MATH 454: Examinable Results

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1. **Monotonicity of outer measure.** $A \subseteq B \subseteq \mathbb{R}^d \implies m_*(A) \leq m_*(B)$.

   Proof. Any covering of B by closed cubes is a covering of A. Thus if V_A and V_B are the sets over which we take the infimum to attain the outer measure of A and B respectively, we must have that $V_B \subseteq V_A$ so that $m_*(A) = \inf V_A \leq \inf V_B = m_*(B)$ as required.

2. **Countable sub-additivity of outer measure.** If $\{A_k\}_{k=1}^{\infty}$ is a sequence of subsets of $\mathbb{R}^d$ with $A := \bigcup_{k=1}^{\infty} A_k$, then $m_*(A) \leq \sum_{k=1}^{\infty} m_*(A_k)$.

   Proof. If there exists a $k \in \mathbb{N}$ such that $m_*(A_k) = \infty$ then there is nothing to prove, hence we suppose otherwise. Let $\varepsilon > 0$ be fixed. For each $k \in \mathbb{N}$, there is a covering of A_k by closed cubes $(Q_{k,j,\varepsilon})_{j \in \mathbb{N}}$ such that $A_k \subseteq \bigcup_{j=1}^{\infty} Q_{k,j,\varepsilon}$ and $\sum_{j=1}^{\infty} \text{vol}(Q_{k,j,\varepsilon}) < m_*(A) + \frac{\varepsilon}{2^k}$. Thus,

   $$\bigcup_{k=1}^{\infty} A_k \subseteq \bigcup_{k=1}^{\infty}\bigcup_{j=1}^{\infty} Q_{k,j,\varepsilon} \implies m_*\left(\bigcup_{k=1}^{\infty} A_k\right) \leq \sum_{k=1}^{\infty}\sum_{j=1}^{\infty} \text{vol}(Q_{k,j,\varepsilon}) < \sum_{k=1}^{\infty}\left(m_*(A) + \frac{\varepsilon}{2^k}\right) = \sum_{k=1}^{\infty} m_*(A) + \varepsilon.$$

   Since ε was arbitrary, we obtain the required result.

3. **Additivity of outer measure with assumption.** Let $A_1, A_2 \subseteq \mathbb{R}^d$ be such that $d(A_1, A_2) = \inf_{x \in A_1, y \in A_2} |x - y| > 0$. Then $m_*(A_1 \cup A_2) = m_*(A_1) + m_*(A_2)$.

   Proof. By sub-additivity, we have $m_*(A_1 \cup A_2) \leq m_*(A_1) + m_*(A_2)$, so we prove the converse inequality. Let $\{Q_k\}_{k=1}^{\infty}$ be a sequence of closed cubes such that $A_1 \cup A_2 \subseteq \bigcup_{k=1}^{\infty} \text{vol}(Q_k)$ and $0 < \delta < d(A_1, A_2)$. By sub-dividing the cubes, we may assume that $\text{diam}(Q_k) = \sup\{|x - y| : x, y \in Q_k\} < \delta$. For $i = 1, 2$ let $K_i := \{k \in \mathbb{N} : Q_k \cap A_i \neq \emptyset\}$. By summing we obtain $m_*(A_1) + m_*(A_2) \leq \sum_{k \in K_1} \text{vol}(Q_k) + \sum_{k \in K_2} \text{vol}(Q_k)$. By the choice of δ, $K_1 \cap K_2 \neq \emptyset$ so that

   $$\sum_{k \in K_1} \text{vol}(Q_k) + \sum_{k \in K_2} \text{vol}(Q_k) = \sum_{k \in K_1 \cup K_2} \text{vol}(Q_k) \leq \sum_{k=1}^{\infty} \text{vol}(Q_k).$$

   By taking the infimum over all coverings Q_k, we then obtain $m_*(A_1) + m_*(A_2) \leq m_*(A_1 \cup A_2)$ which completes the proof.

4. **Countable unions of measurable sets.** If $\{A_k\}_{k=1}^{\infty}$ is a sequence of measurable sets then $\bigcup_{k=1}^{\infty} A_k$ is measurable.

   Proof. Let $\varepsilon > 0$. For each $k \in \mathbb{N}$ there exists an open set $\mathcal{O}_{k,\varepsilon}$ such that $A_k \subseteq \mathcal{O}_{k,\varepsilon}$ and $m_*(\mathcal{O}_{k,\varepsilon} - A_k) < \frac{\varepsilon}{2^k}$. Let $\mathcal{O}_\varepsilon := \bigcup_{k=1}^{\infty} \mathcal{O}_{k,\varepsilon}$, which is open (union of open sets). By construction, $\bigcup_{k=1}^{\infty} A_k \subseteq \mathcal{O}_\varepsilon$ so that $m_*(\mathcal{O}_\varepsilon - \bigcup_{k=1}^{\infty} A_k) \leq m_*(\bigcup_{k=1}^{\infty}(\mathcal{O}_{k,\varepsilon} - A_k))$ by monotonicity since $\bigcup_{k=1}^{\infty} \mathcal{O}_{k,\varepsilon} - \bigcup_{k=1}^{\infty} A_k \subseteq \bigcup_{k=1}^{\infty}(\mathcal{O}_{k,\varepsilon} - A_k)$. Thus, we conclude that

   $$m_*\left(\mathcal{O}_\varepsilon - \bigcup_{k=1}^{\infty} A_k\right) \leq m_*\left(\bigcup_{k=1}^{\infty}(\mathcal{O}_{k,\varepsilon} - A_k)\right) \leq \sum_{k=1}^{\infty} m_*(\mathcal{O}_{k,\varepsilon} - A_k) < \sum_{k=1}^{\infty} \frac{\varepsilon}{2^k} = \varepsilon.$$

5. **Measurability of closed sets.**

Proof. Let $F \subseteq \mathbb{R}^d$ be closed and bounded. There is an open cube Q such that $\mathcal{O} := Q - F$ is open and hence a corresponding sequence of mutually disjoint open cubes $\{Q_k\}_{k=1}^{\infty}$ such that $\mathcal{O} = \bigcup_{k=1}^{\infty} \overline{Q}_k$. For $n \in \mathbb{N}$, let $\mathcal{O}_n := Q - \bigcup_{k=1}^{n} \overline{Q}_k$, which is open since Q is open and each $\overline{Q}_k$ is closed. Moreover, $m_*(\mathcal{O}_n - F) = m_*(\mathcal{O} - \bigcup_{k=n+1}^{\infty} \overline{Q}_k) \leq \sum_{k=n+1}^{\infty} m_*(\overline{Q}_k)$ by monotonicity and sub-addtivitiy. But $\sum_{k=n+1}^{\infty} \operatorname{vol}(\overline{Q}_k) = \sum_{k=n+1}^{\infty} \operatorname{vol}(Q_k) \to 0$ as $n \to \infty$ since $\sum_{k=1}^{\infty} \operatorname{vol}(Q_k) = m_*(O) \leq m_*(Q) < \infty$. Hence compact sets are measurable. Thus, given any closed set A, write $A = \bigcup_{k=1}^{\infty}(A \cap [-k,k]^d)$ so that A is written as a countable union of compact and hence measurable sets (note that $A \cap [-k,k]^d$ is a closed subset of a compact set and hence compact).

6. **Measurability of complements.** Let $A \subseteq \mathbb{R}^d$ be measurable. Then A^c is measurable.

Proof. By the measurability of A, for each $k \in \mathbb{N}$ there exists an open set $\mathcal{O}_k$ such that $A \subseteq \mathcal{O}_k$ and $m_*(\mathcal{O}_k - A) < 1/k$. Let $F_k := \mathcal{O}_k^c$ and $F := \bigcup_{k=1}^{\infty} F_k$ (measurable as countable union of closed sets). Let $N := A^c - F = A^c - \bigcup_{k=1}^{\infty} \mathcal{O}_k^c = A^c \cap \bigcap_{k=1}^{\infty} \mathcal{O}_k \subseteq \mathcal{O}_k \cap A^c = \mathcal{O}_k - A$ so that $m_*(N) \leq m_*(\mathcal{O}_k - A) \leq 1/k \to 0$ as $k \to \infty$. Thus, $A^c = F \cup N$, which is measurable (finite union of closed set and set of measure 0, completes the proof.

7. **Measurability of countable intersections.** Let $\{A_k\}_{k=1}^{\infty}$ be a sequence of measurable subsets of $\mathbb{R}^d$. Then $\bigcap_{k=1}^{\infty} A_k$ is measurable.

Proof. Since for each $k \in \mathbb{N}$ A_k is measurable, so is A_k^c. Thus, $\bigcap_{k=1}^{\infty} A_k = \bigcup_{k=1}^{\infty} A_k^c$ is a union of measurable sets and hence measurable.

8. **Continuity of measure for increasing sets.** Let $\{A_k\}_{k=1}^{\infty}$ be a sequence of measurable subsets of $\mathbb{R}^d$ such that for each $k \in \mathbb{N}$, $A_k \subseteq A_{k+1}$. Then $m_*(\bigcup_{k=1}^{\infty} A_k) = \lim_{k\to\infty} m(A_k)$.

Proof. If there is a $k_0 \in \mathbb{N}$ such that $m(A_{k_0}) = \infty$, then by monotonicity $m(\bigcup_{k=1}^{\infty} A_k) = \infty$ and since $k \geq k_0 \implies A_k \subseteq A_{k_0}$, we likewise have $\lim_k m(A_k) = \infty$ and there is nothing to prove. Hence suppose there is no such k_0. Write $\bigcup_{k=1}^{\infty} A_k$ as $\bigcup_{k=1}^{\infty} B_k$, where $B_1 := A_1$ and for $k \geq 2$, $B_k := A_k - A_{k-1}$. Since the sets $\{B_k\}_k$ are mutually disjoint, by countable additivity, we have

$$m\left(\bigcup_{k=1}^{\infty} A_k\right) = m\left(\bigcup_{k=1}^{\infty} B_k\right) = \sum_{k=1}^{\infty} m(B_k) = \lim_{n\to\infty} \sum_{k=1}^{n} m(B_k) = \lim_{n\to\infty}\left(m(A_1) + \sum_{k=2}^{n} m(A_k - A_{k-1})\right)$$
$$= \lim_{n\to\infty}\left(m(A_1) + \sum_{k=2}^{n}(m(A_k) - m(A_{k-1}))\right) = \lim_{n\to\infty} m(A_n). \qquad (\text{As } \forall k : m(A_k) < \infty)$$

9. **Continuity of measure for decreasing sets.** Let $\{A_k\}_{k=1}^{\infty}$ be a sequence of measurable subsets of $\mathbb{R}^d$ such that $A_{k+1} \supseteq A_k$ for $k \in \mathbb{N}$ and $\exists k_0 \in \mathbb{N}$ such that $m(A_{k_0}) < \infty$.

Proof. Suppose without loss of generality that $m(A_1) < \infty$. Let $B_k := A_k - A_{k+1}$ for each $k \in \mathbb{N}$ so that $A_1 = \bigcap_{k=1}^{\infty} A_k \cup \bigcup_{k=1}^{\infty} B_k$ is a disjoint union of measurable sets. Hence,

$$m(A_1) = m\left(\bigcap_{k=1}^{\infty} A_k\right) + \lim_{N\to\infty} \sum_{k=1}^{N-1} (m(A_k) - m(A_{k+1})) = m\left(\bigcap_{k=1}^{\infty} A_k\right) + m(A_1) - \lim_{N\to\infty} m(A_N).$$

Hence since $m(A_1) < \infty$, it follows that $m\left(\bigcap_{k=1}^{\infty} A_k\right) = \lim_{N\to\infty} m(A_k)$ as required.

10. **Approximation of measurable sets.** Let $\varepsilon > 0$. The following are equivalent:

    (a) $A \subseteq \mathbb{R}^d$ is measurable.

    (b) There exists an open set $\mathcal{O}$ such that $A \subseteq \mathcal{O}$ and $m_*(\mathcal{O} - A) < \varepsilon$.

    (c) There exists a closed set F such that $F \subseteq A$ and $m(A - F) < \varepsilon$.

    (d) There is a G_δ set G and a set $N \subseteq G$ such that $m_*(N) = 0$ and $A = G \setminus N$.

    (e) There is an F_σ set F and a set $N \subseteq A \setminus F$ such that $m_*(N) = 0$ and $A = F \cup N$.

    Proof.
    (a) iff (b) is by definition. (a) implies (d). Since A measurable, $\forall n \in \mathbb{N}$ there exists an open set $\mathcal{O}_k$ such that $A \subseteq \mathcal{O}_k$ and $m_*(\mathcal{O}_k \setminus A) < 1/k$. Let $G := \bigcap_{k=1}^{\infty} \mathcal{O}_k$ be a G_δ set containing A. Let $N := G \setminus A$ so that for $k \in \mathbb{N}$, $m(N) \le m(\mathcal{O}_k \setminus A) < 1/k \to 0$ (monotonicity). (d) implies (a). G is measurable (countable intersection of measurable sets) and N is measurable as its measure is 0. Hence $G \setminus N = G \cap N^c = A$ is measurable.

    (a) iff (c). A measurable $\iff$ A^c measurable $\iff$ for each $\varepsilon > 0$ there exists an open set $\mathcal{O}_\varepsilon$ such that $A^c \subseteq \mathcal{O}_\varepsilon$ and $m_*(\mathcal{O}_\varepsilon \setminus A^c) = m_*(\mathcal{O}_\varepsilon \cap A) = m_*(A \setminus \mathcal{O}_\varepsilon^c) < \varepsilon$, and $F := \mathcal{O}_\varepsilon^c$ contains A since $A^c \subseteq \mathcal{O}_\varepsilon^c$. (d) iff (e). (d) is equivalent to: there exists an F_σ set $F := G^c$ and a set $N \subseteq F^c = G$ such that $A^c = F^c \setminus N$, i.e. $A = F \cup N$.

11. **Continuous functions are measurable.** Let $A \subseteq \mathbb{R}^d$ be measurable and let $f : A \to \mathbb{R}$ be continuous. Then f is measurable.

    Proof. Let $c \in \mathbb{R}$ be fixed. Then $f^{-1}([-\infty, c)) = f^{-1}((-\infty, c))$ since $-\infty$ is never attained by f by continuity. Since f is continuous and $(-\infty, c)$ is open, $f^{-1}((-\infty, c))$ is open and hence measurable. Since c was arbitrary, we conclude that f is measurable.

12. **Measurability of functions equal a.e. to a measurable function.** Let $f, g : A \to \overline{\mathbb{R}}$, where $A \subseteq \mathbb{R}^d$ and f are measurable; $f = g$ a.e. in A. Then g is measurable.

    Proof. Define $N := \{x \in A : f(x) \ne g(x)\}$ so that $m_*(N) = 0$ since $f = g$ a.e. in A. Let $c \in \mathbb{R}$ be fixed so that

    $$g^{-1}([-\infty, c)) = \underbrace{(g^{-1}([-\infty, c) \cap N))}_{\text{(a)}} \cup \underbrace{(g^{-1}([-\infty, c)) \setminus N)}_{\text{(b)}},$$

    where (a) is a subset of N and hence of measure 0 and (b) equals $f^{-1}([-\infty, c)) \setminus N$. Since N and f are measurable, we obtain that $f^{-1}([-\infty, c)) \setminus N$ is measurable so that $g^{-1}([-\infty, c)) \setminus N$ is measurable since $f = g$ everywhere in $A \setminus N$. Thus, $g^{-1}([-\infty, c)) = f^{-1}([-\infty, c)) \setminus N \cup N$ is a union of measurable sets and hence measurable.

13. **Measurability of sum of measurable functions.** Let $A \subseteq \mathbb{R}^d$ and $f, g : A \to \overline{\mathbb{R}}$ be measurable. Then $f+g$ is measurable so long as $g(x) > -\infty$ whenever $f(x) = \infty$ and $g(x) < \infty$ whenever $f(x) = -\infty$.

   Proof. For every $c \in \mathbb{R}$ and $x \in A$, $x \in (f+g)^{-1}([-\infty, c)) \iff f(x)+g(x) < c \iff f(x) < c - g(x)$ (never true if $f(x) = \infty$, always true if $f(x) = -\infty$ and $g(x) < \infty$) $\iff$ by density $\exists\, q \in \mathbb{Q} : f(x) < q < c - g(x) \iff \exists\, q \in \mathbb{Q} : x \in f^{-1}([-\infty, c)) \cap g^{-1}([-\infty, c-q))$ so that

$$(f+g)^{-1}([-\infty, c)) = \bigcup_{q \in \mathbb{Q}} f^{-1}([-\infty, c)) \cap g^{-1}([-\infty, c-q))$$

   is measurable as f, g are and this is a countable union of measurable sets. ■

14. **Measurability of inverse image of Borel sets by measurable functions.** Let $A \subseteq \mathbb{R}^d$ be a measurable set and $f : A \to \overline{\mathbb{R}}$ be a measurable function. Then, for every Borel set $B \subseteq \mathbb{R}$, then $f^{-1}(B)$ is measurable.

   Proof. Define $\Omega := \{B \subseteq \mathbb{R} : f^{-1}(B) \text{ is measurable}\}$. We show that Ω is a σ-algebra containing the open sets. By definition of Borel sets, this suffices to prove that Ω contains the Borel sets.

   (a) $f^{-1}(\mathbb{R}) = f^{-1}(\bigcup_{k=1}^{\infty}[-k,k]) = \bigcup_{k=1}^{\infty} f^{-1}([-k,k]) = \bigcup_{k=1}^{\infty} f^{-1}([-\infty, k] \cap [-k, \infty]) = \bigcup_{k=1}^{\infty}(f^{-1}([-\infty, k]) \cap f^{-1}([-k, \infty]))$ is measurable since countable unions and intersections preserve measurability and f is measurable. It follows that $\mathbb{R} \in \Omega$, since $f^{-1}(\mathbb{R})$ is measurable.

   (b) Suppose $B_1, B_2 \in \Omega$, then $f^{-1}(B_1 \backslash B_2) = f^{-1}(B_1) \backslash f^{-1}(B_2)$ is measurable since $f^{-1}(B_1), f^{-1}(B_2)$ are measurable.

   (c) If (B_k) is a sequence in Ω, then $f^{-1}(\bigcup_{k=}^{\infty} B_k) = \bigcup_{k=1}^{\infty} f^{-1}(B_k)$ is measurable since $f^{-1}(B_k)$ is measurable for each $k \in \mathbb{N}$.

   Let $\mathcal{O} \subseteq \mathbb{R}$ be open. Then there exists a sequence (I_k) of mutually disjoint open intervals in $\mathbb{R}$ such that $\mathcal{O} = \bigcup_{k=1}^{\infty} I_k$. Then $f^{-1}(\mathcal{O}) = f^{-1}(\bigcup_{k=1}^{\infty} I_k) = \bigcup_{k=1}^{\infty} f^{-1}(I_k)$. It remains to be shown that the preimage of each I_k is measurable. Let $a_k, b_k \in \overline{\mathbb{R}}$ be such that $I_k = (a_k, b_k)$ for some $k \geq 1$. We have two cases.

   (a) If $-\infty < a_k < b_k < \infty$, then $f^{-1}(a_k, \infty] \cap [-\infty, b_k)) = f^{-1}((a_k, \infty]) \cap f^{-1}([-\infty, b_k))$ is measurable.

   (b) If $a_k = -\infty$ and $b_k < \infty$, then $f^{-1}((a_k, b_k)) = f^{-1}((-\infty, b_k)) = f^{-1}(\bigcup_{k=1}^{\infty}(-k, b_k)) = \bigcup_{k=1}^{\infty} f^{-1}((-k, b_k))$ is measurable.

   (c) If $b_k = \infty$ and $a_k > -\infty$, then $f^{-1}((a_k, b_k)) = f^{-1}((a_k, \infty)) = f^{-1}(\bigcup_{k=1}^{\infty}(a_k, k)) = \bigcup_{k=1}^{\infty} f^{-1}(a_k, k)$ and then use (1).

   (d) If $a_k = -\infty$ and $b_k = \infty$, then $f^{-1}((-\infty, \infty)) = f^{-1}(\mathbb{R})$ is measurable by (1).

   In all cases, by using the measurability by countable unions and intersections we obtain that $f^{-1}(I_k)$ is measurable. Thus, Ω is a σ-algebra containing the open sets and hence the Borel sets.

15. **Measurable functions equivalences.** Let $A \subseteq \mathbb{R}^d$ be measurable and $f : A \to \overline{\mathbb{R}}$. The following are equivalent.

(a) For all $c \in \mathbb{R}$, $f^{-1}((c, +\infty]) = f^{-1}(\{x \in \overline{\mathbb{R}} : c < x \leq +\infty\})$ is measurable;

(b) For all $c \in \mathbb{R}$, $f^{-1}([c, +\infty]) = f^{-1}(\{x \in \overline{\mathbb{R}} : c \leq x \leq +\infty\})$ is measurable;

(c) For all $c \in \mathbb{R}$, $f^{-1}([-\infty, c)) = f^{-1}(\{x \in \overline{\mathbb{R}} : -\infty \leq x < c\})$ is measurable;

(d) For all $c \in \mathbb{R}$, $f^{-1}([-\infty, c]) = f^{-1}(\{x \in \overline{\mathbb{R}} : -\infty \leq x \leq c\})$ is measurable.

Proof. ($1 \implies 2$) follows from $f^{-1}([c, +\infty]) = f^{-1}\left(\bigcap_{k=1}^{\infty}(c - \frac{1}{k}, +\infty]\right) = \bigcap_{k=1}^{\infty} f^{-1}((c - \frac{1}{k}, +\infty])$ and the fact that countable intersections of measurable sets are measurable. ($2 \implies 3$) follows from $f^{-1}([-\infty, c)) = f^{-1}(\overline{\mathbb{R}} \setminus [c, +\infty]) = A \setminus f^{-1}((c, +\infty])$ and the fact that complements of measurable sets are measurable. ($3 \implies 4$) follows from $f^{-1}([-\infty, c]) = \bigcap_{k=1}^{\infty} f^{-1}([-\infty, c + \frac{1}{k}))$ and ($4 \implies 1$) follows from $f^{-1}((c, +\infty]) = A \setminus f^{-1}([-\infty, c])$.

16. **Measurability of pointwise a.e. limits of measurable functions.** Let $\{f_k\}_{k \in \mathbb{N}}$ be a sequence of measurable functions from a measurable subset $A \subseteq \mathbb{R}^d$ to $\overline{\mathbb{R}}$ converging pointwise a.e. in A to f, i.e. $\lim_{n \to \infty} f_n(x) = f(x)$ for a.e. $x \in A$. Then f is measurable.

   Proof. Let $N = \{x \in A | f_k(x) \not\to f(x)\}$. Then by hypothesis $m_*(N) = 0$. For each $c \in \mathbb{R}$ and $x \in A \setminus N$ $f(x) < c \iff \lim_{k \to \infty} f_k(x) < c \iff \lim_{k \to \infty} f_k(x) < c \iff \exists\, n, K \in \mathbb{N} : \forall k \geq K : f_k(x) < c - \frac{1}{n}$. Thus,
$$f^{-1}([-\infty, c)) \setminus N = \bigcup_{n=1}^{\infty} \bigcup_{K=1}^{\infty} \bigcap_{k=K}^{\infty} f_k^{-1}\left([-\infty, c - 1/n)\right) \setminus N$$
   is measurable by the measurability of f_k (and since measure is preserved via countable unions and complements). Thus, it follows that $f^{-1}([-\infty, c)) = f^{-1}([-\infty, c) \setminus N \cup \underbrace{f^{-1}([-\infty, c)) \cap N}_{\subseteq N, m_*(N) = 0}$ is measurable.

   Hence f is measurable as c was arbitrary.

17. **Measurability of composition of continuous functions with measurable functions.** Let f be measurable and finite-valued g be continuous. Then $g \circ f$ is measurable.

   Proof. Let $c \in \mathbb{R}$ be fixed. Let $\mathcal{O} := g^{-1}((-\infty, c))$ so that $\mathcal{O}$ is open as g is continuous hence preimages of open sets are open. Thus $(g \circ f)^{-1}((-\infty, c)) = f^{-1}(g^{-1}((-\infty, c))) = f^{-1}(\mathcal{O})$ is measurable as $\mathcal{O}$ is Borel.

18. **Simple approximation lemma.** Let $A \subseteq \mathbb{R}^d$ be a measurable set with $m(A) < \infty$ and $f : A \to \mathbb{R}$ be measurable and such that there exists $M > 0$ such that for each $x \in A$, $|f(x)| < M$. Then for each $\varepsilon > 0$ there exist simple functions $\varphi_\varepsilon, \chi_\varepsilon : A \to \mathbb{R}$ such that $\varphi_\varepsilon \leq f \leq \chi_\varepsilon \leq \varphi_\varepsilon + \varepsilon$.

   Proof. Let $m_\varepsilon \in \mathbb{N}$ be large enough so that $\frac{2M}{m_\varepsilon} < \varepsilon$. For each $k \in \{0, 1, \ldots, m_\varepsilon\}$, let $y_{k,\varepsilon} := M\left(\frac{2k}{m_\varepsilon} - 1\right)$ and for $k \in \{0, 1, \ldots, m_\varepsilon - 1\}$ let $A_{k,\varepsilon} := f^{-1}([y_{k,\varepsilon}, y_{k+1,\varepsilon}))$. Then the sets $(A_{k,\varepsilon})_k$ are measurable (inverse image of Borel sets by measurable function), disjoint (since the sets $([y_{k,\varepsilon}, y_{k+1,\varepsilon}))_k$ are), $m(A_{k,\varepsilon}) < \infty$ ($m(A) < \infty$), and $\bigcup_{k=1}^{m_\varepsilon - 1} A_{k,\varepsilon} = f^{-1}(\bigcup_{k=1}^{m_\varepsilon - 1}[y_{k,\varepsilon}, y_{k+1,\varepsilon})) = f^{-1}([-M, M)) = A$ since $|f| < M$ in A.

   Now define
$$\varphi_{k,\varepsilon} := \sum_{k=0}^{m_\varepsilon - 1} y_{k,\varepsilon} \chi_{A_{k,\varepsilon}} \text{ and } \chi_\varepsilon := \sum_{k=0}^{m_\varepsilon - 1} y_{k+1,\varepsilon} \chi_{A_{k,\varepsilon}}.$$

Since $y_{k,\varepsilon} \leq f < y_{k+1,\varepsilon} < y_{k,\varepsilon} + \varepsilon$ in $A_{k,\varepsilon}$ for all $k \in \{0, 1, \dots, m_\varepsilon - 1\}$, it follows that $\varphi_\varepsilon \leq f < \chi_\varepsilon < \varphi_\varepsilon + \varepsilon$.

19. **Simple approximation theorem.** Let $A \subseteq \mathbb{R}^d$ be measurable and $f : A \to \overline{\mathbb{R}}$ be a measurable function. Then there exists a sequence of simple functions $(\varphi_k)_{k\in\mathbb{N}}$ on A such that

   (a) $\forall x \in A, k \in \mathbb{N} : |\varphi_k(x)| \leq |\varphi_{k+1}(x)| \leq |f(x)|$; and

   (b) $\forall x \in A : \lim_{k\to\infty} \varphi_k(x) = f(x)$.

   Moreover, if $f \geq 0$ in A, then we can choose $(\varphi_k)_k$ such that $\varphi_k \geq 0$ in A for each $k \in \mathbb{N}$.

   Proof. We first suppose $f \geq 0$ in A. For each $k \in \mathbb{N}$ let $f_k := \min(f, k)\chi_{V_k(0)}$. Since $f_k \leq k$ and $f_k = 0$ in $\mathbb{R}^d \setminus V_k(0)$, we can apply the simple approximation lemma to f_k, which gives that there exists a simple function $\sim \varphi_k : A \to V_k(0) \to \mathbb{R}$ such that $\tilde{\varphi}_k \leq f_k \leq \tilde{\varphi}_k + \frac{1}{k}$ in $V_k(0)$. By extending $\tilde{\varphi}_k$ by 0 in $\mathbb{R}^d \setminus V_k(0)$, we may consider ϕ_k as a function defined in $\mathbb{R}^d$ such that $\tilde{\varphi}_k \leq f_k \leq \tilde{\varphi}_k + \frac{1}{k}$ in $\mathbb{R}^d$, since $f_k = 0$ in $\mathbb{R}^d \setminus V_k(0)$.

   Define $\varphi_k := \max(\tilde{\varphi}_1, \dots, \tilde{\varphi}_k, 0)$ so that $\varphi_{k+1} \geq \varphi_k \geq 0$ for each $k \in \mathbb{N}$. Moreover, at each $x \in A$,

   - if $f(x) = \infty$ for each $k > |x|$, then $f_k(x) = k$ and $\varphi_k(x) \geq \tilde{\varphi}_k(x) > f_k(x) - \frac{1}{k} = k - \frac{1}{k}$, hence $\lim_{k\to\infty} \varphi_k(x) = \infty$. Moreover, observe that $\tilde{\varphi}_k \leq f_k \leq f$ and $f \geq 0$, hence $\varphi_k \leq f$; and
   - if $f(x) < \infty$ for each $k > \max(|x|, f(x))$, $f_k(x) = f(x)$ and $f(x) - \varphi_k(x) \leq f(x) - \tilde{\varphi}_k(x) = f_k(x) - \tilde{\varphi}_k(x) \leq \frac{1}{k} \to 0$. Thus, $\lim_{k\to\infty} \varphi_k(x) = f(x)$.

   It remains to prove the result holds when f can change sign. In this case, write $f = f_+ - f_-$ where $f_+ := \max(f, 0)$ and $f_- := \max(-f, 0)$ and apply the previous case to f_+ and f_-. For (1), observe that $|f| = f_+ + f_-$.