Problem 1

1.1. Let f, g be continuous functions on (a, b). Show that if $f = g$ a.e. in (a, b), then $f = g$ in (a, b).

Proof. Let f, g be two continuous functions on (a, b) for $a, b \in \mathbb{R}$ with $a < b$. Suppose $f = g$ a.e. in (a, b). We define the set $\mathcal{N} \coloneqq \{x \in (a, b) : f(x) \neq g(x)\}$ and show that it must be empty to assert the claim. Since f and g are continuous, so is their difference $h \coloneqq f - g$. Towards a contradiction, suppose there exists $p \in \mathcal{N}$ so that $f(p) \neq g(p)$. Let $\alpha \coloneqq d(f(p) - g(p), 0) > 0$.

By the continuity of h at p, given $\varepsilon \coloneqq \alpha$ there exists a $\delta > 0$ such that if $x \in (a, b)$ with $d(x, p) < \delta$ then $d(h(x), h(p)) < \varepsilon$. Since $\mathcal{N}$ has measure 0 there exists a point $q \in V_\delta(p) \cap (a, b)$ such that $q \notin \mathcal{N}$.[1] Therefore $h(q) = f(q) - g(q) = 0$. But then $d(q, p) < \delta$, yet $d(h(q), h(p)) = d(0, f(p) - g(p)) = \alpha \not< \alpha$, a contradiction to the continuity of h. Therefore, $p \notin \mathcal{N}$. Since p was arbitrary, we conclude that $\mathcal{N} = \emptyset$ and hence that $f = g$ on (a, b). ■

1.2. Show by providing a counter-example that the assertion in (1.1) is false if (a, b) is replaced by a general measurable set A.

Solution. Let $f : [0, 1] \cup \{2\} \to \{5, 5.1\}$ be given by

$$f(x) = \begin{cases} 5 & x \in [0, 1] \\ 5.1 & x = 2 \end{cases}$$

and $g : [0, 1] \cup \{2\} \to \{5\}$ be given by $g(x) = 5$. Then $A \coloneqq [0, 1] \cup \{2\}$ is measurable (closed sets are measurable and finite unions of measurable sets are measurable). Clearly, $f = g$ a.e. in $[0, 1] \cup \{2\}$ since they differ only on $\{2\}$ which has measure 0 since it is a finite set.

- f is continuous at each $p \in [0, 1] \cup \{2\}$, since if $p \in [0, 1]$ and $\varepsilon > 0$ are fixed, if we let $\delta = \varepsilon$ and suppose $x \in [0, 1] \cup \{2\}$ with $d(x, p) < \delta$, then if $x \in [0, 1]$ then $d(f(x), f(p)) = d(5, 5) = 0 < \varepsilon$, and if $x = 2$ with $d(2, p) < \delta$, then $d(f(2), f(p)) = d(5.1, 5) = 0.1 < d(2, 1) = 1 \leq d(2, p) < \delta = \varepsilon$; likewise, if $p = 2$, given $\varepsilon > 0$ let $\delta = \frac{1}{2}$. Then if $x \in [0, 1] \cup \{2\}$ is such that $d(x, 2) < 1/2$, then $x = p$. Thus, $d(p, p) = 0 < \delta \implies d(f(p), f(p)) = 0 < \varepsilon$. Thus, f is continuous.

- g is continuous since it is uniformly continuous: given $\varepsilon > 0$ choosing $\delta = \varepsilon$ implies that for any $x, y \in [0, 1] \cup \{2\}$ with $d(x, y) < \delta$, we have $d(f(x), f(y)) = d(5, 5) = 0 < \varepsilon$.

Since f and g are both continuous and $f = g$ a.e. on $[0, 1] \cup \{2\}$, however $f \neq g$ on $[0, 1] \cup \{2\}$, these functions serve as a counter example to the generalised version of the claim in (1.1), as needed. ■

[1] This holds as otherwise, $V_\delta(p) \cap (a, b)$ is open (finite intersection of open sets) so that for each $q \in V_\delta(p) \cap (a, b)$ there is an $r > 0$ such that $V_r(q) \subseteq V_\delta(p) \cap (a, b)$; hence $\mathcal{N}$ must have measure at least $q + r - (q - r) = 2r > 0$, since $V_r(q) \subseteq \mathcal{N}$ implies by monotonicity that $0 < 2r = m(V_r(q)) \leq m(\mathcal{N})$, that $m(\mathcal{N}) > 0$ is a contradiction

Problem 2

2. A function $f : A \to \overline{\mathbb{R}}$ is called *Borel measurable* provided its domain $A \subseteq \mathbb{R}^d$ is a Borel set and for each $c \in \mathbb{R}$, the set $\{x \in A : f(x) < c\}$ is a Borel set.

2.1. Prove that every Borel measurable function is Lebesgue measurable.

Proof. Let $A \subseteq \mathbb{R}^d$ be a Borel set and $f : A \to \overline{\mathbb{R}}$ be Borel measurable. Let $c \in \mathbb{R}$ be fixed. Then $\{x \in A : f(x) < c\} = \{x \in A : f(x) \in [-\infty, c)\} = f^{-1}([-\infty, c))$ is a Borel set. Since Borel sets are obtained via countable unions and complements of open sets, all Borel sets are measurable (this was proven in lecture). Thus, $f^{-1}([-\infty, c))$ is measurable, and since c was arbitrary, we conclude that f is Lebesgue measurable. ∎

2.2. If f is Borel measurable and B is a Borel set, then $f^{-1}(B)$ is a Borel set.

Proof. Let $A \subseteq \mathbb{R}^d$ be a Borel set and $f : A \to \overline{\mathbb{R}}$ be Borel measurable. Let Ω denote the collection of all sets B such that $f^{-1}(B)$ is a Borel set. We show that Ω is a σ-algebra containing the open sets.

- $\mathbb{R} \in \Omega$ since

  $$f^{-1}(\mathbb{R}) = f^{-1}\left(\bigcup_{k=1}^{\infty}(-k, k)\right) = \bigcup_{k=1}^{\infty} f^{-1}((-k, k)),$$

  which holds by the properties of the pre-image.[2] Since f is Borel measurable, applying lemma 2.2 for each interval $(-k, k)$, we conclude that $f^{-1}(\mathbb{R})$ is a Borel set, since countable unions of Borel sets are Borel sets.

- If $X, Y \in \Omega$, then $f^{-1}(X)$ and $f^{-1}(Y)$ are Borel sets. Thus, their difference $f^{-1}(X) \setminus f^{-1}(Y) = f^{-1}(X \setminus Y)$ is a Borel set. Thus, $X \setminus Y \in \Omega$.[3]

- If $\{X_k\}_{k\in\mathbb{N}}$ is a sequence of sets in Ω, then for each $k \geq 1$, $f^{-1}(X_k)$ is a Borel set. Since countable unions of Borel sets are Borel sets, we conclude that $\bigcup_{k=1}^{\infty} f^{-1}(X_k) = f^{-1}(\bigcup_{k=1}^{\infty} X_k)$ is a Borel set (cf. footnote 2).

- If $\mathcal{O} \subseteq \mathbb{R}$ is an open set, then by lecture, we can write $\mathcal{O}$ as a countable union of disjoint open intervals $\mathcal{O} = \bigcup_{k=1}^{\infty}(a_k, b_k)$ so that

  $$f^{-1}(\mathcal{O}) = f^{-1}\left(\bigcup_{k=1}^{\infty}(a_k, b_k)\right) = \bigcup_{k=1}^{\infty} f^{-1}((a_k, b_k)).$$

  Applying lemma 2.2 to $f^{-1}((a_k, b_k))$ for each $k \geq 1$, we conclude that each $f^{-1}((a_k, b_k))$ is a Borel set. Since countable unions of Borel sets are Borel sets, we conclude that $f^{-1}(\mathcal{O}) = \bigcup_{k=1}^{\infty} f^{-1}((a_k, b_k))$ is a Borel set. Thus, $\mathcal{O} \in \Omega$.

Therefore, the collection of sets B for which $f^{-1}(B)$ is a Borel set is a σ-algebra Ω containing the open sets. Since the Borel σ-algebra is the smallest σ-algebra containing the open sets, Ω contains the Borel σ-algebra.

[2] Certainly, if $\{A_k\}_{k\in\mathbb{N}}$ is a sequence of sets in $\mathbb{R}$, $x \in \bigcup_{k=1}^{\infty} f^{-1}(A_k) \iff \exists\, m \geq 1 : x \in f^{-1}(A_m) \iff \exists\, m \geq 1 : f(x) \in A_m \iff f(x) \in \bigcup_{k=1}^{\infty} A_k \iff x \in f^{-1}(\bigcup_{k=1}^{\infty} A_k)$.

[3] Indeed, for any sets $F, E \subseteq \mathbb{R}$, $x \in f^{-1}(F) \setminus f^{-1}(E) \iff f(x) \in F, f(x) \notin E \iff f(x) \in F \setminus E \iff x \in f^{-1}(F \setminus E)$.

In particular, we deduce that if B is a Borel set, then $B \in \Omega$ so that $f^{-1}(B)$ is a Borel set by defintion. Since B was arbitrary, the proof is complete. ■

2.3. If f and g are Borel measurable, then $f \circ g$ is Borel measurable.

Proof. Let $f : B \to \overline{\mathbb{R}}$, $g : A \to B$ be Borel measurable functions, where $A \subseteq \mathbb{R}^d$ and $B \subseteq \mathbb{R}$ are Borel sets. We must show that for each $c \in \mathbb{R}$, the set $\{x \in A : (f \circ g)(x) < c\}$ is a Borel set.

Let $c \in \mathbb{R}$ be fixed. Since f is Borel measurable, $E \coloneqq f^{-1}([-\infty, c))$ is a Borel set. Thus,

$$(f \circ g)^{-1}([-\infty, c)) = g^{-1}(f^{-1}([-\infty, c))) = g^{-1}(E)$$

is a Borel set. Indeed, since E is a Borel set and g is Borel measurable, by (2.2) we must have that $g^{-1}(E)$ is a Borel set. Since c was arbitrary, we conclude that $f \circ g$ is Borel measurable. ■

2.4. If f is Borel measurable and g is Lebesgue measurable, then $f \circ g$ is Lebesgue measurable.

Proof. Since f is Borel measurable, by (2.1) it is Lebesgue measurable. By lecture, composition of Lebesgue measurable functions are Lebesgue measurable so that $f \circ g$ is Lebesgue measurable. ■

Lemma 2.1. Let $f : A \to \overline{\mathbb{R}}$ be Borel measurable, where $A \subseteq \mathbb{R}^d$ is a Borel set. Then for each $c \in \mathbb{R}$, $\{x \in A : f(x) > c\}$ is a Borel set.

Let $c \in \mathbb{R}$ be fixed. Since f is Borel measurable, $\{x \in A : f(x) < c\} = f^{-1}([-\infty, c))$ is a Borel set. Since $\mathbb{R}$ is a Borel set, we must have that $\mathbb{R} \setminus f^{-1}([-\infty, c)) = f^{-1}([-\infty, c))^c = f^{-1}([-\infty, c)^c) = f^{-1}([c, \infty])$ is a Borel set (as the difference of two Borel sets is a Borel set).[4]

Thus, $\{x \in A : f(x) \geq c\}$ is a Borel set. Since c was arbitrary, for each $n \in \mathbb{N}$ we must have that $\{x \in A : f(x) \geq c + \frac{1}{n}\}$ is a Borel set. Thus,

$$\bigcup_{n=1}^{\infty} \{x \in A : f(x) \geq c + \frac{1}{n}\} = \bigcup_{n=1}^{\infty} f^{-1}([c + \frac{1}{n}, \infty]) = f^{-1}\left(\bigcup_{n=1}^{\infty} [c + \frac{1}{n}, \infty]\right) = f^{-1}((c, \infty]) \qquad (*)$$

is a Borel set as countable unions of Borel sets are Borel sets.[5] Thus, $f^{-1}((c, \infty]) = \{x \in A : f(x) > c\}$ is a Borel set, thereby completing the lemma. ■

Lemma 2.2. Let $f : A \to \overline{\mathbb{R}}$ be Borel measurable, where $A \subseteq \mathbb{R}^d$ is a Borel set. Then for $a, b \in \mathbb{R}$ with $a < b$, $f^{-1}((a, b))$ is a Borel set. Notice that

$$f^{-1}((a, b)) = f^{-1}(\{x \in \overline{\mathbb{R}} : x > a \text{ and } x < b\}) = f^{-1}((a, \infty] \cap [-\infty, b)) = f^{-1}((a, \infty]) \cap f^{-1}([-\infty, b)).$$

By properties of the inverse image.[6] Hence $f^{-1}((a, \infty]) = \{x \in A : f(x) > a\}$ is a Borel set by lemma 2.1, and $f^{-1}([-\infty, b)) = \{x \in A : f(x) < b\}$ is a Borel set since f is Borel measurable. Hence, $\mathbb{R} \setminus f^{-1}([-\infty, b)) = f^{-1}([-\infty, b))^c$ is a Borel set since $\mathbb{R}$ is (and the difference of two Borel sets is a Borel

[4]Indeed, $x \in (f^{-1}([-\infty, c))^c \iff x \notin f^{-1}([-\infty, c)) \iff f(x) \notin [-\infty, c) \iff f(x) \in [-\infty, c)^c \iff x \in f^{-1}([-\infty, c)^c) = f^{-1}([c, \infty])$.

[5]In $(*)$, we used $(c, \infty] = \bigcup_{n=1}^{\infty} [c + 1/n, \infty]$. This holds as if $x \in (c, \infty]$, then there an $n \in \mathbb{N}$ such that $x \geq c + \frac{1}{n} \implies x \in [c + 1/n, \infty] \subseteq \bigcup_{n=1}^{\infty} [c + 1/n, \infty]$, which holds by Archimedeanity as $x > c$. Conversely, if $x \in \bigcup_{n=1}^{\infty} [c + 1/n, \infty]$, then there is an $m \geq 1$ such that $x \in [c + 1/m, \infty] \subseteq (c, \infty]$ since $c + 1/m > c$. Thus, the result holds.

[6]Indeed, if $X, Y \subseteq \mathbb{R}$, then $x \in f^{-1}(X \cap Y) \iff f(x) \in X \cap Y \iff f(x) \in X, f(x) \in Y \iff x \in f^{-1}(X), x \in f^{-1}(Y) \iff x \in f^{-1}(X) \cap f^{-1}(Y)$.

set). Thus, $f^{-1}((a,\infty])\setminus f^{-1}([-\infty,b))^c = f^{-1}((a,\infty])\cap f^{-1}([-\infty,b)) = f^{-1}((a,b))$ is a Borel set, completing the lemma. ■

Problem 3

3. Let $f(x, y)$ be a function in $\mathbb{R}^2$ that is *separately continuous*, i.e. for each fixed variable, f is continuous in the other variable. Prove that f is measurable in $\mathbb{R}^2$. *Hint. approximate f in the variable x by piecewise-continuous functions.*

Proof. For each $k \in \mathbb{N}$ define $f_k(x, y) \coloneqq f\left(\frac{\lceil kx \rceil}{k}, y\right)$. Then, for each $k \in \mathbb{N}$, f_k is piecewise continuous. To see why, for each integer j, we define the sets

$$A_j \coloneqq \{x \in \mathbb{R} \mid j < kx \leq j + 1\},$$

within which the function $f_k(x, y)$ takes the constant value $f\left(\frac{j+1}{k}, y\right)$ (y fixed) so that f_k is continuous in x on A_j. By the separate continuity of f, f_k is continuous in y on A_j. Thus, f_k is continuous on A_j so that it is piecewise continuous on its entire domain.

Furthermore, $f_k(x, y) \to f(x, y)$ pointwise in $\mathbb{R}^2$ since for fixed y the sequence $\frac{\lceil kx \rceil}{k} \to x$ as $k \to \infty$. Indeed, by definition, $kx \leq \lceil kx \rceil \leq kx + 1 \implies x \leq \frac{\lceil kx \rceil}{k} \leq x + \frac{1}{k}$. Hence sending $k \to 0$ gives $\lim_k x \leq \lim_k \frac{\lceil kx \rceil}{k} \leq \lim_k (x + 1/k)$ so that by the squeeze theorem, $x \leq \lim_k \frac{\lceil kx \rceil}{k} \leq x$ as needed. Hence by the continuity of f in x, $f_k(x, y) = f\left(\frac{\lceil kx \rceil}{k}, y\right) \to f(x, y)$ as $k \to \infty$ (sequential definition of continuity).

Since there exists a sequence of functions $\{f_k\}_{k \in \mathbb{N}}$ such that $f_k \to f$ pointwise for each $x \in \mathbb{R}^2$, f is measurable by lecture, thereby completing the proof. ■

Problem 4

4. Let $(A_\alpha)_{\alpha\in\mathbb{R}}$ be a family of measurable subsets of $\mathbb{R}^d$ such that

$$\bigcup_{\alpha\in\mathbb{R}} A_\alpha = \mathbb{R}^d, \quad \bigcap_{\alpha\in\mathbb{R}} A_\alpha = \emptyset \text{ and } (\forall \alpha, \beta \in \mathbb{R} : \alpha < \beta \implies A_\alpha \subseteq A_\beta).$$

Find a measurable function $f : \mathbb{R}^d \to \mathbb{R}$ such that $f \leq \alpha$ on A_α and $f \geq \alpha$ on $\mathbb{R}^d \setminus A_\alpha$ for each $\alpha \in \mathbb{R}$.

Proof. Define $f : \mathbb{R}^d \to \mathbb{R}$ by $x \overset{f}{\mapsto} \inf\{\alpha \in \mathbb{R} : x \in A_\alpha\}$.

First note that since infima are unique, for each $x \in \mathbb{R}^d$ there exists a unique $\mu \in \mathbb{R}$ such that $f(x) = \mu$. To see why $-\infty < \mu < \infty$, note that $\{\alpha \in \mathbb{R} : x \in A_\alpha\}$ is bounded below. Indeed, since $\bigcup_{\alpha\in\mathbb{R}} A_\alpha = \mathbb{R}^d$, for each $x \in \mathbb{R}^d$ there exists an $\alpha \in \mathbb{R}$ such that $x \in A_\alpha$. Then for each $\alpha' \geq \alpha$, $x \in A_{\alpha'}$ since $A_\alpha \subseteq A_{\alpha'}$. But since $\bigcap_{\alpha\in\mathbb{R}} A_\alpha = \emptyset$, there must be a $\lambda < \alpha$ such that $x \notin A_\lambda$, as otherwise the intersection would contain x. To this end, for $\lambda' < \lambda$, $x \notin A_{\lambda'} \subseteq A_\lambda$ so that λ is a lower bound for $\{\alpha \in \mathbb{R} : x \in A_\alpha\}$. Hence, we conclude that $\inf\{\alpha \in \mathbb{R} : x \in A_\alpha\}$ is a finite number, since we are taking the infimum over a bounded-below subset of $\mathbb{R}$.

Let $\alpha \in \mathbb{R}$ and $x \in A_\alpha$ be fixed. Let $S \coloneqq \{\alpha \in \mathbb{R} : x \in A_\alpha\}$ and $\mu \coloneqq \inf S = f(x)$. By definition of the infimum, since $\alpha \in S$, we have $\mu \leq \alpha$, since μ is a lower bound for S. Therefore, $f(x) = \mu \leq \alpha$. Since α and x were arbitrary, we deduce that $f \leq \alpha$ in A_α for each $\alpha \in \mathbb{R}$.

Now suppose $w \in \mathbb{R}^d \setminus A_\alpha$, with $\rho \coloneqq f(w) = \inf\{\alpha \in \mathbb{R} : w \in A_\alpha\}$. By definition of the infimum, for each $\varepsilon > 0$ there exists a $\rho' \in \mathbb{R}$ such that $w \in A_{\rho'}$ and $\rho' < \rho + \varepsilon$. Since $w \in A_{\rho'}$, we must have that $\alpha < \rho'$, as otherwise $w \in A_{\rho'} \subseteq A_\alpha$ so that $w \in A_\alpha$ and $w \in \mathbb{R}^d \setminus A_\alpha$, a contradiction. But then we have $\alpha < \rho' < \rho + \varepsilon = f(w) + \varepsilon$; and since $\alpha, w,$ and ε were arbitrary, we conclude that $f \geq \alpha$ on $\mathbb{R}^d \setminus A_\alpha$ for each $\alpha \in \mathbb{R}$.

It remains to be shown that f is measurable. Let $c \in \mathbb{R}$ be fixed. Then

$$f^{-1}([-\infty, c]) = f^{-1}((-\infty, c]) = \{x \in \mathbb{R}^d : f(x) \leq c\} = A_c$$

by construction and since f only takes finite values. But A_c is measurable by hypothesis, thus $f^{-1}([-\infty, c])$ is measurable. Since c was arbitrary, we conclude that f is measurable with the desired properties, thereby completing the proof. ∎

Problem 5

5.1. Show that the conclusion of Egorov's theorem can fail if we drop the assumption that the domain has finite measure.

Proof. Let $E := [1,\infty)$ so that $m(E) = \infty$. For $k \in \mathbb{N}$, define $f_k : E \to \{0,1\}$ by $f_k(x) = \chi_{[k,k+1)}(x)$. We have the following claims:

1. *For each $k \in \mathbb{N}$, f_k is measurable.* Let $k \in \mathbb{N}$ be fixed. Since $f_k(E) = \{0,1\}$ is finite and $m(\{x \in [1,\infty) : f_k(x) \neq 0\}) = m(\{x \in [1,\infty) : x \in [k,k+1)\}) = m([k,k+1)) = 1$, (i.e. f_k has finite support), each f_k is a simple function and hence measurable by lecture.

2. *$f_k \to 0$ pointwise in E.* Let $\varepsilon > 0$ be given. For each $x \in [1,\infty)$, let $N := \lfloor x \rfloor \in \mathbb{N}$ so that $x \in [N, N+1)$. Then for all $n > N$ we have $f_n(x) = \chi_{[n,n+1)} = 0$ (since $n > N \implies [N,N+1) \cap [n,n+1) = \emptyset$, i.e. $x \notin [n,n+1)$) so that $|f(x) - f_n(x)| = |0-0| = 0 < \varepsilon$. Thus, $f_k \to 0$ pointwise in E.

3. *There exists an $\varepsilon > 0$ such that for any closed set $F_\varepsilon \subseteq E$, $m(E - F_\varepsilon) > \varepsilon$ or f_k does not converge uniformly to f in F_ε.* Let $\varepsilon := 1/2$ and $F_\varepsilon \subseteq E$ be any closed set.

   (a) Suppose $m_*(E - F_\varepsilon) < \varepsilon = 1/2$. By contradiction suppose also that f_k converges uniformly to 0. Then there exists an $N \in \mathbb{N}$ such that for all $n > N$ and for all $x \in F_\varepsilon$, $|f(x) - f_n(x)| = |0 - f_n(x)| = |\chi_{[n,n+1)}(x)| < 1/2$. But then we must have $x \in F_\varepsilon \implies x \in [1, N+1)$ so that $F_\varepsilon \subseteq [1, N+1)$. Since $x \in E - [1,N+1) \implies x \in E, x \notin [1,N+1) \implies x \in E, x \notin F_\varepsilon \implies x \in E - F_\varepsilon$, this implies that $E - [1,N+1) \subseteq E - F_\varepsilon$. Hence by monotonicity,

   $$m_*(E - [1,N+1)) = m_*([N+1,\infty)) = \infty \leq m_*(E - F_\varepsilon) \implies m_*(E - F_\varepsilon) = \infty, \qquad (*)$$

   a contradiction to the choice of F_ε. Hence, if $m_*(E - F_\varepsilon) < \varepsilon$, there exists no such set F_ε in which $f_k \to 0$ uniformly.

   (b) On the other hand, suppose $f_k \to 0$ uniformly in F_ε. As aforementioned, there exists an $N \in \mathbb{N}$ such that for all $n > N$ and all $x \in F_\varepsilon$ we have $|\chi_{[n,n+1)}(x)| < 1/2 \iff \chi_{[n,n+1)}(x) = 0$. Hence by the exact same reasoning used in $(*)$, $F_\varepsilon \subseteq [1,N+1) \implies E - [1,N+1) \subseteq E - F_\varepsilon \implies m_*((E - F_\varepsilon)) = \infty$.

Therefore, this sequence of functions $\{f_k\}_{k\in\mathbb{N}}$ is a sufficient counter-example to Egorov's theorem without the assumption that the domain of each f_k is of finite measure. ∎

5.2. Let $(f_n)_{n\in\mathbb{N}}$, $f_n : A \to \mathbb{R}$ be a sequence of measurable functions defined on a measurable set $A \subseteq \mathbb{R}^d$ such that $(f_n)_{n\in\mathbb{N}}$ converges pointwise in A to a function $f : A \to \mathbb{R}$. Use Egorov's theorem to show that the set A can be written as the countable union of measurable sets $(A_k)_{k\in\mathbb{N}_{\geq 0}}$ such that $m(A_0) = 0$ and for every $k \geq 1$, $(f_n)_{n\in\mathbb{N}}$ converges uniformly to f in A_k.

Proof. We may first assume that A is a bounded set. By Egorov's theorem, for each $k \in \mathbb{N}$ there exists a closed set $A_k \subseteq A$ such that $m(A \setminus A_k) < 1/k$ and f_n converges uniformly to f in A_k. Define $A_0 := \bigcap_{k=1}^\infty (A \setminus A_k)$ so that $A = \bigcup_{k=0}^\infty A_k$. Indeed, if $x \in A$ then either $\exists\, k \in \mathbb{N} : x \in A_k \implies x \in \bigcup_{k=0}^\infty A_k$ or there is no such k; that is, $\forall k \in \mathbb{N} : x \notin A_k \implies x \in A \setminus A_k$ for each $k \in \mathbb{N} \implies x \in \bigcap_{k=1}^\infty (A \setminus A_k) = A_0 \implies x \in \bigcup_{k=0}^\infty A_k$. On the other hand, if $x \in A_0 \implies x \in \bigcap_{k=1}^\infty (A \setminus A_k) \subseteq A \setminus A_1 \subseteq A$ and if $x \in \bigcup_{k=1}^\infty A_k \implies \exists\, m \in \mathbb{N} : x \in A_m \subseteq A$.

To see why $m(A_0) = 0$, let $\varepsilon > 0$ and choose $N \in \mathbb{N}$ large enough so that $1/N < \varepsilon$. Then $A_0 \subseteq A \setminus A_N$ so that by monotonicity and the construction of A_N, $m(A_0) \leq m(A \setminus A_N) < \frac{1}{N} < \varepsilon$. Since ε was arbitrary, we conclude that $m(A_0) = 0$. Hence, $A = \bigcup_{k=0}^{\infty} A_k$ is a countable union with $m(A_0) = 0$ and for $k \geq 1$, f_k converges uniformly to f in A_k.

Now suppose A is unbounded. We write $A = \bigcup_{m=1}^{\infty} E_m$, where $E_m \coloneqq A \cap [-m, m]^d$ for each $m \in \mathbb{N}$. Then each E_m is measurable (finite intersection of measurable sets), bounded ($E_m \subseteq [-m, m]^d$), and $f_n|_{E_m}$ is measurable (by lecture) and converges pointwise to f in $E_m = A \cap E_m \subseteq A$. Thus, since the claim has been proven when the functions' domains are bounded, we can write $E_m = \bigcup_{j=0}^{\infty} A_{m,j}$ where $A_{m,j}$ is measurable for each $j \geq 0$, $m(A_{m,0}) = 0$, and $j \geq 1 \implies f_n|_{E_m}$ converges uniformly to f in $A_{m,j}$. It follows that

$$A = \bigcup_{m=1}^{\infty} \bigcup_{j=0}^{\infty} A_{m,j} = \bigcup_{m=1}^{\infty} \bigcup_{j=1}^{\infty} A_{m,j} \cup \bigcup_{m=1}^{\infty} A_{m,0}.$$

Now let $A_0 \coloneqq \bigcup_{m=1}^{\infty} A_{m,0}$. By sub-additivity, $m(A_0) = m(\bigcup_{m=1}^{\infty} A_{m,0}) \leq \sum_{m=1}^{\infty} m(A_{m,0}) = 0$ since $m(A_{m,0}) = 0$ for $m \in \mathbb{N}$. For each $m, j \in \mathbb{N}$, we have that $f_n|_{E_m}$ converges uniformly to f in $A_{m,j}$ so that f_n converges uniformly to f in $A_{m,j} \cap E_m = A_{m,j}$. Since $\bigcup_{m=1}^{\infty} \bigcup_{j=1}^{\infty} A_{m,j}$ is a countable union, we can rewrite it as $\bigcup_{k=1}^{\infty} A_k$, a countable union of measurable sets so that for $k \in \mathbb{N}$, f_n converges uniformly to f in A_k.

Thus, $A = \bigcup_{k=0}^{\infty} A_k$ has been written as a countable union of measurable sets (since $k \geq 1 \implies A_k$ is measurable by construction and sets of outer measure zero are measurable); moreover, $m(A_0) = 0$ and for each $k \geq 1$ we have $f_n \to f$ uniformly in A_k as required. Thus, the proof is complete. ■