradicts the choice of G. Thus, e and f indeed do lie on a common cycle, completing the proof. ■

2. Let v be a vertex in a 2-connected graph G. Show that v has a neighbour u such that $G \setminus u \setminus v$ is connected.

Proof. Let G be a 2-connected graph with $v \in V(G)$. Then $\deg v \geq 1$ as otherwise G is not connected. Let H be the minimal connected subgraph of $G \setminus v$ that contains all neighbours of v. That is, if H' is a connected subgraph of $G \setminus v$ and $N(v) \subseteq V(H')$, then $H \subseteq H'$. Then H is a tree: H is connected, and if it contains a cycle we just delete all non-cut-edges (by minimality), preserving connectivity. Hence each $w \in N(v)$ is a leaf in H. Indeed, by contradiction suppose that w is not a leaf in H. So there exists a vertex $z \in V(H)$ distinct from v such that w is adjacent to z. $z \notin N(v)$, as the subgraph C of H given by $V(C) = \{v, w, z\}$ and $E(C) = \{vw, wz, zv\}$ would be a cycle in H, a contradiction since H is a tree. Thus, we must have that $z \notin N(v)$. But then the graph H' obtained from H by removing z and all other vertices so that w is a leaf is connected and $N(v) \subseteq V(H')$, a contradiction to the minimality of H. Thus, each $w \in N(v)$ is a leaf.

Let $u \in V(H)$ be a neighbour of v. Since u is a leaf, by lecture $T := H \setminus u$ is a tree. If we suppose that $G' := G \setminus u \setminus v$ is not connected, then there are at least 2 connected components A_1, A_2 of G'. Since $T \subseteq G'$ and is connected, suppose without loss of generality that $T \subseteq A_2$. Thus all neighbours of v are in A_2 as they are in T. But then, in $G \setminus u$, the vertices in A_1 and v lie in different connected components, since $\deg v \geq 1$ and there is no vertex w in A_1 such that w and v are adjacent, so there is no path from v to a vertex in A_1 (since all neighbours of v are in A_2; note also that v can not be in A_1 since it has neighbours in A_2). By our supposition, A_1 is non-null, hence $G \setminus u$ contains at least 2 connected components. Thus $G \setminus u$ is not connected. But G is 2-connected, so $G \setminus u$ must be connected; this contradicts the choice of G.

Therefore, we conclude that v has a neighbour u such that $G \setminus u \setminus v$ is connected. ■

3. Distinct $u, v \in V(G)$ are ***k*-linked** if there are k paths $P_1, \ldots, P_k$ in G from u to v so that for all i, j with $1 \leq i < j \leq k$, $E(P_i) \cap E(P_j) = \emptyset$. Subsets $X, Y \subseteq V(G)$ are ***k*-joined** if $|X| = |Y| = k$ and there are k paths $P_1, \ldots, P_k$ in G from X to Y so that for all i, j with $1 \leq i < j \leq k$, $V(P_i) \cap V(P_j) = \emptyset$.

3.1. Suppose u, v, w are distinct and u, v are k-linked, as are v, w. Does it follow that u, w are k-linked?

Proof. **Yes!** It follows that u and w are necessarily k-linked.

Let G be a graph. Let $u, v, w \in V(G)$ be distinct vertices so that u, v and v, w are k-linked. Suppose by contradiction that u, w are not k-linked. Then there do not exist k edge-disjoint paths $P_1, \ldots, P_k$ in G from u to w. Thus, by Menger's theorem there exists a set $X \subseteq V(G)$ such that $u \in X$, $w \in V(G) - X$, and $|\delta(X)| < k$. Then, we have two cases:

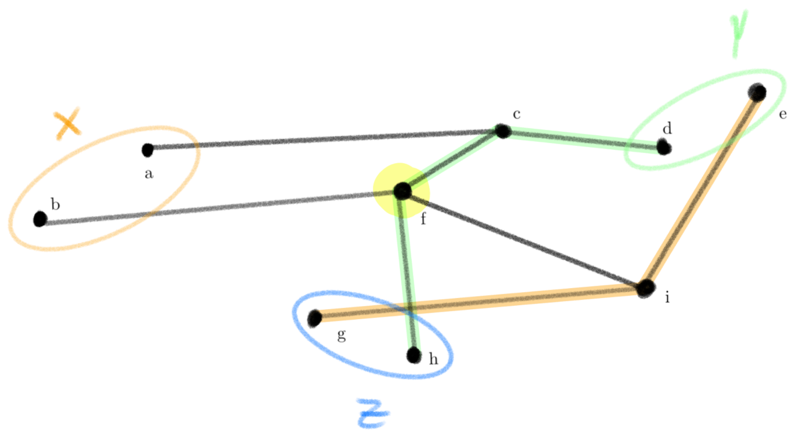
- **Case 1.** If $v \in X$, since v, w are k-linked, there must be at least k edges with an end in X and another in $V(G) - X$. This holds since each path P_i from v to w in G has one end in X (since $v \in X$) and another in $V(G) - X$ (since $w \in V(G) - X$), so for each of such paths, there exists an edge between X and $V(G) - X$, otherwise $V(P_i) \subseteq X$ is a contradiction. However, $|\delta(X)| < k$, a contradiction by the definition of $\delta(X)$. Thus, $v \notin X$.

- **Case 2.** If $v \in V(G) - X$, we attain a contradiction via the exact same reasoning, i.e. we attain k edges with an end in $V(G) - X$ and another in X (via the k edge disjoint paths from w to v), each of which is in $\delta(X)$, a contradiction as $|\delta(X)| < k$. Thus $v \notin V(G) - X$.

Hence, $v \notin V(G)$ is a contradiction. Thus u and w must be k-linked. ■

3.2. Suppose $X, Y, Z \subseteq V(G)$ and X, Y are k-joined, as are Y, Z. Does it follow that X, Z are k-joined?

Proof. **No!** Consider the following counter-example:



Here, $|X| = |Y| = |Z| = 2$. X, Y are 2-linked via the paths with vertices (in order) a, c, d and b, f, i, e. Likewise, Y, Z are 2-linked via the paths with vertices (in order) d, c, f, h and e, i, g. However, notice that any path with an end in X and another in Z must use vertex f (highlighted in yellow). Thus any two paths from X to Z must share the vertex f. Thus, X and Z are not 2-linked. ■