

# MATH 454: Examinable Results

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1. **Monotonicity of outer measure.**  $A \subseteq B \subseteq \mathbb{R}^d \implies m_*(A) \leq m_*(B)$ .

*Proof.* Any covering of  $B$  by closed cubes is a covering of  $A$ . Thus if  $V_A$  and  $V_B$  are the sets over which we take the infimum to attain the outer measure of  $A$  and  $B$  respectively, we must have that  $V_B \subseteq V_A$  so that  $m_*(A) = \inf V_A \leq \inf V_B = m_*(B)$  as required.

2. **Countable sub-additivity of outer measure.** If  $\{A_k\}_{k=1}^\infty$  is a sequence of subsets of  $\mathbb{R}^d$  with  $A := \bigcup_{k=1}^\infty A_k$ , then  $m_*(A) \leq \sum_{k=1}^\infty m_*(A_k)$ .

*Proof.* If there exists a  $k \in \mathbb{N}$  such that  $m_*(A_k) = \infty$  then there is nothing to prove, hence we suppose otherwise. Let  $\varepsilon > 0$  be fixed. For each  $k \in \mathbb{N}$ , there is a covering of  $A_k$  by closed cubes  $(Q_{k,j,\varepsilon})_{j \in \mathbb{N}}$  such that  $A_k \subseteq \bigcup_{j=1}^\infty Q_{k,j,\varepsilon}$  and  $\sum_{j=1}^\infty \text{vol}(Q_{k,j,\varepsilon}) < m_*(A_k) + \frac{\varepsilon}{2^k}$ . Thus,

$$\bigcup_{k=1}^\infty A_k \subseteq \bigcup_{k=1}^\infty \bigcup_{j=1}^\infty Q_{k,j,\varepsilon} \implies m_*\left(\bigcup_{k=1}^\infty A_k\right) \leq \sum_{k=1}^\infty \sum_{j=1}^\infty \text{vol}(Q_{k,j,\varepsilon}) < \sum_{k=1}^\infty \left(m_*(A_k) + \frac{\varepsilon}{2^k}\right) = \sum_{k=1}^\infty m_*(A_k) + \varepsilon.$$

Since  $\varepsilon$  was arbitrary, we obtain the required result.

3. **Additivity of outer measure with assumption.** Let  $A_1, A_2 \subseteq \mathbb{R}^d$  be such that  $d(A_1, A_2) = \inf_{x \in A_1, y \in A_2} |x - y| > 0$ . Then  $m_*(A_1 \cup A_2) = m_*(A_1) + m_*(A_2)$ .

*Proof.* By sub-additivity, we have  $m_*(A_1 \cup A_2) \leq m_*(A_1) + m_*(A_2)$ , so we prove the converse inequality. Let  $\{Q_k\}_{k=1}^\infty$  be a sequence of closed cubes such that  $A_1 \cup A_2 \subseteq \bigcup_{k=1}^\infty Q_k$  and  $0 < \delta < d(A_1, A_2)$ . By sub-dividing the cubes, we may assume that  $\text{diam}(Q_k) = \sup\{|x - y| : x, y \in Q_k\} < \delta$ . For  $i = 1, 2$  let  $K_i := \{k \in \mathbb{N} : Q_k \cap A_i \neq \emptyset\}$ . By summing we obtain  $m_*(A_1) + m_*(A_2) \leq \sum_{k \in K_1} \text{vol}(Q_k) + \sum_{k \in K_2} \text{vol}(Q_k)$ . By the choice of  $\delta$ ,  $K_1 \cap K_2 = \emptyset$  so that

$$\sum_{k \in K_1} \text{vol}(Q_k) + \sum_{k \in K_2} \text{vol}(Q_k) = \sum_{k \in K_1 \cup K_2} \text{vol}(Q_k) \leq \sum_{k=1}^\infty \text{vol}(Q_k).$$

By taking the infimum over all coverings  $Q_k$ , we then obtain  $m_*(A_1) + m_*(A_2) \leq m_*(A_1 \cup A_2)$  which completes the proof.

4. **Countable unions of measurable sets.** If  $\{A_k\}_{k=1}^\infty$  is a sequence of measurable sets then  $\bigcup_{k=1}^\infty A_k$  is measurable.

*Proof.* Let  $\varepsilon > 0$ . For each  $k \in \mathbb{N}$  there exists an open set  $\mathcal{O}_{k,\varepsilon}$  such that  $A_k \subseteq \mathcal{O}_{k,\varepsilon}$  and  $m_*(\mathcal{O}_{k,\varepsilon} - A_k) < \frac{\varepsilon}{2^k}$ . Let  $\mathcal{O}_\varepsilon := \bigcup_{k=1}^\infty \mathcal{O}_{k,\varepsilon}$ , which is open (union of open sets). By construction,  $\bigcup_{k=1}^\infty A_k \subseteq \mathcal{O}_\varepsilon$  so that  $m_*(\mathcal{O}_\varepsilon - \bigcup_{k=1}^\infty A_k) \leq m_*(\bigcup_{k=1}^\infty (\mathcal{O}_{k,\varepsilon} - A_k))$  by monotonicity since  $\bigcup_{k=1}^\infty \mathcal{O}_{k,\varepsilon} - \bigcup_{k=1}^\infty A_k \subseteq \bigcup_{k=1}^\infty (\mathcal{O}_{k,\varepsilon} - A_k)$ . Thus, we conclude that

$$m_*\left(\mathcal{O}_\varepsilon - \bigcup_{k=1}^\infty A_k\right) \leq m_*\left(\bigcup_{k=1}^\infty (\mathcal{O}_{k,\varepsilon} - A_k)\right) \leq \sum_{k=1}^\infty m_*(\mathcal{O}_{k,\varepsilon} - A_k) < \sum_{k=1}^\infty \frac{\varepsilon}{2^k} = \varepsilon.$$

### 5. Measurability of closed sets.

*Proof.* Let  $F \subseteq \mathbb{R}^d$  be closed and bounded. There is an open cube  $Q$  such that  $\mathcal{O} := Q - F$  is open and hence a corresponding sequence of mutually disjoint open cubes  $\{Q_k\}_{k=1}^\infty$  such that  $\mathcal{O} = \bigcup_{k=1}^\infty \overline{Q}_k$ . For  $n \in \mathbb{N}$ , let  $\mathcal{O}_n := Q - \bigcup_{k=1}^n \overline{Q}_k$ , which is open since  $Q$  is open and each  $\overline{Q}_k$  is closed. Moreover,  $m_*(\mathcal{O}_n - F) = m_*(\mathcal{O} - \bigcup_{k=n+1}^\infty \overline{Q}_k) \leq \sum_{k=n+1}^\infty m_*(\overline{Q}_k)$  by monotonicity and sub-additivity. But  $\sum_{k=n+1}^\infty \text{vol}(\overline{Q}_k) = \sum_{k=n+1}^\infty \text{vol}(Q_k) \rightarrow 0$  as  $n \rightarrow \infty$  since  $\sum_{k=1}^\infty \text{vol}(Q_k) = m_*(\mathcal{O}) \leq m_*(Q) < \infty$ . Hence compact sets are measurable. Thus, given any closed set  $A$ , write  $A = \bigcup_{k=1}^\infty (A \cap [-k, k]^d)$  so that  $A$  is written as a countable union of compact and hence measurable sets (note that  $A \cap [-k, k]^d$  is a closed subset of a compact set and hence compact).

### 6. Measurability of complements. Let $A \subseteq \mathbb{R}^d$ be measurable. Then $A^c$ is measurable.

*Proof.* By the measurability of  $A$ , for each  $k \in \mathbb{N}$  there exists an open set  $\mathcal{O}_k$  such that  $A \subseteq \mathcal{O}_k$  and  $m_*(\mathcal{O}_k - A) < 1/k$ . Let  $F_k := \mathcal{O}_k^c$  and  $F := \bigcup_{k=1}^\infty F_k$  (measurable as countable union of closed sets). Let  $N := A^c - F = A^c - \bigcup_{k=1}^\infty \mathcal{O}_k^c = A^c \cap \bigcap_{k=1}^\infty \mathcal{O}_k \subseteq \mathcal{O}_k \cap A^c = \mathcal{O}_k - A$  so that  $m_*(N) \leq m_*(\mathcal{O}_k - A) \leq 1/k \rightarrow 0$  as  $k \rightarrow \infty$ . Thus,  $A^c = F \cup N$ , which is measurable (finite union of closed set and set of measure 0, completes the proof).

### 7. Measurability of countable intersections. Let $\{A_k\}_{k=1}^\infty$ be a sequence of measurable subsets of $\mathbb{R}^d$ . Then $\bigcap_{k=1}^\infty A_k$ is measurable.

*Proof.* Since for each  $k \in \mathbb{N}$   $A_k$  is measurable, so is  $A_k^c$ . Thus,  $\bigcap_{k=1}^\infty A_k = \bigcup_{k=1}^\infty A_k^c$  is a union of measurable sets and hence measurable.

### 8. Continuity of measure for increasing sets. Let $\{A_k\}_{k=1}^\infty$ be a sequence of measurable subsets of $\mathbb{R}^d$ such that for each $k \in \mathbb{N}$ , $A_k \subseteq A_{k+1}$ . Then $m_*(\bigcup_{k=1}^\infty A_k) = \lim_{k \rightarrow \infty} m(A_k)$ .

*Proof.* If there is a  $k_0 \in \mathbb{N}$  such that  $m(A_{k_0}) = \infty$ , then by monotonicity  $m(\bigcup_{k=1}^\infty A_k) = \infty$  and since  $k \geq k_0 \implies A_k \subseteq A_{k_0}$ , we likewise have  $\lim_k m(A_k) = \infty$  and there is nothing to prove. Hence suppose there is no such  $k_0$ . Write  $\bigcup_{k=1}^\infty A_k$  as  $\bigcup_{k=1}^\infty B_k$ , where  $B_1 := A_1$  and for  $k \geq 2$ ,  $B_k := A_k - A_{k-1}$ . Since the sets  $\{B_k\}_k$  are mutually disjoint, by countable additivity, we have

$$\begin{aligned} m\left(\bigcup_{k=1}^\infty A_k\right) &= m\left(\bigcup_{k=1}^\infty B_k\right) = \sum_{k=1}^\infty m(B_k) = \lim_{n \rightarrow \infty} \sum_{k=1}^n m(B_k) = \lim_{n \rightarrow \infty} \left(m(A_1) + \sum_{k=2}^n m(A_k - A_{k-1})\right) \\ &= \lim_{n \rightarrow \infty} \left(m(A_1) + \sum_{k=2}^n (m(A_k) - m(A_{k-1}))\right) = \lim_{n \rightarrow \infty} m(A_n). \quad (\text{As } \forall k : m(A_k) < \infty) \end{aligned}$$

### 9. Continuity of measure for decreasing sets. Let $\{A_k\}_{k=1}^\infty$ be a sequence of measurable subsets of $\mathbb{R}^d$ such that $A_{k+1} \supseteq A_k$ for $k \in \mathbb{N}$ and $\exists k_0 \in \mathbb{N}$ such that $m(A_{k_0}) < \infty$ .

*Proof.* Suppose without loss of generality that  $m(A_1) < \infty$ . Let  $B_k := A_k - A_{k+1}$  for each  $k \in \mathbb{N}$  so that  $A_1 = \bigcap_{k=1}^{\infty} A_k \cup \bigcup_{k=1}^{\infty} B_k$  is a disjoint union of measurable sets. Hence,

$$m(A_1) = m\left(\bigcap_{k=1}^{\infty} A_k\right) + \lim_{N \rightarrow \infty} \sum_{k=1}^{N-1} (m(A_k) - m(A_{k+1})) = m\left(\bigcap_{k=1}^{\infty} A_k\right) + m(A_1) - \lim_{N \rightarrow \infty} m(A_N).$$

Hence since  $m(A_1) < \infty$ , it follows that  $m(\bigcap_{k=1}^{\infty} A_k) = \lim_{N \rightarrow \infty} m(A_N)$  as required.

**10. Approximation of measurable sets.** Let  $\varepsilon > 0$ . The following are equivalent:

- (a)  $A \subseteq \mathbb{R}^d$  is measurable.
- (b) There exists an open set  $\mathcal{O}$  such that  $A \subseteq \mathcal{O}$  and  $m_*(\mathcal{O} - A) < \varepsilon$ .
- (c) There exists a closed set  $F$  such that  $F \subseteq A$  and  $m(A - F) < \varepsilon$ .
- (d) There is a  $G_\delta$  set  $G$  and a set  $N \subseteq G$  such that  $m_*(N) = 0$  and  $A = G \setminus N$ .
- (e) There is an  $F_\sigma$  set  $F$  and a set  $N \subseteq A \setminus F$  such that  $m_*(N) = 0$  and  $A = F \cup N$ .

*Proof.*

(a) iff (b) is by definition. (a) implies (d). Since  $A$  measurable,  $\forall n \in \mathbb{N}$  there exists an open set  $\mathcal{O}_k$  such that  $A \subseteq \mathcal{O}_k$  and  $m_*(\mathcal{O}_k \setminus A) < 1/k$ . Let  $G := \bigcap_{k=1}^{\infty} \mathcal{O}_k$  be a  $G_\delta$  set containing  $A$ . Let  $N := G \setminus A$  so that for  $k \in \mathbb{N}$ ,  $m(N) \leq m(\mathcal{O}_k \setminus A) < 1/k \rightarrow 0$  (monotonicity). (d) implies (a).  $G$  is measurable (countable intersection of measurable sets) and  $N$  is measurable as its measure is 0. Hence  $G \setminus N = G \cap N^c = A$  is measurable.

(a) iff (c).  $A$  measurable  $\iff A^c$  measurable  $\iff$  for each  $\varepsilon > 0$  there exists an open set  $\mathcal{O}_\varepsilon$  such that  $A^c \subseteq \mathcal{O}_\varepsilon$  and  $m_*(\mathcal{O}_\varepsilon \setminus A^c) = m_*(\mathcal{O}_\varepsilon \cap A) = m_*(A \setminus \mathcal{O}_\varepsilon^c) < \varepsilon$ , and  $F := \mathcal{O}_\varepsilon^c$  contains  $A$  since  $A^c \subseteq \mathcal{O}_\varepsilon$ . (d) iff (e). (d) is equivalent to: there exists an  $F_\sigma$  set  $F := G^c$  and a set  $N \subseteq F^c = G$  such that  $A^c = F^c \setminus N$ , i.e.  $A = F \cup N$ .

**11. Continuous functions are measurable.** Let  $A \subseteq \mathbb{R}^d$  be measurable and let  $f : A \rightarrow \mathbb{R}$  be continuous. Then  $f$  is measurable.

*Proof.* Let  $c \in \mathbb{R}$  be fixed. Then  $f^{-1}([-\infty, c)) = f^{-1}((-\infty, c))$  since  $-\infty$  is never attained by  $f$  by continuity. Since  $f$  is continuous and  $(-\infty, c)$  is open,  $f^{-1}((-\infty, c))$  is open and hence measurable. Since  $c$  was arbitrary, we conclude that  $f$  is measurable.

**12. Measurability of functions equal a.e. to a measurable function.** Let  $f, g : A \rightarrow \overline{\mathbb{R}}$ , where  $A \subseteq \mathbb{R}^d$  and  $f$  are measurable;  $f = g$  a.e. in  $A$ . Then  $g$  is measurable.

*Proof.* Define  $N := \{x \in A : f(x) \neq g(x)\}$  so that  $m_*(N) = 0$  since  $f = g$  a.e. in  $A$ . Let  $c \in \mathbb{R}$  be fixed so that

$$g^{-1}([-\infty, c)) = \underbrace{(g^{-1}([-\infty, c)) \cap N)}_{(a)} \cup \underbrace{(g^{-1}([-\infty, c)) \setminus N)}_{(b)},$$

where (a) is a subset of  $N$  and hence of measure 0 and (b) equals  $f^{-1}([-\infty, c)) \setminus N$ . Since  $N$  and  $f$  are measurable, we obtain that  $f^{-1}([-\infty, c)) \setminus N$  is measurable so that  $g^{-1}([-\infty, c)) \setminus N$  is measurable since  $f = g$  everywhere in  $A \setminus N$ . Thus,  $g^{-1}([-\infty, c)) = f^{-1}([-\infty, c)) \setminus N \cup N$  is a union of measurable sets and hence measurable.

13. **Measurability of sum of measurable functions.** Let  $A \subseteq \mathbb{R}^d$  and  $f, g : A \rightarrow \overline{\mathbb{R}}$  be measurable. Then  $f + g$  is measurable so long as  $g(x) > -\infty$  whenever  $f(x) = \infty$  and  $g(x) < \infty$  whenever  $f(x) = -\infty$ .

*Proof.* For every  $c \in \mathbb{R}$  and  $x \in A$ ,  $x \in (f + g)^{-1}([-\infty, c)) \iff f(x) + g(x) < c \iff f(x) < c - g(x)$  (never true if  $f(x) = \infty$ , always true if  $f(x) = -\infty$  and  $g(x) < \infty$ )  $\iff$  by density  $\exists q \in \mathbb{Q} : f(x) < q < c - g(x) \iff \exists q \in \mathbb{Q} : x \in f^{-1}([-\infty, c)) \cap g^{-1}([-\infty, c - q))$  so that

$$(f + g)^{-1}([-\infty, c)) = \bigcup_{q \in \mathbb{Q}} f^{-1}([-\infty, c)) \cap g^{-1}([-\infty, c - q))$$

is measurable as  $f, g$  are and this is a countable union of measurable sets. ■

14. **Measurability of inverse image of Borel sets by measurable functions.** Let  $A \subseteq \mathbb{R}^d$  be a measurable set and  $f : A \rightarrow \overline{\mathbb{R}}$  be a measurable function. Then, for every Borel set  $B \subseteq \mathbb{R}$ , then  $f^{-1}(B)$  is measurable.

*Proof.* Define  $\Omega := \{B \subseteq \mathbb{R} : f^{-1}(B) \text{ is measurable}\}$ . We show that  $\Omega$  is a  $\sigma$ -algebra containing the open sets. By definition of Borel sets, this suffices to prove that  $\Omega$  contains the Borel sets.

- (a)  $f^{-1}(\mathbb{R}) = f^{-1}(\bigcup_{k=1}^{\infty} [-k, k]) = \bigcup_{k=1}^{\infty} f^{-1}([-k, k]) = \bigcup_{k=1}^{\infty} f^{-1}([-\infty, k] \cap [-k, \infty]) = \bigcup_{k=1}^{\infty} (f^{-1}([-\infty, k]) \cap f^{-1}([-k, \infty]))$  is measurable since countable unions and intersections preserve measurability and  $f$  is measurable. It follows that  $\mathbb{R} \in \Omega$ , since  $f^{-1}(\mathbb{R})$  is measurable.
- (b) Suppose  $B_1, B_2 \in \Omega$ , then  $f^{-1}(B_1 \setminus B_2) = f^{-1}(B_1) \setminus f^{-1}(B_2)$  is measurable since  $f^{-1}(B_1), f^{-1}(B_2)$  are measurable.
- (c) If  $(B_k)$  is a sequence in  $\Omega$ , then  $f^{-1}(\bigcup_{k=1}^{\infty} B_k) = \bigcup_{k=1}^{\infty} f^{-1}(B_k)$  is measurable since  $f^{-1}(B_k)$  is measurable for each  $k \in \mathbb{N}$ .

Let  $\mathcal{O} \subseteq \mathbb{R}$  be open. Then there exists a sequence  $(I_k)$  of mutually disjoint open intervals in  $\mathbb{R}$  such that  $\mathcal{O} = \bigcup_{k=1}^{\infty} I_k$ . Then  $f^{-1}(\mathcal{O}) = f^{-1}(\bigcup_{k=1}^{\infty} I_k) = \bigcup_{k=1}^{\infty} f^{-1}(I_k)$ . It remains to be shown that the preimage of each  $I_k$  is measurable. Let  $a_k, b_k \in \overline{\mathbb{R}}$  be such that  $I_k = (a_k, b_k)$  for some  $k \geq 1$ . We have two cases.

- (a) If  $-\infty < a_k < b_k < \infty$ , then  $f^{-1}(a_k, \infty] \cap [-\infty, b_k) = f^{-1}((a_k, \infty]) \cap f^{-1}([-\infty, b_k))$  is measurable.
- (b) If  $a_k = -\infty$  and  $b_k < \infty$ , then  $f^{-1}((a_k, b_k)) = f^{-1}((-\infty, b_k)) = f^{-1}(\bigcup_{k=1}^{\infty} (-k, b_k)) = \bigcup_{k=1}^{\infty} f^{-1}((-k, b_k))$  is measurable.
- (c) If  $b_k = \infty$  and  $a_k > -\infty$ , then  $f^{-1}((a_k, b_k)) = f^{-1}((a_k, \infty)) = f^{-1}(\bigcup_{k=1}^{\infty} (a_k, k)) = \bigcup_{k=1}^{\infty} f^{-1}(a_k, k)$  and then use (1).
- (d) If  $a_k = -\infty$  and  $b_k = \infty$ , then  $f^{-1}((-\infty, \infty)) = f^{-1}(\mathbb{R})$  is measurable by (1).

In all cases, by using the measurability by countable unions and intersections we obtain that  $f^{-1}(I_k)$  is measurable. Thus,  $\Omega$  is a  $\sigma$ -algebra containing the open sets and hence the Borel sets.

15. **Measurable functions equivalences.** Let  $A \subseteq \mathbb{R}^d$  be measurable and  $f : A \rightarrow \overline{\mathbb{R}}$ . The following are equivalent.

- (a) For all  $c \in \mathbb{R}$ ,  $f^{-1}((c, +\infty]) = f^{-1}(\{x \in \overline{\mathbb{R}} : c < x \leq +\infty\})$  is measurable;
- (b) For all  $c \in \mathbb{R}$ ,  $f^{-1}([c, +\infty]) = f^{-1}(\{x \in \overline{\mathbb{R}} : c \leq x \leq +\infty\})$  is measurable;
- (c) For all  $c \in \mathbb{R}$ ,  $f^{-1}([-\infty, c)) = f^{-1}(\{x \in \overline{\mathbb{R}} : -\infty \leq x < c\})$  is measurable;
- (d) For all  $c \in \mathbb{R}$ ,  $f^{-1}([-\infty, c]) = f^{-1}(\{x \in \overline{\mathbb{R}} : -\infty \leq x \leq c\})$  is measurable.

*Proof.* (1  $\implies$  2) follows from  $f^{-1}([c, +\infty]) = f^{-1}(\bigcap_{k=1}^{\infty} (c - \frac{1}{k}, +\infty]) = \bigcap_{k=1}^{\infty} f^{-1}((c - \frac{1}{k}, +\infty])$  and the fact that countable intersections of measurable sets are measurable. (2  $\implies$  3) follows from  $f^{-1}([-\infty, c)) = f^{-1}(\overline{\mathbb{R}} \setminus [c, +\infty]) = A \setminus f^{-1}([c, +\infty])$  and the fact that complements of measurable sets are measurable. (3  $\implies$  4) follows from  $f^{-1}([-\infty, c]) = \bigcap_{k=1}^{\infty} f^{-1}([-\infty, c + \frac{1}{k}))$  and (4  $\implies$  1) follows from  $f^{-1}((c, +\infty]) = A \setminus f^{-1}([-\infty, c])$ .

16. **Measurability of pointwise a.e. limits of measurable functions.** Let  $\{f_k\}_{k \in \mathbb{N}}$  be a sequence of measurable functions from a measurable subset  $A \subseteq \mathbb{R}^d$  to  $\overline{\mathbb{R}}$  converging pointwise a.e. in  $A$  to  $f$ , i.e.  $\lim_{n \rightarrow \infty} f_n(x) = f(x)$  for a.e.  $x \in A$ . Then  $f$  is measurable.

*Proof.* Let  $N = \{x \in A \mid f_k(x) \not\rightarrow f(x)\}$ . Then by hypothesis  $m_*(N) = 0$ . For each  $c \in \mathbb{R}$  and  $x \in A \setminus N$   $f(x) < c \iff \lim_{k \rightarrow \infty} f_k(x) < c \iff \lim_{k \rightarrow \infty} f_k(x) < c \iff \exists n, K \in \mathbb{N} : \forall k \geq K : f_k(x) < c - \frac{1}{n}$ . Thus,

$$f^{-1}([-\infty, c)) \setminus N = \bigcup_{n=1}^{\infty} \bigcup_{K=1}^{\infty} \bigcap_{k=K}^{\infty} f_k^{-1}([-\infty, c - 1/n)) \setminus N$$

is measurable by the measurability of  $f_k$  (and since measure is preserved via countable unions and complements). Thus, it follows that  $f^{-1}([-\infty, c)) = f^{-1}([-\infty, c) \setminus N \cup \underbrace{f^{-1}([-\infty, c)) \cap N}_{\subseteq N, m_*(N)=0})$  is measurable.

Hence  $f$  is measurable as  $c$  was arbitrary.

17. **Measurability of composition of continuous functions with measurable functions.** Let  $f$  be measurable and finite-valued  $g$  be continuous. Then  $g \circ f$  is measurable.

*Proof.* Let  $c \in \mathbb{R}$  be fixed. Let  $\mathcal{O} := g^{-1}((-\infty, c))$  so that  $\mathcal{O}$  is open as  $g$  is continuous hence preimages of open sets are open. Thus  $(g \circ f)^{-1}((-\infty, c)) = f^{-1}(g^{-1}((-\infty, c))) = f^{-1}(\mathcal{O})$  is measurable as  $\mathcal{O}$  is Borel.

18. **Simple approximation lemma.** Let  $A \subseteq \mathbb{R}^d$  be a measurable set with  $m(A) < \infty$  and  $f : A \rightarrow \mathbb{R}$  be measurable and such that there exists  $M > 0$  such that for each  $x \in A$ ,  $|f(x)| < M$ . Then for each  $\varepsilon > 0$  there exist simple functions  $\varphi_\varepsilon, \chi_\varepsilon : A \rightarrow \mathbb{R}$  such that  $\varphi_\varepsilon \leq f \leq \chi_\varepsilon \leq \varphi_\varepsilon + \varepsilon$ .

*Proof.* Let  $m_\varepsilon \in \mathbb{N}$  be large enough so that  $\frac{2M}{m_\varepsilon} < \varepsilon$ . For each  $k \in \{0, 1, \dots, m_\varepsilon\}$ , let  $y_{k,\varepsilon} := M \left( \frac{2k}{m_\varepsilon} - 1 \right)$  and for  $k \in \{0, 1, \dots, m_\varepsilon - 1\}$  let  $A_{k,\varepsilon} := f^{-1}([y_{k,\varepsilon}, y_{k+1,\varepsilon}))$ . Then the sets  $(A_{k,\varepsilon})_k$  are measurable (inverse image of Borel sets by measurable function), disjoint (since the sets  $([y_{k,\varepsilon}, y_{k+1,\varepsilon}))_k$  are),  $m(A_{k,\varepsilon}) < \infty$  ( $m(A) < \infty$ ), and  $\bigcup_{k=1}^{m_\varepsilon-1} A_{k,\varepsilon} = f^{-1}(\bigcup_{k=1}^{m_\varepsilon-1} [y_{k,\varepsilon}, y_{k+1,\varepsilon})) = f^{-1}([-M, M)) = A$  since  $|f| < M$  in  $A$ .

Now define

$$\varphi_{k,\varepsilon} := \sum_{k=0}^{m_\varepsilon-1} y_{k,\varepsilon} \chi_{A_{k,\varepsilon}} \quad \text{and} \quad \chi_\varepsilon := \sum_{k=0}^{m_\varepsilon-1} y_{k+1,\varepsilon} \chi_{A_{k,\varepsilon}}.$$

Since  $y_{k,\varepsilon} \leq f < y_{k+1,\varepsilon} < y_{k,\varepsilon} + \varepsilon$  in  $A_{k,\varepsilon}$  for all  $k \in \{0, 1, \dots, m_\varepsilon - 1\}$ , it follows that  $\varphi_\varepsilon \leq f < \chi_\varepsilon < \varphi_\varepsilon + \varepsilon$ .

**19. Simple approximation theorem.** Let  $A \subseteq \mathbb{R}^d$  be measurable and  $f : A \rightarrow \overline{\mathbb{R}}$  be a measurable function. Then there exists a sequence of simple functions  $(\varphi_k)_{k \in \mathbb{N}}$  on  $A$  such that

- (a)  $\forall x \in A, k \in \mathbb{N} : |\varphi_k(x)| \leq |\varphi_{k+1}(x)| \leq |f(x)|$ ; and
- (b)  $\forall x \in A : \lim_{k \rightarrow \infty} \varphi_k(x) = f(x)$ .

Moreover, if  $f \geq 0$  in  $A$ , then we can choose  $(\varphi_k)_k$  such that  $\varphi_k \geq 0$  in  $A$  for each  $k \in \mathbb{N}$ .

*Proof.* We first suppose  $f \geq 0$  in  $A$ . For each  $k \in \mathbb{N}$  let  $f_k := \min(f, k)\chi_{V_k(0)}$ . Since  $f_k \leq k$  and  $f_k = 0$  in  $\mathbb{R}^d \setminus V_k(0)$ , we can apply the simple approximation lemma to  $f_k$ , which gives that there exists a simple function  $\tilde{\varphi}_k : A \rightarrow V_k(0) \rightarrow \mathbb{R}$  such that  $\tilde{\varphi}_k \leq f_k \leq \tilde{\varphi}_k + \frac{1}{k}$  in  $V_k(0)$ . By extending  $\tilde{\varphi}_k$  by 0 in  $\mathbb{R}^d \setminus V_k(0)$ , we may consider  $\phi_k$  as a function defined in  $\mathbb{R}^d$  such that  $\tilde{\varphi}_k \leq f_k \leq \tilde{\varphi}_k + \frac{1}{k}$  in  $\mathbb{R}^d$ , since  $f_k = 0$  in  $\mathbb{R}^d \setminus V_k(0)$ .

Define  $\varphi_k := \max(\tilde{\varphi}_1, \dots, \tilde{\varphi}_k, 0)$  so that  $\varphi_{k+1} \geq \varphi_k \geq 0$  for each  $k \in \mathbb{N}$ . Moreover, at each  $x \in A$ ,

- if  $f(x) = \infty$  for each  $k > |x|$ , then  $f_k(x) = k$  and  $\varphi_k(x) \geq \tilde{\varphi}_k(x) > f_k(x) - \frac{1}{k} = k - \frac{1}{k}$ , hence  $\lim_{k \rightarrow \infty} \varphi_k(x) = \infty$ . Moreover, observe that  $\tilde{\varphi}_k \leq f_k \leq f$  and  $f \geq 0$ , hence  $\varphi_k \leq f$ ; and
- if  $f(x) < \infty$  for each  $k > \max(|x|, f(x))$ ,  $f_k(x) = f(x)$  and  $f(x) - \varphi_k(x) \leq f(x) - \tilde{\varphi}_k(x) = f_k(x) - \tilde{\varphi}_k(x) \leq \frac{1}{k} \rightarrow 0$ . Thus,  $\lim_{k \rightarrow \infty} \varphi_k(x) = f(x)$ .

It remains to prove the result holds when  $f$  can change sign. In this case, write  $f = f_+ - f_-$  where  $f_+ := \max(f, 0)$  and  $f_- := \max(-f, 0)$  and apply the previous case to  $f_+$  and  $f_-$ . For (1), observe that  $|f| = f_+ + f_-$ .