Pretalk 2: Sites

Exercise 8. Let X be a topological space, and let $\mathbf{Top}_X$ be the category of topological spaces over X.[1] The purpose of this exercise is to study the categories of sheaves on different Grothendieck topologies on $\mathbf{Top}_X$. Consider the sites

$$(\mathbf{Top}_X)_{\mathrm{all}}, \quad (\mathbf{Top}_X)_{\mathrm{surj}}, \quad (\mathbf{Top}_X)_{\text{ét}}, \quad (\mathbf{Top}_X)_{\mathrm{Zar}}$$

all with underlying category $\mathbf{Top}_X$ and whose coverings are as follows: a set $\{f_i \colon U_i \to U\}_{i \in I}$ of morphisms in $(\mathbf{Top}_X)$ with the same target is a covering in

- $(\mathbf{Top}_X)_{\mathrm{all}}$ no matter what.
- $(\mathbf{Top}_X)_{\mathrm{surj}}$ if it is *jointly surjective*, i.e. $U = \bigcup_{i \in I} f(U_i)$.
- $(\mathbf{Top}_X)_{\text{ét}}$ if it is jointly surjective and each f_i is a *local homeomorphism*, i.e. each point in U_i has an open neighborhood which f_i maps homeomorphically onto an open subset of U.
- $(\mathbf{Top}_X)_{\mathrm{Zar}}$ if it is jointly surjective and each f_i is an *open embedding*, i.e. f_i maps U_i homeomorphically onto an open subset of U.

After convincing yourself that each is in fact a site, show the following:

(1) The category of sheaves of sets on $(\mathbf{Top}_X)_{\mathrm{all}}$ is equivalent to the category with one object and one morphism.
(2) The category of sheaves of sets on $(\mathbf{Top}_X)_{\mathrm{surj}}$ is equivalent to the category of sets.
(3) The category of sheaves of sets on $(\mathbf{Top}_X)_{\text{ét}}$ and $(\mathbf{Top}_X)_{\mathrm{Zar}}$ are *equal*: a presheaf on $\mathbf{Top}_X$ is a sheaf on $(\mathbf{Top}_X)_{\text{ét}}$ if and only if it is on $(\mathbf{Top}_X)_{\mathrm{Zar}}$.

Exercise 9. Let X be a topological space.

(1) Show that for any $U \in \mathbf{Top}_X$, the representable functor $h_U \colon \mathbf{Top}_X^{\mathrm{op}} \to \mathbf{Set}$ given by $h_U(V) := \mathrm{Hom}_{\mathbf{Top}_X}(V, U)$ is a sheaf on $(\mathbf{Top}_X)_{\mathrm{Zar}}$ but not $(\mathbf{Top}_X)_{\mathrm{surj}}$. (By the previous exercise, it is also a sheaf on $(\mathbf{Top}_X)_{\text{ét}}$, which is less obvious!)
(2) Conclude that the category of sheaves of sets on $(\mathbf{Top}_X)_{\mathrm{Zar}}$ need not be equivalent to the category of sheaves of sets on X (in the usual sense). (Hint: Take X to be a point.)
(3) * Nevertheless, exhibit a fully faithful functor $B \colon \mathbf{Sh}(X, \mathbf{Set}) \hookrightarrow \mathbf{Sh}((\mathbf{Top}_X)_{\mathrm{Zar}}, \mathbf{Set})$, and show that if $\mathcal{F}$ is a sheaf of Abelian groups on X, then its cohomology equals that of $B(\mathcal{F})$.

Exercise 10. Let $k \to A$ be a ring map, k a field. Prove that the following are equivalent:

(1) A is Noetherian, zero-dimensional, and every local ring $(A \otimes_k \overline{k})_{\mathfrak{p}}$ is regular.
(2) $A \cong \prod_{i \in I} k_i$ for some finite set I and finite separable extensions $k_i \mid k$.

(Hint: The structure theorem for Artinian rings might be useful; see Atiyah–Macdonald, Theorem 8.7.) Thus, under our definitions, "étale" is indeed equivalent to "smooth with zero-dimensional fibers".

[1] That is, an object of $\mathbf{Top}_X$ is a pair (Y, f) consisting of a topological space Y and a continuous map $f \colon Y \to X$; a morphism $(Y, f) \to (Z, g)$ in $\mathbf{Top}_X$ is a continuous map $h \colon Y \to Z$ satisfying $g \circ h = f$. Note that $\mathbf{Top}_X$ is (equivalent to) the category of all topological spaces if X is a point.

Exercise 11. Let X be a scheme, and let $x\colon \mathrm{Spec}(K) \to X$ be a point of X. An *étale neighborhood* of x is a commuting diagram of the form

$$\begin{tikzcd} & U \arrow[d] \\ \mathrm{Spec}(K) \arrow[ur] \arrow[r, "x"] & X \end{tikzcd}$$

in which $U \to X$ is étale. We abbreviate this diagram as "$(U,u) \to (X,x)$". The étale neighborhoods of x form a category in an obvious way.

Show that the category of étale neighborhoods of x is filtered. This means:

(1) Given $(U,u) \to (X,x)$ and $(U',u') \to (X,x)$, there exists a commutative diagram of the form

$$\begin{tikzcd} (U'',u'') \arrow[r] \arrow[d] & (U',u') \arrow[d] \\ (U,u) \arrow[r] & (X,x); \end{tikzcd}$$

(2) Given $f, g\colon (U,u) \to (X,x)$, there exists $h\colon (U',u') \to (U,u)$ with $h \circ f = g \circ f$.

Thus if $\mathcal{F}$ is a presheaf on $\mathbf{\acute{E}t}_X$, the filtered colimit

$$\mathcal{F}_x \coloneqq \operatorname*{colim}_{(U,u)\to(X,x)} \mathcal{F}(U)$$

can be computed in the usual way; it is called the *stalk* of $\mathcal{F}$ at x.