Problem 1. Let S be the set of all closed cubes in $\mathbb{R}^d$, S' be the set of all open cubes in $\mathbb{R}^d$, and S'' be the set of all rectangles R such that for some $a_1 < b_1, \cdots, a_d < b_d \in \mathbb{R}^d$,

$$(a_1, b_1) \times \cdots \times (a_d, b_d) \subseteq R \subseteq [a_1, b_1] \times \cdots \times [a_d, b_d].$$

For every set $A \subseteq \mathbb{R}^d$, let $m_*(A),\ m'_*(A),\ $ and $m''_*(A)$ be defined as:

$$m_*(A) := \inf \sum_{k=1}^{\infty} \operatorname{vol}(Q_k),\ m'_*(A) := \inf \sum_{k=1}^{\infty} \operatorname{vol}(Q'_k),\ \text{and } m''_*(A) := \inf \sum_{k=1}^{\infty} \operatorname{vol}(R_k),$$

where the infima are taken over all counter coverings of A by $Q_k \in S$, $Q'_k \in S'$, and $R_k \in S''$ respectively. Prove that $m''_*(A) = m'_*(A) = m_*(A)$.
As is granted on the assignment sheet, we will use the following facts about rectangles without proof:

1. For every rectangle R and $\varepsilon > 0$, there exists an open rectangle R_ε such that $R \subseteq R_\varepsilon$ and $\operatorname{vol}(R_\varepsilon) \leq \operatorname{vol}(R) + \varepsilon$. Likewise, for every cube Q and $\varepsilon > 0$, there exists an open cube Q_ε such that $Q \subseteq Q_\varepsilon$ and $\operatorname{vol}(Q_\varepsilon) \leq \operatorname{vol}(Q) + \varepsilon$.
2. Every open rectangle R can be written as $R = \bigcup_{k=1}^{\infty} Q_k$ for some cubes $Q_k \in S$ whose interiors are disjoint.
3. If a sequence of disjoint rectangles $(R_k)_{k \geq 1}$ is such that for some rectangle R, $\bigcup_{k=1}^{\infty} R_k \subseteq R$, then $\sum_{k=1}^{\infty} \operatorname{vol}(R_k) \leq \operatorname{vol}(R)$.

Proof. Let $A \subseteq \mathbb{R}^d$ be arbitrary. We note that it suffices to show that $m''_*(A) \leq m'_*(A) \leq m_*(A) \leq m''_*(A)$, since $m''_*(A) \leq m'_*(A), m_*(A) \leq m''_*(A) \implies m'_*(A) = m''_*(A)$ and $m_*(A) = m''(A) \implies m'_*(A) = m_*(A) \implies m''_*(A) = m'_*(A) = m_*(A)$.

Suppose, for 1 - 3 below, that the exterior measure on the left-hand-side of the inequality is finite. We prove the infinite cases below.

1. $m''_*(A) \leq m'_*(A)$ Suppose $m'_*(A) < \infty$, otherwise there is nothing to prove.
We define the following sets:

$$R := \left\{ \sum_{k=1}^{\infty} \operatorname{vol}(R_k) \;\middle|\; A \subseteq \bigcup_{k=1}^{\infty} R_k, R_k \in S'' \right\} \text{ and } C := \left\{ \sum_{k=1}^{\infty} \operatorname{vol}(Q'_k) \;\middle|\; A \subseteq \bigcup_{k=1}^{\infty} Q'_k, Q'_k \in S' \right\}.$$

Notice that $C \subseteq R$; to see why, let $x \in C$ be arbitrary. Hence there exists some sequence of open cubes $(C_k : k \geq 1, C_k \in S')$ such that $A \subseteq \bigcup_{k=1}^{\infty} C_k$ and $x = \sum_{k=1}^{\infty} \operatorname{vol}(C_k)$. But then, by the construction of S'', it follows that each open cube $C_k \in S''$ (since open cubes are indeed rectangles). From this realisation, we obtain that $x \in R$ as it meets the necessary criteria (namely that $A \subseteq \bigcup_{k=1}^{\infty} C_k$ and $C_k \in S''$ for each $k \geq 1$). Hence we have $C \subseteq R \implies m'_*(A) = \inf C \geq \inf R = m''_*(A)$ (cf. Lemma 1.1). Hence, $m''_*(A) \leq m'_*(A)$, as needed.

2. $m'_*(A) \leq m_*(A)$ Suppose $m_*(A) < \infty$, otherwise there is nothing to prove.

We commence this portion of the proof by noting that the volume of a closed cube equals that of its interior by definition. To see this, we use the following argument:

- Firstly, if $C = [a, b] \times \cdots \times [a, b] \in S$ is a closed cube in $\mathbb{R}^d$ (where $a < b$), we note that its interior $C^o = (a, b) \times \cdots \times (a, b)$. To see this, let $x \in C^o$, then there exists an $r > 0$ such that $V_r(x) \subseteq C$.

Obviously, $x \neq a$ and $x \neq b$, since these are boundary points of C, hence $x \in (a,b) \times \cdots \times (a,b) \subseteq C$, since this is an open set and $C^o \subseteq C$. Now if $x \in (a,b) \times \cdots \times (a,b) \implies x \in C^o$, as this is an open subset of C. Hence, the interior of a closed cube is an open cube.

- By the definition of a rectangle, then, we must have that $\mathrm{vol}(C) = \mathrm{vol}([a,b] \times \cdots \times [a,b]) = (b-a)^d = \mathrm{vol}((a,b) \times \cdots \times (a,b)) = \mathrm{vol}(C^o)$.

Using this argument, let $(C_k : k \geq 1, C_k \in S)$ be a sequence of closed cubes such that $A \subseteq \bigcup_{k=1}^{\infty} C_k$. Then, by the argument above, we obtain

$$\sum_{k=1}^{\infty} \mathrm{vol}(C_k) = \sum_{k=1}^{\infty} \mathrm{vol}(C_k^o). \tag{1.2.1}$$

So if $k \coloneqq m_*(A)$ (provided that its finite, otherwise there is nothing to prove) and $\varepsilon > 0$ is arbitrary, by definition of the infimum, there exists some covering of $A \subseteq \bigcup_{k=1}^{\infty} Q_k$ by closed cubes Q_k such that

$$\sum_{k=1}^{\infty} \mathrm{vol}(Q_k) < k + \varepsilon = m_*(A) + \varepsilon.$$

But then we have that $m_*'(A) \leq \sum_{k=1}^{\infty} \mathrm{vol}(Q_k^o) = \sum_{k=1}^{\infty} \mathrm{vol}(Q_k) < m_*(A) + \varepsilon$. Since we can achieve this result for arbitrary $\varepsilon > 0$, we conclude that $m_*'(A) \leq m_*(A)$, as required.

3. $\underline{m_*(A) \leq m_*''(A)}$. Suppose that $m_*(A) < \infty$, otherwise there is nothing to prove.

Let $\{R_k : R_k \in S''\}_{k=1}^{\infty}$ be a sequence of rectangles in $\mathbb{R}^d$ such that $A \subseteq \bigcup_{k=1}^{\infty} R_k$. By fact 1, for each rectangle R_k and $\varepsilon > 0$ there exists an open rectangle $R_{k,\varepsilon}$ such that $R_k \subseteq R_{k,\varepsilon}$ and $\mathrm{vol}(R_{k,\varepsilon}) \leq \mathrm{vol}(R_k) + \frac{\varepsilon}{2^k}$. Therefore,

$$A \subseteq \bigcup_{k=1}^{\infty} R_k \subseteq \bigcup_{k=1}^{\infty} R_{k,\varepsilon}.$$

By fact 2, since each $R_{k,\varepsilon}$ is open, there exists a sequence $\{C_{k,j,\varepsilon} : C_{k,j,\varepsilon} \in S\}_{j=1}^{\infty}$ of closed cubes whose interiors are pairwise disjoint and for which $R_{k,\varepsilon} = \bigcup_{j=1}^{\infty} C_{k,j,\varepsilon}$. It hence follows that

$$\bigcup_{j=1}^{\infty} C_{k,j,\varepsilon}^o \subseteq \bigcup_{j=1}^{\infty} C_{k,j,\varepsilon} = R_{k,\varepsilon},$$

where A^o denotes the interior of a set A. Fact 2 assures that the interiors of the closed cubes $C_{k,j,\varepsilon}$ are pairwise disjoint, hence, by fact 3, we must have that

$$\sum_{j=1}^{\infty} \mathrm{vol}(C_{k,j,\varepsilon}^o) \leq \mathrm{vol}(R_{k,\varepsilon}),$$

but then by (1.2.1) above, we obtain

$$\sum_{j=1}^{\infty} \mathrm{vol}(C_{k,j,\varepsilon}) = \sum_{j=1}^{\infty} \mathrm{vol}(C_{k,j,\varepsilon}^o) \leq \mathrm{vol}(R_{k,\varepsilon}) \leq \mathrm{vol}(R_k) + \frac{\varepsilon}{2^k}.$$

Summing over all k, we have found a covering by closed cubes which admits a sum of volumes less than or equal to that of our arbitrary covering by rectangles:

$$\sum_{k=1}^{\infty}\sum_{j=1}^{\infty}\mathrm{vol}(C_{k,j,\varepsilon}) \leq \sum_{k=1}^{\infty}\left(\mathrm{vol}(R_k)+\frac{\varepsilon}{2^k}\right) = \sum_{k=1}^{\infty}\mathrm{vol}(R_k)+\varepsilon.$$

Since ε and the covering by rectangles was arbitrary, we can conclude the following: for a given $\varepsilon > 0$, there exists a covering by rectangles of $A \subseteq \bigcup_{k=1}^{\infty} Q''_k$, where $Q''_k \in S''$, such that

$$\sum_{k=1}^{\infty}\mathrm{vol}(Q''_k) < m''_*(A)+\varepsilon.$$

Now, we have shown that we can find a covering by closed cubes of $A \subseteq \bigcup_{k=1}^{\infty} Q_k$, where $Q_k \in S$ for which

$$\sum_{k=1}^{\infty}\mathrm{vol}(Q_k) \leq \sum_{k=1}^{\infty}\mathrm{vol}(Q''_k) < m''_*(A)+\varepsilon.$$

Hence, for arbitrary $\varepsilon > 0$, we obtain $m_*(A) \leq \sum_{k=1}^{\infty}\mathrm{vol}(Q_k) \leq m''_*(A)+\varepsilon$; hence, sending $\varepsilon \to 0$, we obtain, as required, that $m_*(A) \leq m''_*(A)$.

As previously stated, inequalities 1, 2, and 3 imply that for a set $A \subseteq \mathbb{R}^d$, $m_*(A) = m'_*(A) = m''_*(A)$, thereby completing the proof. ■

Lemma 1.1. For sets $A, \subseteq \mathbb{R}^d$ with $A \subseteq B$, we have $\inf B \leq \inf A$. This lemma is proven trivially. Note that $A \subseteq B$ implies that for any $a \in A$ there exists a $b \in B$ with $b \leq a$. Let $\varepsilon > 0$ be arbitrary. Then there exists a $y_A \in A$ (and hence a corresponding $y_B \in B : y_B \leq y_A$) such that

$$\inf B \leq y_B \leq y_A < \inf A + \varepsilon.$$

Sending $\varepsilon \to 0$ implies that $\inf B \leq \inf A$ as needed. ■

Infinite case 1. Suppose $m''_*(A) = \infty$. We must show that $m'_*(A) = \infty$. This means that for every covering of A by a sequence of rectangles $\{R_k : R_k \in S''\}_{k=1}^{\infty}$,

$$\sum_{k=1}^{\infty}\mathrm{vol}(R_k) = \infty.$$

Suppose towards contradiction that there is some covering of A by a sequence of open cubes $\{C'_k : C'_k \in S'\}_{k=1}^{\infty}$ such that $\sum_{k=1}^{\infty}\mathrm{vol}(C'_k) < \infty$. But this is an immediate contradiction, since we would have some covering of A by rectangles (since open cubes are rectangles) that is finite, hence $m'_*(A) = \infty$ as needed.

Infinite case 2. Suppose $m'_*(A) = \infty$. We must show that $m_*(A) = \infty$. This means that for every covering of A by a sequence of open cubes $\{C'_k : C'_k \in S'\}_{k=1}^{\infty}$, we have

$$\sum_{k=1}^{\infty}\mathrm{vol}(C'_k) = \infty.$$

Suppose towards contradiction that there is some sequence of closed cubes $\{C_k : C_k \in S\}_{k=1}^{\infty}$ such that $\sum_{k=1}^{\infty}\mathrm{vol}(C_k) < \infty$. But, as argued in (2), there is a corresponding sequence of open cubes (namely the interiors) such that $\sum_{k=1}^{\infty} \mathrm{vol}(C_k) = \sum_{k=1}^{\infty} \mathrm{vol}(C_k^o) < \infty$, a contradiction to the hypothesis that every covering by open cubes leads to an infinite sum of volumes. Thus, it follows that $m_*(A) = \infty$, as required.

Infinite case 3. Suppose $m_*(A) = \infty$. We must show that $m_*''(A) = \infty$. This means that for every covering of A by a sequence of closed cubes $\{C_k : C_k \in S\}_{k=1}^{\infty}$, we have $\sum_{k=1}^{\infty} \mathrm{vol}(C_k) = \infty$. Suppose towards contradiction that there exists a covering of A by a sequence of rectangles $\{R_k : R_k \in S''\}_{k=1}^{\infty}$ such that $\sum_{k=1}^{\infty} \mathrm{vol}(R_k) < \infty$. Yet as argued in (3), we can construct a sequence of closed cubes $\{C_k : C_k \in S\}_{k=1}^{\infty}$ such that $\sum_{k=1}^{\infty} \mathrm{vol}(C_k) \leq \sum_{k=1}^{\infty} \mathrm{vol}(R_k) \implies \sum_{k=1}^{\infty} \mathrm{vol}(R_k) = \infty$, a contradiction. Hence, it follows as well that $m_*''(A) = \infty$, as needed.

Problem 2. Let A be a subset of $[0,\infty)$ and $A^2 \coloneqq \{x^2 \in \mathbb{R} : x \in A\}$.

1. Prove that if $m_*(A) = 0$, then $m_*(A^2) = 0$.
2. Give an example of a set A such that $m_*(A) < \infty$ and $m_*(A^2) = \infty$.

Proof of 2.1. We first suppose that A is bounded, i.e. there is an $M \in \mathbb{N}$ such that $A \subseteq [0, M]$, and such that $m_*(A) = 0$. This means that for each $\varepsilon > 0$ there is a sequence $\{I_n\}_{n=1}^{\infty}$ of open intervals (open cubes; cf. Problem 1) such that

$$A \subseteq \bigcup_{n=1}^{\infty} I_n \coloneqq \bigcup_{n=1}^{\infty} (a_n, b_n) \text{ and } \sum_{n=1}^{\infty} \mathrm{vol}(I_n) < m_*(A) + \varepsilon = \varepsilon.$$

We claim that $A^2 \subseteq \bigcup_{n=1}^{\infty} I_n^2 = \bigcup_{n=1}^{\infty}(a_n^2, b_n^2)$. This holds as if $x^2 \in A^2 \implies x \in A \implies \exists\, n \in \mathbb{N}_{>0} : x \in (a_n, b_n) \implies a_n < x < b_n \implies a_n^2 < x^2 < b_n^2$ (as all non-negative) $\implies x^2 \in (a_n^2, b_n^2) \subseteq \bigcup_{n=1}^{\infty}(a_n^2, b_n^2)$. Now, since A is bounded we let $\ell \coloneqq \sup_{n \geq 1}\{b_n + a_n\} \leq 2M < \infty$. Thus, given $\varepsilon > 0$, find a covering of A by open intervals I_n such that

$$\sum_{n=1}^{\infty} \mathrm{vol}(I_n) < \delta \coloneqq \frac{\varepsilon}{\sup_{n\geq 1}\{b_n + a_n\}} > 0.^{1}$$

Then

$$\begin{aligned}
\sum_{n=1}^{\infty} \mathrm{vol}(I_n^2) &= \sum_{n=1}^{\infty} (b_n^2 - a_n^2) \\
&= \sum_{n=1}^{\infty} (b_n - a_n)(b_n + a_n) \\
&\leq \ell \sum_{n=1}^{\infty} (b_n - a_n) = \ell \sum_{n=1}^{\infty} \mathrm{vol}(I_n) < \ell\delta = \varepsilon.
\end{aligned}$$

Thus, for fixed $\varepsilon > 0$, we have found a covering of A^2 by intervals I_n^2 such that $m_*(A^2) < \sum_{n=1}^{\infty} \mathrm{vol}(I_n^2) < \varepsilon$. Sending $\varepsilon \to 0$ implies that $m_*(A^2) = 0$ for bounded A. Now suppose that A is unbounded, with $m_*(A) = 0$. Then we can write A as

$$A = \bigcup_{n=1}^{\infty} (A \cap [0, n]),$$

a countable union of bounded intervals.[2] So that

$$m_*(A) = m_*(\bigcup_{n=1}^{\infty} (A \cap [0,n])) \leq \sum_{n=1}^{\infty} m_*(A \cap [0,n]) = \sum_{n=1}^{\infty} 0 = 0,$$

by sub-additivity and since $A \cap [0,n] \subseteq A$ for each n implies that $m_*(A \cap [0,n]) \leq m_*(A) = 0 \implies$

[1] Note that if $\sup_{n\geq 1}\{b_n + a_n\} = 0$, then $A^2 \subseteq \{0\}$ so that $m_*(A^2) = 0$, so suppose $\sup_{n \geq 1}\{b_n + a_n\} > 0$.

[2] This holds as $x \in A \implies \exists\, n \geq 0 : x \leq n \implies x \in A \cap [0,n] \subseteq \bigcup_{n=1}^{\infty}(A \cap [0,n])$; and $x \in \bigcup_{n=1}^{\infty}(A \cap [0,n]) \implies \exists\, n \geq 1 : x \in A \wedge x \in [0,n] \implies x \in A$.

$m_*(A \cap [0,n]) = 0$. We can likewise write $A^2 = \bigcup_{n=1}^{\infty}(A \cap [0,n])^2 = \bigcup_{n=1}^{\infty}(A^2 \cap [0,n^2])$ to find that

$$\begin{aligned} m_*(A^2) &\leq \sum_{n=1}^{\infty} m_*(A^2 \cap [0,n^2]) && \text{(applying same argument as above)} \\ &= \sum_{n=1}^{\infty} m_*((A \cap [0,n])^2) && (*) \\ &= \sum_{n=1}^{\infty} m_*(A \cap [0,n]) && \text{(by boundedness)} \\ &= \sum_{n=1}^{\infty} 0 = 0, && \text{(By monotonicty, as argued above)} \end{aligned}$$

where $(*)$ holds as $x^2 \in A^2 \cap [0,n^2] \implies x \in A \cap [0,n] \implies x^2 \in (A \cap [0,n])^2$ and $x^2 \in (A \cap [0,n])^2 \implies x \in A, x \in [0,n] \implies x^2 \in A^2, x \in [0,n^2], \implies x^2 \in A^2 \cap [0,n^2]$ as needed (hence they are subsets of each other). Thus, we have proven the unbounded case as well, since $m_*(A^2) \leq 0 \implies m_*(A^2) = 0$.

Therefore, we conclude that for $A \subseteq [0,\infty)$, $m_*(A) = 0 \implies m_*(A^2) = 0$, thereby completing the proof. ∎

Solution For 2.2. Let

$$A \coloneqq \bigcup_{n=2}^{\infty}(n, n + \frac{1}{n^2}).$$

By problem 1, we use m_* which approximates the volume of A via open cubes. Since each interval is itself an open cube, we must have that

$$\sum_{n=2}^{\infty} \mathrm{vol}((n, n + \frac{1}{n^2})) = \sum_{n=2}^{\infty}\left(n + \frac{1}{n^2} - n\right) = \sum_{n=2}^{\infty}\frac{1}{n^2} \geq m_*(A).$$

By the p-test, $\sum_{n=2}^{\infty}\frac{1}{n^2}$ is a finite number which is an upper bound for $m_*(A)$, hence $m_*(A) < \infty$.

Now notice that $A^2 = \bigcup_{n=2}^{\infty}(n^2, (n + \frac{1}{n^2})^2)$, and by the same reasoning as above, we have

$$\sum_{n=2}^{\infty} \mathrm{vol}((n^2, (n + \frac{1}{n^2})^2)) = \sum_{n=2}^{\infty}(n^2 + \frac{2n}{n^2} + \frac{1}{n^4} - n^2) = \sum_{n=2}^{\infty}(\frac{2}{n} + \frac{1}{n^4}) \geq \sum_{n=2}^{\infty}\frac{1}{n},$$

since the harmonic series diverges, $\sum_{n=2}^{\infty}\mathrm{vol}((n^2, (n + \frac{1}{n^2})^2))$ does too. But notice that

$$m_*(A^2) = \sum_{n=2}^{\infty}\mathrm{vol}((n^2, (n + \frac{1}{n^2})^2)),$$

since we have written A^2 (which is open as it is a union of open intervals) as a union of disjoint (cf. Lemma 2.1) open cubes (the equality thus holds by lecture). Thus, we have found a suitable example where $m_*(A) < \infty$ yet $m_*(A^2) = \infty$. ∎

Lemma 2.1. Indeed, for fixed $n \geq 2$, $(n - 1 + \frac{1}{(n-1)^2})^2 = (\frac{(n-1)^3+1}{(n-1)^2})^2 = \frac{n^2(n^2-3n+3)^2}{(n-1)^4}$, hence $\frac{n^2(n^2-3n+3)^2}{(n-1)^4} - n^2 = \frac{n^2(n^2-3n+3)^2 - n^2(n-1)^4}{(n-1)^4} < 0 \implies n^2((n^2-3n+3)^2 - (n-1)^4) < 0 \implies (n^2-3n+3)^2 - (n-1)^4 < 0 \implies (n-2)(n^2 + \frac{5n}{2} + 2) < 0$ (via tedious factoring). Thus, such holds for $n \geq 2$, i.e. our intervals are disjoint for $n \geq 2$ (as the endpoints do not intersect). ∎

Problem 3. For every $A \subseteq \mathbb{R}^d$, $\delta \coloneqq (\delta_1, \ldots, \delta_d)$, and $y \coloneqq (y_1, \ldots, y_d) \in \mathbb{R}^d$, define

$$A_{\delta,y} \coloneqq \{(\delta_1 x_1 + y_1, \ldots, \delta_d x_d + y_d) : x = (x_1, \ldots, x_d) \in A\}.$$

(1) Prove that $m_*(A_{\delta,y}) = \delta_1 \cdots \delta_d m_*(A)$.

Proof of 1. Let $A \subseteq \mathbb{R}^d$, $\delta \coloneqq (\delta_1, \ldots, \delta_d)$, and $y \coloneqq (y_1, \ldots, y_d) \in \mathbb{R}^d$ be arbitrary. Notice that $A_{\delta,y} = (\delta A) + y$. To see this, let $\delta x + y = (\delta_1 x_1 + y_1, \ldots, \delta_d x_d + y_d) \in A_{\delta,y}$. Then $x \in (\delta A) + y$ as per its definition. The reverse inclusion likewise holds trivially: $\delta x + y \in (\delta A) + y \implies x \in A_{\delta,y}$. Now, we can easily apply lemmas 3.1 and 3.2 to complete the proof:

$$m_*(A_{\delta,y}) = m_*((\delta A) + y) = m_*(\delta A) = \delta_1 \cdots \delta_d m_*(A).$$

Hence, $m_*(A_{\delta,y}) = \delta_1 \cdots \delta_d m_*(A)$, thereby completing the proof. ∎

(2) Prove that A is measurable if and only if $A_{\delta,y}$ is measurable.

Proof.
[$\implies$] Let $A \subseteq \mathbb{R}^d$ be measurable. We show that $A_{\delta,y}$ is measurable as well, for fixed $\delta = (\delta_1, \ldots, \delta_d) \in (0, \infty)^d$ and $y \coloneqq (y_1, \ldots, y_d) \in \mathbb{R}^d$.

Since A is measurable, for each $\varepsilon > 0$ there exists an open set $\mathcal{O}_\varepsilon$ such that $A \subseteq \mathcal{O}_\varepsilon$ and

$$m_*(\mathcal{O}_\varepsilon - A) < \frac{\varepsilon}{\delta_1 \cdots \delta_d}. \qquad (\delta_1 \cdots \delta_d > 0)$$

But notice that

$$\begin{aligned}
\delta_1 \cdots \delta_d \cdot m_*(\mathcal{O}_\varepsilon - A) &= m_*(\delta(\mathcal{O}_\varepsilon - A)) && \text{(Lemma 3.2)}\\
&= m_*(\delta(\mathcal{O}_\varepsilon - A) + y) && \text{(Lemma 3.1)}\\
&= m_*((\delta\mathcal{O}_\varepsilon + y) - (\delta A + y)) && \text{(Lemma 3.4)}\\
&= m_*(\mathcal{O}_{\varepsilon_{\delta,y}} - A_{\delta,y}) < \delta_1 \cdots \delta_d \cdot \frac{\varepsilon}{\delta_1 \cdots \delta_d} = \varepsilon. && \text{(by measurability of } A)
\end{aligned}$$

By lemma 3.3, $\mathcal{O}_{\varepsilon_{\delta,y}}$ is an open set, and $A_{\delta,y} \subseteq \mathcal{O}_{\varepsilon_{\delta,y}}$ as if $\delta x + y \in A_{\delta y} \implies x \in A \implies x \in \mathcal{O}_\varepsilon \implies \delta x + y \in \delta\mathcal{O}_\varepsilon + y \implies \delta x + y \in \mathcal{O}_{\varepsilon_{\delta,y}}$. Hence, since for arbitrary $\varepsilon > 0$ we found an open set $\mathcal{O}_{\varepsilon_{\delta,y}}$ such that $A_{\delta,y} \subseteq \mathcal{O}_{\varepsilon_{\delta,y}}$ and $m_*(\mathcal{O}_{\varepsilon_{\delta,y}} - A_{\delta,y}) < \varepsilon$, we conclude that $A_{\delta,y}$ is measurable.

[$\impliedby$] For our fixed $\delta \coloneqq (\delta_1, \ldots, \delta_d) \in (0, \infty)^d$ and $y \coloneqq (y_1, \ldots, y_d) \in \mathbb{R}^d$, let $\delta' = (\delta'_1, \ldots, \delta'_d) \coloneqq (\frac{1}{\delta_1}, \ldots, \frac{1}{\delta_d})$ (possible as for $1 \le i \le d$, $\delta_i > 0$) and $y' \coloneqq -y$. Let $A \subseteq \mathbb{R}^d$. Suppose $A_{\delta,y}$ is measurable; this means that for each fixed $\varepsilon > 0$ there exists an open set $\mathcal{O}_\varepsilon$ such that $A_{\delta,y} \subseteq \mathcal{O}_\varepsilon$ and $m_*(\mathcal{O}_\varepsilon - A_{\delta,y}) < \frac{\varepsilon}{\delta'_1 \cdots \delta'_d}$. Then notice that

$$
\begin{aligned}
\delta_1' \cdots \delta_d' \cdot m_*(\mathcal{O}_\varepsilon - A_{\delta,y}) &= \delta_1' \cdots \delta_d' \cdot m_*((\mathcal{O}_\varepsilon - A_{\delta,y}) + y') && \text{(Lemma 3.1)} \\
&= \delta_1' \cdots \delta_d' \cdot m_*((\mathcal{O}_\varepsilon + y') - (A_{\delta,y} + y')) && \text{(Lemam 3.4 with } \delta \coloneqq (1,1,\dots,1) \in \mathbb{R}^d\text{)} \\
&= \delta_1' \cdots \delta_d' \cdot m_*((\mathcal{O}_\varepsilon + y') - \delta A) && \text{(By def. of } A_{\delta,y} \text{ and choice of } y'\text{)} \\
&= m_*(\delta'((\mathcal{O}_\varepsilon + y') - \delta A)) && \text{(Lemma 3.2)} \\
&= m_*(\delta'(\mathcal{O}_\varepsilon + y') - A) < \delta_1' \cdots \delta_d' \cdot \frac{\varepsilon}{\delta_1' \cdots \delta_d'} = \varepsilon, && (*)
\end{aligned}
$$

where the last equality holds by the definition of δA and choice of δ', and since if $A, B \in \mathbb{R}^d$, $\delta(A \setminus B) = \{\delta x \in \mathbb{R}^d : x \in A, x \notin B\} = \{\delta x \in \mathbb{R}^d : \delta x \in \delta A, \delta x \notin \delta B\} = \delta A \setminus \delta B$. It remains to be shown that $\delta'(\mathcal{O}_\varepsilon + y')$ is open and contains A.

- $\delta'(\mathcal{O}_\varepsilon + y')$ is open: apply lemma 3.3 to the open set $\mathcal{O}_\varepsilon$ with $\delta \coloneqq (1,1,\dots,1) \in \mathbb{R}^d$ and y' to find that $\mathcal{O}_\varepsilon + y'$ is open. For the sake of clarity, let $\mathcal{U} \coloneqq \mathcal{O}_\varepsilon + y'$. Now apply lemma 3.3 to the open set $\mathcal{U}$ with δ' and $y \coloneqq (0,0,\dots,0) \in \mathbb{R}^d$ to find that $\delta'\mathcal{U} + (0,0,\dots,0) = \delta'(\mathcal{O}_\varepsilon + y')$ is open, as needed.

- A $\subseteq \delta'(\mathcal{O}_\varepsilon + y')$: Let $x = (x_1, \dots, x_d) \in A$. Then $(\delta_1 x_1 + y_1, \dots, \delta_d x_d + y_d) \in A_{\delta,y} \subseteq \mathcal{O}_\varepsilon$. But then $(\delta_1 x_1 + y_1 + (-y_1), \dots, \delta_d x_d + y_d + (-y_d)) = (\delta_1 x_1 + y_1 + y_1', \dots, \delta_d x_d + y_d + y_d') = (\delta_1 x_1, \dots, \delta_d x_d) \in \mathcal{O}_\varepsilon + y'$ by definition of a set's translation; but then $x = (\frac{\delta_1}{\delta_1} x_1, \dots, \frac{\delta_d}{\delta_d} x_d) = (x_1, \dots, x_d) \in \delta'(\mathcal{O}_\varepsilon + y')$. Hence, $A \subseteq \delta'(\mathcal{O}_\varepsilon + y')$.

Therefore, given $\varepsilon > 0$, we have found an open set $\delta'(\mathcal{O}_\varepsilon + y')$ such that $A \subseteq \delta'(\mathcal{O}_\varepsilon + y')$ and $m_*(\delta'(\mathcal{O}_\varepsilon + y') - A) < \varepsilon$. Thus, A is measurable by definition, thereby completing the proof. ■

Lemma 3.1. Translation invariance.
Let $A \subseteq \mathbb{R}^d$ and $y \in \mathbb{R}^d$. Define $A + y \coloneqq \{x + y \in \mathbb{R}^d : x \in A\}$. We will use the definition of exterior measure corresponding to coverings by open cubes (which can be done by problem 1). So suppose $\{C_k\}_{k=1}^\infty$ is a sequence of open cubes in $\mathbb{R}^d$ such that $A \subseteq \bigcup_{k=1}^\infty C_k$. Then,

$$
\begin{aligned}
\sum_{k=1}^\infty \mathrm{vol}(C_k) &= \sum_{k=1}^\infty \mathrm{vol}((a_k, b_k)^d) = \sum_{k=1}^\infty (b_k - a_k)^d \\
&= \sum_{k=1}^\infty ((b_k - a_k)^d + y - y) = \sum_{k=1}^\infty \prod_{i=1}^d (b_k + y_i - (a_k + y_i)) \\
&= \sum_{k=1}^\infty \mathrm{vol}\left((a_k + y_1, b_k + y_1) \times \cdots \times (a_k + y_d, b_k + y_d)\right) = \sum_{k=1}^\infty \mathrm{vol}(C_k + y).
\end{aligned}
$$

Now notice that $A + y \subseteq \bigcup_{k=1}^\infty (C_k + y)$. This holds as if $x \in A + y \implies x - y \in A$, and since $A \subseteq \bigcup_{k=1}^\infty C_k$, there exists some $n \geq 1$ such that $x - y \in C_n$, but then $x \in C_n + y \implies x \in \bigcup_{k=1}^\infty (C_k + y)$. Therefore, we have shown that for any covering of $A \subseteq \bigcup_{k=1}^\infty C_k$ there is a covering of $A + y \subseteq \bigcup_{k=1}^\infty (C_k + y)$ such that $\sum_{k=1}^\infty \mathrm{vol}(C_k) = \sum_{k=1}^\infty \mathrm{vol}(C_k + y)$. Reading the string of equalities in the reverse order implies the exact same statement, but with the covering of $A + y$ being fixed, and the cover of A being derived.

We now define the sets

$$
X \coloneqq \left\{ \sum_{k=1}^\infty \mathrm{vol}(C_k) \;\middle|\; A \subseteq \bigcup_{k=1}^\infty C_k,\ C_k \text{ open cube} \right\},
$$

and

$$Y := \left\{ \sum_{k=1}^{\infty} \operatorname{vol}(C_k + y) \;\middle|\; A + y \subseteq \bigcup_{k=1}^{\infty} (C_k + y),\ C_k + y \text{ open cube} \right\}.$$

By the work above, $x \in X \implies x \in Y$ and $x \in Y \implies x \in X$; therefore, $X = Y \implies m_*(A) = \inf X = \inf Y = m_*(A + y)$. Therefore, the exterior measure is translation invariant. ■

Lemma 3.2. If $A \subseteq \mathbb{R}^d$ and $\delta := (\delta_1, \ldots, \delta_d) \in (0, \infty)^d$, then $m_*(\delta A) = \delta_1 \cdots \delta_d \cdot m_*(A)$, where $\delta A := \{(\delta_1 x_1, \ldots, \delta_d x_d) : (x_1, \ldots, x_d) \in A\}$.

Let $A \subseteq \mathbb{R}^d$ and $\delta := (\delta_1, \ldots, \delta_d) \in (0, \infty)^d$ be fixed. Consider a covering of $A \subseteq \bigcup_{k=1}^{\infty} C_k$ by open cubes C_k (cf. Problem 1). Then notice that

$$\begin{aligned}\prod_{i=1}^{d} \delta_i \sum_{k=1}^{\infty} \operatorname{vol}((a_k, b_k)^d) &= \sum_{k=1}^{\infty} \prod_{i=1}^{d} \delta_i \cdot (b_k - a_k)^d = \sum_{k=1}^{\infty} \prod_{i=1}^{d} (\delta_i b_k - \delta_i a_k) \\ &= \sum_{k=1}^{\infty} \operatorname{vol}((\delta_1 a_k, \delta_1 b_k) \times \cdots \times (\delta_d a_k, \delta_d b_k)) = \sum_{k=1}^{\infty} \operatorname{vol}(\delta C_k).\end{aligned}$$

Therefore, taking the infimum over all coverings by open cubes C_k, we conclude that

$$\inf \prod_{i=1}^{d} \delta_i \sum_{k=1}^{\infty} \operatorname{vol}(C_k) = \prod_{i=1}^{d} \delta_i \inf \sum_{k=1}^{\infty} \operatorname{vol}(C_k) = \delta_1 \cdots \delta_d \cdot m_*(A) = \inf \sum_{k=1}^{\infty} \operatorname{vol}(\delta C_k) = m_*(\delta A);$$

indeed, for every covering by open cubes C_k of A, we have $\delta_1 \cdots \delta_d \sum_{k=1}^{\infty} \operatorname{vol}(C_k) = \sum_{k=1}^{\infty} \operatorname{vol}(\delta C_k)$; thus, looking at all of such possible coverings, we may deduce that the sets over which we take the infimum (when calculating the exterior measure) must be equal.

Therefore, $m_*(\delta A) = \delta_1 \cdots \delta_d \cdot m_*(A)$. ■

Lemma 3.3. For an open set $\mathcal{O} \subseteq \mathbb{R}^d$, $\delta = (\delta_1, \ldots, \delta_d) \in (0, \infty)^d$, and $y := (y_1, \ldots, y_d) \in \mathbb{R}^d$ fixed, $(\delta \mathcal{O}) + y$ is open.

To prove this lemma, suppose $\delta x + y \in (\delta \mathcal{O}) + y$. Clearly, this implies that

$$x \in \mathcal{O} \implies \exists\, \varepsilon > 0 : V_\varepsilon(x) \subseteq \mathcal{O} \implies \delta V_\varepsilon(x) + y \subseteq \delta \mathcal{O} + y,$$

which holds by the definition $\delta \mathcal{O} + y := \{\delta x + y \in \mathbb{R}^d : x \in \mathcal{O}\}$ (and note that $\delta V_\varepsilon(x) + y$ is an open ball that has been scaled and translated, i.e. it is still an open ball). Hence, $\delta \mathcal{O} + y$ is an open set, as x and ε were arbitrary. ■

Lemma 3.4. For an open set $\mathcal{O} \subseteq \mathbb{R}^d$, $\delta = (\delta_1, \ldots, \delta_d) \in (O, \infty)^d$, and $y := (y_1, \ldots, y_d) \in \mathbb{R}^d$ fixed, we have $\delta(\mathcal{O} - A) + y = (\delta \mathcal{O} + y) - (\delta A + y)$.

Suppose $x \in \mathcal{O} - A$. Then $\delta x + y \in \delta(\mathcal{O} - A) + y$, and we know that $x \notin A$, thus $\delta x + y \notin \delta A + y$ yet $\delta x + y \in \delta \mathcal{O} + y$; therefore, we obtain $\delta x + y \in (\delta \mathcal{O} + y) - (\delta A + y)$. Thus $\delta(\mathcal{O} - A) + y \subseteq (\delta \mathcal{O} + y) - (\delta A + y)$. Conversely, if $\delta x + y \in (\delta \mathcal{O} + y) - (\delta A + y)$, then $\delta x + y \in \delta \mathcal{O} + y$, but $\delta x + y \notin \delta A + y$. But this means that $x \notin A \implies x \in \mathcal{O} - A \implies \delta x + y \in \delta(\mathcal{O} - A) + y$, therefore $\delta(\mathcal{O} - A) + y \supseteq (\delta \mathcal{O} + y) - (\delta A + y)$. By definition of set equality, the lemma is complete. ■

Problem 4. Let A be the subset of $[0,1]$ which consists of all numbers which do not have the digit 4 appearing in their decimal expansion. Prove that A is measurable and find $m(A)$.

Proof. We begin by constructing A as follows, writing the set in explicit form:

We define $\tilde{A}_n$ to be the partition of the interval $[0,1]$ into 10^n disjoint intervals of equal length. In other words,

- $\tilde{A}_0 := \{[0,1]\}$,
- $\tilde{A}_1 := \{[0,0.1),[0.1,0.2),[0.2,0.3),\ldots,[0.8,0.9),[0.9,1]\}$,
- $\tilde{A}_2 := \{[0,0.01),[0.01,0.02),\ldots,[0.98,0.99),[0.99,1]\}$,
- For $k>2$: $\tilde{A}_k = \{[0,\frac{1}{10^k}),[\frac{1}{10^k},\frac{2}{10^k}),\ldots,[\frac{10^k-1}{10^k},1]\}$

Define

$$A_n := \bigcup \left\{[a_i,b_i) \in \tilde{A}_n : 4 \text{ is not in the decimal expansion of } a_i\right\} \cup [\frac{10^n-1}{10^n},1].$$

Then it's easy to see that A_n is the set of all points $x \in [0,1]$ such that the first n digits in the decimal expansion of x are not 4, i.e. if $x = x_0.x_1x_2x_3\cdots$, then $x_i \neq 4$ for $1 \le i \le n$. Hence, it is clear that

$$A = \bigcap_{n=0}^{\infty} A_n.$$

Now note that for $k \in \mathbb{N}$ the length of each interval $I \in \tilde{A}_k$ is $\frac{1}{10^k}$ (by construction of $\tilde{A}_k$). From which it follows that the measure[3] of each set A_k is $(1-\frac{1}{10^k})^k = \left(\frac{9}{10}\right)^k$.

This holds by induction on k. Since $A_0 = \tilde{A}_0$, A_0 has measure 1; and $A_1 = [0,0.1)\cup[0.1,0.2)\cup[0.2,0.3)\cup[0.3,0.4)\cup[0.5,0.6)\cup[0.6,0.7)\cup[0.7,0.8)\cup[0.8,0.9)\cup[0.9,1]$ has measure $\frac{9}{10}$. By induction, suppose for some $k \ge 0$ A_k has measure $(\frac{9}{10})^k$. Then the set A_{k+1} corresponds to partitioning each interval from A_k into 10 further sub-intervals; and one of which will be removed (the one containing an endpoint with the digit 4) so that the length of each interval from A_k is decreased to $\frac{9}{10}$ its original length in A_{k+1}. If $m_*(A_k) = \sum_{k=1}^{\infty} \ell_k$, then $m_*(A_{k+1}) = \sum_{k=1}^{\infty} \frac{9}{10}\ell_k = \frac{9}{10}\cdot(\frac{9}{10})^k = (\frac{9}{10})^{k+1}$, by the inductive hypothesis, as required.

But for fixed $k \ge 0$, $A \subseteq A_k$, and $m_*(A_k) = (\frac{9}{10})^k$, so, by monotonicty, $m_*(A) \le m_*(A_k)$. Now let $\varepsilon > 0$ be fixed. By Archimedeanity, there exists a $k \in \mathbb{N}$ with $k > \log_{9/10}\varepsilon$ so that $m_*(A) \le m_*(A_k) = (\frac{9}{10})^k < (\frac{9}{10})^{\log_{9/10}\varepsilon} = \varepsilon$ (as $\frac{9}{10} < 1$, the inequality flips). Since ε was arbitrary, we conclude that $m_*(A) \le m_*(A_k) = 0 \implies m_*(A) = 0$. But this implies that A is measurable by lecture, as we proved that a set $A \subseteq \mathbb{R}^d$ is measurable if it has outer measure 0.

Therefore, A is measurable with $m(A) = 0$, as was to be shown.

∎

[3] This measure equals the sum of the lengths of its intervals. Indeed, A_k is measurable since it is the countable union of measurable sets. So countable additivity applies.

<u>Problem 5.</u> Prove that a set $A \subseteq \mathbb{R}^d$ is measurable if and only if for every set $B \subseteq \mathbb{R}^d$ (not necessarily measurable), we have

$$m_*(B) = m_*(B \cap A) + m_*(B \setminus A).$$

Proof.
[$\implies$] Let $A \subseteq \mathbb{R}^d$ be an arbitrary measurable set with $m(A) < \infty$ and $B \subseteq \mathbb{R}^d$ be any subset. We note that $B \subseteq (B \cap A) \cup (B \setminus A)$,[4] thus, monotonicity and the finite case of sub-additivity (lecture) imply that

$$m_*(B) \leq m_*((B \cap A) \cup (B \setminus A)) \leq m_*(B \cap A) + m_*(B \setminus A).$$

Thus, if $m_*(B) = \infty$, we have ($\leq$) by monotonicty and we also have ($\geq$) trivially, hence suppose $m_*(B) < \infty$. If $m_*(A) = \infty$, then we must have $m_*(B \cap A), m_*(B \setminus A) < \infty$, since $m_*(B) < \infty$ (here we use sub-additivity).

Thus, we must assert the reverse inequality. Let $\varepsilon > 0$ be given. From the hint, there exists an open set $\mathcal{O}$ such that $B \subseteq \mathcal{O}$ and $m_*(\mathcal{O}) < m_*(B) + \varepsilon$, by definition of the infimum. Hence, for any set $C \subseteq \mathbb{R}^d$, $B \subseteq \mathcal{O} \implies B \setminus C \subseteq \mathcal{O} \setminus C$.[5] Thus, we use $B \setminus A \subseteq \mathcal{O} \setminus A$, $B \setminus A^c \subseteq \mathcal{O} \setminus A^c$ and monotonicty to obtain

$$\begin{aligned}
m_*(B \setminus A) + m_*(B \setminus A^c) &\leq m_*(\mathcal{O} \setminus A) + m_*(\mathcal{O} \setminus A^c) \\
&= m_*(\mathcal{O} \cap A^c) + m_*(\mathcal{O} \cap A) \\
&= m(\mathcal{O} \cap A^c) + m(\mathcal{O} \cap A) && \text{(a)} \\
&= m((\mathcal{O} \cap A^c) \cup (\mathcal{O} \cap A)) && \text{(b)} \\
&= m(\mathcal{O}) = m_*(\mathcal{O}) < m_*(B) + \varepsilon, && \text{(c)}
\end{aligned}$$

where (a) holds as finite intersections of measurable sets are measurable (open sets are measurable; A^c is measurable since A is); (b) holds by countable additivity (since these are disjoints sets as $\mathcal{O} \cap A \subseteq A$ and $\mathcal{O} \cap A^c \subseteq A^c$); (c) holds.[6] Thus, letting $\varepsilon \to 0$, we find that $m_*(B \setminus A) + m_*(B \setminus A^c) \leq m_*(B)$, as needed. Hence, we have $m_*(B) = m_*(B \cap A) + m_*(B \setminus A)$, thereby completing the forward implication.

[$\impliedby$] Let $A \subseteq \mathbb{R}^d$ be fixed and suppose that for each subset $B \subseteq \mathbb{R}^d$, we have

$$m_*(B) = m_*(B \cap A) + m_*(B \setminus A).$$

We first assume that A is bounded. Let $\varepsilon > 0$ be given. By the hint, there is an open set $\mathcal{O}$ such that $A \subseteq \mathcal{O}$ and $m_*(\mathcal{O}) < m_*(A) + \varepsilon$, by definition of the infimum. Let $B \coloneqq \mathcal{O}$ so that

$$\begin{aligned}
m_*(B) &= m_*(B \cap A) + m_*(B \setminus A) \\
&= m_*(A) + m_*(B \setminus A). && (A \subseteq B \implies A = B \cap A)
\end{aligned}$$

[4] Since $x \in B \implies x \in A$ or $x \in A^c$; $x \in B, x \in A \implies x \in B \cap A \implies x \in (B \cap A) \cup (B \setminus A)$; and $x \in A^c \implies x \in B \cap A^c = B \setminus A \implies x \in (B \cap A) \cup (B \setminus A)$.

[5] Since $x \in B \setminus C \implies x \in B, x \notin C \implies x \in \mathcal{O}, x \notin C$.

[6] We have two cases: if $x \in \mathcal{O} \cap A^c \implies x \in \mathcal{O}$ if $x \in \mathcal{O} \cap A \implies x \in \mathcal{O}$; the other inclusion is proved in footnote 4.

Thus, since A is bounded, $m_*(A) < \infty$ (as it can be covered by one finite cube), so we have:

$$m_*(B \setminus A) = m_*(B) - m_*(A) < \varepsilon.$$

Since B is open and contains A (and ε was arbitrary), we conclude that A is measurable.

We now must prove that the assertion holds for $A \subseteq \mathbb{R}^d$ which is unbounded. Suppose $A \subseteq \mathbb{R}^d$ is unbounded, hence $m_*(A) = \infty$. Let M be a measurable set with $m_*(M) < \infty$. We need to show that for any set $E \subseteq \mathbb{R}^d$, we have

$$m_*(E) = m_*(E \cap (A \cap M)) + m_*(E \cap (A \cap M)^c).$$

Note that by sub-additivity,

$$m_*(E) = m_*((E \cap (A \cap M)) \cup (E \cap (A \cap M)^c))) \leq m_*(E \cap (A \cap M)) + m_*(M \cap (A \cap M)^c),$$

thus we need only show the converse inequality. Given this, we will be able to prove that A is measurable. So let $E \subseteq \mathbb{R}^d$ be a fixed set with $m_*(E) < \infty$. By hypothesis,

$$m_*(E \cap M) = m_*((E \cap M) \cap A) + m_*((E \cap M) \cap A^c).$$

The measurability of M implies (via the $\implies$ direction) that

$$\begin{aligned}
m_*(E) &= m_*(E \cap M) + m_*(E \cap M^c) \\
&= m_*((E \cap M) \cap A) + m_*((E \cap M) \cap A^c) + m_*(E \cap M^c) \\
&= m_*(E \cap (A \cap M)) + m_*((E \cap M) \setminus A) + m_*(E \setminus M) && \text{(By commutativity of } \cap\text{)} \\
&\geq m_*(E \cap (A \cap M)) + m_*((E \cap M) \setminus A \cup (E \setminus M)) && \text{(By sub-additivity)} \\
&= m_*(E \cap (A \cap M)) + m_*(E \setminus (A \cap M)). && \text{(Lemma 5.1)}
\end{aligned}$$

Thus, since E was arbitrary, we have that for each set $E \subseteq \mathbb{R}^d$, $m_*(E) = m_*(E \cap (A \cap M)) + m_*(E \setminus (A \cap M))$ (since the first inequality holds trivially and we have proven its converse). By $\implies$, this implies that $A \cap M$ is measurable. Now we fix $n \geq 1$ and let $M := [-n, n]$, a measurable set since it is closed (trivially bounded). Hence we know that $A_n := A \cap [-n, n]$ is measurable. As has been argued in problem 2.1, $A = \bigcup_{n=1}^{\infty} A_n$. Since A can be written as a countable union of measurable sets, we conclude that A is measurable, thereby completing the proof.

■

Lemma 5.1. Given sets $E, A, M \subseteq \mathbb{R}^d$, $E \setminus (A \cap M) = (E \cap M) \setminus A \cup (E \setminus M)$.

Let $x \in E \setminus (A \cap M)$ be fixed. For this to happen, we need $x \in E, x \notin A \cap M$; this can happen in two ways, corresponding to if $x \in M$:

- $x \in E, x \in M$ but $x \notin A$, i.e. $x \in (E \cap M) \setminus A$.
- $x \in E$ and $x \notin M$, i.e. $x \in E \setminus M$.

hence, we conclude that $x \in (E \cap M) \setminus A \cup (E \setminus M)$, proving ($\subseteq$). Conversely, suppose $x \in (E \cap M) \setminus A \cup (E \setminus M)$, then

- if $x \in (E \cap M) \setminus A$, then $x \in E$ and $x \in M$, but $x \notin A \implies x \notin A \cap M$. Hence $x \in E \setminus (A \cap M)$.
- if $x \in E \setminus M$, then $x \in E, x \notin M \implies x \notin A \cap M \implies x \in E \setminus (A \cap M)$.

Since this covers all possible cases, we have proven ($\supseteq$). By definition of set equality, we are done. $\blacksquare$