Notes on: *On the complexity of testing for odd holes and odd induced paths*

Jake Horsfield

June 30, 2018

Abstract

Daniel Bienstock proves that the problem of determining whether a graph contains an odd hole which passes through a prescribed vertex is NP-complete by a reduction from 3-SAT. Two problems concerning odd induced paths between prescribed vertices are also shown to be NP-complete, though I only write about the odd hole problem here. An error in the graph construction was found (and fixed) by Bruce Reed. The fixed version is used here.

3-CNF's to Graphs

Let $C = C_1 \wedge \cdots \wedge C_m$ be a 3-CNF formula over propositional variables $x_1, \ldots, x_n$ with each clause being of the form $C_j = (z_1 \vee z_2 \vee z_3)$, where $z_k = x_i$ or $z_k = \bar{x}_i$ ($1 \leq i \leq n$, $1 \leq j \leq m$, and $1 \leq k \leq 3$). We construct a graph G (whose edges are coloured red or blue) from C as follows. For each variable x_i, G will contain an induced subgraph α_i with

$$V(\alpha_i) = \{c_{i,j} \mid 1 \leq j \leq 4\} \cup \{t_{i,j} \mid 1 \leq j \leq 4\} \cup \{f_{i,j} \mid 1 \leq j \leq 4\}.$$

The graph α_i has blue edges

$$\{c_{i,1}, t_{i,1}\}, \{t_{i,1} c_{i,3}\}, \{c_{i,1}, f_{i,1}\}, \{f_{i,1}, c_{i,3}\}, \{c_{i,2}, t_{i,2}\}, \{t_{i,2}, t_{i,3}\},$$
$$\{t_{i,3}, t_{i,4}\}, \{t_{i,4}, c_{i,4}\}, \{c_{i,2}, f_{i,2}\}, \{f_{i,2}, f_{i,3}\}, \{f_{i,3}, f_{i,4}\}, \{f_{i,4}, c_{i,4}\},$$

and red edges $\{f_{i,1}, t_{i,2}\}, \{f_{i,2}, t_{i,1}\}, \{f_{i,1}, t_{i,3}\}, \{t_{i,3}, f_{i,3}\}, \{f_{i,3}, t_{i,1}\}$. For each clause $C_j = (z_1 \vee z_2 \vee z_3)$, G has an induced subgraph β_j ... **to be completed**.

Result

Theorem 1. *Let L be a u-hole. Then, for $1 \leq i \leq n$, exactly one of the following is true:*

1. *L contains the paths $(c_{i,1}, t_{i,1}, c_{i,3})$ and $(c_{i,2}, t_{i,2}, t_{i,3}, t_{i,4}, c_{i,4})$.*

2. *L contains the paths $(c_{i,1}, f_{i,1}, c_{i,3})$ and $(c_{i,2}, f_{i,2}, f_{i,3}, f_{i,4}, c_{i,4})$.*

Proof. It is clear that L must contain the edges $\{u, w\}, \{w, c_{1,1}\}, \{u, c_{1,2}\}$. The proof is by induction. Suppose that $i > 1$, and that the statement holds for $i-1$. Then L contains the vertices $c_{i-1,3}$ and $c_{i-1,4}$ and hence also $c_{i,1}$ and $c_{i,2}$. Note that the subpath of L from $c_{i,1}$ to $c_{i,2}$ has length $2(i-1)+6(i-1)+3$, which is odd. To show that both (1) and (2) cannot both be true at the same time, suppose (by symmetry) that L contains $\{c_{i,1}, t_{i,1}\}$. Then L cannot contain $\{c_{i,2}, f_{i,2}\}$, for otherwise L would contain the red edge $\{f_{i,2}, t_{i,1}\}$, making L even, not odd. So L must contain $\{c_{i,2}, t_{i,2}\}$. It is easy to see that L cannot contain any red edge that has both endpoints in α_i, so L must contain $\{t_{i,2}, t_{i,3}\}$. We have already determined that $t_{i,1}$ is in L, so we must show that $\{t_{i,1}, c_{i,3}\}$ is an edge of L. Suppose that L contains the edge $\{t_{i,1}, f_j(\bar{x}_i)\}$ (i.e. x_i appears negated in the clause C_j). Then L must also contain the edge $\{f_j(\bar{x}_i), t_{i,3}\}$, which contradicts our assumption that L is odd. Thus L contains $\{t_{i,1}, c_{i,3}\}, \{t_{i,3}, t_{i,4}\}$, and $\{t_{i,4}, c_{i,4}\}$. The base case, $i = 1$, is similar. □

Corollary 1. *Let L be a u-hole. Then for $1 \leq j \leq m$, L contains exactly one blue path $(d_{j,2}, f_j(z), d_{j,4})$ for some literal z appearing in C_j.*

Theorem 2. *Let L be a u-hole. Then*

1. *L contains no red edges.*
2. *L contains v.*
3. *$C_1 \wedge \cdots \wedge C_m$ is satisfiable.*

Proof. (1) and (2) follow from Theorem 1 and Corollary 1. □