

# The Perfect Matching Polytope

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Honours Research Project  
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# Agenda

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- Quick review of linear programming & polytopes

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  - Application to cubic graphs

# Linear Programming & Polytopes



# Linear Programming

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For example:

$$\begin{array}{ll}\text{maximize} & z = 5x_1 + 3x_2 - 7x_3 \\ \text{subject to} & x_1 + x_2 + x_3 \leq 12 \\ & 4x_1 + 5x_3 \leq 50 \\ & x_1, x_2, x_3 \geq 0\end{array}$$

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- ( $x, c \in \mathbb{R}^n$ ,  $A \in \mathbb{R}^{m \times n}$ , and  $b \in \mathbb{R}^m$ )

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The dual LP is formulated as:

$$\begin{array}{ll}\text{minimize} & b^T y \\ \text{subject to} & A^T y \geq c \\ & y \geq 0.\end{array}$$

## The Dual of an LP

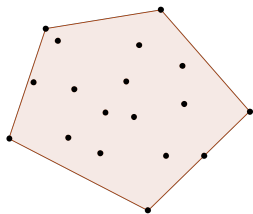
The primal and dual LPs provide bounds on each other's optimal values. If an LP has an optimum at  $x = \hat{x}$ , then its dual also has this optimum (Strong duality theorem).



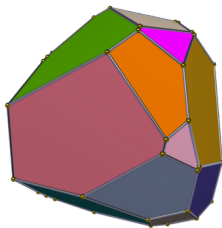
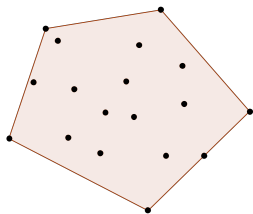
# Polytopes

A **polytope** is the convex hull of a finite collection of vectors.

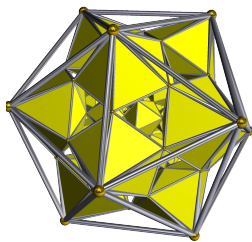
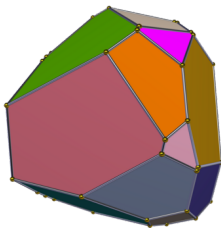
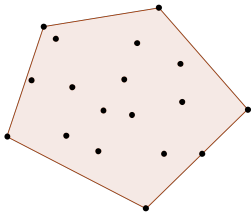
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How are polytopes related to LP?

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*via an equivalent definition...*

## Polytopes & LP

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- If the set of all feasible solutions to an LP is a polytope, then one of its corners is an optimum.

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- Finding min-weight perfect matchings in graphs.

If we can reduce these problems to solving linear programs, we can leverage efficient LP solvers to obtain optimal solutions.

This talk is about the connection between linear programming, polytopes, and perfect matchings.

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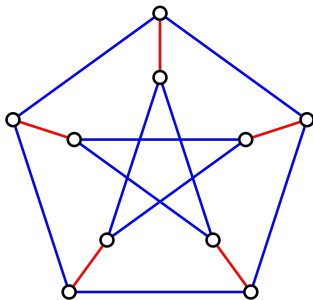
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A matching  $M$  is **perfect** if every vertex  $v \in V$  is incident to *exactly* one edge in  $M$ .

# Perfect Matchings

A perfect matching (red edges) in the Petersen graph:



# The Perfect Matching Polytope

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- E.g. If  $F \subseteq E$ , define  $\chi_F \in \mathbb{R}^E$  by  $\chi_F(e) = 1$  if  $e \in F$  and  $\chi_F(e) = 0$  otherwise.

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$$x(F) := x \cdot \chi_F = \sum_{e \in F} x(e).$$

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where  $\text{conv}(A)$  is the smallest convex set containing  $A \subseteq \mathbb{R}^E$ .

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- Let's do that now!

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*Proof.*

We may write  $x$  as a convex combination of characteristic vectors of perfect matchings in  $G$ :

$$x = \lambda_1 \chi_{M_1} + \lambda_2 \chi_{M_2} + \cdots + \lambda_n \chi_{M_n}.$$

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Since  $\lambda_j \in [0, 1]$  and  $\chi_{M_j} \geq 0$  for each  $j \in [n]$ , it follows that  $x$  has non-negative components. □



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$$\begin{aligned} x(\delta(v)) &= \sum_{e \in \delta(v)} x(e) = \sum_{e \in \delta(v)} \sum_{j=1}^n \lambda_j \chi_{M_j}(e) \\ &= \sum_{j=1}^n \sum_{e \in \delta(v)} \lambda_j \chi_{M_j}(e) = \sum_{j=1}^n \lambda_j = 1, \end{aligned}$$

since each  $M_j$  is a perfect matching. □

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- *This is really starting to look like a linear program...*

# The Fractional Perfect Matching Polytope

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Then, it follows from observation 3 that  $\mathcal{PM}(G) \subseteq \mathcal{FPM}(G)$ .

Are there any graphs  $G$  with  
 $\mathcal{PM}(G) = \mathcal{FPM}(G)$ ?

Yes!

If  $G$  is **bipartite**, then  $\mathcal{PM}(G) = \mathcal{FPM}(G)$ .

**Lemma 1.** *Let  $A$  be the incidence matrix of a bipartite graph  $G$ . Then  $A$  is totally unimodular; that is, every square submatrix  $B$  of  $A$  satisfies  $\det B \in \{0, \pm 1\}$ .*

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- Otherwise, every column of  $B$  has exactly two non-zero entries. Since any two non-zero entries from the same column are in different partite sets, the sum of all rows pertaining to vertices from one partite set equals the sum of all rows from the other. So  $\det B = 0$  since its rows are linearly dependent.



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**Proof.**

The proof is long and somewhat off-topic. Given the time constraint, see week 8 from [this webpage](#) for a careful and detailed consideration.



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**Theorem.** *If  $G$  is a bipartite graph, then  $\mathcal{PM}(G) = \mathcal{FPM}(G)$ .*

*Proof.*

We have already seen that  $\mathcal{PM}(G) \subseteq \mathcal{FPM}(G)$ . From Lemma 1, the incidence matrix  $A$  of  $G$  is totally unimodular; and so Lemma 2 implies that every corner of  $\mathcal{FPM}(G)$  is integral. Then, Lemma 3 implies that all corners of  $\mathcal{FPM}(G)$  are in  $\mathcal{PM}(G)$ . By the convexity of  $\mathcal{PM}(G)$ , we conclude that  $\mathcal{FPM}(G) \subseteq \mathcal{PM}(G)$ . □

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We have  $Ax \leq 1$  (rather than  $Ax = 1$ ) because every vertex is incident to *at most* 1 edge in a matching.

At this point, you're probably thinking...

*“Who cares? When am I ever gonna use this?”*

Well...

We can use this to prove **König's Theorem!** :)

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From the previous theorem (and Lemmas 1-3), one of the corners of

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$$\begin{aligned} &\text{maximize} && z(x) = \sum_{e \in E} x(e) \\ &\text{subject to} && \sum_{e \in \delta(v)} x(e) \leq 1, \quad \forall v \in V \\ &&& x(e) \geq 0, \quad \forall e \in E. \end{aligned}$$

*(proof continues on next slide.)*

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(Remark: the dual LP is integral. Indeed, the transpose of a totally unimodular matrix is totally unimodular, so Lemma 2 asserts the claim.)



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since each vertex in  $C$  has exactly two neighbours. So  $c \in \mathcal{FPM}(C)$ . But  $c \notin \mathcal{PM}(C)$ , since  $\mathcal{PM}(C) = \emptyset$ . □



Can we extend this theory to *general* graphs?

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Please see Theorem 5 [here](#).



Application: Perfect matchings in cubic  
(3-regular) graphs

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By Edmonds' theorem,  $x = (\frac{1}{3}, \frac{1}{3}, \dots, \frac{1}{3}) \in \mathcal{PM}(G) \neq \emptyset$ . □

In fact, every  $d$ -regular,  $(d - 1)$ -edge-connected graph has a perfect matching.



**Thanks for listening! 😊**  
Let me know if you have any questions!



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References: We followed Lovász and Plummers' book, "Matching Theory" (1985), and used information from the notes linked [here](#) (week 9) and [here](#).