Definition. A function $f : [a, b] \to \mathbb{R}$ is of **bounded variation** if

$$T_f(a,b) \coloneqq \sup \left\{ \sum_{i=1}^{k} |f(x_i) - f(x_{i-1})| : a = x_1 < \cdots < x_k = b \right\} < \infty.$$

Problem 1. Show that for every pair of functions $f, g : [a, b] \to \mathbb{R}$ of bounded variation, the functions $f + g$ and fg are of bounded variation on $[a, b]$.

Proof. Let $a = x_1 < \cdots < x_k = b$ be a subdivision of $[a, b]$. Then

$$\begin{aligned}
\sum_{i=1}^{k} |(f+g)(x_i) - (f+g)(x_{i-1})| &= \sum_{i=1}^{k} |f(x_i) + g(x_i) - f(x_{i-1}) - g(x_{i-1})| \\
&\leq \sum_{i=1}^{k} (|f(x_i) - f(x_{i-1})| + |g(x_i) - g(x_{i-1})|) \\
&= \sum_{i=1}^{k} |f(x_i) - f(x_{i-1})| + \sum_{i=1}^{k} |g(x_i) - g(x_{i-1})| \\
&\leq T_f(a,b) + T_g(a,b) < \infty.
\end{aligned}$$

Moreover,

$$\begin{aligned}
\sum_{i=1}^{k} |(fg)(x_i) - (fg)(x_{i-1})| &= \sum_{i=1}^{k} |f(x_i)g(x_i) - f(x_{i-1})g(x_{i-1})| \\
&= \sum_{i=1}^{k} |g(x_i)(f(x_i) - f(x_{i-1})) + f(x_{i-1})(g(x_i) - g(x_{i-1}))| \\
&\leq \sum_{i=1}^{k} |g(x_i)(f(x_i) - f(x_{i-1}))| + \sum_{i=1}^{k} |f(x_{i-1})(g(x_i) - g(x_{i-1}))| \\
&\leq \sup_{1 \leq i \leq k} |g(x_i)| \sum_{i=1}^{k} |f(x_i) - f(x_{i-1})| + \sup_{1 \leq i \leq k} |f(x_{i-1})| \sum_{i=1}^{k} |g(x_i) - g(x_i - 1)| \\
&\leq \max\{ \sup_{1 \leq i \leq k} |g(x_i)|, \sup_{1 \leq i \leq k} |f(x_{i-1})| \} \left(\sum_{i=1}^{k} |f(x_i) - f(x_{i-1})| + \sum_{i=1}^{k} |g(x_i) - g(x_{i-1})| \right) \\
&\leq \max\{ \sup_{1 \leq i \leq k} |g(x_i)|, \sup_{1 \leq i \leq k} |f(x_{i-1})| \} (T_f(a,b) + T_g(a,b)) < \infty.
\end{aligned}$$

Therefore, if $\ell \coloneqq T_{f+g}(a,b)$, then for every $\varepsilon > 0$ there exists a subdivision $a = x_1 < \cdots < x_k = b$ of $[a, b]$ such that

$$\ell < \sum_{i=1}^{k} |(f+g)(x_i) - (f+g)(x_{i-1})| + \varepsilon < \infty$$

so that $f + g$ is of bounded variation. Likewise, if $\jmath \coloneqq T_{fg}(a,b)$, then there is a subdivision such that

$$\jmath < \sum_{i=1}^{k} |(fg)(x_i) - (fg)(x_{i-1})| + \varepsilon < \infty.$$

Therefore, fg is also of bounded variation, thereby completing the proof. ∎

Problem 3. Let $f : [a, b] \to \mathbb{R}$ be a function that fails to be of bounded variation on $[a, b]$. Show that there exists a point $x_0 \in [a, b]$ such that for every $\delta > 0$, f fails to be of bounded variation on the interval $[x_0 - \delta, x_0 + \delta] \cap [a, b]$.

Proof. Suppose by contradiction that for every $x \in [a, b]$ there exists $\delta_x > 0$ such that f is of bounded variation on $[x - \delta, x + \delta] \cap [a, b]$. By the compactness of $[a, b]$,

$$[a,b] \subseteq \bigcup_{x \in [a,b]} (x - \delta_x, x + \delta_x) \implies \exists K_0 \in \mathbb{N} : [a,b] \subseteq \bigcup_{k=1}^{K_0} (x_k - \delta_{x_k}, x_k + \delta_{x_k}) \subseteq \bigcup_{k=1}^{K_0} [x_k - \delta_{x_k}, x_k + \delta_{x_k}] \cap [a,b].$$

For $1 \leq k \leq K_0$ define $F_k \coloneqq [x_k - \delta_{x_k}, x_k + \delta_{x_k}] \cap [a, b]$ and $f_k : [a, b] \to \mathbb{R}$ by $f_k = f\chi_{F_k}$. Then f_k is of bounded variation on F_k by hypothesis, but it is also of bounded variation on $[a, b]$ since it vanishes on $[a, b] \setminus F_k$. Using (1), $f = \sum_{k=1}^{K_0} f_k$ is of bounded variation on $[a, b]$, a contradiction. ∎

Problem 4. Let f be of bounded variation on $[a, b]$ and define $v(x) = T_f(a, x)$ for all $x \in [a, b]$. Show that $|f'| \leq v'$ a.e. on $[a, b]$ and infer from which that

$$\int_{[a,b]} |f'| \leq T_f(a, b).$$

Proof. For all reals $x, y \in [a, b]$ with $x < y$, $|f(y) - f(x)| \leq T_f(x, y) = v(y) - v(x)$. Thus,

$$\frac{|f(x) - f(y)|}{y - x} \leq \frac{v(x) - v(y)}{y - x}.$$

Sending $y \to x$, it follows that $|f'| \leq v'$, whenever f' and v' exist. Since f is of bounded variation, f' exists *a.e.* in (a, b), and since v is monotone increasing, v' exists a.e. in (a, b). Thus, $|f'| \leq v'$ a.e. in $[a, b]$.

Since v is increasing, so is v', thus by monotonicity,

$$\int_{[a,b]} |f'| \leq \int_{[a,b]} v' \leq v(b) - v(a) = v(b) = T_f(a, b).$$

∎