The Perfect Matching Polytope

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Honours Research Project
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Agenda

Plan for today:

- Quick review of linear programming & polytopes
- Fundamentals of matching theory
- The perfect matching polytope
  - Bipartite & general graphs
  - Application to cubic graphs

Linear Programming & Polytopes

Linear Programming

The goal of **linear programming** is to maximize (or minimize) a linear objective function subject to a collection of linear constraints.

For example:

$$\begin{aligned} \text{maximize} \quad & z = 5x_1 + 3x_2 - 7x_3 \\ \text{subject to} \quad & x_1 + x_2 + x_3 \leq 12 \\ & 4x_1 + 5x_3 \leq 50 \\ & x_1, x_2, x_3 \geq 0 \end{aligned}$$

Linear Programming

The goal of **linear programming** is to maximize (or minimize) a linear objective function subject to a collection of linear constraints.

We can express this compactly using matrices:

- Goal: Find a vector x such that $c^T x$ is maximized and x satisfies the constraints $Ax \leq b$ and $x \geq 0$.
- $(x, c \in \mathbb{R}^n$, $A \in \mathbb{R}^{m \times n}$, and $b \in \mathbb{R}^m)$

The Dual of an LP

Given a linear program in standard form:

$$\begin{aligned} \text{maximize} \quad & c^T x \\ \text{subject to} \quad & Ax \leq b \\ & x \geq 0. \end{aligned}$$

The dual LP is formulated as:

$$\begin{aligned} \text{minimize} \quad & b^T y \\ \text{subject to} \quad & A^T y \geq c \\ & y \geq 0. \end{aligned}$$

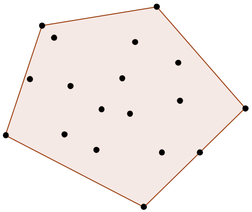
The Dual of an LP

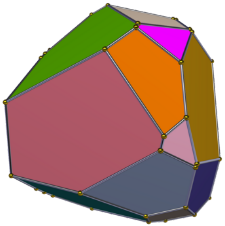
The primal and dual LPs provide bounds on each other's optimal values. If an LP has an optimum at $x = \hat{x}$, then its dual also has this optimum (Strong duality theorem).

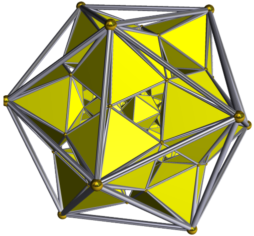
Polytopes

A **polytope** is the convex hull of a finite collection of vectors.

Polytopes







How are polytopes related to LP?

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via an equivalent definition...

Polytopes & LP

A vector $\mathbf{x}$ is called a **feasible solution** if it satisfies the linear constraints of the LP, i.e. if $A\mathbf{x} \leq \mathbf{b}$ and $\mathbf{x} \geq \mathbf{0}$.

The set $P = \{\mathbf{x} \geq \mathbf{0} : A\mathbf{x} \leq \mathbf{b}\}$ of all feasible solutions is called a **polyhedron**.

- Bounded polyhedra are called **polytopes**.
- If the set of all feasible solutions to an LP is a polytope, then one of its corners is an optimum.

The Big Picture

We would like to solve combinatorial optimization problems using algorithms.

Some examples:

- Finding min-weight edge or vertex covers in graphs.
- Finding (pure, mixed, correlated) Nash equilibria in games.
- Finding max flows and min cuts in flow networks.
- Finding min-weight perfect matchings in graphs.

If we can reduce these problems to solving linear programs, we can leverage efficient LP solvers to obtain optimal solutions.

This talk is about the connection between linear programming, polytopes, and perfect matchings.

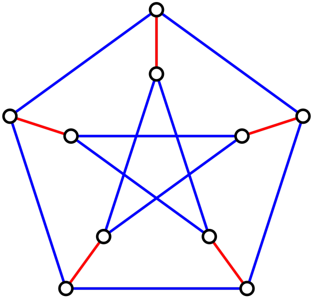
Perfect Matchings

A **matching** in a simple graph $G = (V, E)$ is a subset $M \subseteq E$ of edges such that no two edges in M share an end. So every vertex $v \in V$ is incident to *at most* one edge in M.

A matching M is **perfect** if every vertex $v \in V$ is incident to *exactly* one edge in M.

Perfect Matchings

A perfect matching (red edges) in the Petersen graph:



The Perfect Matching Polytope

Preliminaries

Let $G = (V, E)$. We will work in the vector space $\mathbb{R}^E := \mathbb{R}^{|E|}$.

- Vectors in $\mathbb{R}^E$ have components indexed by E.
- E.g. $x = (x(e) : e \in E) \in \mathbb{R}^E$

As such, each component of a vector in $\mathbb{R}^E$ contains information about an edge $e \in E$.

- E.g. If $F \subseteq E$, define $\chi_F \in \mathbb{R}^E$ by $\chi_F(e) = 1$ if $e \in M$ and $\chi_F(e) = 0$ otherwise.

Preliminaries

We must cover one final preliminary:

Given $x \in \mathbb{R}^E$ and $F \subseteq E$, define

$$x(F) := x \cdot \chi_F = \sum_{e \in F} x(e).$$

The Perfect Matching Polytope

Let $\mathcal{M}_G$ denote the collection of perfect matchings of G. Then, we define the **perfect matching polytope** $\mathcal{PM}(G)$ of G by

$$\mathcal{PM}(G) \coloneqq \operatorname{conv}(\{\chi_M \in \mathbb{R}^E : M \in \mathcal{M}_G\}),$$

where $\operatorname{conv}(A)$ is the smallest convex set containing $A \subseteq \mathbb{R}^E$.

The Perfect Matching Polytope

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- By definition, $\mathcal{PM}(G)$ is a polytope, so it would be really nice if we could *find a linear program* whose optimum occurs at one of its corners.
- Let's do that now!

The Perfect Matching Polytope

Observation 1: *If $x \in \mathcal{PM}(G)$, then $x(e) \geq 0$ for every $e \in E$.*

Proof.

We may write x as a convex combination of characteristic vectors of perfect matchings in G:

$$x = \lambda_1 \chi_{M_1} + \lambda_2 \chi_{M_2} + \cdots + \lambda_n \chi_{M_n}.$$

Since $\lambda_j \in [0, 1]$ and $\chi_{M_j} \geq 0$ for each $j \in [n]$, it follows that x has non-negative components. $\square$

The Perfect Matching Polytope

Observation 2: *If $x \in \mathcal{PM}(G)$, then $x(\delta(v)) = 1$ for all $v \in V$.* *(note: $\delta(X)$ is the set edges with exactly one end in X)*

Proof.

Again, write $x = \sum_{j=1}^{n} \lambda_j \chi_{M_j}$ as a convex combination. Then

$$
\begin{aligned}
x(\delta(v)) &= \sum_{e \in \delta(v)} x(e) = \sum_{e \in \delta(v)} \sum_{j=1}^{n} \lambda_j \chi_{M_j}(e) \\
&= \sum_{j=1}^{n} \sum_{e \in \delta(v)} \lambda_j \chi_{M_j}(e) = \sum_{j=1}^{n} \lambda_j = 1,
\end{aligned}
$$

since each M_j is a perfect matching. $\square$

The Perfect Matching Polytope

Observation 1: *If* $x \in \mathcal{PM}(G)$, *then* $x(e) \geq 0$ *for every* $e \in E$.

Observation 2: *If* $x \in \mathcal{PM}(G)$, *then* $x(\delta(v)) = 1$ *for all* $v \in V$.

Observations 1 and 2 can be written compactly as the following:

Observtion 3: *If* $x \in \mathcal{PM}(G)$, *then* $x \geq 0$ *and* $Ax = 1$, *where* $A = (a_{ve})_{v \in V, e \in E}$ *is the incidence matrix of* G.

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- *This is really starting to look like a linear program...*

The Fractional Perfect Matching Polytope

Observtion 3: *If* $x \in \mathcal{PM}(G)$*, then* $x \geq 0$ *and* $Ax = 1$*, where* $A = (a_{ve})_{v \in V, e \in E}$ *is the incidence matrix of* G*.*

Define the **fractional perfect matching polytope** $\mathcal{FPM}(G)$ of a graph G by

$$\mathcal{FPM}(G) \coloneqq \{x \in \mathbb{R}^E : x \geq 0 \text{ and } Ax = 1\}.$$

Then, it follows from observation 3 that $\mathcal{PM}(G) \subseteq \mathcal{FPM}(G)$.

The Fractional Perfect Matching Polytope

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Then, it follows from observation 3 that $\mathcal{PM}(G) \subseteq \mathcal{FPM}(G)$.

- *This raises a natural question...*

Are there any graphs G with
$\mathcal{PM}(G) = \mathcal{FPM}(G)$?

Yes!

If G is **bipartite**, then $\mathcal{PM}(G) = \mathcal{FPM}(G)$.

Lemma 1. *Let A be the incidence matrix of a bipartite graph G. Then A is totally unimodular; that is, every square submatrix B of A satisfies* $\det B \in \{0, \pm 1\}$.

Proof.

Let B be an $m \times m$ submatrix of A. The proof is by induction on m. If $m = 1$ then clearly $\det B \in \{0, 1\}$, so fix $m \geq 2$.

- If B has a column with at most one non-zero entry, then we may cofactor-expand along this column and use the IH to obtain $\det B \in \{0, \pm 1\}$.
- Otherwise, every column of B has exactly two non-zero entries. Since any two non-zero entries from the same column are in different partite sets, the sum of all rows pertaining to vertices from one partite set equals the sum of all rows from the other. So $\det B = 0$ since its rows are linearly dependent.

$\square$

Lemma 2. *If A is totally unimodular and b is a vector with integer components, then all corners of $P = \{x \geq 0 : Ax \leq b\}$ have integer components.*

Proof.

The proof is long and somewhat off-topic. Given the time constraint, see section n here. $\square$

Lemma 3. *If $x \in \mathcal{FPM}(G)$ is integral, then $x \in \mathcal{PM}(G)$.*

Proof.

We know that $x(e) \geq 0$ for each $e \in E$ and $\sum_{e \in \delta(v)} x(e) = 1$ for each $v \in V$. Since x is integral (i.e. has integer components), there is exactly one edge $e \in \delta(v)$ with $x(e) = 1$. Since this holds for each $v \in V$, x is a characterstic vector of a perfect matching in G. So $x \in \mathcal{PM}(G)$. $\square$

Theorem. *If G is a bipartite graph, then $\mathcal{PM}(G) = \mathcal{FPM}(G)$.*

Proof.

We have already seen that $\mathcal{PM}(G) \subseteq \mathcal{FPM}(G)$. From Lemma 1, the incidence matrix A of G is totally unimodular; and so Lemma 2 implies that every corner of $\mathcal{FPM}(G)$ is integral. Then, Lemma 3 implies that all corners of $\mathcal{FPM}(G)$ are in $\mathcal{PM}(G)$. By the convexity of $\mathcal{PM}(G)$, we conclude that $\mathcal{FPM}(G) \subseteq \mathcal{PM}(G)$. $\square$

We can define the **matching polytope** of G analagously:

$$\mathcal{M}(G) \coloneqq \mathrm{conv}(\{\chi_M : M \text{ is a matching in } G\}.$$

We can also define its fractional matching polytope:

$$\mathcal{FM}(G) \coloneqq \{x \in \mathbb{R}^E : x \geq 0 \text{ and } Ax \leq 1\}.$$

We have $Ax \leq 1$ (rather than $Ax = 1$) because every vertex is incident to *at most* 1 edge in a matching.

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- By the exact same reasoning as before, $\mathcal{M}(G) = \mathcal{FM}(G)$ whenever G is bipartite.

At this point, you're probably thinking...

“Who cares? When am I ever gonna use this?”

Well...

We can use this to prove **König's Theorem**! :)

Let $G = (V, E)$. Then, we define

Note that $\tau(G) \leq \nu(G)$ for every graph G. Indeed, given any maximal matching M, let X be a set consisting of one end from each edge in M. Then X is a vertex cover and so

$$\tau(G) \leq |X| \leq |M| = \nu(G).$$

$\square$

Let $G = (V, E)$. Then, we define

- $\nu(G) :=$ max size of a matching in G.

Note that $\tau(G) \le \nu(G)$ for every graph G. Indeed, given any maximal matching M, let X be a set consisting of one end from each edge in M. Then X is a vertex cover and so

$$\tau(G) \le |X| \le |M| = \nu(G).$$

$\square$

Let $G = (V, E)$. Then, we define

- $\nu(G) \coloneqq$ max size of a matching in G.
- $\tau(G) \coloneqq$ min size of a vertex cover in G.

Note that $\tau(G) \leq \nu(G)$ for every graph G. Indeed, given any maximal matching M, let X be a set consisting of one end from each edge in M. Then X is a vertex cover and so

$$\tau(G) \leq |X| \leq |M| = \nu(G).$$

$\square$

König's Theorem. The quantities $\nu(G) = \tau(G)$ for every bipartite graph G.

Proof.

From the previous theorem (and Lemmas 1-3), one of the corners of

$$\mathcal{M}(G) = \mathcal{FM}(G) = \{x \in \mathbb{R}^E : x \geq 0 \text{ and } Ax \leq 1\}$$

gives $\nu(G)$ as an optimum to the following (integral) LP:

$$\begin{aligned} \text{maximize} \quad & z(x) = \sum_{e \in E} x(e) \\ \text{subject to} \quad & \sum_{e \in \delta(v)} x(e) \leq 1, \ \forall v \in V \\ & x(e) \geq 0, \ \forall e \in E. \end{aligned}$$

(*proof continues on next slide.*)

Using a bit of algebra, one can check that the dual of the LP on the previous slide is:

$$\begin{aligned} \text{minimize} \quad & \sum_{v \in V} y(v) \\ \text{subject to} \quad & y(u) + y(v) \geq 1, \ \forall uv \in E \\ & y(v) \geq 0, \ \forall v \in V. \end{aligned}$$

From the strong duality theorem, it follows that the dual has optimum value $\nu(G)$. But notice that by construction, the dual LP computes $\tau(G)$, so $\nu(G) = \tau(G)$.[1] $\square$

[1] Remark: the dual LP is integral. Indeed, the transpose of a totally unimodular matrix is totally unimodular, so Lemma 2 asserts the claim.

Remark. There are graphs with $\mathcal{FPM}(G) \nsubseteq \mathcal{PM}(G)$.

Proof.
Consider any odd cycle $C = (V, E)$. Define the vector $c \in \mathbb{R}^E$ given by $c(e) = 1/2$ for each $e \in E$. Then $c \geq 0$ and

$$c(\delta(v)) = \sum_{e \in \delta(v)} c(e) = 1/2 + 1/2 = 1,$$

since each vertex in C has exactly two neighbours. So $c \in \mathcal{FPM}(C)$. But $c \notin \mathcal{PM}(C)$, since $\mathcal{PM}(C) = \emptyset$. $\square$

Can we extend this theory to *general* graphs?

Edmonds' Theorem. *For any graph G, the polytope $\mathcal{PM}(G)$ is precisely the set of vectors $x \in \mathbb{R}^E$ satisfying:*

1. $x \geq 0$;
2. $x(\delta(v)) = \sum_{e \in \delta(v)} x(e) = 1$, for each $v \in V$;
3. $x(\delta(X)) = \sum_{e \in \delta(X)} x(e) \geq 1$, for each odd subset $X \subseteq V$.

Proof.

See n in here. $\square$

Application: Perfect matchings in cubic (3-regular) graphs

Theorem. *Every cubic bridgeless graph has a perfect matching.*

Proof.

It suffices to prove that its perfect matching polytope $\mathcal{PM}(G)$ is non-empty. Put $x = (1/3 : e \in E)$. Then $x \geq 0$, and since G is cubic, $x(\delta(v)) = 3 \cdot 1/3 = 1$. Finally, fix an odd subset $X \subseteq V$ and let $\ell = |\delta(X)|$ be the number of edges leaving X. Then

$$3|X| = \sum_{v \in X} \deg v = 2|E(X)| + \ell.$$

Now observe that $3|X|$ is odd and $2|E(X)|$ is even, so ℓ is odd. Further, $\ell > 1$ since G is bridgeless, and so $\ell \geq 3$. Hence

$$x(\delta(X)) = \sum_{e \in \delta(X)} x(e) = \ell/3 \geq 1.$$

By Edmonds' theorem, $x = (\frac{1}{3}, \frac{1}{3}, \ldots, \frac{1}{3}) \in \mathcal{PM}(G) \neq \emptyset$. $\square$

In fact, every d-regular, $(d-1)$-edge-connected graph has a perfect matching.

Thanks for listening!

Let me know if you have any questions!

References: We followed Lovász and Plummers' book, "Matching Theory" (1985), and used information from the notes linked here (week 9) and here.