Problem 1. Let A be a measurable subset of $\mathbb{R}^d$ and $f : A \to \overline{\mathbb{R}}$. For every $\delta = (\delta_1, \ldots, \delta_d) \in (0, \infty)^d$ and $y = (y_1, \ldots, y_d) \in \mathbb{R}^d$, let

$$A_{\delta,y} := \{(\delta_1 x_1 + y_1, \ldots, \delta_d x_d + y_d) : x = (x_1, \ldots, x_d) \in A\}$$

and $f_{\delta,y} : A_{\delta,y} \to \overline{\mathbb{R}}$ be the function defined by

$$f_{\delta,y}(x) = f\left(((x_1 - y_1)/\delta_1, \ldots, (x_d - y_d)/\delta_d)\right), \ \forall x \in A_{\delta,y}.$$

Show that:

1. f is measurable if and only if $f_{\delta,y}$ is measurable.
2. f is integrable over A if and only if $f_{\delta,y}$ is integrable over $A_{\delta,y}$. Furthermore, if f is integrable over A, then
$$\int_{A_{\delta,y}} f_{\delta,y} = \delta_1 \cdots \delta_d \int_A f.$$

Proof of 1.1. We note that throughout the solution, by $1/\delta$ we mean $(1/\delta_1, \ldots, 1/\delta_d)$ and by $\delta x + y$ we mean $(\delta_1 x_1 + y_1, \ldots, \delta_d x_d + y_d)$. For "$\Rightarrow$", suppose f is measurable. Let $c \in \mathbb{R}$ be given. Let $\delta x + y \in A_{\delta,y}$ be such that $\delta x + y \in f_{\delta,y}^{-1}([-\infty, c))$. Then

$$\begin{aligned}
\delta x + y \in f_{\delta,y}^{-1}([-\infty, c)) &\iff f\left(\left(\frac{\delta_1 x_1 + y_1 - y_1}{\delta_1}, \ldots, \frac{\delta_d x_d + y_d - y_d}{\delta_d}\right)\right) < c \\
&\iff f(x_1, \ldots, x_d) < c \iff x \in f^{-1}([-\infty, c)) \\
&\iff \delta x + y \in \delta(f^{-1}([-\infty, c))) + y := \{\delta x + y : x \in f^{-1}([-\infty, c))\}.
\end{aligned}$$

Thus, it follows that $f_{\delta,y}^{-1}([-\infty, c)) = \delta(f^{-1}([-\infty, c))) + y$. Since f is measurable, $f^{-1}([-\infty, c))$ is measurable by definition. By question three of assigment one, it follows that $\delta(f^{-1}([-\infty, c))) + y = f_{\delta,y}^{-1}([-\infty, c))$ is measurable as we showed that translations and dilations of measurable sets preserve measurability. Since c was arbitrary, we conclude that $f_{\delta,y}^{-1}([-\infty, c))$ is measurable.

On the other hand, for "$\Leftarrow$", suppose $f_{\delta,y}$ is measurable and let $c \in \mathbb{R}$ be arbitrary. Then

$$\begin{aligned}
x = (x_1, \ldots, x_d) \in f^{-1}([-\infty, c)) &\iff \left(\frac{\delta_1 x_1 + y_1 - y_1}{\delta_1}, \ldots, \frac{\delta_d x_d + y_d - y_d}{\delta_d}\right) \in f^{-1}([-\infty, c)) \\
&\iff f\left(\left(\frac{(\delta_1 x_1 + y_1) - y_1}{\delta_1}, \ldots, \frac{(\delta_d x_d + y_d) - y_d}{\delta_d}\right)\right) < c \\
&\iff \delta x + y \in f_{\delta,y}^{-1}([-\infty, c)) \iff \delta x \in f_{\delta,y}^{-1}([-\infty, c)) - y \\
&\iff x \in \frac{1}{\delta}(f_{\delta,y}^{-1}([-\infty, c)) - y),
\end{aligned}$$

where $\frac{1}{\delta}(f_{\delta,y}^{-1}([-\infty, c)) - y) := \{\frac{1}{\delta}(x - y) : x \in f_{\delta,y}^{-1}([-\infty, c))\}$. Thus since $1/\delta \in (0, \infty)^d$ and $-y \in \mathbb{R}^d$, the measurability of $f_{\delta,y}^{-1}([-\infty, c))$ implies that $\frac{1}{\delta}(f_{\delta,y}^{-1}([-\infty, c)) - y) = f^{-1}([-\infty, c))$ is measurable (this likewise follows via question one of assigment 3). Since c was arbitrary, f is measurable.

Therefore, we conclude that f is measurable if and only if $f_{\delta,y}$ is measurable. ■

Proof of 1.2. For "$\Rightarrow$" , suppose f is integrable on A. Let $E \subseteq A$ be measurable. We first consider the case where f is a characteristic function, so we define $f := \chi_E$. It follows that $f_{\delta,y} = \chi_{E_{\delta,y}}$ since for $\delta x + y \in E_{\delta,y}$ we have $f_{\delta,y}(\delta x + y) = f(\frac{(\delta x+y)-y}{\delta}) = f(x) = 1$ since $\delta x + y \in E_{\delta,y} \implies x \in E$; on the other hand, if $\delta x + y \notin E_{\delta,y}$, then $x \notin E$ so that $f_{\delta,y}(\delta x + y) = f(x) = 0$. Furthermore, $E_{\delta,y}$ is measurable and $m(E_{\delta,y}) = \delta_1 \cdots \delta_d m(E)$ (cf. Assigment 1, Question 3). Thus, by definition of the Lebesgue integral we have

$$\int_{E_{\delta,y}} f_{\delta,y} = \int_{E_{\delta,y}} \chi_{E_{\delta,y}} = m(E_{\delta,y}) = \delta_1 \cdots \delta_d m(E) = \delta_1 \cdots \delta_d \int_E \chi_E = \delta_1 \cdots \delta_d \int_E f.$$

Now suppose f is any characteristic function with canonical form $f = \sum_{k=1}^N c_k \chi_{E_k}$. Thus, $f_{\delta,y} = \sum_{k=1}^N c_k \chi_{E_{k,\delta,y}}$, where $E_{k,\delta,y} := \{\delta x + y : x \in E_k\}$ and $m(E_{k,\delta,y}) = \delta_1 \cdots \delta_d m(E_k)$ (cf. Assignment 1, Question 3). Therefore, by the construction of the Lebesgue integral of simple functions, we have

$$\int_{E_{\delta,y}} f_{\delta,y} = \int_{E_{\delta,y}} \sum_{k=1}^N c_k \chi_{E_{k,\delta,y}} = \sum_{k=1}^N c_k m(E_{k,\delta,y}) = \delta_1 \cdots \delta_d \sum_{k=1}^N c_k m(E_k) = \delta_1 \cdots \delta_d \int_E \sum_{k=1}^N c_k \chi_E = \delta_1 \cdots \delta_d \int_E f.$$

Thus, we have the desired equality for simple functions.

We now suppose that f is non-negative on E. In this case, so is $f_{\delta,y}$ since for each $\delta x + y \in E_{\delta,y}$, $f_{\delta,y}(\delta x + y) = f(x) \geq 0$. By the simple approximation theorem, there exists a sequence $\{\varphi_k\}_{k\in\mathbb{N}}$ of simple functions such that $\lim_{k\to\infty} \varphi_k = f$ and for each $k \in \mathbb{N}$, $\varphi_k \geq 0$ and $\varphi_k \leq \varphi_{k+1}$. Define for each $k \in \mathbb{N}$ and $\delta x + y \in E_{\delta,y}$ the function $\varphi_{k,\delta,y}(\delta x + y) = \varphi_k(\frac{(\delta x+y)-y}{\delta}) = \varphi_k(x)$ so that:

1. For each $\delta x + y \in E_{\delta,y}$, $\varphi_{k,\delta,y}(\delta x + y) \to f_{\delta,y}(\delta x + y)$. Indeed, $\varphi_{k,\delta,y}(\delta x + y) = \varphi_k(x) \to f(x) = f_{\delta,y}(\delta x + y)$.
2. For each $k \in \mathbb{N}$, $0 \leq \varphi_{k,\delta,y} \leq \varphi_{k+1,\delta,y}$. Indeed, for each $k \in \mathbb{N}$ and $\delta x + y \in E_{\delta,y}$ we have

$$0 \leq \varphi_k(x) = \varphi_{k,\delta,y}(\delta x + y) \leq \varphi_{k+1}(x) = \varphi_{k+1,\delta,y}(\delta x + y).$$

Thus, we can use the monotone convergence theorem as well as the previous argument regarding simple functions to obtain

$$\int_{E_{\delta,y}} f_{\delta,y} = \lim_{k\to\infty} \int_{E_{\delta,y}} \varphi_{k,\delta,y} = \lim_{k\to\infty} \left(\delta_1 \cdots \delta_d \int_E \varphi_k \right) = \delta_1 \cdots \delta_d \lim_{k\to\infty} \int_E \varphi_k = \delta_1 \cdots \delta_d \int_E f.$$

Finally, if f can change sign, we write $f = f_+ - f_-$. It follows that for $\delta x + y \in E_{\delta,y}$ we have that $f_{\delta,y}(\delta x+y) = f(\frac{(\delta x+y)-y}{\delta}) = f_+(\frac{(\delta x+y)-y}{\delta}) - f_-(\frac{(\delta x+y)-y}{\delta}) =: f_{+,\delta,y}(\delta x+y) - f_{-,\delta,y}(\delta x+y)$, where $f_{+,\delta,y}(\delta x+y) := f_+(\frac{(\delta x+y)-y}{\delta})$ and $f_{-,\delta,y}(\delta x + y) := f_-(\frac{(\delta x+y)-y}{\delta})$. Thus, by the previous argument regarding non-negative functions, since $f_+, f_{+,\delta,y}, f_-, f_{-,\delta,y} \geq 0$ on their domains. Since $\int_E f = \int_E f_+ - \int_E f_-$, it follows that

$$\int_{E_{\delta,y}} f_{\delta,y} = \int_{E_{\delta,y}} f_{+,\delta,y} - \int_{E_{\delta,y}} f_{-,\delta,y} = \delta_1 \cdots \delta_d \int_E f_+ - \delta_1 \cdots \delta_d \int_E f_- = \delta_1 \cdots \delta_d \int_E f.$$

Thus, in all of such cases, we have that $f_{\delta,y}$ is also integrable since $\int_{E_{\delta,y}} f_{\delta,y} = \delta_1 \cdots \delta_d \int_E f < \infty$, thereby completing the forward implication of the proof, taking $E = A$.

On the other hand, for "$\Leftarrow$", suppose $f_{\delta,y}$ is integrable over $A_{\delta,y}$. For the sake of clarity, let $E := A_{\delta,y}$, and we denote $f_{\delta,y}$ by g. By the forward implication, we know that for any $\alpha = (\alpha_1, \ldots, \alpha_d) \in (0,\infty)^d$ and $z = (z_1, \ldots, z_d) \in \mathbb{R}^d$, the function $g \circ h$ is integrable, where $h : E_{\alpha,z} \to \overline{\mathbb{R}}$ is a function given by $h(x) = \frac{x-z}{\alpha}$. Thus, for the given δ and y, let $\alpha \coloneqq 1/\delta$ and $z \coloneqq -y/\delta$ so that by "$\Rightarrow$" we obtain that $g \circ h$ is integrable. But notice that $A = E_{\alpha,z}$ as

$$\alpha(\delta x + y) + z \in E_{\alpha,z} \iff \frac{1}{\delta}(\delta x + y) - \frac{y}{\delta} = x + \frac{y}{\delta} - \frac{y}{\delta} = x \in E_{\alpha,z} \iff \delta x + y \in E = A_{\delta,y} \iff x \in A.$$

Thus, $h : A \to \overline{\mathbb{R}}$ and $g \circ h = f_{\delta,y} \circ h$ is integrable. But for $x \in A$,

$$f_{\delta,y} \circ h(x) = f_{\delta,y}(h(x)) = f_{\delta,y}\left(\frac{x-z}{\alpha}\right) = f_{\delta,y}\left(\frac{x + \frac{y}{\delta}}{\frac{1}{\delta}}\right) = f\left(\frac{\left(\frac{x+\frac{y}{\delta}}{\frac{1}{\delta}}\right) - y}{\delta}\right) = f\left(\frac{\delta x + y - y}{\delta}\right) = f(x).$$

Thus, we conclude that $g \circ h = f$ is integrable, thereby completing the proof. ■

Problem 2. Let f be integrable over $A \subseteq \mathbb{R}^d$. Show that:

1. If $\int_B f \geq 0$ for all measurable sets $B \subseteq A$, then $f \geq 0$ a.e. in A.
2. If $\int_B f = 0$ for all measurable sets $B \subseteq A$, then $f = 0$ a.e. in A.

Proof of 2.1. Define $N = \{x \in A : f(x) < 0\} = f^{-1}([-\infty, 0)) \subseteq A$. Since f is integrable, it is measurable; thus N is measurable by the measurability of f. Since $N \subseteq A$, by hypothesis

$$\int_N f \geq 0.$$

Notice that for each $k \in \mathbb{N}$, $k \cdot \chi_N f \leq f$ on A since for $x \in N$, the $k \cdot \chi_N(x) f(x) = -k|f(x)| \leq -|f(x)|$, and for $x \notin N$, $k \cdot \chi_N(x) f(x) = 0 \leq f(x)$, since $f \geq 0$ on $A \setminus N$. Thus, by monotonicity, for every $k \in \mathbb{N}$ we have

$$\begin{aligned} \int_A k \cdot \chi_N f \leq \int_A f &\iff k \int_A \chi_N f \leq \int_A f && \text{(by linearity)} \\ &\iff \int_N f = \int_A \chi_N f \leq \frac{1}{k} \int_A f. \end{aligned}$$

Thus, we have that for every $k \in \mathbb{N}$, $0 \leq \int_N f \leq \frac{1}{k} \int_A f$. Since f is integrable, there exists $\ell \in \mathbb{R}$ such that $\int_A f = \ell$. Thus, $0 \leq \int_N f \leq \ell/k$. Since ℓ is fixed, sending $k \to \infty$ gives $\int_N f \leq 0$ so that $\int_N f = 0$, since we also have $\int_N f \geq 0$. But by definition, $f < 0$ on N so that $f_+ = \max(f, 0) = 0$ on N. Hence,

$$0 = \int_N f = \int_N f_+ - \int_N f_- = \int_N 0 - \int_N f_- = -\int_N f_- = -\left(\underbrace{\int_{A \setminus N} f_-}_{=0} + \int_N f_- \right) = -\int_A f_-,$$

which holds as $f_- = 0$ on $A \setminus N$ (as $f \geq 0$ on $A \setminus N$) and since $(A \setminus N) \cap N = \emptyset$ and the sets $A \setminus N$, N are measurable (difference of measurable sets; for $B_1, B_2 \subseteq \mathbb{R}^d$ disjoint, measurable, f integrable: $\int_{B_1 \cup B_2} f = \int_{B_1} f + \int_{B_2} f$). Thus, we conclude that $-\int_A f_- = 0$ so that $\int_A f_- = 0$. Since $f_- : A \to [0, \infty]$ is measurable and $\int_A f_- = 0$, by lecture we have $f_- = 0$ a.e. in A. Thus, $\max(-f, 0) = 0$ a.e. in A so that $f \geq 0$ a.e. in A as desired. ■

Proof of 2.2. By (2.1), we already know that $f \geq 0$ a.e. in A. Thus, it remains to be shown that $f \leq 0$ a.e. in A. Define $N = \{x \in A : f(x) > 0\} = f^{-1}((0, \infty]) \subseteq A$, which is measurable since f is. Thus, by hypothesis, $\int_N f = 0$ so that, since $f_- = 0$ on N (since f is positive on N) and $f_+ = 0$ on $A \setminus N$ (as f non-positive on $A \setminus N$),

$$\int_N f = \int_N f_+ - \int_N f_- = \int_N f_+ = \left(\underbrace{\int_{A \setminus N} f_+}_{=0} + \int_N f_+ \right) = \int_A f_+ = 0,$$

since $(A \setminus N) \cap N = \emptyset$ and the sets $A \setminus N$, N are measurable (difference of measurable sets; property of integral over disjoint, measurable sets as above). Thus, since $f_+ : A \to [0, \infty]$ is measurable and $\int_A f_+ = 0$, it follows by lecture that $f_+ = 0$ a.e. in A so that $f \leq 0$ a.e. in A.

Since $f \geq 0$ a.e. in A, the set $N_1 \coloneqq \{x \in A : f(x) < 0\}$ has measure zero, and since $f \leq 0$ a.e. in A, the set $N_2 \coloneqq \{x \in A : f(x) > 0\}$ has measure zero. Thus, the set $M = N_1 \cup N_2$ has measure zero since by finite sub-additivity,

$$m(M) = m(N_1 \cup N_2) \leq m(N_1) + m(N_2) = 0 \implies m(M) = 0.$$

Thus, the set on which $f \neq 0$ has measure zero. Hence, we conclude that $f = 0$ a.e. in A. ■

Problem 3. Let $f : \mathbb{R}^d \to \mathbb{R}$ be measurable, non-negative on $\mathbb{R}^d$, and finite almost everywhere. For each $k \in \mathbb{Z}$, let $F_k \coloneqq \{x : 2^k < f(x) \leq 2^{k+1}\}$.

1. Show that f is integrable on $\mathbb{R}^d$ if and only if $\sum_{k=-\infty}^{\infty} 2^k m(F_k) < \infty$.
2. Use (1) to verify that
$$f(x) \coloneqq \begin{cases} |x|^{-a} & |x| \leq 1 \\ 0 & \text{otherwise} \end{cases}$$
is integrable over $\mathbb{R}^d$ if and only if $a < d$.
3. Use (1) to verify that
$$g(x) \coloneqq \begin{cases} |x|^{-b} & |x| > 1 \\ 0 & \text{otherwise} \end{cases}$$
is integrable over $\mathbb{R}^d$ if and only if $b > d$.

Note that $|x| = \sqrt{x_1^2 + \cdots + x_d^2}$ for all $x \in (x_1, \ldots, x_d) \in \mathbb{R}^d$.

Proof of 3.1.

For "$\Rightarrow$", suppose f is integrable on $\mathbb{R}^d$. Let $\varphi \coloneqq \sum_{k=-\infty}^{\infty} 2^k \chi_{F_k}$. Then it is clear by the definition of F_k that for $k \in \mathbb{Z}$ the F_k's are mutually disjoint.[1] We note that $F_{-\infty} = \{x : 0 < f(x) \leq 0\} = \emptyset$ and $F_\infty = \{x : \infty < f(x) \leq \infty\} = \emptyset$ as well; thus, we write $\sum_{k=-\infty}^{\infty} 2^k m(F_k) = \sum_{k\in\mathbb{Z}} 2^k m(F_k)$, since $m(\emptyset) = 0$.

Let $x \in \mathbb{R}^d$ be arbitrary. If there does not exist a k for which $x \in F_k$, then $\varphi(x) = 0 \leq f(x)$ by non-negativity. Otherwise, there is a unique F_k containing x so that $\varphi(x) = 2^k < f(x)$ by definition of F_k. Thus, $\varphi \leq f$ on $\mathbb{R}^d$ so that by monotonicity,
$$\int \varphi \leq \int f < \infty. \tag{3.1}$$

Notice that by Lemma 3.1, since $2^k > 0$ for every $k \in \mathbb{Z}$,
$$\int \varphi = \int \sum_{k\in\mathbb{Z}} 2^k \chi_{F_k} = \sum_{k\in\mathbb{Z}} \int 2^k \chi_{F_k} = \sum_{k\in\mathbb{Z}} 2^k \int \chi_{F_k} = \sum_{k\in\mathbb{Z}} 2^k m(F_k) = \sum_{k=-\infty}^{\infty} 2^k m(F_k) \leq \int f < \infty,$$
using (3.1), linearity, and the definition of the integral of a characteristic function. Thus, we have that $\sum_{k=-\infty}^{\infty} 2^k m(F_k) < \infty$, completing the forward implication.

On the other hand, for "$\Leftarrow$" , suppose $\sum_{k=-\infty}^{\infty} 2^k m(F_k) = \sum_{k\in\mathbb{Z}} 2^k m(F_k) < \infty$. Then
$$2 \sum_{k=-\infty}^{\infty} 2^k m(F_k) = \sum_{k\in\mathbb{Z}} 2^{k+1} m(F_k) < \infty. \tag{3.2}$$

Define $\psi \coloneqq \sum_{k\in\mathbb{Z}} 2^{k+1} \chi_{F_k}$. It follows for almost every $x \in \mathbb{R}^d$, $\psi \leq f$. Indeed, for $x \in \mathbb{R}^d$, if there is no k such that $x \in F_k$, then $f(x) = \infty$ or $f(x) = 0$ (such holds as $(0, \infty) = \bigcup_{k\in\mathbb{Z}} (2^k, 2^{k+1}] \subseteq [0, \infty]$, which is the image of f). Since f is finite almost everywhere, we need not consider such x for which $f(x) = \infty$; and if

[1] Indeed, if $x \in F_i \cap F_j$ for $i, j \in \mathbb{Z}$ then $x \in f^{-1}((2^i, 2^{i+1}])$ and $x \in f^{-1}((2^j, 2^{j+1}])$. Thus $f(x) \in (2^i, 2^{i+1}] \cap (2^j, 2^{j+1}]$. Suppose towards a contradiction that $i \neq j$; without loss of generality, take $i < j$. Then $2^{i+1} < 2^{i+2} \leq 2^j$ implies that $f(x) \in (2^i, 2^{i+1}] \cap (2^j, 2^{j+1}] = \emptyset$, a contradiction.

$f(x) = 0$ then $\psi(x) = 0 = f(x)$ as needed. On the other hand, if there is an F_k containing x, by disjointness F_k is unique so that $\psi(x) = 2^{k+1} \geq f(x)$ by definition of F_k. Thus, by monotonicity,

$$\int f \leq \int \psi. \tag{3.3}$$

Again, using Lemma 3.1, since $2^{k+1} > 0$ for $k \in \mathbb{Z}$, we must have that

$$\int \psi = \int \sum_{k\in\mathbb{Z}} 2^{k+1}\chi_{F_k} = \sum_{k\in\mathbb{Z}} \int 2^{k+1}\chi_{F_k} = \sum_{k\in\mathbb{Z}} 2^{k+1} \int \chi_{F_k} = \sum_{k\in\mathbb{Z}} 2^{k+1} m(F_k) < \infty,$$

using (3.2), linearity, and the definition of the integral of a characteristic function. Thus, using (3.3), we conclude that f is integrable since

$$\int f \leq \int \psi = \sum_{k\in\mathbb{Z}} 2^{k+1} m(F_k) < \infty,$$

thereby completing the proof. ■

Proof of 3.2.

We first handle the trivial case where $a \leq 0$. Then $a < d$ is a tautology as $d \geq 1$ so that f integrable $\implies a < d$. On the other hand, we now show that f is integrable whenever $a \leq 0 < d$. Let $k := |a|$ and note that for $x \in \overline{B(0,1)}$, $|x| \leq 1 \implies |x|^{k-1} \leq 1 \implies |x|^k \leq |x| \leq 1$ so that by monotonicity and the disjointness of a set with its complement,

$$\int_{\mathbb{R}^d} f = \underbrace{\int_{\mathbb{R}^d \setminus \overline{B(0,1)}} f}_{=0} + \int_{\overline{B(0,1)}} f = \int_{\overline{B(0,1)}} |x|^k \leq \int_{\overline{B(0,1)}} 1 = \int_{\overline{B(0,1)}} \chi_{\overline{B(0,1)}} = m(\overline{B(0,1)}) < \infty$$

so that f is integrable as needed.

We now suppose that $a > 0$ and commence by noting that

$$\begin{aligned} x \in F_k &\iff f(x) \in (2^k, 2^{k+1}] \iff 0 < 2^k < |x|^{-a} \leq 2^{k+1} \text{ and } |x| \leq 1 \\ &\iff 0 < 2^k < |x|^{-a} \leq 2^{k+1} \text{ and } x \in \overline{B(0,1)} \end{aligned}$$

so that

$$F_k = \{x \in \overline{B(0,1)} : 2^k < |x|^{-a} \leq 2^{k+1}\}.$$

We must calculate $m(F_k)$ for each $k \in \mathbb{Z}$. Notice that if $k \in \mathbb{Z}$ then any $x \in \overline{B(0,1)}$ satisfies $x \in F_k$ if and only if

$$\begin{aligned} 2^k < |x|^{-a} \leq 2^{k+1} &\iff 2^k < \left(\frac{1}{|x|}\right)^a \leq 2^{k+1} \iff 2^{k/a} < \frac{1}{|x|} \leq 2^{\frac{k+1}{a}} \\ &\iff 2^{-k/a} > |x| \geq 2^{-\frac{k+1}{a}}. \end{aligned}$$

For $k \geq 0$ we have $F_k = B(0, 2^{-k/a}) \setminus B(0, 2^{-\frac{k+1}{a}})$. Indeed, $B(0, 2^{-k/a}) \setminus B(0, 2^{-\frac{k+1}{a}}) \subseteq B(0, 2^{-k/a}) \subseteq \overline{B(0,1)}$ since $2^{-k/a} \leq 1$ (the exponent is non-positive). Furthermore, for $k \leq -1$ we can bound $m(F_k) \leq m(\overline{B(0,1)}) < \infty$ by monotonicity as $F_k \subseteq \overline{B(0,1)}$.

Moreover, we claim that for $k \geq 0$, $B(0, 2^{-\frac{k+1}{a}}) \subseteq B(0, 2^{-k/a})$ since $x \in B(0, 2^{-\frac{k+1}{a}}) \implies |x| < 2^{-\frac{k+1}{a}} = \frac{1}{2^{\frac{k+1}{a}}} \leq \frac{1}{2^{\frac{k}{a}}} = 2^{-k/a}$ so that $x \in B(0, 2^{-k/a})$. Since these open balls have finite radii, it follows that they have finite measure. Thus, by the excision property,

$$m(F_k) = m(B(0, 2^{-k/a}) \setminus B(0, 2^{-\frac{k+1}{a}})) = m(B(0, 2^{-k/a})) - m(B(0, 2^{-\frac{k+1}{a}})).$$

Now notice that

$$B(0, 2^{-k/a}) = \{2^{-k/a}x : x \in B(0,1)\} \text{ and } B(0, 2^{-\frac{k+1}{a}}) = \{2^{-\frac{k+1}{a}}x : x \in B(0,1)\}. \qquad \text{(by scaling of balls)}$$

Thus, using question 3 of assignment 1, we can use the dilation property of measure (with $\delta = (2^{-k/a}, \ldots, 2^{-k/a})$ or $\delta = (2^{-\frac{k+1}{a}}, \ldots, 2^{-\frac{k+1}{a}})$) to obtain:

$$\begin{aligned} m(F_k) &= m(B(0, 2^{-k/a})) - m(B(0, 2^{-\frac{k+1}{a}})) = (2^{(-k/a)})^d m(B(0,1)) - (2^{(-\frac{k+1}{a})})^d m(B(0,1)) \\ &= m(B(0,1)) \cdot (2^{-dk/a} - 2^{(-dk-d)/a}) = m(B(0,1)) \cdot (2^{-dk/a} - 2^{-dk/a-d/a}) \\ &= m(B(0,1)) \cdot (2^{-dk/a} - 2^{-dk/a} \cdot 2^{-d/a}) = 2^{-dk/a} m(B(0,1))(1 - 2^{-d/a}). \end{aligned}$$

For the sake of clarity, let $b_d \coloneqq m(B(0,1))$ and $\overline{b_d} \coloneqq m(\overline{B(0,1)})$. For "$\Rightarrow$", suppose f is integrable over $\mathbb{R}^d$. It follows from (1) that

$$\sum_{k=-\infty}^{\infty} 2^k m(F_k) = \sum_{k \in \mathbb{Z}} 2^k m(F_k) < \infty.$$

But notice that

$$\begin{aligned} \sum_{k \in \mathbb{Z}} 2^k m(F_k) &= \sum_{k \in \mathbb{Z}_{\leq -1}} 2^k m(F_k) + \sum_{k=0}^{\infty} 2^k m(f_k) \\ &\leq \sum_{k \in \mathbb{Z}_{\leq -1}} 2^k \cdot \overline{b_d} + \sum_{k=0}^{\infty} 2^k m(F_k) && \text{(as } m(F_k) \leq m(\overline{B(0,1)})) \\ &= \overline{b_d} \cdot \sum_{k=1}^{\infty} 2^{-k} + \sum_{k=0}^{\infty} 2^k m(F_k) = \overline{b_d} + \sum_{k=0}^{\infty} 2^k m(F_k), \end{aligned}$$

and since $\sum_{k=0}^{\infty} 2^k m(F_k) \leq \sum_{k \in \mathbb{Z}_{\leq -1}} 2^k m(F_k) + \sum_{k=0}^{\infty} 2^k m(F_k) = \sum_{k=-\infty}^{\infty} 2^k m(F_k) < \infty$, it follows that

$$\begin{aligned} \sum_{k=0}^{\infty} 2^k m(F_k) &= \sum_{k=0}^{\infty} 2^k b_d (2^{-dk/a})(1 - 2^{-d/a}) = b_d(1 - 2^{-d/a}) \sum_{k=0}^{\infty} 2^{k - dk/a} \\ &= b_d(1 - 2^{-d/a}) \sum_{k=0}^{\infty} 2^{k(1-d/a)} \end{aligned}$$

so that $\sum_{k=0}^{\infty} 2^{k(1-d/a)} < \infty$. But then we must have that $1 - d/a$ is negative, otherwise the sum would diverge as $c \coloneqq 1 - d/a \geq 0$ is fixed and $\lim_{k\to\infty} 2^{ck} = \infty \neq 0$. Thus $1 < d/a$ so that $a < d$.

On the other hand, for "$\Leftarrow$", if $a < d$ then $\sum_{k=0}^{\infty} 2^{k(1-d/a)}$ converges, since $a < d$ means that $\sum_{k=0}^{\infty} 2^{k(1-d/a)} = \sum_{k=0}^{\infty} \frac{1}{2^{k(d/a-1)}} < \infty$. Then since $c \coloneqq d/a - 1 > 0$ is fixed, by the ratio test we have convergence:

$$\lim_{k\to\infty} \left| \frac{\frac{1}{2^{ck+c}}}{\frac{1}{2^{ck}}} \right| = \frac{2^{ck}}{2^{ck} \cdot 2^c} = \frac{1}{2^c} < 1.$$

Thus,

$$\sum_{k=-\infty}^{\infty} 2^k m(F_k) \leq \overline{b_d} + \sum_{k=0}^{\infty} 2^k m(F_k) < \infty,$$

thus f is integrable by (1). Thus, the proof is complete. ■

Proof of 3.3. Let $F_k = \{x : 2^k < g(x) \leq 2^{k+1}\}$. We first note that if b is negative then g is unbounded on $\mathbb{R}^d \setminus \overline{B(0,1)}$. This means by monotonicity that

$$\int_{\mathbb{R}^d \setminus \overline{B(0,1)}} g \geq \int_{\mathbb{R}^d \setminus \overline{B(0,1)}} \chi_{\mathbb{R}^d \setminus \overline{B(0,1)}} = m(\mathbb{R}^d \setminus \overline{B(0,1)}) = \infty,$$

since $|x| > 1 \implies g(x) = |x|^{|b|} > 1 = \chi_{\mathbb{R}^d \setminus \overline{B(0,1)}}(x) = 1$, for $x \in \mathbb{R}^d \setminus \overline{B(0,1)}$. Thus, g is not integrable over $\mathbb{R}^d \setminus \overline{B(0,1)}$ as $\int_{\mathbb{R}^d \setminus \overline{B(0,1)}} g = \infty$. It follows that g can not be integrable on $\mathbb{R}^d$. Hence, we may assume that $b \geq 0$.

Using (2), we see that for $k \in \mathbb{Z}$,

$$F_k = \{x \in \mathbb{R}^d \setminus \overline{B(0,1)} : 2^k < |x|^{-b} \leq 2^{k+1}\} = \{x \in \mathbb{R}^d \setminus \overline{B(0,1)} : 2^{-k/b} > |x| \geq 2^{-\frac{k+1}{b}}\}.$$

Thus, since $x \in F_k \iff x \in B(0, 2^{-k/b}) \setminus (\overline{B(0,1)} \cup B(0, 2^{-\frac{k+1}{b}}))$, we have that $F_k = \emptyset$ for $k \geq 0$ since $B(0, 2^{-k/b}) \subseteq \overline{B(0,1)}$ since $2^{-k/b} \leq 1$. For $k \leq -1$ we have $2^{-\frac{k+1}{b}} \geq 1$ (since the exponent is non-negative) so that $F_k = B(0, 2^{-k/b}) \setminus B(0, 2^{-\frac{k+1}{b}})$. Thus, using the work in (2) to calculate $m(F_k) = 2^{-dk/b} b_d (1 - 2^{-d/b})$, we obtain

$$\begin{aligned}
\sum_{k=-\infty}^{\infty} 2^k m(F_k) &= \sum_{k \leq -1} 2^k m(F_k) + \sum_{k \geq 0} 2^k \cdot 0 && (k \geq 0 \implies F_k = \emptyset) \\
&= b_d (1 - 2^{-d/b}) \sum_{k \leq -1} 2^{k(1-d/b)} = b_d (1 - 2^{-d/b}) \sum_{k=1}^{\infty} \frac{1}{2^{k(1-d/b)}}.
\end{aligned}$$

Now, if $b > d$, letting $c := 1 - d/b > 0$ so that $\sum_{k=1}^{\infty} \frac{1}{2^{ck}}$ converges by the ratio test (this was proven in (2)). Thus $\sum_{k=-\infty}^{\infty} 2^k m(F_k) = b_d (1 - 2^{-d/b}) \sum_{k=1}^{\infty} 2^{-k(1-d/b)} < \infty$ so that g is integrable.

On the other hand, if g is integrable, then the above sum converges by (1). In this case, we have that $1 - d/b$ is positive, otherwise the sum would diverge since if $c := 1 - d/b < 0$ is fixed then the sum diverges since $\lim_{k \to \infty} 1/2^{kc} = \lim_{k \to \infty} 2^{|kc|} = \infty \neq 0$. Thus, $1 - d/b > 0 \iff d/b < 1 \iff b > d$. Therefore, the proof is complete. ■

Lemma 3.1. Let $\{F_k\}_{k \in \mathbb{Z}}$ be a sequence of mutually disjoint measurable sets and $(c_k)_{k \in \mathbb{Z}}$ be a sequence of non-negative reals. Then,

$$\int \sum_{k \in \mathbb{Z}} c_k \cdot \chi_{F_k} = \sum_{k \in \mathbb{Z}} \int c_k \chi_{F_k}.$$

Proof. For $k \in \mathbb{Z}$ define $a_k(x) := c_k \cdot \chi_{F_k}(x)$. Then a_k is non-negative for each k, and it is measurable since it is a simple function. Since $\mathbb{Z}$ is countable, without loss of generality, we can re-index the sequence and the sets F_k to both start at $k = 1$. Therefore, we can apply the corollary of Fatou's lemma regarding series (covered in lecture), we conclude that

$$\int \sum_{k \in \mathbb{Z}} c_k \cdot \chi_{F_k} = \int \sum_{k=1}^{\infty} a_k = \sum_{k=1}^{\infty} \int a_k = \sum_{k \in \mathbb{Z}} \int c_k \cdot \chi_{F_k}.$$

■

Problem 4. Let f be measurable on $A \subseteq \mathbb{R}^d$ and $(A_k)_{k\in\mathbb{N}}$ be a sequence of measurable subsets of A. Show that:

1. If f is non-negative and $A_k \subseteq A_{k+1}$ for all $k \in \mathbb{N}$ then
$$\int_{\bigcup_{k=1}^{\infty} A_k} f = \lim_{k\to\infty} \int_{A_k} f.$$
2. If f is integrable and $A_k \subseteq A_{k+1}$ for all $k \in \mathbb{N}$ then
$$\int_{\bigcup_{k=1}^{\infty} A_k} f = \lim_{k\to\infty} \int_{A_k} f.$$
3. If f is integrable over A_1 and $A_{k+1} \subseteq A_k$ for all $k \in \mathbb{N}$, then
$$\int_{\bigcap_{k=1}^{\infty} A_k} f = \lim_{k\to\infty} \int_{A_k} f.$$

Proof of 4.1. Let f be non-negative. We note that if f is integrable over $\bigcup_{k=1}^{\infty} A_k$, then we obtain from (2) that
$$\int_{\bigcup_{k=1}^{\infty} A_k} f = \lim_{k\to\infty} \int_{A_k} f$$
as needed, thus we suppose that $\int_{\bigcup_{k=1}^{\infty} A_k} f = \infty$, as this is all that is left to show by non-negativity. To show that $\lim_{k\to\infty} \int_{A_k} f = \int_{\bigcup_{k=1}^{\infty} A_k} = \infty$, we show that it is unbounded. Thus, let $M \in \mathbb{N}$ be arbitrary. Since $A_k \subseteq A_{k+1}$ for each $k \in \mathbb{N}$, it follows that there is a large enough $n \in \mathbb{N}$ such that
$$\int_{\bigcup_{k=1}^{n} A_k} f > M;$$
indeed, if there is no such n, then since limits respect order
$$\lim_{n\to\infty} \int_{\bigcup_{k=1}^{n} A_k} f = \int_{\bigcup_{k=1}^{\infty} A_k} f \leq M < \infty$$
is a contradiction to the choice of f. We note that $\bigcup_{k=1}^{n} A_k = A_n$ (since $A_n \subseteq \bigcup_{k=1}^{n} A_n$ trivially and $x \in \bigcup_{k=1}^{n} A_k \implies \exists m = 1, 2, \ldots, n : x \in A_m \subseteq A_{m+1} \subseteq \cdots \subseteq A_n \implies x \in A_n$ as needed.) Thus, for this given M, we have found a set A_n such that
$$\int_{\bigcup_{k=1}^{n} A_k} f = \int_{A_n} f > M.$$
Since M was arbitrary, we conclude as required that
$$\int_{\bigcup_{k=1}^{\infty} A_k} f = \lim_{k\to\infty} \int_{A_k} f = \infty.$$

■

Proof of 4.2. Let f be integrable over $A = \bigcup_{k=1}^\infty A_k$ and define $E_1 \coloneqq A_1$ and for $k \in \mathbb{N}_{\geq 2}$, let $E_k \coloneqq A_k \setminus A_{k-1}$. Then each E_k is a difference of measurable sets and hence measurable, and $\bigcup_{k=1}^\infty A_k = \bigcup_{k=1}^\infty E_k$.[2] By construction, the E_k's are mutually disjoint. Thus, it follows by disjointness that for each $n \in \mathbb{N}$

$$\int_{\bigcup_{k=1}^n A_k} f = \int_{\bigcup_{k=1}^n E_k} f = \sum_{k=1}^n \int_{E_k} f = \int_{A_n} f \qquad \text{(Lemma 4.1 by disjointness of } E_k\text{'s.)}$$

since $A_k \subseteq A_{k+1}\ \forall k \in \mathbb{N} \implies A_n = \bigcup_{k=1}^n A_k$ for $n \in \mathbb{N}$. Thus, again by disjointness and Lemma 4.2, we have

$$\int_{\bigcup_{k=1}^\infty A_k} f = \int_{\bigcup_{k=1}^\infty E_k} f = \lim_{n\to\infty} \int_{\bigcup_{k=1}^n E_k} f = \lim_{n\to\infty} \sum_{k=1}^n \int_{E_k} f = \lim_{n\to\infty} \int_{A_n} f.$$

Replacing n with k gives $\int_{\bigcup_{k=1}^\infty A_k} f = \lim_{k\to\infty} \int_{A_k} f$, completing the proof. ■

Proof of 4.3.

Let f be integrable over A_1 so that $\int_{A_1} f < \infty$. Define for all integers $k \geq 1$ $E_k \coloneqq A_1 \setminus A_k$. Then since $A_{k+1} \subseteq A_k$ for each $k \in \mathbb{N}$, we must have that the E_k's are mutually disjoint.[3] Moreover, since for each $k \in \mathbb{N}$ we have $A_{k+1} \subseteq A_k$, we obtain $E_k = A_1 \setminus A_k \subseteq A_1 \setminus A_{k+1} = E_{k+1}$ for every $k \in \mathbb{N}$. Furthermore,

$$\bigcup_{k=1}^\infty E_k = \bigcup_{k=1}^\infty (A_1 \setminus A_k) = \bigcup_{k=1}^\infty (A_1 \cap A_k^c) = A_1 \cap \bigcup_{k=1}^\infty A_k^c = A_1 \setminus \left(\bigcup_{k=1}^\infty A_k^c\right)^c = A_1 \setminus \bigcap_{k=1}^\infty A_k$$

via DeMorgan's law. Since f is integrable on A_1, it is also integrable on $\bigcup_{k=1}^\infty E_k = A_1 \setminus \bigcap_{k=1}^\infty A_k \subseteq A_1$. By (2), since the E_k's increase towards their union, we obtain

$$\int_{A_1 \setminus \bigcap_{k=1}^\infty A_k} f = \int_{\bigcup_{k=1}^\infty E_k} f = \lim_{k\to\infty} \int_{E_k} f = \lim_{k\to\infty} \int_{A_1 \setminus A_k} f. \qquad (*)$$

Furthermore, since f is integrable on A_1, it is integrable on $\bigcap_{k=1}^\infty A_k \subseteq A_1$ so that

$$\int_{A_1 \setminus \bigcap_{k=1}^\infty A_k} f = \int_{A_1} f - \int_{\bigcap_{k=1}^\infty A_k} f,$$

and since $A_k \subseteq A_1$ for each $k \in \mathbb{N}$, we also have $\int_{A_1 \setminus A_k} f = \int_{A_1} f - \int_{A_k} f$ (cf. Lemma 4.3). Therefore, using $(*)$ we obtain that

$$\int_{A_1} f - \int_{\bigcap_{k=1}^\infty A_k} f = \int_{A_1 \setminus \bigcap_{k=1}^\infty A_k} = \lim_{k\to\infty} \left(\int_{A_1 \setminus A_k} f\right) = \lim_{k\to\infty} \left(\int_{A_1} f - \int_{A_k} f\right) = \int_{A_1} f - \lim_{k\to\infty} \int_{A_k} f.$$

Since $\int_{A_1} f < \infty$, we obtain (by subtracting and multiplying by -1)

$$\int_{\bigcap_{k=1}^\infty A_k} f = \lim_{k\to\infty} \int_{A_k} f,$$

[2] "$\supseteq$" clearly holds by construction. On the other hand, if $x \in A_k$ for some $k \in \mathbb{N}_{\geq 2}$ then $x \in A_k \setminus A_{k-1}$ or $x \in A_{k-1}$; in the former case, $x \in E_k \subseteq \bigcup_{k=1}^\infty E_k$ and in the latter case, $x \in A_{k-1} \setminus A_{k-2}$ or $x \in A_{k-2}$; eventually, $x \in E_j \subseteq \bigcup_{k=1}^\infty E_k$ for some $j \geq 2$ (otherwise $x \in A_1 = E_1$). And if $x \in A_1$ then $x \in E_1 \subseteq \bigcup_{k=1}^\infty E_k$. This set equality also holds by the exact same reasoning with $\bigcup_{k=1}^n A_k = \bigcup_{k=1}^n E_k$ for each $n \in \mathbb{N}$.

[3] Indeed, if $1 \leq i < j$ are integers, then $E_i = A_1 \setminus A_i$ and $E_j = A_1 \setminus A_j$; hence $i < j \implies A_j \subseteq A_i$ so that $E_i \cap E_j = \emptyset$.

thereby completing the proof. ■

Lemma 4.1. Let $X \subseteq \mathbb{R}^d$ and f be an integrable function on X. If $\{X_k\}_{k=1}^n$ is a finite collection of mutually disjoint measurable subsets of X, then

$$\int_{X_1 \cup \cdots \cup X_n} f = \int_{X_1} f + \cdots + \int_{X_n} f.$$

Proof. We proceed by induction on n. For $n = 1$ there is nothing to prove. If $n = 2$, then $\int_{X_1 \cup X_2} f = \int_{X_1} f + \int_{X_2} f$ was proven in lecture. Suppose the claim holds for some $n \geq 1$ and let $X_1, \ldots, X_n, X_{n+1}$ be a collection of mutually disjoint measurable subsets of X. Then letting $Y_k := X_k$ for $k = 1, 2, \ldots, n-1$ and $Y_n := X_n \cup X_{n+1}$, the inductive hypothesis gives:

$$\begin{aligned}\int_{Y_1 \cup \cdots \cup Y_n} f &= \int_{Y_1} f + \cdots + \int_{Y_n} f = \int_{X_1} f + \cdots + \int_{X_{n-1}} f + \int_{X_n \cup X_{n+1}} f \\ &= \sum_{k=1}^{n-1} \int_{X_k} f + \int_{X_n \cup X_{n+1}} f = \sum_{k=1}^{n+1} \int_{X_k} f. \qquad \text{(by disjointness and case } n = 2\text{)}\end{aligned}$$

Furthermore, notice that

$$\int_{X_1 \cup \cdots X_{n+1}} f = \int_{Y_1 \cup \cdots \cup Y_n} f = \sum_{k=1}^{n+1} \int_{X_k} f,$$

which completes the inductive step and hence the proof.

Lemma 4.2. Let $X \subseteq \mathbb{R}^d$ and f be a integrable function on X. If $\{X_k\}_{k=1}^\infty$ is a sequence of mutually disjoint measurable subsets of X such that

$$X = \bigcup_{k=1}^\infty X_k, \text{ then } \int_X f = \sum_{k=1}^\infty \int_{X_k} f.$$

Proof. Define for $j \in \mathbb{N}$

$$f_j := f \cdot \chi_{\bigcup_{k=1}^j X_k}.$$

Then for each $j \in \mathbb{N}$, f_j is measurable by lecture as it is a product of measurable functions (note that characteristic functions are measurable by lecture). Furthermore, by the construction of f_j, $|f_j| \leq |f|$ since for $x \in \bigcup_{k=1}^j X_k$, $|f_j(x)| = |f(x)|$ and for $x \notin \bigcup_{k=1}^j X_k$, $|f_j(x)| = 0 \leq |f(x)|$; and $|f|$ is integrable by lecture since f is. Moreover, $f_j \to f$ pointwise: given $\varepsilon > 0$ and $x \in X$, since $X = \bigcup_{k=1}^\infty X_k$ there is a large enough j so that $x \in \bigcup_{k=1}^j X_k$ so that for $j' \geq j$, $|f_{j'}(x) - f(x)| = |f(x) \cdot 1 - f(x)| = 0 < \varepsilon$. Thus, by the dominated convergence theorem we have

$$\lim_{j \to \infty} \int_X f_j = \int_X f.$$

Notice that for each $j \in \mathbb{N}$, since $f_j = f$ on $\bigcup_{k=1}^j X_k$ and using the disjointness (and measurability) of $\bigcup_{k=1}^j X_k$ and $X \setminus \bigcup_{k=1}^j X_k$,

$$\int_X f_j = \int_{\bigcup_{k=1}^j X_k} f_j + \underbrace{\int_{X \setminus \bigcup_{k=1}^j X_k} f_j}_{=0} = \int_{\bigcup_{k=1}^j X_k} f.$$

By Lemma 4.1, since the X_k's are mutually disjoint by hypothesis, we additionally have:

$$\int_X f_j = \int_{\bigcup_{k=1}^j X_k} f = \sum_{k=1}^j \int_{X_k} f.$$

Thus, we conclude that

$$\int_X f = \lim_{j\to\infty} \int_X f_j = \lim_{j\to\infty} \sum_{k=1}^j \int_{X_k} f_j = \sum_{k=1}^\infty \int_{X_k} f.$$

■

Lemma 4.3. Let f be integrable on A and $B \subseteq A$. Then

$$\int_{A\setminus B} f = \int_A f - \int_B f.$$

Proof. Notice that $B \subseteq A \implies A = A \setminus B \cup B$ is a disjoint union. Since f is integrable over A, it is integrable over B so that

$$\int_A f = \int_{A\setminus B} f + \int_B f \implies \int_{A\setminus B} = \int_A f - \int_B f,$$

using $\int_B f < \infty$ to subtract it on both sides. ■

Problem 5. Let $(f_k)_{k\in\mathbb{N}}$ be a sequence of integrable functions over $A \subseteq \mathbb{R}$ which converges pointwise almost everywhere in A to a function f which is also integrable over A. Show that

$$\lim_{k\to\infty} \int_A |f_k - f| = 0 \iff \lim_{k\to\infty} \int_A |f_k| = \int_A |f|.$$

Hint. If may be useful to consider the functions $g_k = |f_k| + |f| - |f_k - f|$.

Proof.

For "$\Rightarrow$", suppose $\lim_{k\to\infty} \int_A |f_k - f| = 0$ and let $\varepsilon > 0$. To show that

$$\lim_{k\to\infty} \int_A |f_k| = \int_A |f|,$$

we must find an $N \in \mathbb{N}$ such that for all $m \geq N$,

$$\left| \int_A |f| - \int_A |f_m| \right| < \varepsilon.$$

Since $\lim_{k\to\infty} \int_A |f_k - f| = 0$, there exists an $N \in \mathbb{N}$ such that for all $m \geq N$, we have $\int_A |f_m - f| < \varepsilon$. But notice that

$$\begin{aligned}
\left| \int_A |f| - \int_A |f_n| \right| &= \left| \int_A (|f| - |f_m|) \right| && \text{(linearity)} \\
&\leq \int_A ||f| - |f_m|| && (|\textstyle\int_A f| \leq \int_A |f|) \\
&\leq \int_A |f - f_m| < \varepsilon,
\end{aligned}$$

by monotonicity since $||a| - |b|| \leq |a - b|$ for $a, b \in \mathbb{R}$ by the triangle inequality. Since ε was arbitrary, we conclude that

$$\lim_{k\to\infty} \int_A |f_k| = \int_A |f|.$$

For "$\Leftarrow$", let $N = \{x \in A : f_k(x) \not\to f(x)\}$. It suffices to show that

$$\limsup_{k\to\infty} \int_{A\setminus N} |f_k - f| \leq 0. \tag{$*$}$$

Indeed, we already know that

$$\liminf_{k\to\infty} \int_{A\setminus N} |f_k - f| \leq \limsup_{k\to\infty} \int_{A\setminus N} |f_k - f| \tag{5.1}$$

holds in general; and by Fatou's lemma (since $|f_k|$ is non-negative)

$$\int_{A\setminus N} \liminf_{k\to\infty} |f_k - f| = \int_{A\setminus N} 0 = 0 \leq \liminf_{k\to\infty} \int_{A\setminus N} |f_k - f|, \tag{5.2}$$

where $\liminf_{k\to\infty} |f_k - f| = 0$ since $f_k \to f$ everywhere in $A \setminus N$. Combining (5.1) and (5.2), we attain that

$$\liminf_{k\to\infty} \int_{A\setminus N} |f_k - f| \le \limsup_{k\to\infty} \int_{A\setminus N} |f_k - f| \le 0 \le \liminf_{k\to\infty} \int_{A\setminus N} |f_k - f| \le \limsup_{k\to\infty} \int_{A\setminus N} |f_k - f|$$

so that $\liminf_{k\to\infty} \int_{A\setminus N} |f_k - f| = \limsup_{k\to\infty} \int_{A\setminus N} |f_k - f| = 0$. Thus, since N has measure 0, it would follow that

$$\liminf_{k\to\infty} \int_{A\setminus N} |f_k - f| = 0 \le \lim_{k\to\infty} \int_{A\setminus N} |f_k - f| = \lim_{k\to\infty} \int_{A} |f_k - f| \le 0 = \limsup_{k\to\infty} \int_{A\setminus N} |f_k - f|$$

so that $\lim_{k\to\infty} \int_A |f_k - f| = 0$ as required. Thus, we prove $(*)$ now to complete the proof.

Following the hint, for each $k \in \mathbb{N}$ we define $g_k := |f_k| + |f| - |f_k - f|$. Since $|a+b| \le |a| + |b|$ for each $a, b \in \mathbb{R}$ (triangle inequality), $|f_k| + |f| \ge |f_k - f|$ so that $g_k \ge 0$ on A. Since $\lim_{k\to\infty} \int_A |f_k| = \lim_{k\to\infty} \int_{A\setminus N} |f_k| = \int_A |f| = \int_{A\setminus N} |f|$, we have also that $\liminf_{k\to\infty} \int_A |f_k| = \liminf_{k\to\infty} \int_{A\setminus N} |f_k| = \int_A |f| = \int_{A\setminus N} |f|$; so, using this, we obtain via Fatou's lemma that

$$\begin{aligned}
\int_{A\setminus N} \liminf_{k\to\infty} g_k &= \int_{A\setminus N} \left(|f| + \liminf_{k\to\infty}(|f_k| - |f_k - f|)\right) \\
&= \int_{A\setminus N} (|f| + |f| + 0) && (f_k \to f \text{ in } A) \\
&= 2\int_{A\setminus N} |f| && \text{(by linearity)} \\
&\le \liminf_{k\to\infty} \int_{A\setminus N} g_k \\
&= \liminf_{k\to\infty} \left(\int_{A\setminus N} |f| + \int_{A\setminus N} |f_k| - \int_{A\setminus N} |f_k - f| \right) && \text{(by linearity)} \\
&= \liminf_{k\to\infty} \int_{A\setminus N} |f| + \liminf_{k\to\infty} \int_{A\setminus N} |f_k| + \liminf_{k\to\infty} \left(-\int_{A\setminus N} |f_k - f| \right) \\
&= 2\int_{A\setminus N} |f| + \liminf_{k\to\infty} \left(-\int_{A\setminus N} |f_k - f| \right).
\end{aligned}$$

Thus, it follows that

$$\liminf_{k\to\infty} \left(-\int_{A\setminus N} |f_k - f| \right) \ge 0 \iff -\liminf_{k\to\infty} \left(-\int_{A\setminus N} |f_k - f| \right) = \limsup_{k\to\infty} \int_{A\setminus N} |f_k - f| \le 0,$$

since $-\liminf_{k\to\infty}(-x_k) = \limsup_{k\to\infty} x_n$ for any sequence x_n of reals. Thus, this proves $(*)$ and hence completes the proof. ■

Problem 6. For each $\lambda \in (-\infty, 1)$, find

$$\lim_{k\to\infty} \int_{(0,k)} \left(1 - \frac{x}{k}\right)^k e^{\lambda x}.$$

Proof.
Fix $\lambda \in (-\infty, 1)$. We first note that the sequence $(1-(x/k))^k$ converges to e^{-x} as $k \to \infty$ and is monotonically increasing (cf. Lemma 6.1). Thus, for each fixed $k \in \mathbb{N}$ and $x \in (0, \infty)$,

$$|f_k(x)| = \left(1 - \frac{x}{k}\right)^k e^{\lambda x} < e^{-x} e^{\lambda x} = e^{x(\lambda-1)}.$$

We must show that $e^{x(\lambda-1)}$ is integrable in order to apply the dominated convergence theorem. To do so, we calculate

$$\int_{(0,\infty)} e^{x(\lambda-1)} = \lim_{t\to\infty} \int_{(0,t)} e^{x(\lambda-1)} = \lim_{t\to\infty} \int_{[0,t]} e^{x(\lambda-1)},$$

since $\{0, t\}$ is finite and hence of measure 0, by lecture we have

$$\int_{(0,t)} e^{x(\lambda-1)} = \int_{(0,t)} e^{x(\lambda-1)} + \underbrace{\int_{\{0,t\}} e^{x(\lambda-1)}}_{=0}.$$

Thus, by lecture, we can take the Reimann integral to find the Lebesgue integral $\int_{(0,t)} e^{x(\lambda-1)}$ (since $\lambda < 1$ we have that $e^{x(\lambda-1)}$ is bounded and continuous, and hence Reimann integrable on [0, t] for $t \in \mathbb{R}$ with $t > 0$). Using calculus,

$$\int_0^t e^{x(\lambda-1)} = \left[\frac{e^{(\lambda-1)x}}{\lambda - 1}\right]_0^t = \frac{e^{t(\lambda-1)} - e^0}{\lambda - 1} = \frac{e^{t(\lambda-1)} - 1}{\lambda - 1}, \qquad \text{(this is the Reimann integral)}$$

so that

$$\int_{(0,\infty)} e^{x(\lambda-1)} = \lim_{t\to\infty} \int_{[0,t]} e^{x(\lambda-1)} = \lim_{t\to\infty} \frac{e^{t(\lambda-1)} - 1}{\lambda - 1} = \lim_{t\to\infty} \frac{\frac{1}{e^{t(1-\lambda)}} - 1}{\lambda - 1} = -\frac{1}{\lambda - 1} < \infty,$$

since $1 - \lambda > 0$ so that $e^{t(1-\lambda)} \to \infty$ as $t \to \infty$; hence $t/e^{t(1-\lambda)} \to 0$ as $t \to \infty$.

Therefore, $e^{x(\lambda-1)}$ is integrable. Furthermore, since for $x \in (0, \infty)$ we have

$$\lim_{k\to\infty} ((1 - (x/k))^k e^{\lambda x}) = e^{\lambda x} \lim_{k\to\infty} (1 - (x/k))^k = e^{\lambda x} \cdot e^{-x}.$$

Thus, by the dominated convergence theorem, we have that

$$\lim_{k\to\infty} \int_{(0,k)} \left(1 - \frac{x}{k}\right)^k e^{\lambda x} = \int_{(0,\infty)} \left(\lim_{k\to\infty} \left(1 - \frac{x}{k}\right)^k e^{\lambda x}\right) = \int_{(0,\infty)} e^{x(\lambda-1)} = -\frac{1}{\lambda - 1}.$$

Therefore, we conclude that for each $\lambda \in (-\infty, 1)$,

$$\boxed{\lim_{k\to\infty} \int_{(0,k)} \left(1 - \frac{x}{k}\right)^k e^{\lambda x} = \frac{1}{1 - \lambda}}$$

■

Lemma 6.1. *The sequence defined by* $x_k \coloneqq (1-\frac{x}{k})^k$ *is increasing and converges to* e^{-x} *for* $x \in \mathbb{R}$.

Proof. The convergence of x_k is trivially proven using techniques from calculus. Let $f(k) = (1-\frac{x}{k})^k$ for each $k \in \mathbb{R}$. Then for each $k \in \mathbb{N}$, $x_k = f(k)$ so it remains to be shown that $\lim_{k\to\infty} f(k) = e^{-x}$. If we let $L = \lim_{k\to\infty} f(k)$, by the continuity of the natural logarithm, we obtain

$$\begin{aligned}\ln L &= \ln \lim_{k\to\infty} \left(1-\frac{x}{k}\right)^k = \lim_{k\to\infty} \ln\left(\left(1-\frac{x}{k}\right)^k\right) = \lim_{k\to\infty} \frac{\ln\left(1-\frac{x}{k}\right)}{\frac{1}{k}} \\ &= \lim_{k\to\infty} \frac{\frac{\frac{x}{k^2}}{1-\frac{x}{k}}}{\frac{-1}{k^2}} = \lim_{k\to\infty} \frac{-x}{1-\frac{x}{k}} = -x, && \text{(Indt. } \tfrac{0}{0} \text{ (H))}\end{aligned}$$

so that $\ln L = -x \implies e^{\ln L} = L = e^{-x}$ as needed. It remains to be shown that x_k is increasing, that is, that for each $k \in \mathbb{N}$, $x_{k+1}/x_k \geq 1$. Let $k \in \mathbb{N}$ be fixed. Then,

$$\begin{aligned}\frac{x_{k+1}}{x_k} &= \frac{\left(1-\frac{x}{k+1}\right)^{k+1}}{\left(1-\frac{x}{k}\right)^k} = \left(\frac{1-\frac{x}{k+1}}{1-\frac{x}{k}}\right)^{k+1}\left(1-\frac{x}{k}\right) = \left(\frac{\frac{k+1-x}{k+1}}{\frac{k-x}{k}}\right)^{k+1}\left(1-\frac{x}{k}\right) \\ &= \left(\frac{k(k+1-x)}{(k+1)(k-x)}\right)^k\left(1-\frac{x}{k}\right) = \left(\frac{k^2+-kx+k-x+x}{(k+1)(k-x)}\right)^{k+1}\left(1-\frac{x}{k}\right) \\ &= \left(\frac{(k+1)(k-x)+x}{(k+1)(k-x)}\right)^{k+1}\left(1-\frac{x}{k}\right) = \left(1+\frac{x}{k-x}\cdot\frac{1}{k+1}\right)^{k+1}\left(1-\frac{x}{k}\right) \\ &\geq \left(1+\frac{x(k+1)}{(k-x)(k+1)}\right)\left(1-\frac{x}{k}\right) = \left(1+\frac{x}{k-x}\right)\left(1-\frac{x}{k}\right) \\ & \qquad\qquad \text{(Bernoulli inequality with } r=k+1,\ x=\tfrac{x}{(k-x)(k+1)}) \\ &= 1+\frac{x}{k-x}-\frac{x}{k}-\frac{x^2}{k(k-x)} = 1+\frac{kx-x(k-x)-x^2}{k(k-x)} = 1.\end{aligned}$$

Thus x_k is increasing and converges to e^{-x} as needed. ■