

Ramsey Theory and Applications to Analysis

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Agenda

Plan for today:

- ▶ The big picture of Ramsey Theory
- ▶ Classical examples
 - ▶ Diagonal Ramsey number & $R(3) = 6$
 - ▶ Van der Waerden numbers
- ▶ Ramsey's theorem (infinite version)
- ▶ A *super elegant* application of Ramsey's theorem to analysis

What is Ramsey Theory?

Ramsey Theory: The Big Picture

Ramsey Theory explores the underlying structure emerging in “large enough” complex systems.

Main idea: In search of a particular kind of structure in a complex system, we ask: *how large must our system be to guarantee the existence of this structure?*

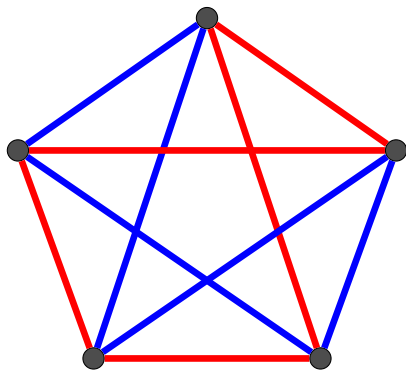
Some classical examples

Diagonal Ramsey Numbers

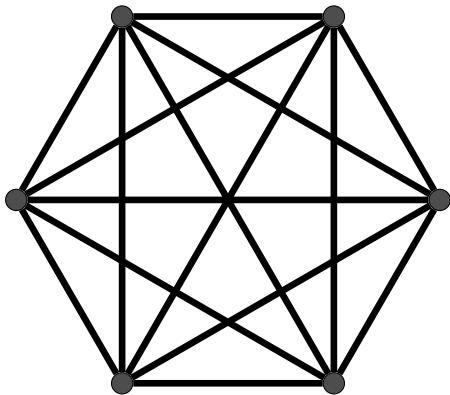
The k 'th **diagonal Ramsey number** $R(k)$ is the smallest integer N such that in any colouring of the edges of the complete graph K_N in either **red** or **blue**, there is always a set of k vertices joined by all red edges or all blue edges.

- ▶ *Ramsey's Theorem (finite version):* $R(k)$ exists for every positive integer k .
- ▶ $R(1) = 1$, $R(2) = 2$, $R(3) = 6$.

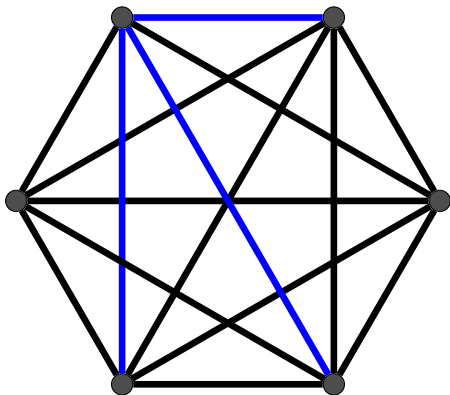
K_5 with No Monochrome $K_3 \Leftrightarrow R(3) \geq 6$



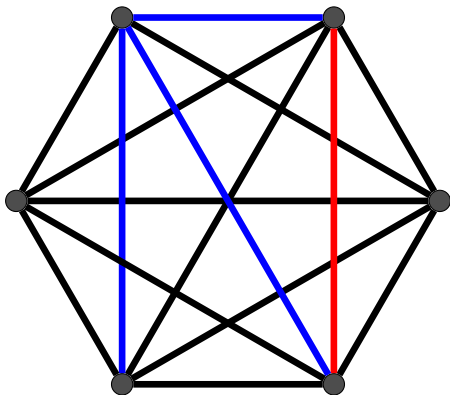
$$R(3) \leq 6$$



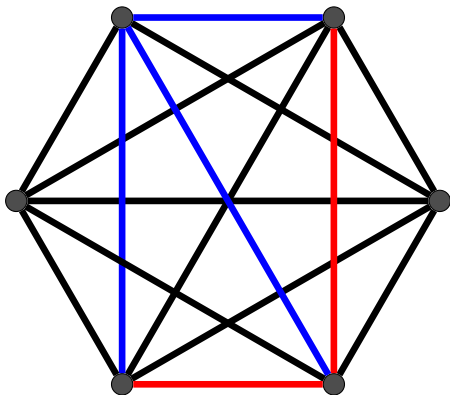
$$R(3) \leq 6$$



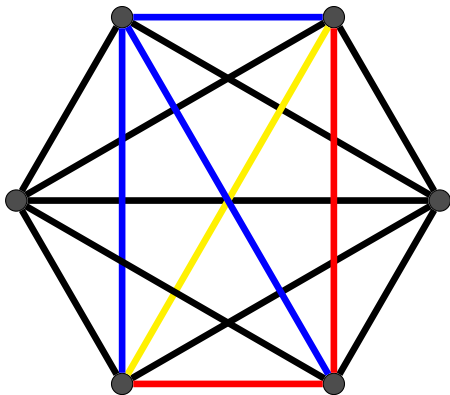
$$R(3) \leq 6$$



$$R(3) \leq 6$$



$$R(3) \leq 6$$



Therefore $R(3) \geq 6$ and $R(3) \leq 6 \Leftrightarrow R(3) = 6$. \square

Van der Waerden Numbers

For $r, k \in \mathbb{N}$, the **Van der Waerden number** $W(k, r)$ is the smallest integer N such that in any coloring of $[N]$ in k colors there is a set of r monochromatic integers in arithmetic progression.

- ▶ *Van der Waerden's Theorem:* $W(k, r)$ exists for all positive integers r, k .
- ▶ Positive integers a_1, a_2, \dots, a_r are in **arithmetic progression** if there is an integer $d \geq 0$ such that the difference of consecutive terms is d ; that is, for $1 \leq k \leq N - 1$,

$$a_{k+1} - a_k = d.$$

$$W(2,3) = 9$$

1 2 3 4 5 6 7 8 9

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1 2 3 4 5 6 7 8 9

Then $\{4, 6, 8\}$ is a monochromatic arithmetic progression of length 3 with $d = 2$.

Ramsey's Theorem (infinite version)

Notation

We fix some notation: For a set X , let $[X]^2$ denote the family of all two element subsets of X , i.e. the *edges* between points in X .

Ramsey's Theorem (infinite version)

In every coloring of $[\mathbb{N}]^2$ in either red or blue there is an infinite subset A of \mathbb{N} such that all edges in $[A]^2$ are the same color.

Proof of Ramsey's Theorem (infinite version)

Statement: In every red-blue coloring of $[\mathbb{N}]^2$ there is an infinite subset A of \mathbb{N} such that all edges in $[A]^2$ are the same color.

Proof.

Set $A_0 = \mathbb{N}$, fix $x_0 \in A_0$. By Pigeonhole, $\exists c_0 \in \{\text{red}, \text{blue}\}$ with $A_1 = \{x \in A_0 : \{x_0, x\} \text{ has color } c_0\}$ infinite. Fix $x_1 \in A_1$ and apply the same argument: $\exists c_1 \in \{\text{red}, \text{blue}\}$ so that the set $A_2 = \{x \in A_1 : \{x_1, x\} \text{ has color } c_1\}$ is infinite. Continue to obtain the following for every $k \in \mathbb{N}$:

1. $x_k \in A_k$,
2. $A_{k+1} \subseteq A_k$,
3. $\forall x \in A_{k+1} : \{x_k, x\} \text{ has color } c_k$.

Let $c \in \{\text{red}, \text{blue}\}$ be such that $c = c_k$ for infinitely many k .

Then $A = \{x_k : c_k = c\}$ has edges only with color c . □

Case 1: Unbounded

Suppose that (x_n) is unbounded, we have that: $\forall \epsilon \in \mathbb{R} \exists r \in \mathbb{N}$ such that $x_r > \epsilon$

choose $\epsilon_1 = \epsilon_f$, we have that: $\exists r_1 \in \mathbb{N}$ such that $x_{r_1} > \epsilon_f$

Consider: $S_1(n) = \{x_n\} \setminus \{x_1, x_2, \dots, x_{r_1}\}$, This is also unbounded, for if it were bounded, the union of $S_1(n)$ with $\{x_1, x_2, \dots, x_{r_1}\}$ will give me a bounded set, but this contradicts the initial statement that (x_n) is unbounded. Now for this set, choose $\epsilon_2 = x_{r_1}$. Then we have, $\exists r_2 \in \mathbb{N}$ such that $x_{r_2} > x_{r_1} > \epsilon_f$ and $r_2 > r_1$

Consider: $\{x_n\} \setminus \{x_1, x_2, \dots, x_{r_1}, \dots, x_{r_2}\}$, This is also unbounded. Now for this set, choose $\epsilon_3 = x_{r_2}$. Then we have, $\exists r_3 \in \mathbb{N}$ such that $x_{r_3} > x_{r_2} > x_{r_1} > \epsilon_f$ and $r_3 > r_2 > r_1$. As such, this can be carried on forever to construct a monotone increasing sequence of x_r .

Case 2: Bounded

Suppose that (x_n) is bounded. Start by defining the set $S_k = \{x_n : k \leq n\}$.

$S_1 = \{x_1, x_2, \dots\}$, $S_2 = \{x_2, x_3, \dots\}$. We can see that $S_{k+1} \subseteq S_k \subseteq \dots \subseteq S_1$. These sets S_i are all bounded, and hence they all have supremums, which we can call $\sup(S_i) = U_i$

Subcase 1: finitely many $i \in \mathbb{N}$ such that $U_i \in S_i$

Suppose the sets $\{S_{r_1}, S_{r_2}, \dots, S_{r_p}\}$ contain their own supremum, and the remaining S_i don't contain their supremums. This means that, for $i > r_p$, $U_i = U_{i+1} = U_{i+2}, \dots = U$ to see this, consider $S_i = \{x_i, x_{i+1}, x_{i+2}, \dots\}$ and $S_{i+1} = \{x_{i+1}, x_{i+2}, \dots\}$ and since none of $x_j : j \geq i$ is the supremum of S_i , it is clear that none of $x_j : j \geq i+1$ is the supremum of S_{i+1}

In this case, consider the (infinite) bounded set $S_{r_p+1} = \{x_{r_p+1}, x_{r_p+2}, \dots\}$. The supremum of this set is U . Hence, $\exists s_1 \in S_{r_p+1}$ such that $U - \frac{1}{n_1} < s_1$

Moreover, $\forall j \in \mathbb{N}, \exists s(j) \in S_{r_p+1}$ such that $U - \frac{1}{j} < s(j)$

assertion:

There exists $n_2 > n_1$ such that $s_1 < U - \frac{1}{n_2}$

proof:

Suppose not, i.e., for a particular j , no matter what $i \in \mathbb{N}$ we use, $s(j) \geq U - \frac{1}{j}$

$$\Leftrightarrow \forall i \in \mathbb{N} s(j) + \frac{1}{j} \geq U$$

$$\Leftrightarrow \lim_{i \rightarrow \infty} (s(j) + \frac{1}{j}) = s(j) \geq U \text{ Which is absurd, hence, assumption is false.}$$

Therefore, for all $j \in \mathbb{N} \exists i > j : i \in \mathbb{N}$ and $s(j) \in S_{r_p+1}$ such that $U - \frac{1}{j} < s(j) < U - \frac{1}{i}$

Choose $j=1$, then:

$$\exists i_1 > 1 : i_1 \in \mathbb{N} \text{ and } s_1 \in S_{r_p+1} \text{ such that } U - 1 < s_1 < U - \frac{1}{i_1}$$

Choose $j = i_1$, then:

$$\exists i_2 > i_1 : i_2 \in \mathbb{N} \text{ and } s_2 \in S_{r_p+1} \text{ such that } U - 1 < s_1 < U - \frac{1}{i_1} < s_2 < U - \frac{1}{i_2}$$

and continue this way, with $j = i_k$ to get s_{k+1} , hence forming an infinite sequence of s_i which is monotone increasing.

Subcase 2: infinitely many $i \in \mathbb{N}$ such that $U_i \in S_i$

We have that, $\{S_{k_1}, S_{k_2}, \dots, S_{k_r}, \dots\}$ for all $i \in \mathbb{N}$ is an infinite set of S_i such that $U_i \in S_i$

notice then, that, for any i and $r \in \{k_1, k_2, \dots\}$, $U_i \geq U_r$ if $r > i$. Choose $U_{k_{i_0}} = s_{k_i}$, and see that $s_k \geq s_{k_{i+1}}$, for all $i \in \mathbb{N}$. This gives us a monotonic decreasing sequence of s_i

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answered Sep 2, 2023 at 5:30



nickbros123
423 3 12

A Lemma

We need the following lemma:

Lemma. *Every sequence (x_n) in \mathbb{R} has a monotone subsequence.*

Proof.

Given natural numbers n, m with $n < m$, color the edge $e = \{n, m\}$ **red** if $x_n \leq x_m$ and color e **blue** otherwise.

- ▶ From Ramsey's Theorem, there is an infinite subset $A \subseteq \mathbb{N}$ such that $[A]^2$ is either red or blue. Write $A = (n_k)_{k \in \mathbb{N}}$ so that $n_k \leq n_{k+1}$ for each $k \in \mathbb{N}$.
- ▶ Then the subsequence $(x_{n_k})_{k \in \mathbb{N}}$ is monotone: it is non-decreasing if $[A]^2$ is red and it is non-increasing if $[A]^2$ is blue.



Proving The Bolzano-Weierstrass Theorem

Theorem (Bolzano-Weierstrass). *Every bounded sequence (x_n) of real numbers has a convergent subsequence.*

Proof.

From the lemma, (x_n) has a monotone subsequence which is bounded as (x_n) is. Since bounded monotone sequences converge, this subsequence is convergent. □

Acknowledgements

- ▶ I would like to give my sincere thanks to Professor Jakobson for giving me the opportunity to give this talk.
- ▶ I also dearly thank Professor Anush Tserunyan for teaching me about this clever combinatorial proof of the BW theorem. I learnt this during my first semester in one of her Math 254 lectures, and it has stuck with me ever since.

Thank you all for listening to my talk!

Any and all questions are appreciated. 😊