Problem 1. Let f be integrable over $\mathbb{R}^d$ and $\alpha > 0$. Show that

$$\int_{\mathbb{R}^d} |f(x)|\ dx = \int_0^\infty m(E_\alpha)\ d\alpha,$$

where $E_\alpha \coloneqq \{x \in \mathbb{R}^d : |f(x)| > \alpha\}$.

Proof. If $\alpha > 0$ is fixed, we note that

$$\int_{\mathbb{R}^d} \chi_{E_\alpha}(x)\ dx = m(E_\alpha),$$

which holds by the definition of the Lebesgue integral of a characteristic function. From which it follows that

$$\int_0^\infty m(E_\alpha)\ d\alpha = \int_0^\infty \left(\int_{\mathbb{R}^d} \chi_{E_\alpha(x)}\ dx \right) d\alpha.$$

Then, since characteristic functions are integrable, Fubini's theorem yields

$$\int_0^\infty \left(\int_{\mathbb{R}^d} \chi_{E_\alpha}(x)\ dx \right) d\alpha = \int_{\mathbb{R}^d} \left(\int_0^\infty \chi_{E_\alpha}(x)\ d\alpha \right) dx = \int_{\mathbb{R}^d} \left(\int_0^{|f(x)|} \chi_{E_\alpha}(x)\ d\alpha + \int_{|f(x)|}^\infty \chi_{E_\alpha}(x)\ d\alpha \right) dx,$$

breaking up the interval over which we integrate. Since $x \in E_\alpha$ if and only if $|f(x)| > \alpha$ by definition, if $|f(x)| \leq \alpha$ then $\chi_{E_\alpha}(x) = 0$. Thus, since x is fixed, $\int_{|f(x)|}^\infty \chi_{E_\alpha}(x)\ d\alpha = \int_{|f(x)|}^\infty 0\ d\alpha = 0$ (since $|f(x)| \leq \alpha$ always). Furthermore, we have

$$\begin{aligned}
\int_0^{|f(x)|} \chi_{E_\alpha}(x)\ d\alpha &= \int_{[0,|f(x)|)} \chi_{E_\alpha}(x)\ d\alpha && \text{(remove } |f(x)| \text{ from interval as } m(\{|f(x)|\}) = 0) \\
&= \int_{[0,|f(x)|)} 1\ d\alpha && \text{(as } |f(x)| > \alpha \text{ for } \alpha \in [0, |f(x)|)) \\
&= \int_{[0,|f(x)|)} \chi_{[0,|f(x)|)}(\alpha)\ d\alpha \\
&= m([0, |f(x)|)) = |f(x)| - 0 = |f(x)|,
\end{aligned}$$

by definition of the Lebesgue integral of a characteristic function. Thus,

$$\int_0^\infty m(E_\alpha)\ d\alpha = \int_{\mathbb{R}^d} \left(\int_0^{|f(x)|} \chi_{E_\alpha}(x)\ d\alpha + \int_{|f(x)|}^\infty \chi_{E_\alpha(x)}\ d\alpha \right) dx = \int_{\mathbb{R}^d} |f(x)|\ dx.$$

Therefore, $\int_{\mathbb{R}^d} |f(x)|\ dx = \int_0^\infty m(E_\alpha)\ d\alpha$ completes the proof. ∎

Problem 2. Let $f : \mathbb{R}^d \to \overline{\mathbb{R}}$ be measurable and $\Gamma = \{(x, y) \in \mathbb{R}^d \times \mathbb{R} : y = f(x)\}$. Show that Γ is measurable and $m(\Gamma) = 0$.

Proof. We show first that Γ is measurable. Define the functions $\hat{h}, \hat{g} : \mathbb{R}^d \times \mathbb{R} \to \mathbb{R}$ by

$$\hat{h} : (x, y) \mapsto f(x) \text{ and } \hat{g} : (x, y) \mapsto y.$$

Then $\hat{h}$ and $\hat{g}$ are measurable. Indeed, for every $c \in \mathbb{R}$,

$$\begin{aligned}\hat{h}^{-1}([-\infty, c)) &= \{(x, y) \in \mathbb{R}^d \times \mathbb{R} : \hat{h}(x, y) < c\} = \{(x, y) \in \mathbb{R}^d \times \mathbb{R} : f(x) < c\} \\ &= \{x \in \mathbb{R}^d : f(x) < c\} \times \mathbb{R} = f^{-1}([-\infty, c)) \times \mathbb{R}\end{aligned}$$

and

$$\hat{g}^{-1}([-\infty, c)) = \{(x, y) \in \mathbb{R}^d \times \mathbb{R} : \hat{g}(x, y) < c\} = \{(x, y) \in \mathbb{R}^d \times \mathbb{R} : y < c\} = \mathbb{R}^d \times (-\infty, c).$$

Then $\hat{h}^{-1}([-\infty, c)) = f^{-1}([-\infty, c)) \times \mathbb{R}$ is a cartesian product of measurable sets ($f^{-1}([-\infty, c))$ is measurable since f is measurable; $\mathbb{R}$ is measurable as it is a Borel set, which are measurable by lecture) and so is $\hat{g}^{-1}([-\infty, c)) = \mathbb{R}^d \times (-\infty, c)$ ($\mathbb{R}^d$ and $(-\infty, c)$ are measurable as they are Borel sets). From lecture we know that cartesian products of measurable sets are measurable so that $\hat{h}^{-1}([-\infty, c))$ and $\hat{g}^{-1}([-\infty, c))$ are measurable. Since c was arbitrary we conclude that $\hat{h}$ and $\hat{g}$ are measurable.

Since $\hat{h}$ and $\hat{g}$ are measurable and finite valued, we attain from lecture that $\hat{f} = \hat{h} - \hat{g}$ is a measurable function since it is the sum of two measurable functions.[1] Then

$$\begin{aligned}\hat{f}^{-1}(\{0\}) &= \{(x, y) \in \mathbb{R}^d \times \mathbb{R} : \hat{f}(x, y) = 0\} = \{(x, y) \in \mathbb{R}^d \times \mathbb{R} : \hat{h}(x, y) - \hat{g}(x, y) = 0\} \\ &= \{(x, y) \in \mathbb{R}^d \times \mathbb{R} : \hat{h}(x, y) = \hat{g}(x, y)\} = \{(x, y) \in \mathbb{R}^d \times \mathbb{R} : f(x) = y\} = \Gamma.\end{aligned}$$

But then $\Gamma = \hat{f}^{-1}(\{0\}) = \hat{f}^{-1}([-\infty, 0] \cap [0, \infty]) = \hat{f}^{-1}([-\infty, 0]) \cap \hat{f}([0, \infty])$ is a finite intersection of measurable sets (by the measurability of $\hat{f}$). Thus Γ is measurable.

Then, using corollary 1 of Fubini's theorem from lecture, since Γ is measurable, defining the set

$$\Gamma_x \coloneqq \{y \in \mathbb{R} : (x, y) \in \Gamma\} = \{y \in \mathbb{R} : f(x) = y\} = \{f(x)\}$$

yields

$$m(\Gamma) = \int_{\mathbb{R}} m(\Gamma_x) = \int_{\mathbb{R}} 0 = 0,$$

since singleton sets have measure 0. Therefore, this proves that Γ is measurable and of measure 0. ■

[1] For the sake of completeness, we note that $-\hat{g}$ is measurable as for $c \in \mathbb{R}$, $-\hat{g}^{-1}([-\infty, c)) = \mathbb{R}^d \times (-c, \infty)$ is the cartesian product of two Borel (and hence measurable) sets and thus is measurable.

Problem 3. Let F be a closed subset of $\mathbb{R}$ whose complement has finite measure. Let $I : \mathbb{R} \to [0, \infty]$ be the function defined for every $x \in \mathbb{R}$ by

$$I(x) = \int_{\mathbb{R}} \frac{d(y, F)}{|x - y|^2}\ dy,$$

where $d(y, F) = \inf\{|y - z| : z \in F\}$.

1. Show that $d(\cdot, F)$ is Lipshitz continuous in $\mathbb{R}$, i.e.

$$|d(x, F) - d(y, F)| \leq |x - y|\ \forall x, y \in \mathbb{R}.$$

2. Show that $I(x) = \infty$ for each $x \notin F$ and $I(x) < \infty$ for a.e. $x \in F$. You may use the results of Questions 1 and 3 in Assignment 4.

Hint for 3.2 (b). Use a double integration and observe that $F \cap (y - d(y, F), y + d(y, F)) = \emptyset$ *for every* $y \in \mathbb{R} \setminus F$.

Proof of 3.1. Let $x, y \in \mathbb{R}$ be fixed and $\varepsilon > 0$ be given. By the definition of $d(\cdot, F)$, for every $\varepsilon > 0$ there exists a $z_x \in F$ such that $d(x, F) > |x - z_x| - \varepsilon$. Using this, notice that

$$\begin{aligned} d(y, F) &\leq |y - z_x| = |y - z_x + x - x| \leq |x - y| + |x - z_x| && \text{(triangle inequality)} \\ &< |x - y| + d(x, F) + \varepsilon && \text{(definition of infimum)} \\ &\iff d(y, F) - d(x, F) < |x - y| + \varepsilon \iff d(x, F) - d(y, F) > -(|x - y| + \varepsilon). && (3.1) \end{aligned}$$

Similarly, for every $\varepsilon > 0$ there exists a $z_y \in F$ such that $d(y, F) > |y - z_y| - \varepsilon$. Using this once more, it follows that

$$\begin{aligned} d(x, F) &\leq |x - z_y| = |x - z_y + y - y| \leq |x - y| + |y - z_y| && \text{(triangle inequality)} \\ &< |x - y| + d(y, F) + \varepsilon && \text{(definition of infimum)} \\ &\iff d(x, F) - d(y, F) < |x - y| + \varepsilon. && (3.2) \end{aligned}$$

Thus, combining (3.1) and (3.2), it follows that

$$-(|x - y| + \varepsilon) < d(x, F) - d(y, F) < |x - y| + \varepsilon \iff |d(x, F) - d(y, F)| < |x - y| + \varepsilon.$$

since ε was arbitrary, we conclude that

$$|d(x, F) - d(y, F)| \leq |x - y|,$$

and since x and y were arbitrary, we conclude that $d(\cdot, F)$ is Lipschitz continuous. ■

Proof of 3.2 (a). We show that $I(x) = \infty$ for each $x \in \mathbb{R} \setminus F$. Let $\ell := d(x, F)$. Note that $\ell \neq 0$ otherwise $x \in F$ since x would be a cluster point of F (as $\forall \varepsilon > 0 : \exists z_\varepsilon \in F : |x - z_\varepsilon| < \ell + \varepsilon = \varepsilon \iff x \in V_\varepsilon(z_\varepsilon)$) and closed sets (like F) contain all of their cluster points by definition.

Let $c > 1$ be fixed. From (1), we know that if $y \in B(x, \ell/c)$ (i.e. $|x - y| < \ell/c$) then $d(y, F) \in B(d(x, F), \ell/c)$ (i.e. $|d(x, F) - d(y, F)| < \ell/c$) $\iff d(y, F) \in B(\ell, \ell/c) \iff d(y, F) \in (\ell - \ell/c, \ell + \ell/c)$ so that $d(y, F) > \ell - \ell/c$. Thus, since $B(x, \ell/c) \subseteq \mathbb{R}$ and $I \geq 0$ on $\mathbb{R}$, we have

$$I(x) = \int_{\mathbb{R}} \frac{d(y,F)}{|x-y|^2}\ dy \geq \int_{B(x,\ell/c)} \frac{d(y,F)}{|x-y|^2}\ dy > \int_{B(x,\ell/c)} \frac{\ell - \ell/c}{|x-y|^2}\ dy = (\ell - \ell/c)\int_{B(x,\ell/c)} \frac{1}{|x-y|^2}\ dy. \quad (*)$$

Now notice that $B(x, \ell/c) = B(0, \ell/c) + x = \{a + x \in \mathbb{R} : a \in B(0, \ell)\}$. Then if $f(y) = \frac{1}{|x-y|^2}$ for $y \in B(x, \ell/c)$, we define for $y \in B(0, \ell/c)$ the function $f_x(y) = f(y + x) = \frac{1}{|x-(y+x)|^2} = \frac{1}{y^2}$. Then, applying the results from Assignment 4, Question 1, we attain from $(*)$ that

$$I(x) \geq (\ell - \ell/c)\int_{B(x,\ell/c)} \frac{1}{|x-y|^2}\ dy = (\ell - \ell/c)\int_{B(0,\ell/c)} \frac{1}{y^2}\ dy = (\ell - \ell/c)\int_{[0,\ell/c]} \frac{1}{y^2}\ dy,$$

where we can add these endpoints since $m(\{0, \ell/c\}) = 0$. Then, notice that

$$\int_{[0,\ell/c]} \frac{1}{y^2}\ dy = \lim_{t\to 0^+}\int_{[t,\ell/c]} \frac{1}{y^2}\ dy.$$

Since $y \mapsto 1/y^2$ is Reimann integrable (as it is continuous) and bounded for any fixed $t > 0$, we can equivalently evaluate the Reimann integral to find that

$$I(x) \geq \lim_{t\to 0^+}\int_{[t,\ell/c]} \frac{1}{y^2}\ dy = \lim_{t\to 0^+}\left[-\frac{1}{y}\right]_t^{\ell/c} = \lim_{t\to 0^+}\left(-\frac{1}{\ell/c} + \frac{1}{t}\right) = \infty.$$

Thus, we conclude as needed that $I(x) = \infty$ since we showed that $I(x) \geq \infty$. ■

Proof of 3.2 (b). We show that for a.e. $x \in F$, $I(x) < \infty$. We start by noting that $I(x) = \int_{\mathbb{R}} \frac{d(y,F)}{|x-y|^2}\ dy = \int_F \frac{d(y,F)}{|x-y|^2}\ dy + \int_{\mathbb{R}\setminus F} \frac{d(y,F)}{|x-y|^2}\ dy = \int_{\mathbb{R}\setminus F} \frac{d(y,F)}{|x-y|^2}\ dy$, since for $y \in F$, $d(y, F) = |y - y| = 0$ so that $\int_F \frac{d(y,F)}{|x-y|^2}\ dy = \int_F 0\ dy = 0$. By lecture, we know that if a function f is integrable over $A \subseteq \mathbb{R}^d$, then $f(x) < \infty$ for a.e. $x \in A$. Thus, it suffices to show that $\int_F I(x)\ dx < \infty$. Since $\int_F I(x)\ dx = \int_{F\times\mathbb{R}\setminus F} f$ (where $f : \mathbb{R}\times\mathbb{R} \to \mathbb{R}$ is given by $f(x, y) = \frac{d(y,F)}{|x-y|^2}$), and $f \geq 0$ everywhere and is measurable[2], we can swap the integrals using Tonelli's theorem to obtain

$$\begin{aligned}\int_F I(x)\ dx &= \int_F \left(\int_{\mathbb{R}\setminus F} \frac{d(y,F)}{|x-y|^2}\ dy\right) dx = \int_{\mathbb{R}\setminus F}\left(\int_F \frac{d(y,F)}{|x-y|^2}\ dx\right) dy \\ &= \int_{\mathbb{R}\setminus F}\left(d(y,F)\int_F \frac{1}{|x-y|^2}\ dx\right) dy. \quad (*)\end{aligned}$$

Now, following the hint, note that for every $y \in \mathbb{R}\setminus F$, $F \cap (y - d(y, F), y + d(y, F)) = \emptyset$. Indeed, if $x \in F$, then $d(y, F) \leq |x - y|$ so that $x \notin (y - d(y, F), y + d(y, F))$. Thus, $F \subseteq (-\infty, y - d(y, F)) \cup (y + d(y, F), \infty)$

[2]Firstly, $f \geq 0$ on $\mathbb{R}^2$ as $d(y, F), |x - y|^2 \geq 0$. Secondly, f is measurable; indeed, using (1) $d(\cdot, F)$ is Lipschitz continuous (and hence continuous) and finite-valued so that it is measurable and $1/|x - y|^2$ is continuous for every $x, y \in \mathbb{R}^2$ such that $x \neq y$, i.e. the set of discontinuities is $\Gamma \coloneqq \{(x, y) \in \mathbb{R}\times\mathbb{R} : y = \mathrm{id}(x)\}$. Since the identity function is measurable, using (2) Γ has measure 0. Thus, $1/|x - y|^2$ is continuous almost everywhere and finite-valued and hence measurable by lecture. Since these functions are finite valued and since the product of measurable functions is measurable, $f = d(y, F) \cdot \frac{1}{|x-y|^2}$ is measurable as needed.

so that

$$\int_F \frac{1}{|x-y|^2}\ dx \le \int_{-\infty}^{y-d(y,F)} \frac{1}{|x-y|^2}\ dx + \int_{y+d(y,F)}^{\infty} \frac{1}{|x-y|^2}\ dx$$
$$= \int_{-\infty}^{-d(y,F)} \frac{1}{x^2}\ dx + \int_{d(y,F)}^{\infty} \frac{1}{x^2}\ dx,$$

again using Question 1 of Assignment 4 as we did above: for every $x \in (-\infty, y-d(y,F)] \cup [y+d(y,F), \infty)$, if $f(x) = \frac{1}{|x-y|^2}$ then define for $x \in (-\infty, -d(y,F)] \cup [d(y,F), \infty)$ $f_y(x) = f(x+y) = \frac{1}{|x+y-y|^2} = \frac{1}{x^2}$.

Since $1/x^2$ is Reimann integrable (as it is continuous) and bounded on $[t, -d(y,F)]$ and $[d(y,F), t]$ for any fixed t, we can evaluate these integrals as Reimann integrals:

$$\begin{aligned}
\int_F \frac{1}{|x-y|^2}\ dx &\le \int_{-\infty}^{-d(y,F)} \frac{1}{x^2}\ dx + \int_{d(y,F)}^{\infty} \frac{1}{x^2}\ dx \\
&= \lim_{t\to-\infty} \int_t^{-d(y,F)} \frac{1}{x^2}\ dx + \lim_{t\to\infty} \int_{d(y,F)}^{t} \frac{1}{x^2}\ dx \\
&= \lim_{t\to-\infty} \left(\int_{[t,-d(y,F)]} \frac{1}{x^2}\ dx \right) + \lim_{t\to\infty} \left(\int_{[d(y,F),t]} \frac{1}{x^2}\ dx \right) \\
&\qquad\qquad \text{(add endpoints as finite sets have measure 0)} \\
&= \lim_{t\to-\infty} \int_t^{-d(y,F)} x^{-2}\ dx + \lim_{t\to\infty} \int_{d(y,F)}^{t} x^{-2}\ dx \\
&= \lim_{t\to-\infty} \left[\frac{-1}{x} \right]_t^{-d(y,F)} + \lim_{t\to\infty} \left[\frac{-1}{x} \right]_{d(y,F)}^{t} \\
&= \frac{1}{d(y,F)} + \lim_{t\to-\infty} \frac{1}{t} + \lim_{t\to\infty} \frac{-1}{t} + \frac{1}{d(y,F)} \\
&= \frac{2}{d(y,F)}.
\end{aligned}$$

Now, applying this result to $(*)$ yields

$$\int_F I(x)\ dx = \int_{\mathbb{R}\setminus F} \left(d(y,F) \int_F \frac{1}{|x-y|^2}\ dx \right) dy \le \int_{\mathbb{R}\setminus F} d(y,F) \frac{2}{d(y,F)}\ dy$$
$$= 2 \int_{\mathbb{R}\setminus F} \chi_{\mathbb{R}\setminus F}\ dy = 2m(\mathbb{R}\setminus F) < \infty,$$

which holds by the hypothesis that the complement of F has finite measure and by definition of the Lebesgue integral of a characteristic function. Therefore, $\int_F I(x)\ dx < \infty$ implies that for almost every $x \in F$, $I(x) < \infty$, thus the proof is complete. ∎