The Perfect Matching Polytope

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Honours Research Project
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Agenda

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- Quick review of linear programming & polytopes

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  - Application to cubic graphs

Linear Programming & Polytopes

Linear Programming

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For example:

$$\begin{aligned} \text{maximize} \quad & z = 5x_1 + 3x_2 - 7x_3 \\ \text{subject to} \quad & x_1 + x_2 + x_3 \leq 12 \\ & 4x_1 + 5x_3 \leq 50 \\ & x_1, x_2, x_3 \geq 0 \end{aligned}$$

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- ($x, c \in \mathbb{R}^n$, $A \in \mathbb{R}^{m \times n}$, and $b \in \mathbb{R}^m$)

The Dual of an LP

Given a linear program in standard form:

$$
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The dual LP is formulated as:

$$\begin{aligned} \text{minimize} \quad & b^T y \\ \text{subject to} \quad & A^T y \geq c \\ & y \geq 0. \end{aligned}$$

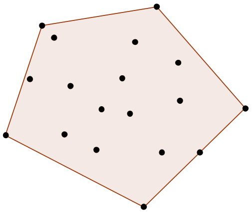
The Dual of an LP

The primal and dual LPs provide bounds on each other's optimal values. If an LP has an optimum at $x = \hat{x}$, then its dual also has this optimum (Strong duality theorem).

Polytopes

A **polytope** is the convex hull of a finite collection of vectors.

Polytopes



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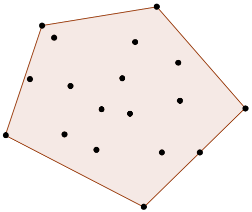

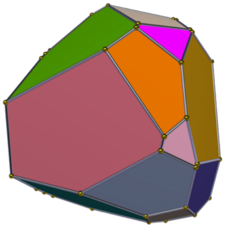

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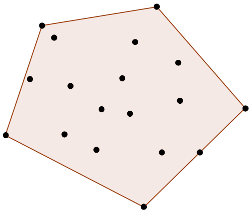
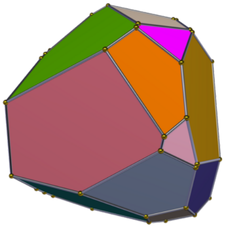
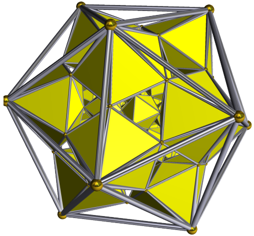

How are polytopes related to LP?

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via an equivalent definition...

Polytopes & LP

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- Bounded polyhedra are called **polytopes**.
- If the set of all feasible solutions to an LP is a polytope, then one of its corners is an optimum.

The Big Picture

We would like to solve combinatorial optimization problems using algorithms.

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- Finding min-weight perfect matchings in graphs.

If we can reduce these problems to solving linear programs, we can leverage efficient LP solvers to obtain optimal solutions.

This talk is about the connection between linear programming, polytopes, and perfect matchings.

Perfect Matchings

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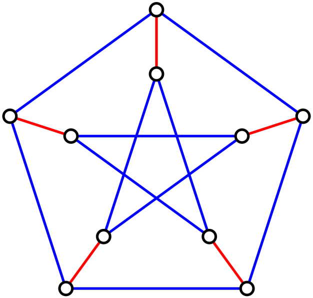
A matching M is **perfect** if every vertex $v \in V$ is incident to *exactly* one edge in M.

Perfect Matchings

A perfect matching (red edges) in the Petersen graph:



The Perfect Matching Polytope

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As such, each component of a vector in $\mathbb{R}^E$ contains information about an edge $e \in E$.

- E.g. If $F \subseteq E$, define $\chi_F \in \mathbb{R}^E$ by $\chi_F(e) = 1$ if $e \in M$ and $\chi_F(e) = 0$ otherwise.

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$$x(F) := x \cdot \chi_F = \sum_{e \in F} x(e).$$

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where $\mathrm{conv}(A)$ is the smallest convex set containing $A \subseteq \mathbb{R}^E$.

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- By definition, $\mathcal{PM}(G)$ is a polytope, so it would be really nice if we could *find a linear program* whose optimum occurs at one of its corners.
- Let's do that now!

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Observation 1: *If* $x \in \mathcal{PM}(G)$*, then* $x(e) \geq 0$ *for every* $e \in E$*.*

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Proof.

We may write x as a convex combination of characteristic vectors of perfect matchings in G:

$$x = \lambda_1 \chi_{M_1} + \lambda_2 \chi_{M_2} + \cdots + \lambda_n \chi_{M_n}.$$

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Since $\lambda_j \in [0, 1]$ and $\chi_{M_j} \geq 0$ for each $j \in [n]$, it follows that x has non-negative components. $\square$

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$$\begin{aligned} x(\delta(v)) &= \sum_{e \in \delta(v)} x(e) = \sum_{e \in \delta(v)} \sum_{j=1}^{n} \lambda_j \chi_{M_j}(e) \\ &= \sum_{j=1}^{n} \sum_{e \in \delta(v)} \lambda_j \chi_{M_j}(e) = \sum_{j=1}^{n} \lambda_j = 1, \end{aligned}$$

since each M_j is a perfect matching. $\square$

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- *This is really starting to look like a linear program...*

The Fractional Perfect Matching Polytope

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Define the **fractional perfect matching polytope** $\mathcal{FPM}(G)$ of a graph G by

$$\mathcal{FPM}(G) := \{x \in \mathbb{R}^E : x \geq 0 \text{ and } Ax = 1\}.$$

Then, it follows from observation 3 that $\mathcal{PM}(G) \subseteq \mathcal{FPM}(G)$.

Are there any graphs G with $\mathcal{PM}(G) = \mathcal{FPM}(G)$?

Yes!

If G is **bipartite**, then $\mathcal{PM}(G) = \mathcal{FPM}(G)$.

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- Otherwise, every column of B has exactly two non-zero entries. Since any two non-zero entries from the same column are in different partite sets, the sum of all rows pertaining to vertices from one partite set equals the sum of all rows from the other. So $\det B = 0$ since its rows are linearly dependent.

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Lemma 2. *If A is totally unimodular and b is a vector with integer components, then all corners of $P = \{x \geq 0 : Ax \leq b\}$ have integer components.*

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Proof.

The proof is long and somewhat off-topic. Given the time constraint, see week 8 from this webpage for a careful and detailed consideration. □

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The Bipartite Case

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$$\mathcal{FM}(G) \coloneqq \{x \in \mathbb{R}^E : x \geq 0 \text{ and } Ax \leq 1\}.$$

We have $Ax \leq 1$ (rather than $Ax = 1$) because every vertex is incident to *at most* 1 edge in a matching.

At this point, you're probably thinking...

"Who cares? When am I ever gonna use this?"

Well...

We can use this to prove **König's Theorem**! :)

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- $\nu(G) \coloneqq$ max size of a matching in G.
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Note that $\tau(G) \leq \nu(G)$ for every graph G.

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(*proof continues on next slide.*)

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(Remark: the dual LP is integral. Indeed, the transpose of a totally unimodular matrix is totally unimodular, so Lemma 2 asserts the claim.) $\square$

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since each vertex in C has exactly two neighbours. So $c \in \mathcal{FPM}(C)$. But $c \notin \mathcal{PM}(C)$, since $\mathcal{PM}(C) = \emptyset$. $\square$

Can we extend this theory to *general* graphs?

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Please see Theorem 5 here. $\square$

Application: Perfect matchings in cubic (3-regular) graphs

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$$x(\delta(X)) = \sum_{e \in \delta(X)} x(e) = \ell/3 \geq 1.$$

By Edmonds' theorem, $x = (\frac{1}{3}, \frac{1}{3}, \ldots, \frac{1}{3}) \in \mathcal{PM}(G) \neq \emptyset$. $\square$

In fact, every d-regular, $(d-1)$-edge-connected graph has a perfect matching.

Thanks for listening! ☺

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References: We followed Lovász and Plummers' book, "Matching Theory" (1985), and used information from the notes linked here (week 9) and here.