Problem 1. Let $(A_k)_{k\in\mathbb{N}}$ be a sequence of measurable sets such that

$$\sum_{k=1}^{\infty} m(A_k) < \infty.$$

Let A be the set of all $x \in \mathbb{R}^d$ such that $x \in A_k$ for infinitely many k. Show that A is measurable and $m(A) = 0$.

Proof. It suffices to show that $m_*(A) = 0$. By lemma 1.1, since $\sum_{k=1}^{\infty} m(A_k) = \ell < \infty$ for some $\ell \in \mathbb{R}_{\geq 0}$, given $\varepsilon > 0$ there exists an $N \in \mathbb{N}$ such that $\sum_{k=N}^{\infty} m(A_k) < \varepsilon$.

Now we claim that $A \subseteq \bigcup_{k=N}^{\infty} A_k$. Indeed, if we fix $x \in A$ and suppose towards contradiction that $x \notin \bigcup_{k=N}^{\infty} A_k$, then x can only be an element of at most each A_k with $1 \leq k \leq N-1$, a contradiction to the choice of A as $x \in A_k$ for only finitely many k.

Thus, by monotonicty and sub-additivity,

$$m_*(A) \leq m_*\left(\bigcup_{k=N}^{\infty} A_k\right) \leq \sum_{k=N}^{\infty} m(A_k) < \varepsilon.$$

Since ε was arbitrary, we conclude that $m_*(A) = 0$. Thus, since we proved in lecture that sets of outer measure zero are measurable, we conclude that A is measurable with $m(A) = 0$.

■

Lemma 1.1. Let $(a_n)_{n\in\mathbb{N}}$ be a sequence of non-negative reals with $\sum_{n=1}^{\infty} a_n = \ell < \infty$ for some $\ell \in \mathbb{R}_{\geq 0}$. Then for every $\varepsilon > 0$ there exists an $M \in \mathbb{N}$ such that $\sum_{n=M}^{\infty} a_n < \varepsilon$.

Let $(a_n)_{n\in\mathbb{N}}$ be given as above. Since $\sum_{n=1}^{\infty} a_n = \ell \in \mathbb{R}_{\geq 0}$, we have that

$$\lim_{N\to\infty} \sum_{n=1}^{N} a_n = \ell.$$

Thus, given $\varepsilon > 0$ there is an $M - 1 \in \mathbb{N}$ such that

$$\begin{aligned} \left| \ell - \sum_{n=1}^{M-1} a_n \right| &= \ell - \sum_{n=1}^{M-1} a_n && (a_n \geq 0 \ \forall n \geq 1) \\ &= \ell - a_1 - a_2 - \cdots - a_{M-1} = \sum_{n=M}^{\infty} a_n < \varepsilon, \end{aligned}$$

as required. ■

<u>Problem 2.</u> Let $A \subseteq \mathbb{R}^d$ be such that $m_*(A) < \infty$. Show that A is non-measurable if and only if there exists a G_δ set G such that $A \subseteq G$, $m(G) = m_*(A)$, and $m_*(G \setminus A) > 0$.

Proof.
[$\Longrightarrow$] Let $A \subseteq \mathbb{R}^d$ be a non-measurable set with $m_*(A) < \infty$. We construct an appropriate G_δ set G.

We use the following fact from lecture: $m_*(A) = \inf\{m_*(\mathcal{O}) : A \subseteq \mathcal{O},\ \mathcal{O} \text{ open}\}$. Thus, by definition of the infimum, given $\varepsilon > 0$ there exists an open set $\mathcal{O}_\varepsilon$ containing A such that $m_*(\mathcal{O}_\varepsilon) < m_*(A) + \varepsilon$. We use this to define a sequence $\{\mathcal{O}_n : n \in \mathbb{N}_+\}$ of open sets containing A. Let $\mathcal{O}_1$ be an open set such that $m_*(\mathcal{O}_1) < m_*(A) + 1$; more generally, for any $n \geq 1$, define $\mathcal{O}_n$ to be an open set such that $m_*(\mathcal{O}_n) < m_*(A) + \frac{1}{n}$. We define our G_δ set by

$$G \coloneqq \bigcap_{n=1}^{\infty} \mathcal{O}_n.$$

Property 1. $A \subseteq G$. This is trivial, since by construction, for each $n \geq 1$, $A \subseteq \mathcal{O}_n$, we must have that $A \subseteq \bigcap_{n=1}^{\infty} \mathcal{O}_n = G$.

Property 2. $m(G) = m_*(A)$. Since $A \subseteq G$, by monotonicity, $m_*(A) \leq m_*(G) = m(G)$. It remains to be shown that $m(G) \leq m_*(A)$. Let $\varepsilon > 0$ be fixed. By Archimedeanity, there is an $m \in \mathbb{N}$ with $1/m < \varepsilon$ and hence a corresponding open set $\mathcal{O}_m$ from the sequence such that $m_*(\mathcal{O}_m) < m_*(A) + \frac{1}{m} < m_*(A) + \varepsilon$. Since $G = \bigcap_{n=1}^{\infty} \mathcal{O}_n \subseteq \mathcal{O}_m$, monotonicity imples that $m(G) = m_*(G) \leq m_*(\mathcal{O}_m) < m_*(A) + \varepsilon$. Hence $m(G) < m_*(A) + \varepsilon$ for every $\varepsilon > 0$. Since ε was arbitrary, we conclude that $m(G) \leq m_*(A)$ and hence $m(G) = m_*(A)$.

Property 3. $m_*(G \setminus A) > 0$. Since A is non-measurable, there exists an $\varepsilon > 0$ such that for every open set $\mathcal{O} \subseteq \mathbb{R}^d$ with $A \subseteq \mathcal{O}$, $m_*(\mathcal{O} \setminus A) > \varepsilon$. For this ε, since $m_*(G \setminus A) = \inf\{m_*(\mathcal{O}) : G \setminus A \subseteq \mathcal{O}, \mathcal{O} \text{ open}\}$, there is an open set $\mathcal{O}$ such that $G \setminus A \subseteq \mathcal{O}$ and $m_*(\mathcal{O}) < m_*(G \setminus A) + \varepsilon$. But then $\mathcal{O} \setminus A \subseteq \mathcal{O}$, thus by monotonicity

$$m_*(\mathcal{O} \setminus A) \leq m_*(\mathcal{O}) < m_*(G \setminus A) + \varepsilon.$$

But A being non-measurable implies $m_*(\mathcal{O} \setminus A) > \varepsilon$, so we obtain

$$\varepsilon < m_*(\mathcal{O} \setminus A) < m_*(G \setminus A) + \varepsilon \implies m_*(G \setminus A) > 0,$$

subtracting ε on both sides. Thus, there is a G_δ set G such that $A \subseteq G,\ m(G) = m_*(A),$ and $m_*(G \setminus A) > 0$, thereby completing the forward implication.

[$\Longleftarrow$] Suppose there is a G_δ set G such that $A \subseteq G,\ m(G) = m_*(A),$ and $m_*(G \setminus A) > 0$. We must show that A is not measurable. Hence, suppose towards contradiction that A is measurable. Thus, since $A \subseteq G$, we can write $G = (G \setminus A) \cup A$, a disjoint union, so that $m(G) = m((G \setminus A) \cup A) = m(G \setminus A) + m(A)$, which holds by the finite case of countable additivity.[1] Hence $m_*(G \setminus A) = m(G \setminus A) = m(G) - m(A) = 0$, since $m(G) = m_*(A) = m(A)$ by supposition; note that we can subtract $m(A)$ on both sides since we are given that $m_*(A) = m(A) = m(G) < \infty$. Thus, $m_*(G \setminus A) = 0$, a contradiction. Thus, A is not measurable. ∎

[1] Note that since G and A are measurable, $G \setminus A = G \cap A^c$ is measurable as finite intersections of measurable sets are measurable and complements of measurable sets are measurable.

<u>Problem 3.</u> Let $A, B \subseteq \mathbb{R}^d$ and $A + B := \{x + y : x \in A, y \in B\}$.

1. Show that if A is closed and B is compact, then $A + B$ is closed.

2. If A and B are closed, show that $A + B$ is measurable.

Proof of 1. Let $(z_n)_{n\in\mathbb{N}}$ be a sequence in $A + B$, with $z_n \to z \in \mathbb{R}^d$ as $n \to \infty$. Then for each $n \geq 1$ we can write $z_n := a_n + b_n$, where $(a_n)_n \subseteq A$ and $(b_n)_n \subseteq B$. To show that $A + B$ is closed, it suffices to prove that $z \in A + B$, as this would show that $A + B$ contains all of its cluster points, since (z_n) was an arbitrary sequence.

Since B is compact, it is sequentially compact, thus there is a subsequence $(b_{n_k})_{k\in\mathbb{N}}$ of (b_n) such that $b_{n_k} \to b \in B$ as $k \to \infty$. Since $z_n = a_n + b_n \to z$ as $n \to \infty$, we must have that all of its subsequences do too. In particular, $z_{n_k} = a_{n_k} + b_{n_k} \to z$ as $k \to \infty$. Thus, $a_{n_k} + b \to z$ as $k \to \infty$, so that $a_{n_k} \to z - b$ as $k \to \infty$.[2] Hence, we have found a convergent sequence $(a_{n_k})_{k\in\mathbb{N}} \subseteq A$; by closedness, $a_{n_k} \to z - b \in A$. Thus, $b \in B$ and $z - b \in A$ means $z_{n_k} \to (z - b) + b = z \in A + B$, and $z_n \to z$ as well, hence $z_n \to z \in A + B$. Therefore, $A + B$ is closed, since $(z_n)_{n\in\mathbb{N}}$ was an arbitrary sequence. ■

Proof of 2. We note that the result holds whenever B is compact. Indeed, this implies by (1) that $A + B$ is closed; since closed sets are measurable, $A + B$ is measurable as needed. Let $n \geq 1$ be a fixed positive integer. Then $B_n := B \cap [-n, n]^d$ is compact, since $B \cap [-n, n]^d \subseteq [-n, n]^d$ is a closed subset of a compact set (since finite intersections of closed sets are closed; cf. Lemma 3.1). Thus, $A + B_n$ is closed and hence measurable. Since we can write

$$A + B = \bigcup_{n=1}^{\infty} (A + B_n),$$

as a countable union of measurable sets (cf. Lemma 3.2), we conclude that $A + B$ is measurable. ■

Lemma 3.1. Let $X \subseteq \mathbb{R}^d$ be compact, with a closed subset $Y \subseteq X$. Then Y is compact.

Let $\{U_\alpha\}_{\alpha\in I}$ be a collection of open sets for some index set I such that $Y \subseteq \bigcup_{\alpha\in I} U_\alpha$ is an open cover of Y. Since Y is closed, Y^c is open, so that $X \subseteq \bigcup_{\alpha\in I} U_\alpha \cup Y^c$ is an open cover of X. Indeed, if $x \in X$, then if $x \in Y \implies \exists\, \alpha \in I : x \in U_\alpha \subseteq \bigcup_{\alpha\in I} U_\alpha \cup Y^c$ and if $x \notin Y \implies x \in Y^c \subseteq \bigcup_{\alpha\in I} U_\alpha \cup Y^c$.

By compactness, there exists an $N \in \mathbb{N}$ such that $X \subseteq \bigcup_{k=1}^{N} U_{\alpha_k} \cup Y^c$. But then $Y \subseteq \bigcup_{k=1}^{N} U_{\alpha_k}$, since Y^c does not contribute to the cover of Y trivially. Thus, since $\{U_\alpha\}_{\alpha\in I}$ was arbitrary, we conclude that Y is compact. ■

Lemma 3.2. Let $A, B \subseteq \mathbb{R}^d$. Then $A + B = \bigcup_{n=1}^{\infty}(A + B_n)$, where $B_n := B \cap [-n, n]^d$.

[$\subseteq$] Let $x = a + b = (a_1 + b_1, \ldots, a_d + b_d) \in A + B$ be arbitrary. Since $\mathbb{R}^d = \bigcup_{n=1}^{\infty}[-n, n]^d$, there exists an $n \geq 1$ such that $b \in [-n, n]^d$. Hence, $a + b \in A + B_n$ as $b \in B \cap [-n, n]^d$; thus $x \in \bigcup_{n=1}^{\infty}(A + B_n)$. Hence $A + B \subseteq \bigcup_{n=1}^{\infty}(A + B_n)$.

[$\supseteq$] Let $x = a + b = (a_1 + b_1, \ldots, a_d + b_d) \in \bigcup_{n=1}^{\infty}(A + B_n)$. Then there exists an $n \geq 1$ such that $a + b \in A + B_n \implies a \in A$ and $b \in B \cap [-n, n]^d \subseteq B \implies a + b \in A + B$. Hence $\bigcup_{n=1}^{\infty}(A + B_n) \subseteq A + B$.

By definition of set equality, it follows that $A + B = \bigcup_{n=1}^{\infty}(A + B_n)$, thereby completing the proof. ■

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[2] Indeed, since $b_{n_k} \to b$ and $a_{n_k} + b_{n_k} \to z$ as $k \to \infty$, given $\varepsilon > 0$, there is an $M_1 \in \mathbb{N}$ such that $m > M_1$ implies $d(a_{n_m} + b_{n_m}, z) < \varepsilon/2$ and an $M_2 \in \mathbb{N}$ such that $m > M_2$ implies $d(b_{n_m}, b) = d(b, b_{n_m}) < \varepsilon/2$, so that for $m > M := \max\{M_1, M_2\}$, $d(a_{n_m} + b, z) \leq d(a_{n_m} + b, a_{n_m} + b_{n_m}) + d(a_{n_m} + b_{n_m}, z) = d(b, b_{n_m}) + d(a_{n_m} + b_{n_m}, z) < 2 \cdot \varepsilon/2 = \varepsilon$, since $d(a_{n_m} + b_{n_m}, a_{n_m} + b) = \sqrt{\sum_{i=1}^{d}(a_{n_m i} + b_{n_m i} - a_{n_m i} - b_i)^2} = \sqrt{\sum_{i=1}^{d}(b_{n_m i} - b_i)^2} = d(b_{n_m}, b)$. It follows that $a_{n_k} \to z - b$ as $k \to \infty$, as $m > M \implies d(a_{n_m} + b, z) = \sqrt{\sum_{i=1}^{d}(a_{n_m i} + b_i - z_i)^2} = \sqrt{\sum_{i=1}^{d}(a_{n_m i} - (z_i - b_i))^2} = d(a_{n_m}, z - b) < \varepsilon$.

Problem 4

1. Show that a strictly increasing function that is defined on an interval has a continuous inverse.
2. Let A and B be two Borel sets of $\mathbb{R}$ and $f : A \to \mathbb{R}$ be a continuous function. Show that $f^{-1}(B)$ is a Borel set.

   Hint. Show that the collection of sets B where $f^{-1}(B)$ is Borel is a σ-algebra containing the open sets.
3. Use (1) and (2) to show that a strictly increasing continuous function defined on an interval maps Borel sets to Borel sets.

Proof of 1. Let $f : I \to \mathbb{R}$ be a strictly increasing function, where I is some interval. This means that for any $x, y \in I$ with $x < y$, $f(x) < f(y)$. Define

$$f^{-1} : f(I) \to I \text{ by } f(i) \xmapsto{f^{-1}} i,$$

for $i \in I$ (this function is well-defined by injectivity, which we prove in footnote 3). We show that f^{-1} is continuous by showing that if $\mathcal{O} \subseteq I$ is open, then its pre-image $(f^{-1})^{-1}(\mathcal{O})$ is open. But notice that if $f(x) \in (f^{-1})^{-1}(\mathcal{O}) \iff f^{-1}(f(x)) \in \mathcal{O} \iff x \in \mathcal{O} \iff f(x) \in f(\mathcal{O})$, where the second to last $\iff$ holds since f is injective.[3] This means that $(f^{-1})^{-1}(\mathcal{O}) = f(\mathcal{O})$, as these sets are subsets of each other. Thus, to show that f^{-1} is continuous, it suffices to show that for any open subset $\mathcal{O} \subseteq I$, $f(\mathcal{O})$ is open.

To this end, let $\mathcal{O} \subseteq I$ be any open set, with $x \in \mathcal{O}$. Then there exists an $\varepsilon > 0$ such that $V_\varepsilon(x) = (x-\varepsilon, x+\varepsilon) \subseteq \mathcal{O}$. Then $f(V_\varepsilon(x)) \subseteq f(\mathcal{O})$; but notice that for any $y \in V_\varepsilon(x)$, $f(y) \in f(V_\varepsilon(x)) \implies f(x-\varepsilon) < f(y) < f(x+\varepsilon)$, since f is strictly increasing. Thus, $f(x) \in (f(x-\varepsilon), f(x+\varepsilon)) \subseteq f(\mathcal{O})$, an open interval, hence we conclude that $f(\mathcal{O})$ is open.

Thus, we have shown that the preimages of open sets under f^{-1} are open. Equivalently, this means that f has a continuous inverse, as was to be shown. ■

Proof of 2. Let $A, B \subseteq \mathbb{R}$ be two Borel sets and $f : A \to \mathbb{R}$ be a continuous function. We now use lemma 4.1 to complete the proof.

By definition, the Borel σ-algebra is the intersection of all σ-algebras containing the open sets in $\mathbb{R}^d$ ($d = 1$ in this case). Lemma 4.1 implies that Ω, the σ-algebra of sets B such that $f^{-1}(B)$ is a Borel set is a σ-algebra containing the open sets. Thus, the Borel σ-algebra is a subset of Ω, since the Borel σ-algebra is the smallest σ-algebra containing the open sets by definition. Hence, if B is a Borel set, then $B \in \Omega$, which means that $f^{-1}(B)$ is a Borel set. Since B was arbitrary, the proof is complete. ■

Proof of 3. Let $f : I \to \mathbb{R}$ be a strictly increasing continuous function, where I is an interval. Let $B \subseteq I$ be any Borel set. We must show that $f(B)$ is a Borel set.

Since f is strictly increasing, by (1) we have that $f^{-1} : f(I) \to I$ is continuous, where $f^{-1}(f(i)) = i$ for $i \in I$. Note also that since f is strictly increasing, $f(I)$ is an interval since I is. Hence, by lemma 4.2, $f(I)$ is a Borel set. Define $g : f(I) \to \mathbb{R}$, where $g(f(i)) = f^{-1}(f(i))$ for each $i \in I$. Then g is also continuous, since $g(x) = f^{-1}(x)$ for each $x \in f(I)$ (i.e. we just extended the codomain of f^{-1}). Thus, by (2), $g^{-1}(B) = (f^{-1})^{-1}(B)$ is a Borel set. But $(f^{-1})^{-1}(B) = f(B)$. Indeed, we showed that this is true in (1), since f is injective. Thus, $f(B)$ is a Borel set, as needed. ■

[3] f is clearly injective as if $x, y \in I$ are such that $x \neq y \implies x < y$ or $x > y$; in either case, $f(x) < f(y)$ or $f(x) > f(y)$ respectively, so that $f(x) \neq f(y)$, as needed. Of course, $\mathcal{O} \subseteq f^{-1}(f(\mathcal{O}))$, since if $x \in \mathcal{O} \implies f(x) \in f(\mathcal{O}) \implies x \in f^{-1}(f(\mathcal{O}))$. Conversely, suppose $x \in f^{-1}(f(\mathcal{O})) \implies f(x) \in f(\mathcal{O})$. Thus, there exists an $o \in \mathcal{O}$ such that $f(x) = f(o) \implies x = o$ by injectivity so that $x \in \mathcal{O}$ as required.

Lemma 4.1. Let $A \subseteq \mathbb{R}$ be a Borel set and $f : A \to \mathbb{R}$ be a continuous function. Then the collection of sets B where $f^{-1}(B)$ is a Borel set is a σ-algebra containing the open sets.

Denote this collection of such sets B by Ω. $\mathbb{R} \in \Omega$ since $f^{-1}(\mathbb{R}) = A$ is a Borel set. Suppose $E, F \in \Omega$; then both $f^{-1}(E), f^{-1}(F)$ are Borel sets so that $f^{-1}(F) \setminus f^{-1}(E)$ is a Borel set (since $f^{-1}(E), f^{-1}(F)$ are in the Borel σ-algebra), yet $f^{-1}(F) \setminus f^{-1}(E) = f^{-1}(F \setminus E)$ so that $F \setminus E \in \Omega$.[4] Finally, if $\{A_k\}_{k\in\mathbb{N}} \subseteq \Omega$, then $\{f^{-1}(A_k)\}_{k\in\mathbb{N}}$ is a sequence of Borel sets so that

$$\bigcup_{k=1}^{\infty} f^{-1}(A_k) = f^{-1}\left(\bigcup_{k=1}^{\infty} A_k\right)$$

is a Borel set (as each A_k belongs to the Borel σ-algebra) so that $\bigcup_{k=1}^{\infty} A_k \in \Omega$.[5] Thus Ω is a σ-algebra and it remains to be shown that if $\mathcal{O} \subseteq \mathbb{R}$ is an open set, then $\mathcal{O} \in \Omega$. But this is clear: by the continuity of f, $f^{-1}(\mathcal{O})$ is open and hence a Borel set so that $\mathcal{O} \in \Omega$ as required, completing the lemma. ■

Lemma 4.2. Let $I \subseteq \mathbb{R}$ be any interval. Then I is a Borel set.

To prove this lemma, we use the following two facts:

1. *Singleton sets are Borel sets.* Let $a \in \mathbb{R}$. Then $\{a\}$ is closed so that $\{a\}^c$ is open and hence a Borel set. Thus, $\mathbb{R} \setminus \{a\}^c = \{a\}$ is a Borel set by the difference property of the Borel σ-algebra.

2. *Finite unions of Borel sets are Borel sets.* This follows immediately from the countable-union property of the Borel σ-algebra, i.e. if $A_1, \ldots, A_N$ are Borel sets for some $N \in \mathbb{N}$, then $A_1 \cup A_2 \cup \cdots \cup A_N = \bigcup_{i=1}^{\infty} B_i$, where $1 \le i \le N \implies B_i = A_i$ and $i > N \implies B_i = \emptyset$ is a Borel set, since countable unions of Borel sets are Borel sets.

Now, we have the following cases. Let $a, b \in \mathbb{R}$ with $a < b$.

1. If $I = \mathbb{R}$ then I is a Borel set by definition of the Borel σ-algebra. If $I = (-\infty, b)$ or (a, ∞), then I is open and hence a Borel set. If $I = (-\infty, b]$ or $I = [a, \infty)$, then facts 1 and 2 imply that $I = (-\infty, b) \cup \{b\}$ and $I = (a, \infty) \cup \{a\}$ are Borel sets.

2. $I = [a, b] = \{a\} \cup (a, b) \cup \{b\}$, or $I = (a, b] = (a, b) \cup \{b\}$, or $I = [a, b) = \{a\} \cup (a, b)$. In all of these cases, facts 1 and 2 imply that I is a Borel set.

Since we have covered all possible cases, we conclude that I is a Borel set. ■

[4] Indeed, $x \in f^{-1}(F) \setminus f^{-1}(E) \iff f(x) \in F, f(x) \notin E \iff f(x) \in F \setminus E \iff x \in f^{-1}(F \setminus E)$.

[5] Certainly, $x \in \bigcup_{k=1}^{\infty} f^{-1}(A_k) \iff \exists\, m \ge 1 : x \in f^{-1}(A_m) \iff \exists\, m \ge 1 : f(x) \in A_m \iff f(x) \in \bigcup_{k=1}^{\infty} A_k \iff x \in f^{-1}(\bigcup_{k=1}^{\infty} A_k)$.

Problem 5.

1. Show that every closed subset of $\mathbb{R}^d$ is a G_δ set and every open subset of $\mathbb{R}^d$ is a F_σ set.

   Hint. If $F \subseteq \mathbb{R}^d$ is closed, consider $O_n \coloneqq \{x \in \mathbb{R} : d(x, F) < 1/n\}$.

2. Show that $\mathbb{Q}$ is an F_σ set in $\mathbb{R}$ but not a G_δ set.

   Hint. You may argue by contradiction: assume that $\mathbb{Q}$ is both an F_σ set and a G_δ, then show that there exist open sets $(O_n)_{n\in\mathbb{N}}$ which are all dense in $\mathbb{R}$ and whose intersection is empty, and finally derive a contradiction with a well known property of $\mathbb{R}$.

Proof of 1.

1.1. Closed subsets of $\mathbb{R}^d$ are G_δ sets.

Let $E \subseteq \mathbb{R}^d$ be closed. Consider for each $n \in \mathbb{N}_+$ the set $O_n \coloneqq \{x \in \mathbb{R}^d : \exists\, p \in E : d(x, p) < 1/n\}$. Clearly, for $n \geq 1$, $E \subseteq O_n$, since if $x \in E$, then $d(x, x) = 0 < 1/n$. Since $n \geq 1$ was arbitrary, $E \subseteq O \coloneqq \bigcap_{n=1}^{\infty} O_n$. Now suppose $x \in O$. Then, for $k = 1, 2, \ldots$, there exists a point $x_k \in E : d(x, x_k) < \frac{1}{k}$. Let $\varepsilon > 0$ be fixed. By Archimedeanity, choose an $N \in \mathbb{N}$ such that $1/N < \varepsilon$. Then if $n > N$, $d(x, x_n) < \frac{1}{n} < \frac{1}{N} < \varepsilon$. Thus, the sequence $(x_k)_{k\in\mathbb{N}} \subseteq E$ converges to x. Since E is closed, we must have that $x \in E$. Thus,

$$E = O = \bigcap_{n=1}^{\infty} O_n.$$

It remains to be shown that O_n is open for each $n \geq 1$. So fix $n \geq 1$ and consider $x \in O_n$. Then there exists $p \in E : d(x, p) < \frac{1}{n}$. Then $x \in V_{1/n}(p) \subseteq O_n$. Indeed, if $y \in V_{1/n}(p) \implies d(y, p) < \frac{1}{n} \implies y \in O_n$. Hence, we have written E as a countable intersection of open sets, hence E is a G_δ set.

1.2. Open subset of $\mathbb{R}^d$ are F_σ sets.

Let $U \subseteq \mathbb{R}^d$ be open. Then U^c is closed. By 1.1, we can write $U^c = \bigcap_{n=1}^{\infty} O_n$, where each O_n is some open set. But then, for $n \geq 1$, O_n^c must be closed so that

$$U = (U^c)^c = \left(\bigcap_{n=1}^{\infty} O_n\right)^c = \bigcup_{n=1}^{\infty} O_n^c,$$

via DeMorgan's law. Hence, we have written U as a countable union of closed sets O_n^c, hence U is an F_σ set.[6] ■

Proof of 2. By lemma 5.1, $\mathbb{Q}$ is an F_σ set in $\mathbb{R}$. Suppose towards contradiction that $\mathbb{Q}$ is also a G_δ set in $\mathbb{R}$. Then, there exists a sequence of open sets $\{\mathcal{U}_n : n \geq 1\}$ such that

$$\mathbb{Q} = \bigcap_{n=1}^{\infty} \mathcal{U}_n.$$

Clearly, for each $n \geq 1$, $\mathcal{U}_n$ is dense in $\mathbb{R}$. Indeed, since $\mathbb{Q}$ is dense in $\mathbb{R}$ and $\mathbb{Q} = \bigcap_{n=1}^{\infty} \mathcal{U}_n \subseteq \mathcal{U}_n$, given any $a, b \in \mathbb{R} : a < b$, there is a $q \in \mathbb{Q} \subseteq \mathcal{U}_n$ such that $a < q < b$ so that $\mathcal{U}_n$ is dense in $\mathbb{R}$. Since $\mathbb{Q}$ is countable, we can enumerate $\mathbb{Q}$ as a sequence $\{q_n : n \geq 1\}$.

For each $n \geq 1$, define the open set $\mathcal{O}_n \coloneqq \mathcal{U}_n \backslash \{q_n\}$. Then $\mathcal{O}_n$ is still dense in $\mathbb{R}$, since given $a, b \in \mathbb{R} : a < b$, there were infinitely-many $q \in \mathbb{Q}$ satisfying $a < q < b$, i.e. removing q_n is insignificant. Furthermore, $\mathcal{O}_n$ is

[6] For the sake of completeness, $U = (U^c)^c$ since $x \in U \iff x \notin U^c \iff x \in (U^c)^c$.

open since $\{q_n\}$ being closed implies that $\{q_n\}^c$ is open so that $\mathcal{O}_n \cap \{q_n\}^c$ is open because finite intersections of open sets are open. Also note that $\bigcap_{n=1}^{\infty} \mathcal{O}_n = \emptyset$; to see why, suppose otherwise: if $x \in \bigcap_{n=1}^{\infty} \mathcal{O}_n$, then since for $n \geq 1$ $\mathcal{O}_n = \mathcal{U}_n \setminus \{q_n\} \subseteq \mathcal{U}_n$, $x \in \bigcap_{n=1}^{\infty} \mathcal{U}_n = \mathbb{Q}$, a contradiction, since this means there is a $k \geq 1$ such that $x = q_k$ so that $x \notin \mathcal{O}_k = \mathcal{U}_k \setminus \{x\} \implies x \notin \bigcap_{n=1}^{\infty} \mathcal{O}_n$.

To complete the proof, we construct a sequence $\{F_j : j \geq 1\}$ of compact nested intervals.

- By construction, $\mathcal{O}_1 \neq \emptyset \implies \exists\, x \in \mathcal{O}_1$. By openness, there exists an $\varepsilon_1 > 0$ such that $(x_1 - \varepsilon_1, x_1 + \varepsilon_1) \subseteq \mathcal{O}_1$. Thus, $[x_1 - \frac{\varepsilon_1}{2}, x_1 + \frac{\varepsilon_1}{2}] \subseteq (x_1 - \varepsilon_1, x_1 + \varepsilon_1) \subseteq \mathcal{O}_1$. We let $F_1 \coloneqq [x_1 - \frac{\varepsilon_1}{2}, x_1 + \frac{\varepsilon_1}{2}] \subseteq \mathcal{O}_1$.
- For $k \geq 1$, we define F_{k+1} as follows. By the density of $\mathcal{O}_{k+1}$ in $\mathbb{R}$, there exists a point $x_{k+1} \in \mathcal{O}_{k+1}$ such that $x_{k+1} \in F_k^o$. Since a compact interval's interior is non-empty and open, there exists an $\varepsilon'_{k+1} > 0$ such that $(x_{k+1} - \varepsilon'_{k+1}, x_{k+1} + \varepsilon'_{k+1}) \subseteq F_k^o \subseteq F_k$. By the openness of $\mathcal{O}_{k+1}$, there is an $\varepsilon''_{k+1} > 0$ such that $(x_{k+1} - \varepsilon''_{k+1}, x_{k+1} + \varepsilon''_{k+1}) \subseteq \mathcal{O}_{k+1}$. Letting $\varepsilon_{k+1} \coloneqq \min\{\varepsilon'_{k+1}, \varepsilon''_{k+1}\}$, it follows that $(x_{k+1} - \varepsilon_{k+1}, x_{k+1} + \varepsilon_{k+1}) \subseteq F_k, \mathcal{O}_{k+1}$. Thus, we define $F_{k+1} \coloneqq [x_{k+1} - \frac{\varepsilon_{k+1}}{2}, x_{k+1} + \frac{\varepsilon_{k+1}}{2}] \subseteq F_k, \mathcal{O}_{k+1}$.

■

By construction, each set $(F_j)_j$ is compact and we have $F_1 \supseteq F_2 \supseteq \cdots \supseteq F_j \supseteq F_{j+1} \supseteq \cdots$. Thus, by the nested interval property of $\mathbb{R}$, there exists an $x \in \bigcap_{j=1}^{\infty} F_j$. However, by construction,

$$x \in \bigcap_{j=1}^{\infty} F_j \subseteq \bigcap_{n=1}^{\infty} \mathcal{O}_n = \emptyset,$$

which holds as $x \in \bigcap_{j=1}^{\infty} F_j \implies x \in F_j\ \forall j \geq 1$, so that $x \in F_1 \implies x \in \mathcal{O}_1$, and for each $k \geq 1$, $x \in F_k \implies x \in \mathcal{O}_k$. Thus, $x \in \bigcap_{n=1}^{\infty} \mathcal{O}_n$. But this is a contradiction as we have shown $x \in \emptyset$. Thus, $\mathbb{Q}$ is not a G_δ set.

Lemma 5.1. $\mathbb{Q}$ is an F_σ set in $\mathbb{R}$.

Write $\mathbb{Q} = \bigcup_{q \in \mathbb{Q}} \{q\}$. Since finite sets are closed and $\mathbb{Q}$ is countable, $\mathbb{Q}$ is an F_σ set in $\mathbb{R}$ since it has been written as a countable union of closed sets. ■