Notes on: *Structure and algorithms for (cap, even hole)-free graphs*

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September 6, 2018

1 Overview

The class of (cap, even hole)-free graphs is studied through the more general class of (cap, 4-hole)-free odd-signable graphs. It is proved that every such graph has a vertex of degree at most $\frac{3}{2}\omega(G) - 1$, giving the bound $\chi(G) \leq \frac{3}{2}\omega(G)$ through degeneracy. Algorithms that run in $\mathcal{O}(nm)$ time are given for q-colouring and for maximum weight stable set for these graphs. A polynomial-time algorithm is given for minimum vertex colouring. These algorithms are based on results that triangle-free odd-signable graphs have treewidth at most 5 and thus clique-width at most 48, and that (cap, 4-hole)-free odd-signable graphs without clique cutsets have treewidth at most $6\omega(G) - 1$ and clique-width at most 48.

2 Odd-signable graphs

A graph G is *odd-signable* if there exists an assignment of 0,1 weights to the edges of G such that the weight of every chordless cycle (i.e. including triangles) is odd. Clearly the class of odd-signable graphs contains the even-hole-free graphs, for we can just assign 1 to every edge of an even-hole-free graph.

Now for a characterisation of odd-signable graphs. First, we need to state Truemper's theorem.

Theorem 1 (Truemper). *Let β be a $\{0, 1\}$-vector whose entries are in one-to-one correspondence with the chordless cycles of a graph G. Then there exists a set $F \subseteq E(G)$ such that $|F \cap C| \equiv \beta_C \pmod 2$ for all chordless cycles C of G, if and only if for every induced subgraph G' of G that is a Truemper configuration or K_4, there exists a set $F' \subseteq E(G')$ such that $|F' \cap C| \equiv \beta_C \pmod 2$ for all chordless cycles C of G'.*

So what this Theorem essentially says is that we can sign the chordless cycles of a graph G in a particular way if and only if we can do it for the chordless cycles in the Truemper configurations (and K_4) that are contained in a graph G. For odd-signable graphs, β is an all-ones vector.

Theorem 2. *A graph is odd-signable if and only if it does not contain an even wheel, a theta, or a prism.*

Proof. By Truemper's theorem, we can prove the statement by showing that even wheels, thetas, and prisms are not odd-signable, and that odd wheels, pyramids, and graphs isomorphic to K_4 are odd-signable.

It is easy to see that even wheels, thetas, and prisms are not odd-signable. For an odd wheel (H, x), assign 1 to all edges having x as an endnode. For every subpath of H with endnodes adjacent to x and no intermediate node adjacent to x, assign 1 to one edge and 0 to all others. For a pyramid with triangle xyz, assign 1 to the edges of this triangle, and 0 to all other edges. For a K_4, simply assign 1 to all edges. $\square$

3 A decomposition theorem for (cap, 4-hole)-free graphs

First we define the 'basic' class of cap-free graphs. A *basic cap-free graph* is either a chordal graph (i.e. has no holes) or a 2-connected triangle-free graph together with at most one universal vertex (i.e. has holes).

An *amalgam* in a graph G is a tuple (V_1, V_2, A_1, A_2, K) such that:

- (V_1, V_2, K) partitions $V(G)$.
- $|V_1|, |V_2| \geq 2$.
- For $i = 1, 2$, $\varnothing \neq A_i \subseteq V_i$.
- A_1 is complete to A_2 and there are no other V_1-V_2 edges.
- K is a possibly empty clique that is complete to $A_1 \cup A_2$.

The presence of an amalgam sometimes implies the existence of a clique cutset or a proper clique module.

Theorem 3 (Conforti, Cornuéjols, Kapoor, Vušković)**.** *A connected cap-free graph that is not basic has an amalgam.*

Let us apply this theorem to cap-free graphs that are also 4-hole-free. Let G be such a graph that has an amalgam (V_1, V_2, A_1, A_2, K). Without loss of generality we may assume that A_1 induces a clique, for otherwise if both A_1 and A_2 were not cliques then $G[A_1 \cup A_2]$ would contain a 4-hole. Now if $V_1 \setminus A_1 = \varnothing$ then A_1 is a proper clique module, otherwise $A_1 \cup K$ is a clique cutset that separates $V_1 \setminus A_1$ from V_2. All together, we obtain the following decomposition theorem:

Theorem 4. *If G is a (cap, 4-hole)-free graph, then either G (is not basic in which case it) has a clique cutset or a proper clique module, or G (is basic and so) is a complete graph or a (triangle, 4-hole)-free graph together with at most one universal vertex.*

Note: the fact that G is complete if it does not have a clique cutset follows from Dirac's characterisation of chordal graphs.

4 The structure of (cap, 4-hole)-free graphs with no clique cutset

First let's introduce a type of induced subgraph called an expanded hole. An *expanded hole* consists of nonempty disjoint sets of vertices $S_1, \ldots, S_{k \geq 4} \subseteq V(G)$, not all singletons, such that for all $1 \leq i \leq k$, $G[S_i]$ is connected and for $i \neq j$, S_i is complete to S_j if $j = i-1$ or $j = i+1 \pmod k$ and anticomplete otherwise. Note that if an expanded hole is 4-hole-free, then S_i is a clique for every $1 \leq i \leq k$.

Lemma 5. *Let G be a (cap, 4-hole)-free graph. Suppose that $S = \cup_{i=1}^{k} S_i$ is an inclusion-wise maximal expanded hole of G such that $|S_2| \geq 2$. Let U be the set of vertices of G that are complete to S. Then G has an amalgam (V_1, V_2, A_1, A_2, K) where $S_2 = A_2$ and $K \subseteq U$. In particular, either $K \cup S_2$ is a clique cutset or S_2 is a proper clique module.*

So if a (cap, 4-hole)-free graph contains an expanded hole, then it contains an amalgam.

The block of decomposition of G with respect to a module M is the graph $G' = G \setminus (M \setminus \{u\})$ for any vertex $u \in M$.

Lemma 6. *Let M be a proper clique module of a graph G, and let G' be the block of decomposition with respect to this module. If G does not have a clique cutset, then G' does not have a clique cutset.*

Proof. We prove the contrapositive. Let K be a clique cutset of G'. Let u be the vertex of M that is in G'. If $u \notin K$ then $N_{G'}(u) \setminus K$ is in the same connected component of $G' \setminus K$ as u and so K is a clique cutset of G. If $u \in K$ then $M \cup K$ is a clique cutset of G. $\square$

Now for a big technical theorem.

Theorem 7. *Let G be a (cap, 4-hole)-free graph that contains a hole. Let F be a maximal vertex subset of $V(G)$ that induces a 2-connected triangle-free graph (we know this exists because G has a hole), U the set of vertices of $V(G) \setminus F$ that are complete to F, D the set of vertices of $V(G) \setminus F$ that have at least two neighbours in F but are not complete to F, and $S = V(G) \setminus (F \cup U \cup D)$. The following hold:*

1. *U is a clique.*
2. *U is complete to $D \cup F$.*
3. *If G does not have a clique cutset, then for every $d \in D$, there is a vertex $u \in F$ and $D' \subseteq D$ that contains d such that $D' \cup \{u\}$ is a clique module of G. In particular, for every $d' \in D$, $N[d'] = N[u]$.*
4. *If G does not have a clique cutset, then $S = \varnothing$.*
5. *If G does not have a clique cutset, F does not have a clique cutset.*

Proof. G is 4-hole-free therefore (1) holds. Now we prove a claim that is used to prove (2) and (3).

Claim. For every $d \in D$, $G[F]$ contains a hole H such that $G[V(H) \cup \{d\}]$ is an expanded hole.

Proof of Claim. Let d be any vertex of D, and assume that there is no hole H in $G[F]$ such that $G[V(H) \cup \{d\}]$ is an expanded hole. Since F was chosen to be minimal, $G[F \cup \{d\}]$ contains a triangle dxy, and since $G[F]$ is 2-connected and triangle-free, x and y lie on some hole H of $G[F]$. As G is cap-free and $G[V(H) \cup \{d\}]$ is not an expanded hole, d is complete to $V(H)$. Let F' be a maximal subset of F such that $G[F']$ contains H, is 2-connected, and d is complete to F'. By definition, vertices of D are not complete to F, so $F \neq F'$. Since both $G[F]$ and $G[F']$ are 2-connected, some $z \in F \setminus F'$ belongs to a hole H' that contains an edge of $G[F']$. By the same argument that d is complete to $V(H)$, d is also complete to $V(H')$. But then $F' \cup V(H')$ contradicts the maximality of F', thus completing the proof of the claim.

By the Claim, every vertex $d \in D$ has two nonadjacent neighbours in F, say x and y. Let u be any vertex of U. But then $\{x, d, y, u\}$ is a 4-hole, and G is 4-hole-free. So U is complete to d, hence (2) holds.

Suppose that G does not have a clique cutset. By the Claim, for every $d \in D$ there is a hole H contained in F such that $G[V(H) \cup \{d\}]$ is an expanded hole $S = \cup_{i=1}^{k} S_i$ with $|S_2| \geq 2$ (say $S_2 = \{u, d\}$). Let $D' = \{d\} \subseteq D$. Lemma 5, together with the fact that G has no clique cutset gives us that $S_2 = D' \cup \{u\}$ is a (proper) clique module, so (3) holds.

Let $D' \cup \{u\}$ be a proper clique module (which exists by (3)). The block of decomposition with respect to this module is the graph $G' = G \setminus D'$. By performing a sequence of such clique module decompositions, we obtain $G' = G \setminus D$. By Lemma 6, G' has no clique cutset. Suppose $S \neq \varnothing$. Note that every vertex in S has at most one neighbour in F, otherwise they would be in either D or U. Let C be a connected component of $G[S]$. Since F is maximal, at most one vertex in F, say y, has a neighbour in C. But then $U \cup \{y\}$ is a clique cutset of G' (hence G has a clique cutset), a contradiction. If no component of $G[S]$ is adjacent to a vertex of F, then u is a clique cutset of G', a contradiction. So $S = \varnothing$ and (4) holds.

For (5), suppose that F has a clique cutset K. But by (1) and (4), $K \cup U$ is a clique cutset of G', a contradiction. □

Next is a theorem that tells us how to construct (cap, 4-hole)-free graphs with no clique cutset. This will come in handy for bounding the chromatic number of such graphs.

Theorem 8. *Let G be a (cap, 4-hole)-free graph that contains a hole and has no clique cutset. Let F be any maximal induced subgraph of G with at least 3 vertices that is triangle-free and has no clique cutset. Then G is obtained from F by first blowing up vertices of F into cliques and then adding a universal clique. Furthermore, any graph obtained by this sequence of operations starting from a (triangle, 4-hole)-free graph with at least 3 vertices and no clique cutset is (cap, 4-hole)-free and has no clique cutset.*

Proof. Let F' be a maximal 2-connected triangle-free induced subgraph of G that contains F. By Theorem 7 (5), F' does not have a clique cutset so $F' = F$. So the first statement follows from Theorem 7. The second statement follows from the fact that blowing up vertices into cliques and adding a universal clique preserves being (cap, 4-hole)-free and having no clique cutset. □

Question. So is it the case that $V(K_u) \setminus \{u\} \subseteq D$ for every $u \in F$?
Yes. Blowing up a vertex to a K_n ($n \geq 2$) creates triangles, so the new vertices cannot be in F. Since $S = \varnothing$, the new vertices must therefore be in D.

5 Building triangle-free odd-signable graphs

Note that the class of triangle-free odd-signable graphs contains the class of (cap, even hole)-free graphs.

The following describes a construction of triangle-free odd-signable graphs. A chordless xz-path P is an *ear* of a hole H contained in a graph G if $V(P) \setminus \{x, z\} \subseteq V(G) \setminus V(H)$, vertices $x, z \in V(H)$ have a common neighbour y in H, and $(V(H) \setminus \{y\}) \cup V(P)$ induces a hole H' in G. A graph G is said to be obtained from a graph G' by an *ear addition.* An ear addition is *good* if:

1. y has an odd number of neighbours in P
2. G' contains no wheel (H_1, v), where $x, y, z \in V(H_1)$ and v is adjacent to y
3. G' contains no wheel (H_2, y), where x, z are neighbours of y in H_2.

Theorem 9. *Let G be a triangle-free graph with at least three vertices that is not the cube and has no clique cutset. Then, G is odd-signable if and only if it can be obtained, starting from a hole, by a sequence of good ear additions.*

As cubes cannot be contained in 4-hole-free/even-hole-free graphs, we don't have to worry about them.

6 χ-bound

A bound on the chromatic number for (cap, 4-hole)-free odd-signable graphs is obtained by showing that every such graph has a vertex of degree at most $\frac{3}{2}\omega(G) - 1$. Since this class of graphs is hereditary, the property of having a vertex of degree at most $\frac{3}{2}\omega(G) - 1$ is inherited by all subgraphs, thus every (cap, 4-hole)-free odd-signable graph is $(\frac{3}{2}\omega(G) - 1)$-degenerate. The χ-bound then follows.

Theorem 10. *Every (cap, 4-hole)-free odd-signable graph G has a vertex of degree at most $\frac{3}{2}\omega(G) - 1$.*

Proof. Say a vertex v is nice if it has degree at most $\frac{3}{2}\omega(G) - 1$. It is proved that if G is a (cap, 4-hole)-free odd-signable graph, then either G is complete or it has at least two nonadjacent nice vertices. Assume that this does not hold and let G be a minimum counterexample.

Suppose K is a clique cutset of G. Let $C_1, \ldots, C_k$ be the connected components of $G \setminus K$, and $G_i = G[C_i \cup K]$ for every i. Since G is a minimum countexample, every G_i is either complete or has at least two nonadjacent nice vertices. But then G has at least two nonadjacent nice vertices (it is certain that at least one vertex from each C_i is nice – both nice vertices cannot be in K since they must be nonadjacent). This is a contradiction, so G has no clique cutset.

Since G is neither complete or has a clique cutset, G is not chordal. So G contains a hole. As such, there is some induced subgraph of G that is triangle-free and has no clique cutset. Let F be a maximal such induced subgraph. By Theorem 8, G is obtained from F by blowing up vertices of F into cliques and adding a universal clique U. Note that if a vertex u is nice in $G \setminus U$ then it is nice in G (*), so by the choice of G, $U = \varnothing$ (otherwise, by the contrapositive of (*), we could get a counterexample smaller than G). So we don't need to worry about a universal clique.

For $u \in V(F)$, let K_u be the clique that u is blown up into. We now prove a claim that is applied to subpaths of ears.

Claim. If u', u, v, v' is a path of F such that u and v are each of degree 2 in F, then u or v is nice in G.

Proof of Claim. Since $|K_u| + |K_v| \leq \omega(G)$, we can assume that $|K_u| \leq \frac{1}{2}\omega(G)$. But then

$$d_G(v) = \underbrace{|K_u|}_{\leq \frac{1}{2}\omega(G)} + \underbrace{|K_v| - 1 + |K_{v'}|}_{\leq \omega(G)-1} \leq \frac{3}{2}\omega(G) - 1.$$

By Theorem 9, we know that F may be obtained from a hole by a sequence of good ear additions. Let P be the last ear added, with attachments x, z onto a hole H. Since P is a good ear, y has an odd number of neighbours in P thus there is at least one such neighbour. Let $y_1, \ldots, y_k$ be the neighbours of y on P in the order when traversing P from x to z (so $y_1 = x$). Both the y_1y_2-subpath and the $y_{k-1}y_k$-subpath have an edge whose ends have degree 2 (since G is (triangle, 4-hole)-free). Applying the Claim to these subpaths completes the proof. $\square$

It then follows by degeneracy that $\chi(G) \leq \frac{3}{2}\omega(G)$ for (cap, 4-hole)-free odd-signable graphs.