Pretalk 3: Cohomology theories

The Weil conjectures. — *In the following exercises, you will show that the existence of Weil cohomology theories over finite fields implies the Weil conjectures minus the Riemann hypothesis.*

Exercise 14. Let k and F be fields with k algebraically closed and $\mathrm{char}(F) = 0$, and let $\mathbf{V}_k$ be the category of *connected* smooth projective k-varieties. Let H^* be a functor from $\mathbf{V}_k^{\mathrm{op}}$ to the category of $\mathbb{Z}_{\geq 0}$-graded-commutative F-algebras equipped with, for each $X \in \mathbf{V}_k$, a "trace map" $\mathrm{tr}_X\colon H^{2\dim(X)}(X) \xrightarrow{\sim} F$ which is an isomorphism. Assume that H^* satisfies *Poincaré duality*, meaning that for each X and i, the pairing

$$H^i(X) \times H^{2\dim(X)-i}(X) \xrightarrow{\smile} H^{2\dim(X)}(X) \xrightarrow{\mathrm{tr}_X} F$$

is perfect, where $\smile$ denotes the ("cup") product on $H^*(X)$.

(1) Construct a "pushforward" map $f_*\colon H^i(X) \to H^{i+2(\dim(Y)-\dim(X))}(Y)$ for any morphism $f\colon X \to Y$ in $\mathbf{V}_k$.

(2) Show that the pushforward is functorial and satisfies the projection/adjunction formula, i.e. we have

$$(g \circ f)_* = g_* \circ f_* \qquad \text{and} \qquad f_*(\alpha \smile f^*(\beta)) = f_*(\alpha) \smile \beta$$

whenever these identities make sense. (*Hint:* Show that $f_*(\eta)$ is uniquely characterized by the formula "$\mathrm{tr}_Y(f_*(\eta) \smile \theta) = \mathrm{tr}_X(\eta \smile f^*(\theta))$".)

(3) For a morphism $f\colon X \to Y$ in $\mathbf{V}_k$, let $\gamma_f \in H^*(X \times Y)$ denote the image of 1 under $(\mathrm{id}, f)_*\colon H^0(X) \to H^{\dim(Y)}(X \times Y)$. Show that we have

$$f_*(\alpha) = \mathrm{pr}_{2*}(\gamma_f \smile \mathrm{pr}_1^*(\alpha)) \qquad \text{and} \qquad f^*(\beta) = \mathrm{pr}_{1*}(\gamma_f \smile \mathrm{pr}_2^*(\beta))$$

for all α and β. This should look very familiar! (*Hint:* Use the projection formula.)

(4) More generally, for a closed embedding $\iota\colon Z \hookrightarrow X$ in $\mathbf{V}_k$, we define $\mathrm{cl}_X(Z)$ to be the image of 1 under ι_*. Assume that H^* is *normalized* in the sense that $\mathrm{tr}_{\mathrm{Spec}(k)}(1) = 1$. Show that if $x \in X$ is a closed point, then $\mathrm{tr}_X(\{x\}) = 1$. (Recall we assumed k to be algebraically closed.)

Exercise 15. Let k, F, $\mathbf{V}_k$, H^*, and cl_X be as in the previous exercise. Assume, in addition to Poincaré duality and normalization, the following axioms:

- (Künneth). The natural map $H^*(X) \otimes_F H^*(Y) \xrightarrow{\sim} H^*(X \times Y)$ given by $\alpha \otimes \beta \mapsto \mathrm{pr}_1^*(\alpha) \smile \mathrm{pr}_2^*(\beta)$ is an isomorphism of graded F-algebras.
- We have $\mathrm{tr}_{X\times Y}(\mathrm{pr}_1^*(\alpha) \smile \mathrm{pr}_2^*(\beta)) = \mathrm{tr}_X(\alpha) \cdot \mathrm{tr}_Y(\beta)$.
- If Z and Z' are smooth closed subschemes of X which satisfy $\dim(Z) + \dim(Z') = \dim(X)$ and are such that the scheme-theoretic intersection of Z and Z' in X is 0-dimensional and reduced, then $\mathrm{cl}_X(Z) \smile \mathrm{cl}_X(Z') = \sum_{x \in Z \cap Z'} \mathrm{cl}_X(\{x\})$.

Now let $f\colon X \to X$ be a morphism in $\mathbf{V}_k$, and assume that the scheme-theoretic intersection of Γ_f and Δ_X in $X \times X$ is 0-dimensional and reduced. Then "Lefschetz's fixed-point theorem" says that

$$\#\{x \in X(k) \mid f(x) = x\} = \sum_{i=0}^{2\dim(X)} (-1)^i \,\mathrm{tr}(f^*|_{H^*(X)}).$$

Complete the following steps to prove this fact.

(1) Show that the number of fixed points of f on $X(k)$ equals $\mathrm{tr}_{X\times X}(\gamma_f \smile \gamma_{\mathrm{id}})$, in the notation of part (c) of the previous exercise.
(2) Let $(e_{i,j})_j$ be an F-basis for $H^i(X)$, and let $(e^\vee_{2d-i,j})_j$ be the Poincaré-dual basis for $H^{2d-i}(X)$, where $d := \dim(X)$. By the Künneth axiom, we may write
$$\gamma_f = \sum_{i,j} \mathrm{pr}_1^*(\alpha_{i,j}) \smile \mathrm{pr}_2^*(e^\vee_{i,j})$$
for *unique* $\alpha_{i,j} \in H^*(X)$. Show that $\alpha_{i,j} = f^*(e_{2d-i,j})$. (*Hint:* Use part (c) of the previous exercise.)
(3) Show that
$$\gamma_{\mathrm{id}} = \sum_{i,j} (-1)^i \,\mathrm{pr}_1^*(e^\vee_{2d-i,j}) \smile \mathrm{pr}_2^*(e_{i,j}).$$
(4) Prove Lefschetz's fixed-point theorem.

Exercise 16. Let k, F, $\mathbf{V}_k$, H^*, etc. be as in the previous exercise, but now assume k is the algebraic closure of $\mathbb{F}_p$. Let X be a geometrically connected smooth projective variety over $\mathbb{F}_q$, where q is a power of p. Recall that the *zeta function* of X is the power series
$$Z_X(t) := \exp\left(\sum_{n\geq 1} \frac{\#X(\mathbb{F}_{q^n})}{n}\cdot t^n\right).$$
Verify the Weil conjectures, minus the Riemann hypothesis, in the following steps. (The important arguments take place in parts (2), (4), and (5); the reader could solve only these, taking the statements of the other parts for granted.)

(1) Let $X_k \in \mathbf{V}_k$ be the base-change of X to k. Let $\varphi\colon X \to X$ be the "absolute q-Frobenius", which is the identity on the underlying topological space of X and acts on $\mathcal{O}_X(U)$ as $\varphi(f) := f^q$, and let $\mathrm{Fr} = \varphi \times_{\mathbb{F}_q} \mathrm{id}_{\mathrm{Spec}(k)}\colon X_k \to X_k$ be the base-change of φ to k. Show that the scheme-theoretic intersection of Γ_{Fr} and Δ_X in $X\times X$ is 0-dimensional and reduced. (*Hint:* Reduce to the affine setting.)
(2) Use Lefschetz's trace formula to show that
$$Z_X(t) = \frac{P_1(t)P_3(t)\cdots P_{2d-1}(t)}{P_0(t)P_2(t)\cdots P_{2d}(t)},$$
where $d := \dim(X)$ and $P_i(t) = \det(1 - t\cdot \mathrm{Fr}^*|_{H^i(X_k)})$. Also, show that $P_0(t) = 1-t$.
(3) Show that $\mathbb{Q}[\![t]\!] \cap F(t) = \mathbb{Q}(t)$. Thus, by part (2), $Z_X(t) \in \mathbb{Q}(t)$.
(4) Show that Fr^* acts on $H^{2d}(X_k)$ as multiplication by q^d and that $P_{2d}(t) = 1 - q^d t$.[2] (*Hint:* Use the formula "$\mathrm{tr}_Y(f_*(\eta)\smile\theta) = \mathrm{tr}_X(\eta \smile f^*(\theta))$".)
(5) Prove the following functional equation, where $\chi := \sum_{i=1}^{2d} \dim_F(H^i(X_k))$:
$$Z_X\left(\frac{1}{q^d t}\right) = \pm q^{d\chi/2} t^\chi \cdot Z_X(t).$$
The number χ is the *Euler characteristic* of X.[3] (*Hint:* How does Poincaré duality relate the eigenvalues of Frobenius acting on $H^i(X_k)$ and on $H^{2d-i}(X_k)$?)

[2] In fact, Fr is flat (this is even equivalent to the smoothness of X) and $[\mathrm{Fr}^{-1}(\{x\})] = d\cdot[\{x\}]$ in $\mathrm{CH}^d(X_k)$ for any closed point x of X_k. These facts can be proven without much difficulty by passing to complete local rings.

[3] Note that by Lefschetz's fixed point theorem and the "full" cycle-class axiom, one has $\chi = \deg([\Delta]\cdot[\Delta])$, where $[\Delta] \in \mathrm{CH}^d(X_k\times X_k)$ is the class of the diagonal.