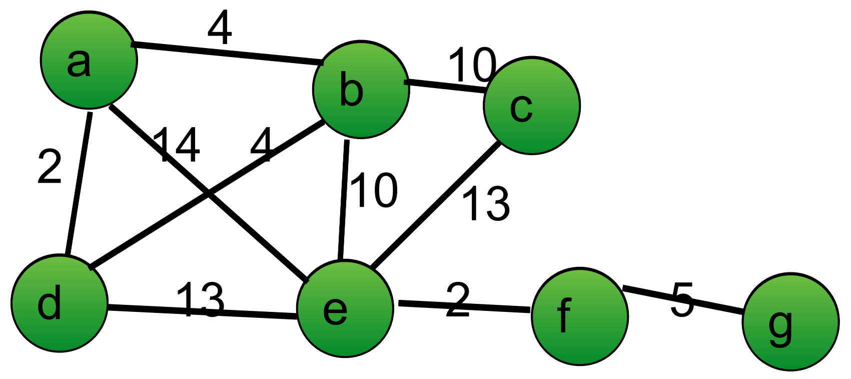
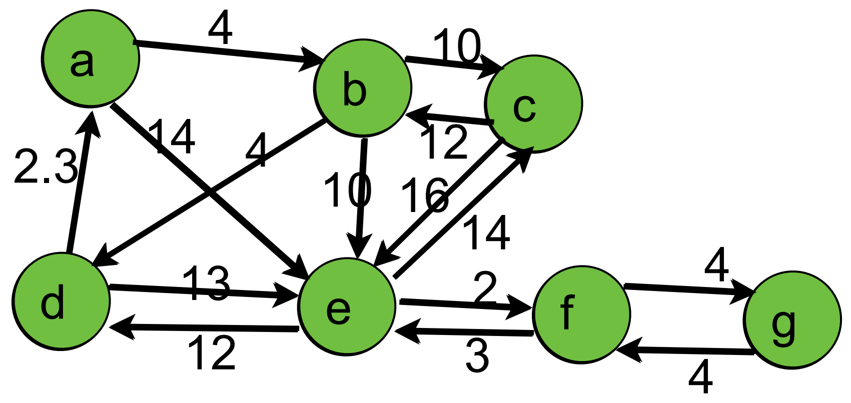




Lecture 22

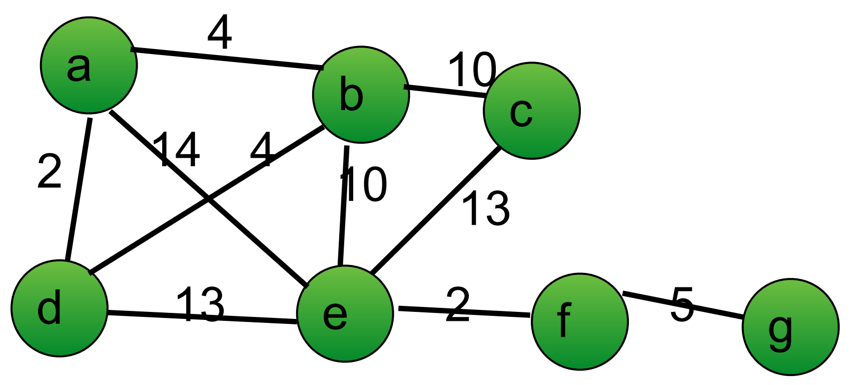
Weighted Graphs and Dijkstra's Algorithm

If the graph was a computer network where the edge weights were the time it took for a packet to traverse the edge, how could we find the path (route) our message (packets) as fast as possible?

Check this out! https://qiao.github.io/PathFinding.js/visual/

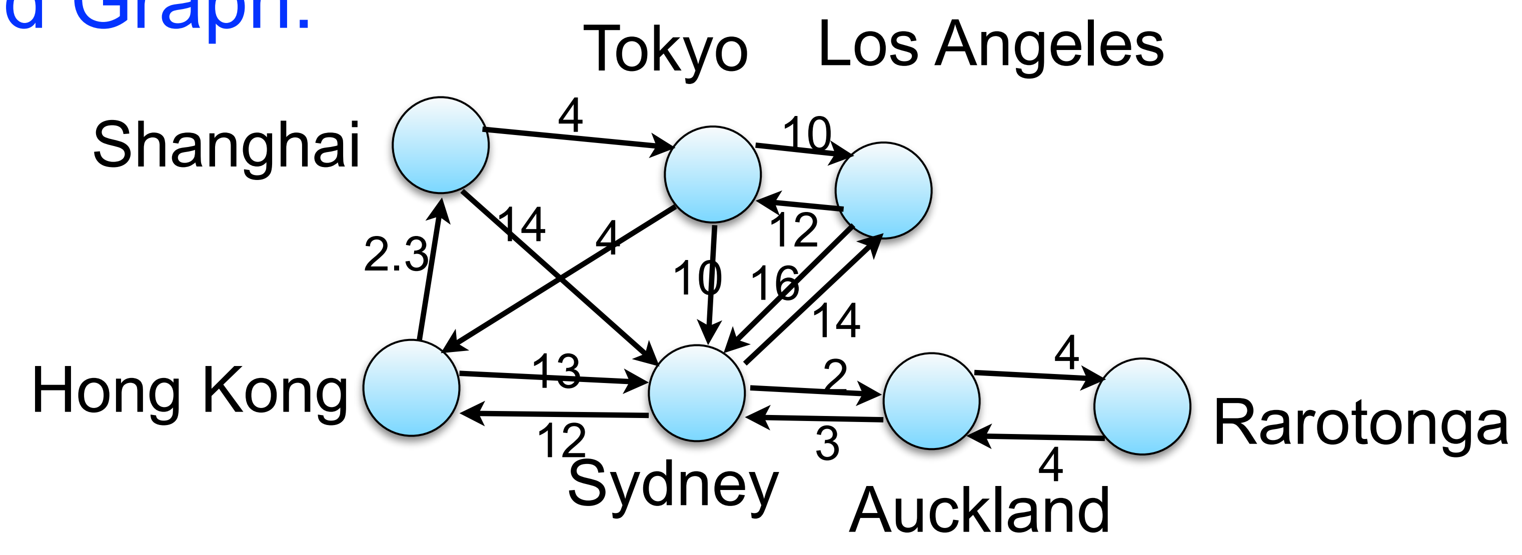
Weighted Graphs

- For many applications, it’s useful to associate a “weight” (cost) with each edge:
  - Transportation: price of ticket between two cities, or physical distance, or travel time, or …
  - Communication: congestion on link between two routers
  - …
- Adjacency list
  - Store weight along with target node for each edge
- Adjacency matrix
  - Store weight, rather than true/false for each pair of vertices with some default value indicating “no edge”



$$\begin{tikzcd}
a \arrow[r] & b\ (4) \arrow[r] & e\ (14) \arrow[r] & d\ (2) & \\
b \arrow[r] & a(4) \arrow[r] & d\ (4) \arrow[r] & e\ (10) \arrow[r] & c\ (10) \\
c \arrow[r] & e\ (13) \arrow[r] & b\ (10) & & \\
\dots & & & &
\end{tikzcd}$$

Directed Graph:



Adjacency List:

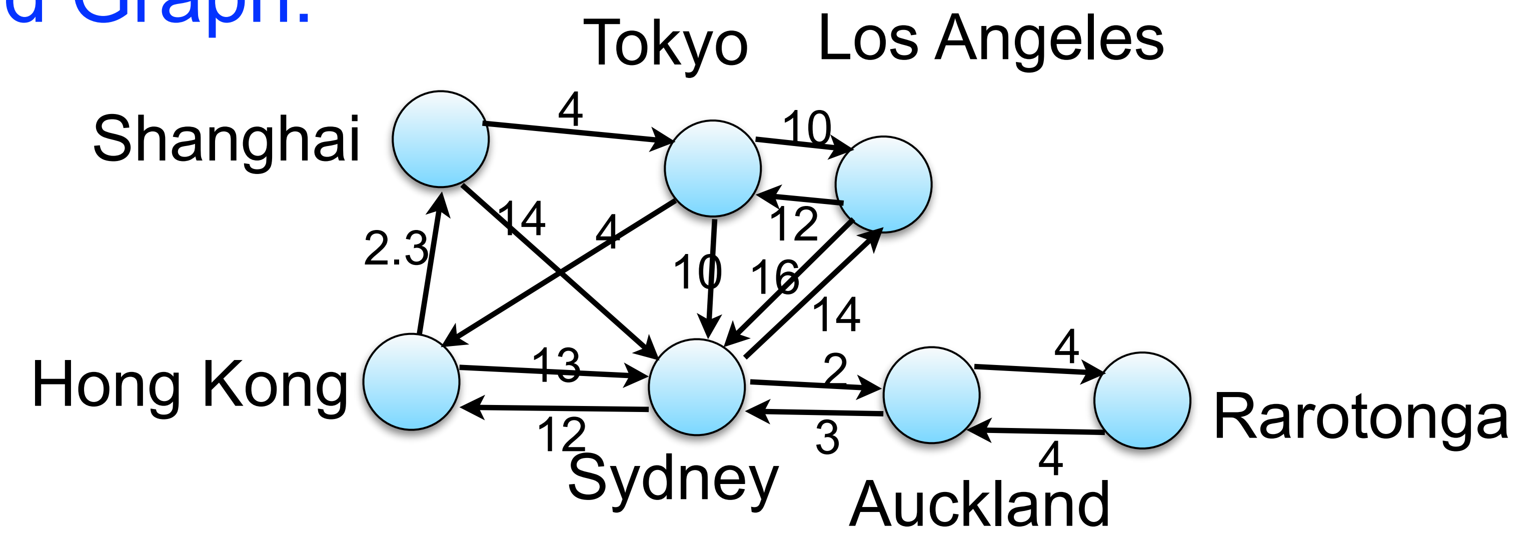
Shanghai → Tokyo (4) → Sydney (14)

Tokyo → Hong Kong (4) → Sydney (10) → Los Angeles (10)

Hong Kong → Sydney (13) → Shanghai (2.3)

...

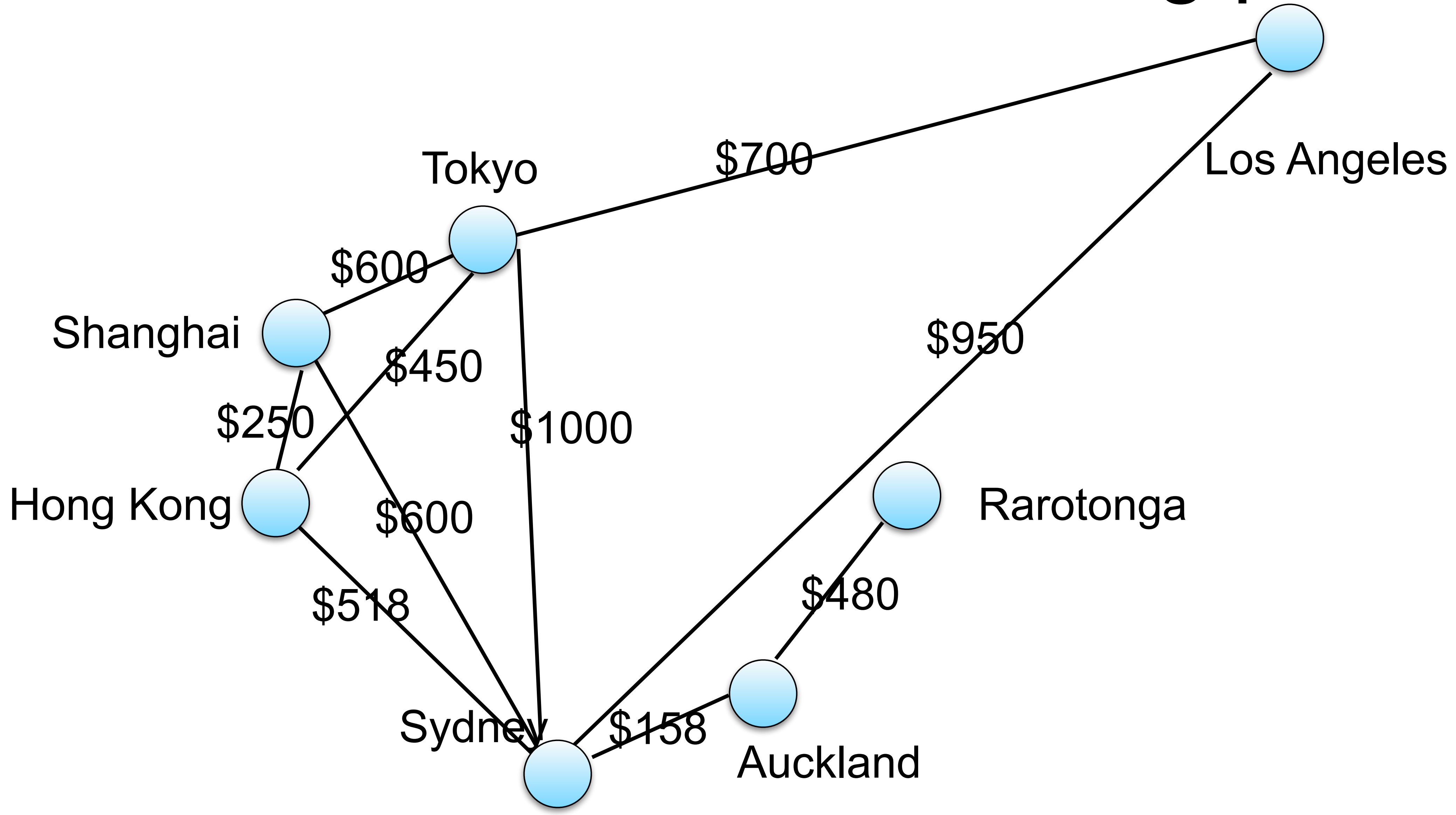
Directed Graph:



Adjacency Matrix:

	Shanghai	Hong Kong	Sydney	Tokyo	Auckland	Los Angeles	Rarotonga
Shanghai			14	4			
Hong Kong	2.3		13				
Sydney		12			2	14	
Tokyo		4	10			10	
Auckland			3				4
Los Angeles			16	12			
Rarotonga					4		

What is the cost of the following path?



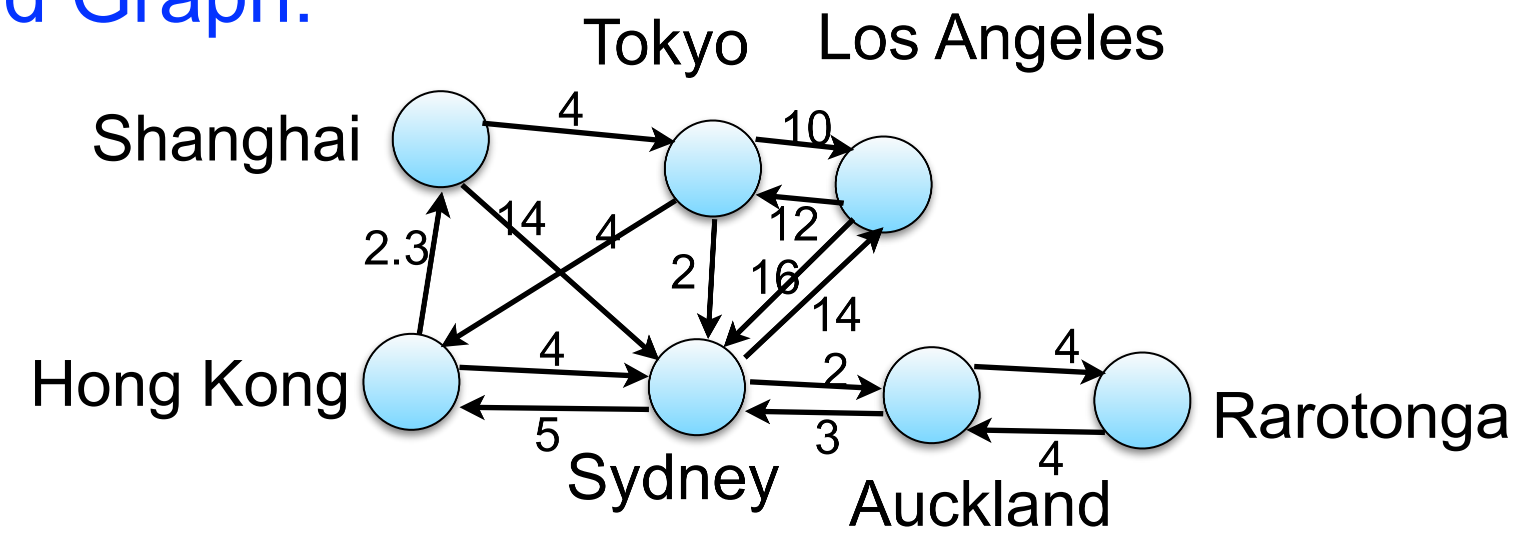
A path: Los Angeles,Tokyo, Shanghai, Sydney, Auckland   Not the best path!

This weighted path has length 2058.

Shortest Path Problem in Weighted Graphs

- The weight/cost of a path is the sum of the weights/costs of its edges
- Shortest path between vertices v and w is the path connecting v and w with minimal weight/cost
- Note: shortest path from v to w is not necessarily the path with fewest edges.

Directed Graph:



The shortest unweighted path from Shanghai to Auckland has length 2.

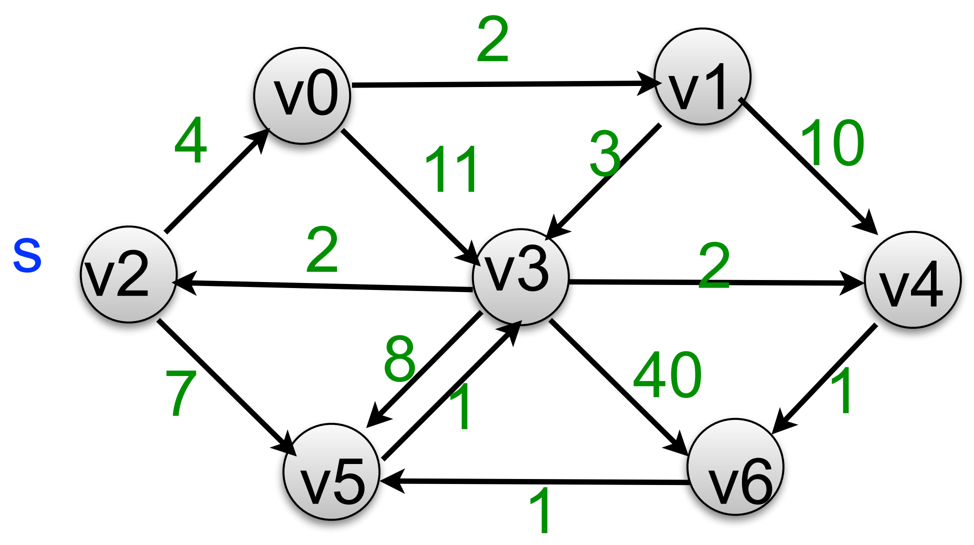
The path is Shanghai, Sydney, Auckland

This path has cost 16.

The shortest weighted path from Shanghai to Auckland has length 3.
The path is Shanghai, Tokyo, Sydney, Auckland

This path has cost 8.

Weighted Shortest-path



What is the shortest unweighted path from v2 to v6?

What is the shortest weighted path from v2 to v6?

Dijkstra's Algorithm

Finds shortest weighted path

- requires all edge weights to be positive
- finds the shortest path from a source node **s** to all the other nodes in **G**
- **G** can be directed or undirected (also true for unweighted shortest graph algorithm)

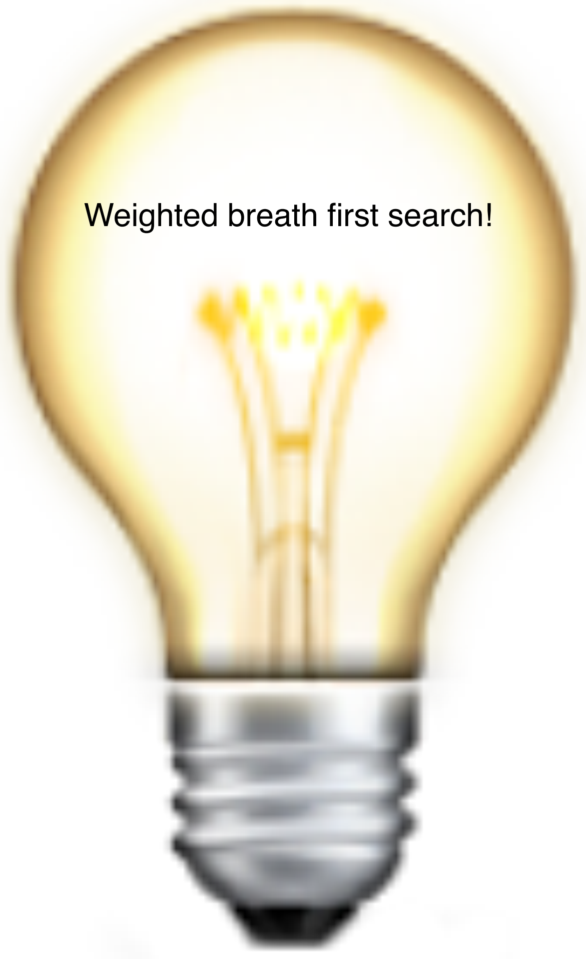
Dijkstra's Algorithm

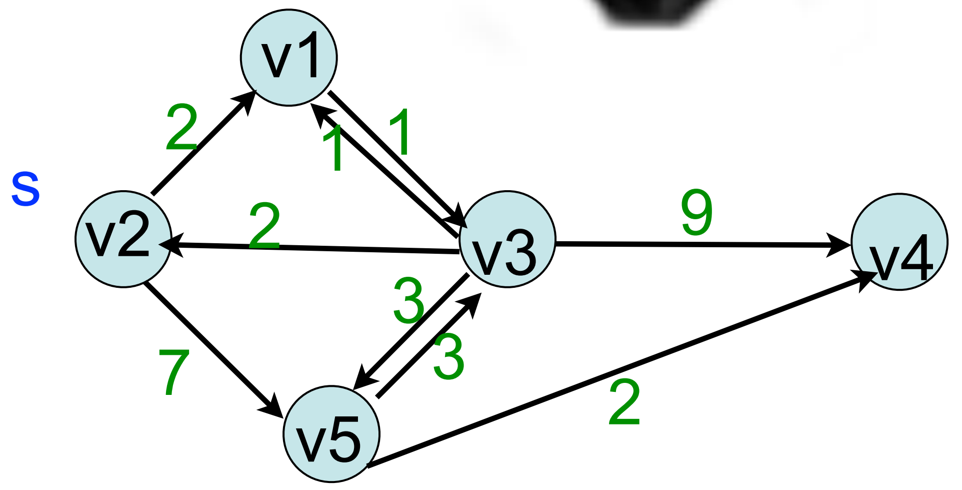
- Modification of BFS:
  - Three categories of nodes:
    - visited (finished),
    - discovered (but not visited),
    - undiscovered;
  - initially s is discovered and has distance 0. And all other nodes are undiscovered and have infinite distance.
  - At each stage, choose the discovered/unvisited node, v, that has smallest distance (total cost) from s:
    - For each edge adjacent to v (that is not finished)
      - Discover neighbor OR
      - Reduce cost of neighbor already discovered, if a shorter path has been found
    - set v to having been visited (finished)

Pseudocode for Dijkstra's Algorithm

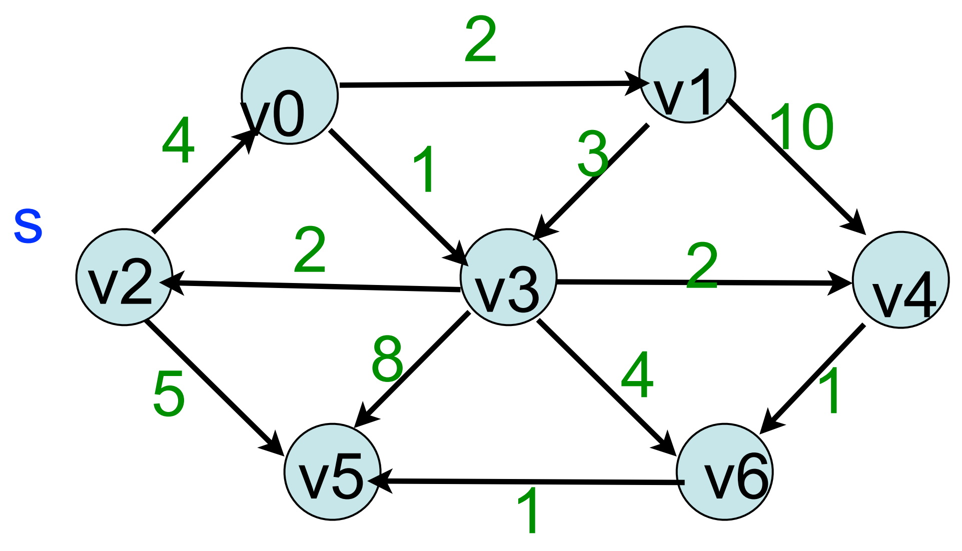
```
Input: Graph G, source vertex s
Initialization
  For every vertex, v, set dist[v] to ∞
  For every vertex, v, set prev[v] to sentinel
  Mark all nodes as undiscovered

  Mark s as discovered
  dist[s] = 0
main loop
  While there exists a discovered (and not visited) node {
      Find a discovered vertex v with min dist[v]
      For each neighbor w of v that has not been visited {
         If (dist(v) + weight(v,w) < dist(w)){
              update dist(w) to be dist(v) + weight(v,w)
              update prev[w] to be v
              mark w as discovered
         }
       mark v as visited (v is no longer marked as discovered)
      }
  }
```



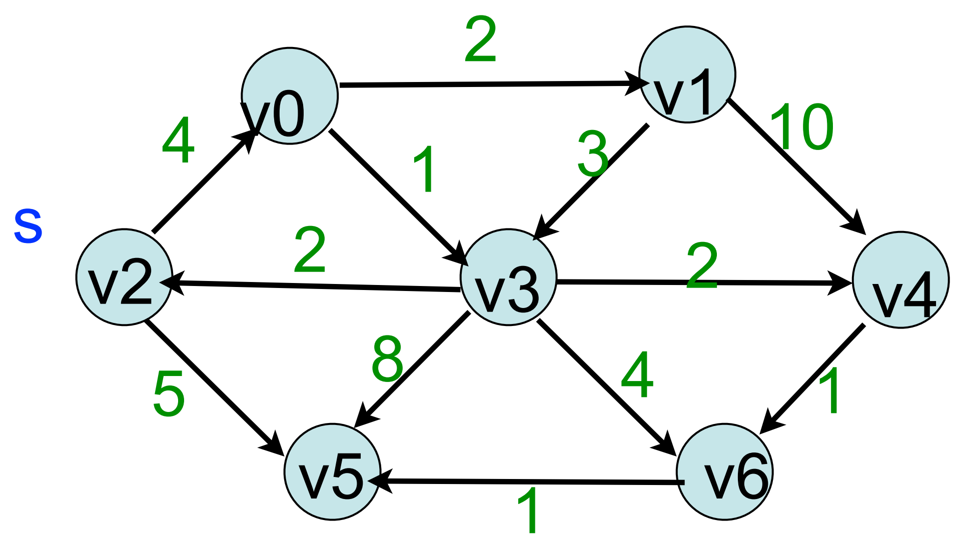


Weighted Shortest-path
Initialization



	distance/predecessor								
phase	v0	v1	v2	v3	v4	v5	v6	visiting	discovered&not visited
init	∞	∞	0/-	∞	∞	∞	∞		v2

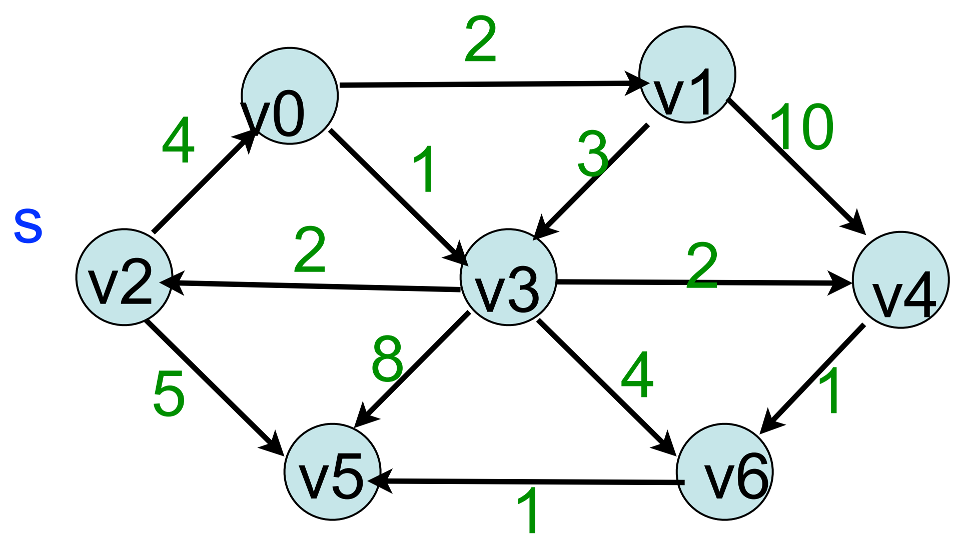
Weighted Shortest-path



distance/predecessor

phase	v0	v1	v2	v3	v4	v5	v6	visiting	discovered&not visited
init	∞	∞	0/-	∞	∞	∞	∞		v2
1	4/v2	∞	0/-	∞	∞	5/v2	∞	v2	v0 , v5

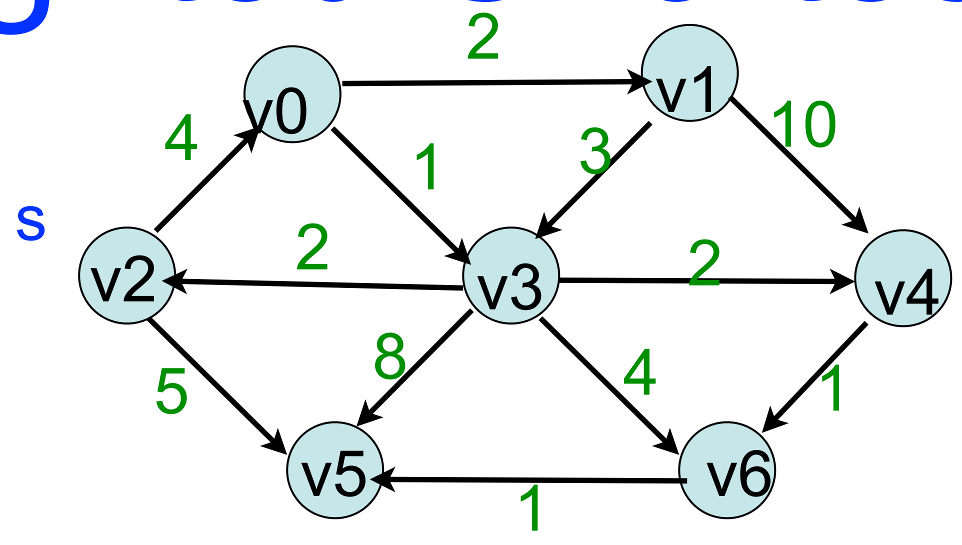
Weighted Shortest-path



distance/predecessor

phase	v0	v1	v2	v3	v4	v5	v6	visiting	discovered&not visited
init	∞	∞	0/-	∞	∞	∞	∞		v2
1	4/v2	∞	0/-	∞	∞	5/v2	∞	v2	v0, v5
2	4/v2	6/v0	0/-	5/v0	∞	5/v2	∞	v0	v5 v1 v3

Weighted Shortest-path



		distance/predecessor							
phase	v0	v1	v2	v3	v4	v5	v6	visiting	discovered&not visited
init	∞	∞	0/-	∞	∞	∞	∞		v2
1	4/v2	∞	0/-	∞	∞	5/v2	∞	v2	v0, v5
2	4/v2	6/v0	0/-	5/v0	∞	5/v2	∞	v0	v5 v1 v3
3									

Why does it work?

Invariants:

- Whenever a node is finished, the shortest estimated distance cannot be improved. A shortest path from s to this node has been found.
- At any stage the distances to discovered nodes represent the shortest distance via path consisting only of finished nodes.
  - Can prove this by induction.

At the final stage, all nodes are finished

=> algorithm is correct

Details/Variations

Data structure for finding min weight discovered nodes?

Version 0

- Just use the distance vector
  - Mark nodes when finished
  - At each stage, find discovered node v with minimum dist(v) and process its neighbors
- Running time $O(|V|^2)$
  - Find discovered node with smallest distance |V| times, O|V| each time = $O(|V|^2)$
  - Total for neighbor processing is O(|E|)
  - Running time is $O(|V|^2 + |E|) = O(|V|^2)$
- Can we do better for sparse graphs?

Variation 1

- Use data structure that supports insert, deleteMin, and decreaseKey efficiently
- When discover a node, insert into data structure
- Need to decrease distances of neighbors
- Our binary heap doesn’t support decreaseKey ☹
- Fancy data structures: Fibonacci heap (Omit details)

Variation 2

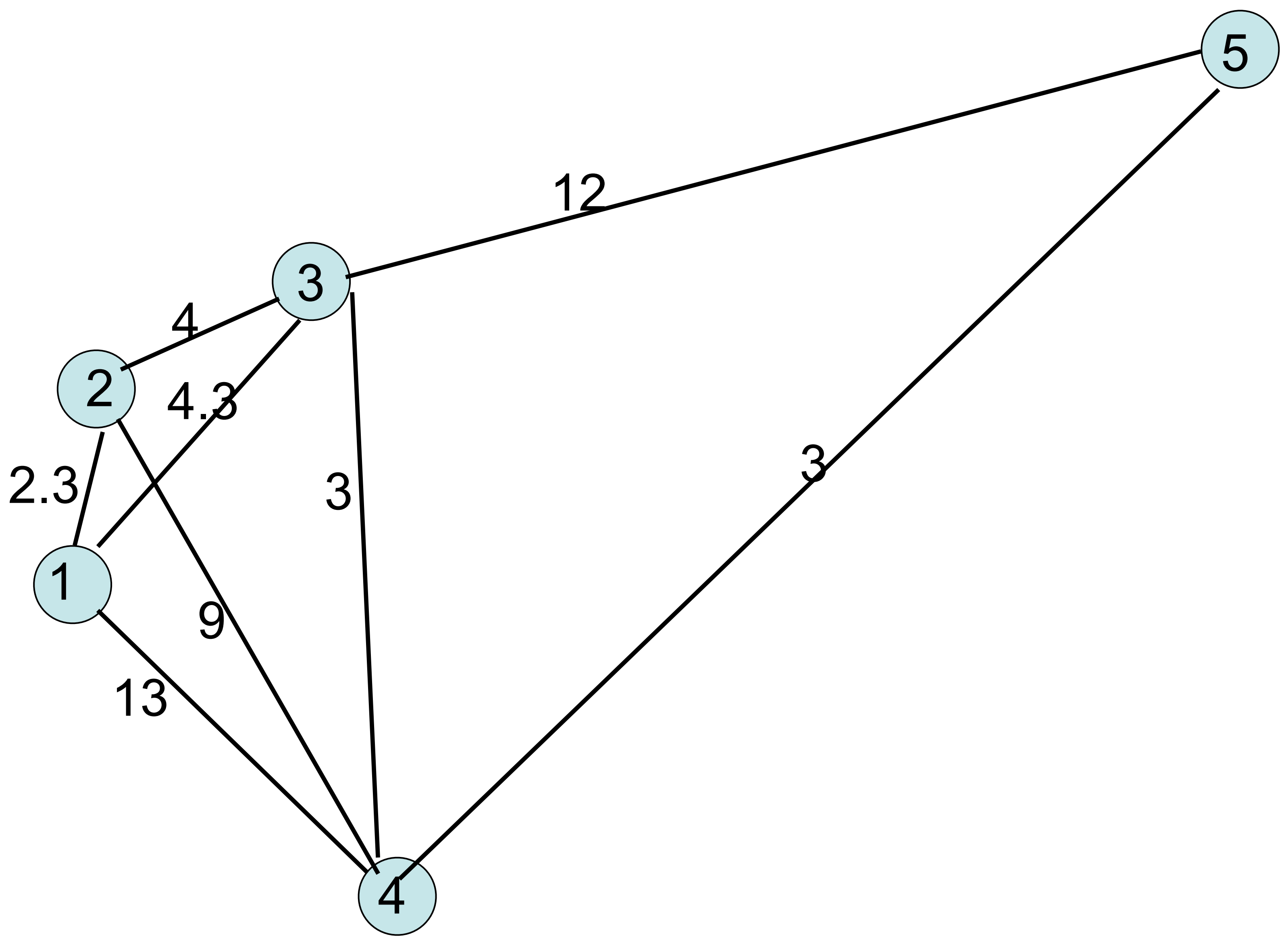
- Store (node, distance) pairs in ordinary binary heap, with distances as keys
- Insert node (with distance) when discovered
- Instead of decreasing key of w from d1 to d2, leave old pair (w, d1) in heap and insert new pair (w, d2)
- When do deleteMin, check if node already finished. If so, discard and try again.
  - Note that if (w,d1) and (w, d2) are both in the heap, with d2 < d1, item (w,d2) will be returned earlier, so (w,d1) will eventually be discarded.

Running time of Variation 2

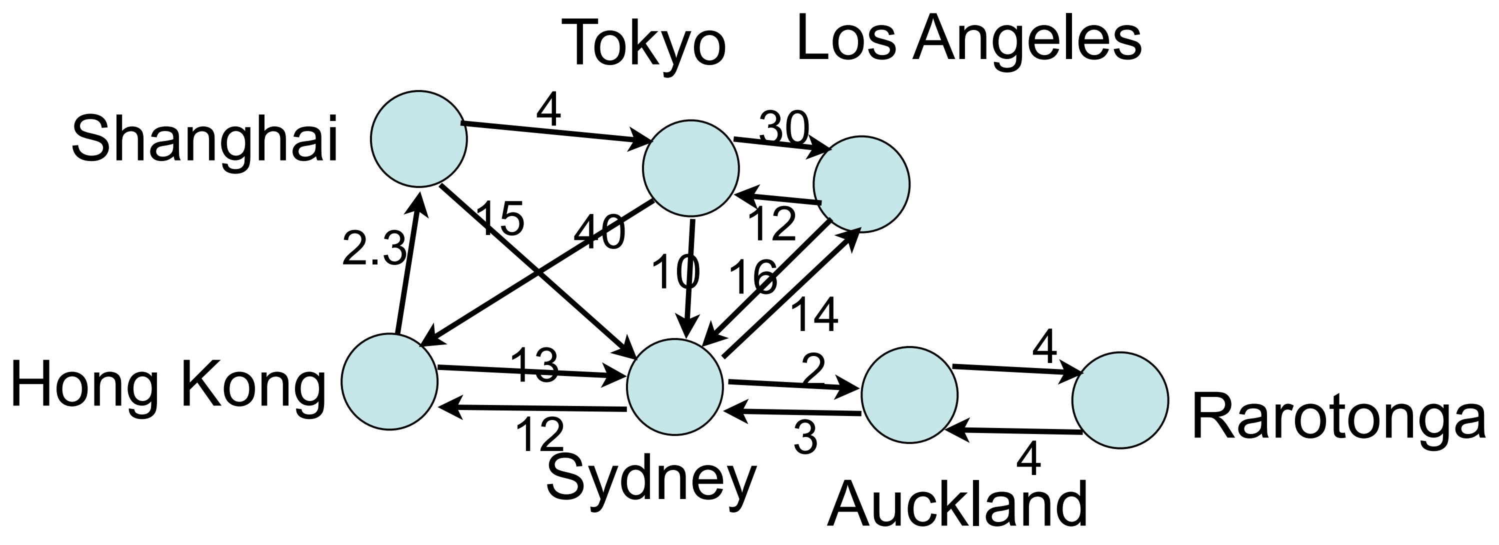
$O(|E| \log |V|)$

- Binary heap size is at most $|E|$, so insert and deleteMin take $O(\log |E|)$.
- At most $|E|$ inserts, and $|E|$ deleteMins
- Initialization is $O(|V|)$
- Assuming all nodes reachable from s, so $|E| >= |V|-1$, total is $O(|E| \log |E|)$

  $= O(|E| \log |V|)$ since $|E|$ is $O(|V|^2)$


5
12
3
4
2
4.3
3
3
2.3
1
9
13
4

Weighted Shortest Path From Shanghai?



Why does it work?

Invariants:

- Whenever a node is finished, the shortest estimated distance cannot be improved. A shortest path from s to this node has been found.
- At any stage the distances to discovered nodes represent the shortest distance via path consisting only of finished nodes.
  - Can prove this by induction.

At the final stage, all nodes are finished

=> algorithm is correct

Loop Invariant (aka what is true at the top of the loop):

At the start of every iteration of the while loop, $\text{dist}(v) = \text{optimal dist}(s,v)$ for all v which are marked as discovered.

Maintenance (proving the loop invariant is true at the top of the loop):

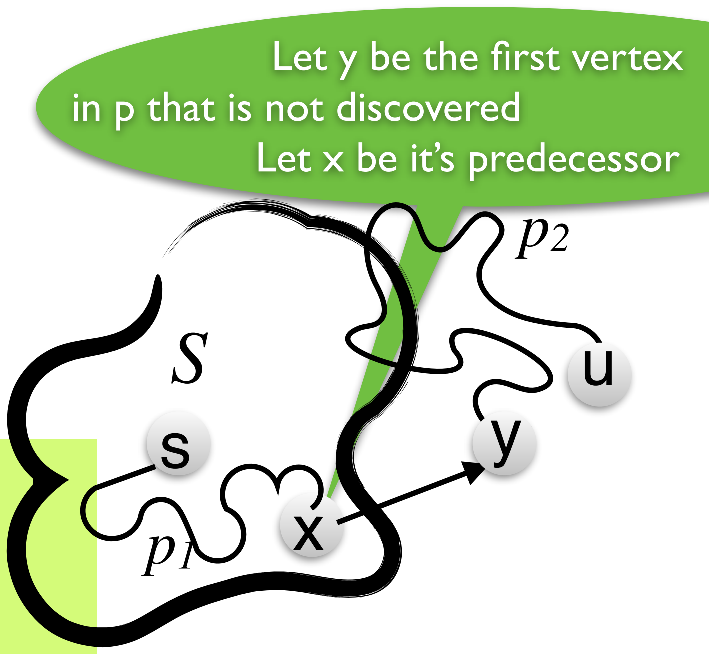
Need to show that $\text{dist}(u) = \text{optimal dist}(s,u)$ when u is marked as discovered in the loop

Suppose not! Suppose u is marked discovered, but $\text{dist}(u) \neq \text{optimal dist}(s,u)$

WLOG let u be the first vertex marked as discovered that does not have the optimal path

$u \neq s$ Let p be a shortest path $s \overset{p}{\rightsquigarrow} u$

Decompose p into subpaths: $s \overset{p_1}{\rightsquigarrow} x \to y \overset{p_2}{\rightsquigarrow} u$


Let y be the first vertex in p that is not discovered Let x be it's predecessor
p_2
S
s
u
y
x
p_1

Claim: $\text{dist}(y) = \text{optimal dist}(s,y)$ when u was marked discovered

Proof: $\text{dist}(x) = \text{optimal dist}(s,x)$ when u was marked discovered

dist(x,y) was updated when x was discovered ◗◖

y is on shortest path $s \rightsquigarrow u$, and all edges are non-negative ☞ $\text{optimal dist}(s,y) \leq \text{optimal dist}(s,u)$

☞ $\text{dist}(y) = \text{optimal dist}(s,y) \leq \text{optimal dist}(s,u) \leq \text{dist}(u)$

But, u and y were in Q, and we chose u so,

$\text{dist}(u) \leq \text{dist}(y)$ ☞ $\text{dist}(u) = \text{dist}(y)$ ☞ $\text{optimal dist}(s,y) = \text{optimal dist}(s,u)$ ◗◖