MATH 454: Final Exam Practise

McGill University (Fall 2023)
Jake R. Gameroff

Let $A \subseteq \mathbb{R}^d$ be measurable.

1. Let $f : A \to [0, \infty]$ be integrable. Then for every $\varepsilon > 0$ there exists a $\delta > 0$ such that for every measurable subset $E \subseteq A$ with $m(E) < \delta$, $\int_E f < \varepsilon$.

2. Let $E \subseteq \mathbb{R}^d$. Show that $m_*(E) = \inf\{m(\mathcal{O}) : E \subseteq \mathcal{O}, \mathcal{O} \text{ open}\}$. Use this result to show that $m_*(E) = \sup\{m(F) : F \subseteq E, F \text{ closed}\}$.

3. Let $f : A \to [0, \infty]$ be measurable. Let $\{E_k\}_{k \in \mathbb{N}}$ be a sequence of mutually disjoint measurable subsets of A whose union is E. Show that
$$\int_E f = \sum_{k \in \mathbb{N}} \int_{E_k} f.$$

4. Let $\{f_k\}_{k \in \mathbb{N}}$ be a sequence of non-negative, measurable functions on A. Show that $\sum_{k \in \mathbb{N}} f_k$ is measurable and
$$\int_A \sum_{k \in \mathbb{N}} f_k = \sum_{k \in \mathbb{N}} \int_A f_k.$$

5. Show that the conclusion of Egorov's theorem can fail if the domain of f has infinite measure.

6. Provide a counter-example for each of the following:
   (a) Show that the bounded convergence theorem does not hold for the Reimann integral.
   (b) Show that the bounded convergence theorem does not hold if $\{f_k\}_{k \in \mathbb{N}}$ is a sequence of functions such that for each $k \in \mathbb{N}$ there exists an $m_k \in \mathbb{N}$ such that $|f_k| \leq m_k$ (there is not one m for every k).
   (c) Show that $\leq$ can be strict in Fatou's lemma
   (d) Let $E = E_1 \times E_2 \subseteq \mathbb{R}^{d_1} \times \mathbb{R}^{d_2}$ be measurable. Are E_1 and E_2 measurable?

7. State and prove Chebyshev's inequality. Show that $\int_A f = 0 \iff f = 0$ a.e. in A. Show that $\int_A f < \infty \implies f < \infty$ a.e. in A.

8. Let $f : A \to \mathbb{R}$ be bounded and measurable. Show that there exists a sequence of simple functions that converge uniformly to f in A.

9. Let $A \subseteq \mathbb{R}^{d-1}$ be measurable and $f : A \to [0, \infty]$ be a function. Show that f is measurable if and only if
$$\Gamma \coloneqq \{(x, y) \in \mathbb{R}^{d-1} \times \mathbb{R} : 0 \leq y \leq f(x)\}$$
is measurable. Also, if f is measurable, show that $m(E) = \int_A f$.

10. Let $A \subseteq \mathbb{R}^d$ be measurable. Show that for almost every $y \in \mathbb{R}^{d_2}$, (a) A^y is measurable, (b) $y \mapsto m(A^y)$ is measurable, and (c) $\int_{\mathbb{R}^{d_2}} m(A^y) = m(A)$.

11. State the definition of Lipschitz continuity. Show that if $f : [a, b] \to \mathbb{R}$ is continuous and differentiable in (a, b), and f' is bounded, then f is of bounded variation on $[a, b]$.

12. Show that $f : [a, b] \to \mathbb{R}$ is of bounded variation if and only if it can be written as the difference of two increasing functions.

13. State both the simple approximation lemma and simple approximation theorem. Show that the simple approximation gives a sequence of simple functions that converge *uniformly* to f on its domain.