Ramsey's Theorem and Applications to Analysis

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ABSTRACT. This document contains the notes for my talk on June 12, 2024 on Ramsey Theory. We will explore some classical results from the theory and use them to prove the famous Bolzano-Weierstrass theorem.

1. Introduction

Ramsey Theory explores the underlying structure emerging in "large enough" complex systems. For example, in 1930 Frank Ramsey proved that for each $k \in \mathbb{N}$ there is a sufficiently large $n \in \mathbb{N}$ such that in any red-blue coloring of the edges of the complete graph K_n there is a set of k vertices joined by edges of the same color [3].

Another seminal result is Van der Waerden's theorem [1], which states that for all positive integers $r, k \in \mathbb{N}$, there is a large enough $n \in \mathbb{N}$ such that if we color the integers in $[n] \coloneqq \{1, 2, \ldots, n\}$ using k colors, one can always find a set of r monochromatic integers in arithmetic progression (i.e. there is a number $d \geq 0$ such that the difference of all consecutive terms is d).

Therefore, Ramsey Theory essentially implies that *complete disorder is impossible* in large enough systems.

2. Ramsey's Theorem

We have already seen the finite version of Ramsey's theorem in the introduction. Here is its statement:

Theorem 2.1. *For each $k \in \mathbb{N}$ there is a positive integer $n = R(k)$ such that in any red-blue coloring of the edges of K_n there is a set X of k vertices such that all edges between vertices in X are the same color.*

Pondering a little bit, a natural question may arise: *is there a similar kind of structure when coloring the edges of an infinite graph?* It turns out that the answer is yes, and this result is an infinite version of Ramsey's theorem.

For a set X, let $[X]^2 = \{A \subseteq X : |A| = 2\}$ be the collection of two element subsets of X. We are now ready to state and prove the (infinite) Ramsey theorem.

Theorem 2.2 (Ramsey's Theorem [2])**.** *For every red-blue coloring of $[\mathbb{N}]^2$ there is an infinite subset $A \subseteq N$ of naturals such that every pair in $[A]^2$ is the same color.*

Proof. Define $A_0 = \mathbb{N}$ and fix a point $x_0 \in A_0$. Since x_0 is incident to infinitely many edges which are each either red or blue, it follows from the Pigeonhole principle that there is a color $c_0 \in \{\text{red, blue}\}$ such that x_0 is incident to infinitely many edges with color c_0.

We define the set $A_1 = \{x \in A_0 : \{x_0, x\} \text{ has color } c_0\}$, which is infinite by the choice of c_0. Now fix $k \geq 2$ and assume that the sets $A_1, A_2, \ldots, A_{k-1}$ have been defined inductively and are infinite. Fix a point $x_{k-1} \in A_{k-1}$. Note that A_{k-1} is infinite, so by this same Pigeonhole argument we can find a color $c_{k-1} \in \{\text{red, blue}\}$ so that x_{k-1} has infinitely many neighbours $y \in A_{k-1}$ so that $\{x_{k-1}, y\}$ has color c_{k-1}. Correspondingly, we define the set

$$A_k = \{x \in A_{k-1} : \{x_{k-1}, x\} \text{ has color } c_{k-1}\},$$

which is infinite by the choice of c_{k-1}.

Continuing this way, we obtain sequences (A_k), (x_k), and (c_k) with the following three properties for every $k \in \mathbb{N}$:

(1) $x_k \in A_k$
(2) A_k is infinite
(3) $A_{k+1} \subseteq A_k$

Now let $c \in \{\text{red, blue}\}$ be a color such that $c = c_k$ for infinitely many k (applying the Pigeonhole argument one last time). Hence the set

$$A = \{x_k : c_k = c\}$$

is infinite. We claim that all edges between points in A are of color c. Indeed, if $\{x_i, x_j\} \in [A]^2$ with $i < j$ then from property (3) we have $A_j \subseteq A_{j-1} \subseteq \cdots \subseteq A_{i+1}$ so that $x_j \in A_{i+1}$. By the construction of $A_{i+1} = \{x \in A_i : \{x_i, x\} \text{ has color } c_i\}$, the edge $\{x_i, x_j\}$ has color $c_i = c$. Since x_i and x_j were arbitrary, the proof is complete. □

3. Applications to Analysis

Although this theorem is interesting in its own right, it provides us with a really elegant way to prove the Bolzano-Weierstrass theorem. I must note that I learnt about this technique in Professor Anush Tserunyan's analysis class Math 254 during my first semester here at McGill.

We will first use the following lemma:

Lemma 3.1. *Every sequence of reals has a monotone subsequence.*

Proof. Given a sequence $(x_n) \subseteq \mathbb{R}$ and natural numbers n, m with $n < m$, we color the edge $e = \{n, m\}$ red if $x_n \leq x_m$ and we color e blue otherwise. Then Theorem 2.2 implies that there is an infinite subset $A \subseteq \mathbb{N}$ such that $[A]^2$ is either red or blue. Write $A = (n_k)_{k\in\mathbb{N}}$. Then the subsequence $(x_{n_k})_{k\in\mathbb{N}}$ is monotone: it is non-decreasing if $[A]^2$ is red and it is non-increasing if $[A]^2$ is blue. $\square$

We now state and prove the Bolzano-Weierstrass theorem.

Theorem 3.2 (Bolzano-Weierstrass)**.** *Every bounded sequence of real numbers has a convergent subsequence.*

Proof. Let (x_n) be a bounded sequence of real numbers. Then from Lemma 3.1, it has a monotone subsequence $(x_{n_k})_{k\in\mathbb{N}}$. Since (x_n) is bounded, so is $(x_{n_k})_{k\in\mathbb{N}}$. Since bounded monotone sequences converge, $(x_{n_k})_{k\in\mathbb{N}}$ is a convergent subsequence. $\square$

4. Final Remarks

Ramsey Theory is a fundamental branch of combinatorics. Not only are Ramsey-theoretic problems exciting on their own, but they also have some really nice applications to other branches of math such as analysis. In particular, the classical infinite Ramsey theorem offers an especially elegant proof of the Bolzano-Weierstrass theorem.

References

[1] B. L. Van der Waerden. Beweis einer baudetschen vermutung. *Nieuw Arch. Wiskunde*, 15:212–216, 1927.
[2] J. Misra. A proof of infinite ramsey theorem. July 2012.
[3] F. P. Ramsey. On a problem of formal logic. *Proceedings of the London Mathematical Society*, s2-30(1):264–286, 1930.