Ramsey Theory and Applications to Analysis

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Agenda

Plan for today:

- The big picture of Ramsey Theory
- Classical examples
  - Diagonal Ramsey number + $R(3) = 6$
  - Van der Waerden numbers
- Ramsey's theorem (infinite version)
- A *very elegant* application of Ramsey's theorem to analysis

What is Ramsey Theory?

Ramsey Theory: The Big Picture

Ramsey Theory explores the underlying structure emerging in "large enough" complex systems.

Main idea: In search of a particular kind of structure in a complex system, we ask: *how large must our system be to guarantee the existence of this structure?*

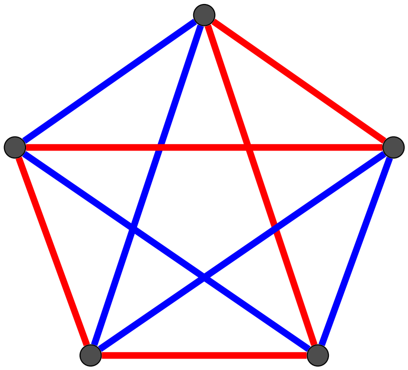
Some classical examples

Diagonal Ramsey Numbers

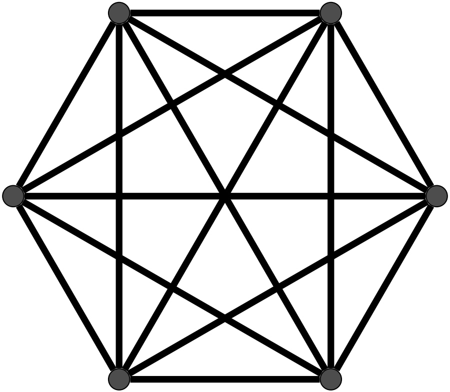
The k'th **diagonal Ramsey number** $R(k)$ is the smallest integer N such that in any colouring of the edges of the complete graph K_N in either red or blue, there is always a set of k vertices joined by all red edges or all blue edges.

- *Ramsey's Theorem (finite version):* $R(k)$ exists for every positive integer k.
- $R(1) = 1$, $R(2) = 2$, $R(3) = 6$.

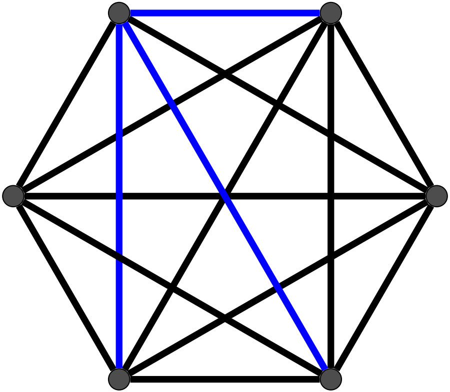
K_5 with No Monochrome K_3



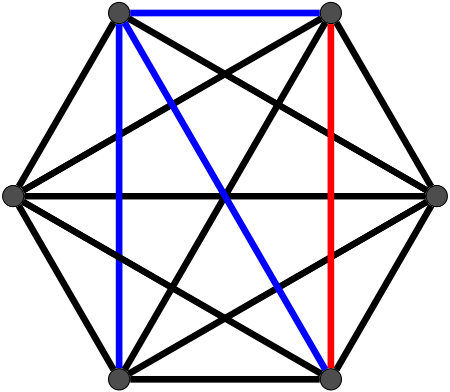
$R(3) = 6$



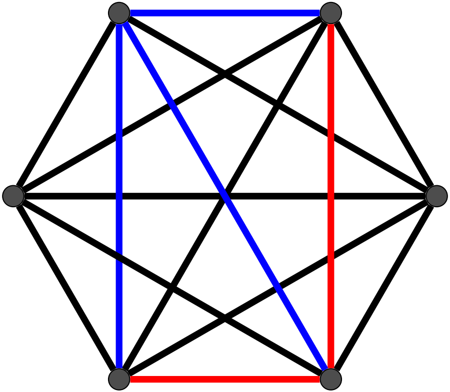
$R(3) = 6$



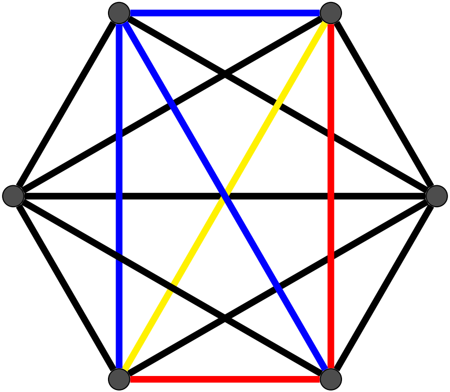
$R(3) = 6$



$R(3) = 6$



$R(3) = 6$



Van der Waerden Numbers

For $r, k \in \mathbb{N}$, the **Van der Waerden number** $W(k, r)$ is the smallest integer N such that in any coloring of $[n]$ in k colors there is a set of r monochromatic integers in arithmetic progression.

- *Van der Waerden's Theorem:* $W(k, r)$ exists for all positive integers r, k.
- Positive integers $a_1, a_2, \ldots, a_r$ are in **arithmetic progression** if there is an integer $d \geq 0$ such that the difference of consecutive terms is d; that is, for $1 \leq k \leq n-1$,

$$a_{k+1} - a_k = d.$$

Ramsey's Theorem (infinite version)

Notation

We fix some notation: For a set X, let $[X]^2$ denote the family of all two element subsets of X, i.e. the *edges* between points in X.

Ramsey's Theorem (infinite version)

In every coloring of $[\mathbb{N}]^2$ in either red or blue there is an infinite subset A of $\mathbb{N}$ such that all edges in $[A]^2$ are the same color.

Proof of Ramsey's Theorem (infinite version)

Statement: In every red-blue coloring of $[\mathbb{N}]^2$ there is an infinite subset A of $\mathbb{N}$ such that all edges in $[A]^2$ are the same color.

Proof.

Set $A_0 = \mathbb{N}$, fix $x_0 \in A_0$. By Pigeonhole, $\exists c_0 \in \{\text{red}, \text{blue}\}$ with $A_1 = \{x \in A_0 : \{x_0, x\} \text{ has color } c_0\}$ infinite. Fix $x_1 \in A_1$ and apply the same argument: $\exists c_1 \in \{\text{red}, \text{blue}\}$ so that the set $A_2 = \{x \in A_1 : \{x_1, x\} \text{ has color } c_1\}$ is infinite. Continue to obtain the following for every $k \in \mathbb{N}$:

1. $x_k \in A_k$,
2. $A_{k+1} \subseteq A_k$,
3. $\forall x \in A_{k+1} : \{x_k, x\}$ has color c_k.

Let $c \in \{\text{red}, \text{blue}\}$ be such that $c = c_k$ for infinitely many k. Then $A = \{x_k : c_k = c\}$ edges only with color c. □

Case 1: Unbounded

Suppose that (x_n) is unbounded, we have that: $\forall\ \epsilon \in \mathbb{R}\ \exists\ r \in \mathbb{N}$ such that $x_r > \epsilon$

choose $\epsilon_1 = \epsilon_f$, we have that: $\exists\ r_1 \in \mathbb{N}$ such that $x_{r_1} > \epsilon_f$

Consider: $S_1(n) = \{x_n\} \backslash \{x_1, x_2, \ldots\ldots, x_{r_1}\}$, This is also unbounded, for if it were bounded, the union of $S_1(n)$ with $\{x_1, x_2 \ldots\ldots, x_{r_1}\}$ will give me a bounded set, but this contradicts the initial statement that (x_n) is unbounded. Now for this set, choose $\epsilon_2 = x_{r_1}$. Then we have,
$\exists\ r_2 \in \mathbb{N}$ such that $x_{r_2} > x_{r_1} > \epsilon_f$ and $r_2 > r_1$

Consider: $\{x_n\} \backslash \{x_1, x_2, \ldots\ldots, x_{r_1}, \ldots, x_{r_2}\}$, This is also unbounded. Now for this set, choose $\epsilon_3 = x_{r_2}$. Then we have, $\exists\ r_3 \in \mathbb{N}$ such that $x_{r_3} > x_{r_2} > x_{r_1} > \epsilon_f$ and $r_3 > r_2 > r_1$. As such, this can be carried on forever to construct a monotone increasing sequence of x_{r_i}

Case 2: Bounded

Suppose that (x_n) is bounded. Start by defining the set $S_k = \{x_n : k \leq n\}$.
$S_1 = \{x_1, x_2, \ldots\ldots\}$, $S_2 = \{x_2, x_3, \ldots\ldots\}$. We can see that $S_{k+1} \subseteq S_k \subseteq \ldots \subseteq S_1$. These sets S_i are all bounded, and hence they all have supremums, which we can call $Sup(S_i) = U_i$

Subcase 1: finitely many i $\in \mathbb{N}$ such that $U_i \in S_i$

Suppose the sets $\{S_{r_1}, S_{r_2} \ldots S_{r_p}\}$ contain their own supremum, and the remaining S_i dont contain their supremums. This means that, for $i > r_p$, $U_i = U_{i+1} = U_{i+2} \ldots = U$ to see this, consider $S_i = \{x_i, x_{i+1}, x_{i+2} \ldots\}$ and $S_{i+1} = \{x_{i+1}, x_{i+2} \ldots\}$ and since none of $x_j : j \geq i$ is the supremum of S_i, it is clear that none of $x_j : j \geq i+1$ is the supremum of S_{i+1}

In this case, consider the (infinite) bounded set $S_{r_p+1} = \{x_{r_p+1}, x_{r_p+2}, \ldots\}$. The supremum of this set is U. Hence, $\exists s_1 \in S_{r_p+1}$ such that $U - \frac{1}{n_1} < s_1$

Moreover, $\forall j \in \mathbb{N}, \exists s(j) \in S_{r_p+1}$ such that $U - \frac{1}{j} < s(j)$

assertion:

There exists $n_2 > n_1$ such that $s_1 < U - \frac{1}{n_2}$

proof:

Suppose not, i.e, for a particuar j, no matter what $i \in \mathbb{N}$ we use, $s(j) \geq U - \frac{1}{i}$

$\iff\ \forall i \in \mathbb{N}\ s(j) + \frac{1}{i} \geq U$

$\iff\ \lim_{i\to\infty}(s(j) + \frac{1}{i}) = s(j) \geq U$ Which is absurd, hence, assumption is false.

Therefore, for all $j \in \mathbb{N}\ \exists i > j : i \in \mathbb{N}$ and $s(j) \in S_{r_p+1}$ such that $U - \frac{1}{j} < s(j) < U - \frac{1}{i}$

Choose j=1, then:

$\exists i_1 > 1 : i_1 \in \mathbb{N}$ and $s_1 \in S_{r_p+1}$ such that $U - 1 < s_1 < U - \frac{1}{i_1}$

Choose $j = i_1$, then:
$\exists i_2 > i_1 : i_2 \in \mathbb{N}$ and $s_2 \in S_{r_p+1}$ such that $U - 1 < s_1 < U - \frac{1}{i_1} < s_2 < U - \frac{1}{i_2}$

and continue this way, with $j = i_k$ to get s_{k+1}, hence forming an infinite sequence of s_i which is monotone increasing.

Subcase 2: infinitely many $i \in \mathbb{N}$ such that $U_i \in S_i$

We have that, $\{S_{k_1}, S_{k_2}, \ldots, S_{k_i} \ldots\}$ for all $i \in \mathbb{N}$ is an infinite set of S_i such that $U_i \in S_i$

notice then, that, for any l and $r \in \{k_1, k_2 \ldots\}$, $U_l \geq U_r$ if $r > l$. Choose $U_{k+i} = s_{k_i}$, and see that $s_{k_i} \geq s_{k_{i+1}}$, for all $i \in \mathbb{N}$. This gives us a monotonic decreasing sequence of s_i

A Lemma

We need the following lemma:
Lemma. *Every sequence* (x_n) *in* $\mathbb{R}$ *has a monotone subsequence.*

Proof.
Given natural numbers n, m with $n < m$, color the edge $e = \{n, m\}$ red if $x_n \leq x_m$ and color e blue otherwise.

- From Ramsey's Theorem, there is an infinite subset $A \subseteq \mathbb{N}$ such that $[A]^2$ is either red or blue. Write $A = (n_k)_{k \in \mathbb{N}}$ so that $n_k \leq n_{k+1}$ for each $k \in \mathbb{N}$.
- Then the subsequence $(x_{n_k})_{k \in \mathbb{N}}$ is monotone: it is non-decreasing if $[A]^2$ is red and it is non-increasing if $[A]^2$ is blue.

$\square$

Proving The Bolzano-Weierstrass Theorem

Theorem (Bolzano-Weierstrass)**.** *Every bounded sequence* (x_n) *of real numbers has a convergent subsequence.*

Proof.

From the lemma, (x_n) has a monotone subsequence which is bounded as (x_n) is. Since bounded monotone sequences converge, this subsequence is convergent. $\square$

Acknowledgements

- I would like to give my most sincere thanks Professor Jakobson for his kindness in helping me prepare this presentation.
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Thank you all for listening to my talk!

Any and all questions are appreciated. ☺