Problem 1. Let $g(n)$ be the number of pairwise non-isomorphic graphs with n vertices. Show that for every positive integer n, there are at least $g(n)/2$ pairwise non-isomorphic *connected* graphs on n vertices.

Proof. Let $n \geq 1$ be fixed and consider $g(n)$, the number of pairwise non-isomorphic graphs on n vertices. Suppose there do not exist at least $g(n)/2$ pairwise non-isomorphic connected graphs. So there exist more than $g(n)/2$ pairwise non-isomorphic graphs that are not connected. Taking the complement of each of such graphs, it follows that there exist more than $g(n)/2$ pairwise non-isomorphic graphs that are connected (cf. Lemma 1.1). Note that these graphs are likewise pairwise non-isomorphic, since their complements were[1]. Hence, we have contradicted the supposition that there were less than $g(n)/2$ pairwise non-isomorphic connected graphs. Hence, there are indeed at least $g(n)/2$ pairwise non-isomorphic connected graphs, thereby completing the proof.

Lemma 1.1. Given a graph G, either G or $\overline{G}$ is connected.

We show that if G is not connected, then $\overline{G}$ is connected. By simply noting that $\overline{\overline{G}} = G$, we see that $\overline{G}$ being not connected implies that G is connected. Furthermore, we can suppose that G and $\overline{G}$ are not both connected, otherwise there is nothing to prove.

Suppose G is a graph which is not connected. Since G is not connected, there must exist at least 2 and at most $|V(G)|$ connected components of G. So let $k \in \{2, 3, \ldots, V(G)\}$ be the number of connected components of G and denote their vertex sets by $V(G_1), V(G_2), \ldots, V(G_k)$. For each $v \in V(G)$, there exists exactly one $j \in \{1, 2, \ldots, k\}$ such that $v \in V(G_j)$, this follows since from lecture, we know that for any vertex $v \in V(G)$, there exists a unique connected component of G containing v. Furthermore, if $v \in V(G_i)$ $(1 \leq i \leq k)$, then $v \in V(G)$ by definition of a connected component. Hence

$$\bigcup_{i=1}^{k} V(G_i) = V(G) \text{ and } \mathrm{V(G}_i) \cap V(G_j) = \emptyset \text{ for } 1 \leq i < j \leq k$$

implies that $(V(G_1), V(G_2), \ldots, V(G_k))$ partitions $V(G)$. To show that $\overline{G}$ must be connected, let $u, v \in V(\overline{G})$ be arbitrary vertices. Since u and v were arbitrary, if we show that there must exist a walk from u to v in $\overline{G}$, we will have proven that $\overline{G}$ is connected by definition. As such, we are presented with two cases: ■

- **<u>Case 1</u>**. The vertices u and v lie in the same connected component's vertex set $V(G_j)$ for some $j \in \{2, 3, \ldots, k\}$. By definition, then, there exists a walk from u to v, since the connected component G_j of G is a connected subgraph (i.e. there exists a walk between any two vertices within which). But since G is not connected, there exists some vertex $q \in V(G)$ such that $q \in V(G_i)$, where $1 \leq i \leq k$ and $i \neq j$. It follows that there does not exist a walk from u to q and there does not exist a walk from q to v (otherwise, these three vertices would lie in the same connected component of G by maximality). Thus, $uq \notin E(G)$ and $qv \notin E(G)$ imply by definition that $uq \in E(\overline{G})$ and $qv \in E(\overline{G})$. Hence, there exists the walk $\{u, q, v\}$ from u to v in $\overline{G}$.

- **<u>Case 2</u>**. The vertices u and v do not lie in the same connected component's vertex set. As has been argued earlier, this implies that there does not exist a walk from u to v in G, which implies that $uv \notin E(G)$, which implies that $uv \in E(\overline{G})$. Thus, there is a walk $\{u, v\}$ from u to v in $\overline{G}$.

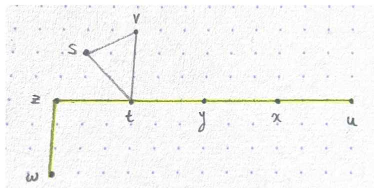
[1]This holds as if G and H are two non-isomorphic graphs, then there exists no relation-preserving bijection between $V(G)$ and $V(H)$. Hence if $f : V(\overline{G}) \to V(\overline{H})$ is some bijection, then there is some pair of vertices $u, v \in V(G) = V(\overline{G})$, such that $f(u)f(v) \in E(H)$ but $uv \notin E(G)$ (since G and H are not isomorphic) $\implies$ $uv \in E(\overline{G})$, but $f(u)f(v) \notin E(\overline{H})$, hence $\overline{G}$ and $\overline{H}$ are not isomorphic. Thus, the statement holds taking G and H to be any pair of the non-isomorphic graphs that are not connected.

Since all possible cases imply that there is a walk from u to v in $\overline{G}$, we conclude that $\overline{G}$ is connected. ■

Problem 2. For each of the following statements decide if it is true or false, and either prove it or give a counter example.

(a) If u, v, w are vertices of G and there is an even length path from u to v and an even length path from v to w, then there is an even length path from u to w.

Proof. The assertion is **false**. Consider the following counter example:



Notice that there is a path of even length from u to v (that which goes: $u \to x \to y \to t \to v$ of length 4) and a path of even length from v to w (that which goes: $v \to s \to t \to z \to w$ of length 4). However, the only path between u and w is that which goes: $u \to x \to y \to t \to z \to w$ of odd length 5. Such holds as once we reach t from u, we can only follow the highlighted yellow line, otherwise we would have to cross the vertex t twice to get to vertex w, which would not be a path by definition. ■

(b) If G is connected and has no path with length larger than k, then every two paths in G of length k have at least one vertex in common.

Proof. The assertion is **true**. Suppose G is connected and such that no path of which has length larger than some positive integer k. Let A_k and B_k be two paths in G, and suppose towards a contradiction that they share no common vertices. For the sake of clarity, we let

- $V(A_k) := \{a_0, a_1, a_2, \ldots, a_k\}$, $E(A_k) := \{a_i a_{i+1} : 0 \le i \le k-1\}$;
- $V(B_k) := \{b_0, b_1, b_2, \ldots, b_k\}$, $E(B_k) := \{b_i b_{i+1} : 0 \le i \le k-1\}$;

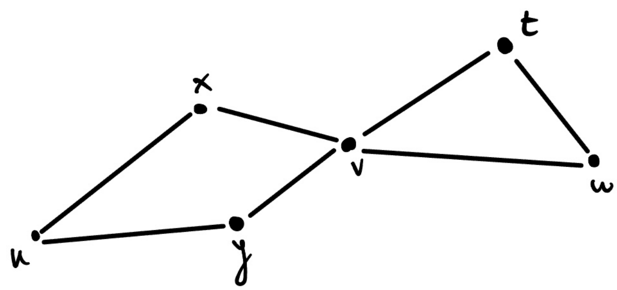
and we note that by hypothesis, $a_i \neq b_i$ for $i = 0, 1, \ldots, k$. But G is connected; and we use this fact to construct a path of length greater than k, a contradiction. Namely, we use this fact: Since G is connected, we know by lecture that there exists a path from between any two vertices $u, v \in V(G)$ in G.

So let P be the shortest path with one end in $a_i \in V(A_k)$ and one end in $b_j \in V(B_k)$. We suppose that both $i, j \ge k/2$, otherwise, we can relabel the vertices so that this holds[2]. Then the path from a_0 to a_i has length $\ge k/2$, and likewise for the path from b_0 to b_j; note that the path P must have length at least one, since it connects at least two vertices. Thus, the union of these paths forms a path from a_0 to a_i to b_j to b_0 of length $\ge k/2 + k/2 + 1 = k + 1$ (there is a path from b_j to b_0 by[2]), a contradiction to the construction of G, completing the proof. ■

[2] In other words, for an arbitrary path A_k define as above, there also exists a path from a_k to a_0 in G; to see why, note that for any $a_i a_{i+1} \in E(A_k)$, $a_{i+1} a_i \in E(A_k)$, which holds since the edge set is composed of *unordered* pairs of vertices in G.

(c) If u, v, w are vertices of G, and there is a cycle of G containing u and v, and a cycle containing v and w, then there is a cycle containing u and w.

Proof. The assertion is **false**. Consider the graph below, where there is a cycle containing u and v as well as a cycle containing v and w, but there is no cycle containing both u and w, since there is no way to start at a vertex, go to u (or w) via a path, and then go to w (or u) via a path, and then get back to that starting vertex without crossing vertex v twice.



■

(d) If e, f, g are edges of G, and there is a cycle containing e and f, and a cycle containing f and g, then there is a cycle containing e and g.

Proof. The assertion is **true**. Suppose there is a cycle C containing edges e, f in a graph G and a cycle D containing edges f, g in G. If the cycles C and D have equal vertex and edge sets, then there is nothing to prove, so suppose otherwise. For the sake of clarity, we let

- $V(C) := \{c_1, c_2, \ldots, c_n\}$ and $E(C) := \{c_1c_2, c_2c_3, \ldots c_{n-1}c_n, c_nc_1\}$;
- $V(D) := \{d_1, d_2, \ldots, d_m\}$ and $E(D) := \{d_1d_2, d_2d_3, \ldots d_{m-1}d_m, d_md_1\}$.

Suppose, for $1 \leq i \leq m$, the edge g has ends d_i, d_{i+1}. Consider the path P_1 from the vertex d_i to a vertex d_j (along the cycle D) such that $1 \leq j \leq m$ and d_j is a vertex $c_\ell \in V(C)$. Likewise, consider the path P_2 from the vertex d_{i+1} to a vertex d_p such that $1 \leq p \leq m$ and d_p is a vertex $c_q \in V(C)$. We are assured that these paths exist since D is a cycle, and we know that these distinct vertices in $V(C)$ must also exist, since we know that the cycles at least share the edge f. Since C is a cycle, we know that the edge e must lie either on one of the two paths from c_ℓ to c_q (corresponding to the direction in which we traverse toward c_q from c_ℓ on the cycle; and we know these paths must exist trivially since C is a cycle). So e is on a path from c_ℓ to c_q which we denote by P_3, hence $e \in E(P_3)$.

Then, $\mathcal{C}$ is a cycle in G containing the edges e and g, where

$$V(\mathcal{C}) = V(P_1) \cup V(P_2) \cup V(P_3) \text{ and } E(\mathcal{C}) = E(P_1) \cup \{d_id_{i+1}\} \cup E(P_2) \cup E(P_3).$$

Such holds, as we can travel via these paths P_1, P_2, P_3 from the vertex d_i to c_ℓ to c_q to d_{i+1} and back to d_i (via g), and we can do this without repetitions since each P_i, $1 \leq i \leq 3$ is a distinct path. Therefore, we conclude that $\mathcal{C}$ is a cycle in G containing the edges e and g, thereby completing the proof. ■

Problem 3. Let G be a non-null graph, and let k be the maximum size of a set of pairwise non-adjacent vertices in G. Show that there exist paths $P_1, P_2, \ldots, P_k$ in G such that $(V(P_1), V(P_2), \ldots, V(P_k))$ is a partition of $V(G)$.

Proof. Let G be a non-null graph, and let k be the maximum size of a set of pairwise non-adjacent vertices in G. Consider a collection of paths $P_1, P_2, \ldots, P_\ell$ in G such that $(V(P_1), V(P_2), \ldots, V(P_\ell))$ is a partition of $V(G)$, $\ell \geq k$, and ℓ is the minimal positive integer with these properties. Note that if it is possible to do this with $\ell < k$, then there is nothing to prove as we can just break up each path until we get k of them. We show that $\ell = k$. We note that we are assured that such a partition exists, taking $\ell \coloneqq V(G) \geq k$, where each path consists of a unique vertex of G.

Towards a contradiction, suppose that $\ell > k$, where ℓ is minimal, i.e. suppose that we *need* more than k paths in order to partition $V(G)$ with the paths' vertex sets. By the minimality of ℓ, we can deduce that for any starting vertex v of a path P_i $(1 \leq i \leq \ell)$, v is not adjacent to any other starting vertices of paths P_j $(i \neq j$ and $1 \leq j \leq \ell)$, otherwise we could merge two paths into one (by taking the union of these paths' vertex sets, and the same for their edge sets but including the edge between the two starting vertices), resulting in $\ell - 1$ paths with the desired property, thereby contradicting the minimality of ℓ. But since this holds for every starting vertex of the ℓ paths, we have constructed a set of pairwise non-adjacent vertices of size $l > k$, a contradiction to the construction of G.

Hence ℓ can not be greater than k, so we must have that $\ell = k$, proving that there exist paths $P_1, P_2, \ldots, P_k$ in G such that $(V(P_1), V(P_2), \ldots, V(P_k))$ is a partition of $V(G)$. ■