MATH 454: Counter Examples

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Jake R. Gameroff

1. **Continuity of measure.** If $m_*(A_1) < \infty$ is dropped in the continuity of measure (i.e. $m(\bigcap_{k\in\mathbb{N}} A_k) \to m(A_k)$), we can take $A_k = [k, \infty)$ so that $\bigcap_{k\in\mathbb{N}} A_k = \emptyset$ but $m(A_k) = \infty$ for each $k \in \mathbb{N}$.

2. **Existence of non-measurable set.** Let $A \subseteq \mathbb{R}^d$ have positive outer measure. Then there exists a subset $D \subseteq A$ that is not measurable.

3. **Existence of non-Borel measurable set.** Let φ be the Cantor-Lebesgue function, $\mathcal{C}$ be the Cantor set, $\psi : x \mapsto \varphi(x) + x$, and $E \subseteq \psi(\mathcal{C})$ be a non-measurable subset (since $m_*(\psi(\mathcal{C})) > 0$). Then $D = \psi^{-1}(E)$ is measurable but not a Borel set.

4. **Existence of non-measurable pre-image of measurable set by measurable function.** Let $f = \psi^{-1}$ and E, be as above from 3. Then $f^{-1}(D) = (\psi^{-1})^{-1}(D) = \psi(D) = E$ is not measurable.

5. **Composition of measurable functions is not measurable.** If $g : A \to B$ is continuous and $f : B \to \mathbb{R}$ is measurable, $A \subseteq \mathbb{R}^d$ is measurable and $B \subseteq \mathbb{R}$ is a Borel set: let $\psi,\ D,\ E$ be defined as in 3. Let $g = \psi^{-1}$ and $f = \chi_D$. Then
$$(f \circ g)^{-1}([1, \infty]) = g^{-1}(f^{-1}([1, \infty])) = g^{-1}(D) = \psi^{-1}(D) = E$$
is not measurable.

6. **Egorov's theorem, finite measure of domain requirement.** Take $f_k = \chi_{[k,k+1)}$ on $[1, \infty)$ to attain pointwise convergence to 0; uniform convergence is impossible.

7. **Uniform boundedness in BCT.** Define $f_k := k \cdot \chi_{(0,1/k)}(x)$ in $[0, 1]$. Then, $\int_{[0,1]} f_k = k \cdot m(0, 1/k) = 1$ but $f_k \to 0$ so that $\int_{[0,1]} 0 = 0$.

8. **Fatou's Lemma with strict inequality.** Let $f_k = k \cdot \chi_{(0,1/k)}$. Then $\int_{(0,1)} \liminf_{k\to\infty} f_k = \int_{(0,1)} 0 < \liminf_{k\to\infty} \int_{(0,1)} f_k = 1$.

9. **Non-measurability of slices everywhere.** Let $D \subseteq [0, 1]$ be a non-measurable set. Let $C = \{0\}$. Then $A = D \times C$ is measurable in $\mathbb{R}^2$ (D has measure 0). But $A^0 = \{x \in \mathbb{R} : (x, 0) \in A\} = D$ is not measurable.

10. **Fubini interchanging.** It is not always true that
$$\int_{\mathbb{R}^{d_1}} \int_{A_x} f(x, y)\ dy\ dx = \int_{\mathbb{R}^{d_2}} \int_{A^y} f(x, y)\ dx\ dy.$$
Just take $f(x, y) = \frac{x^2-y^2}{(x^2+y^2)^2}$.

11. **Example of:** $\int_a^b f' < f(b) - f(a)$. Let $f = \chi_{[1/2,1]}$ in $[0, 1]$. Then f' exists everywhere except at $1/2$, and $\int f' = 0 < f(1) - f(0) = 1$. Another example is the Cantor-Lebesgue function, it is a.e. locally constant.

12. **Vitali $c < 3$.** Take $F = \{[-1, 0], [0, 1]\}$.

13. **Function not of bounded variation.** Take $f(x) = x \cos \frac{\pi}{2x}$ for $x \in (0, 1]$ and 0 when $x = 0$.