Lecture 1: Risk and Uncertainty

Jacob Kohlhepp

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Introduction

- In Econ 11, we mostly assumed things were *certain*.
  - **Example:** Walrasian Eq. where Rosy has apples and Jamie has oranges and they each sell at a price. Everyone knows exactly how much they will enjoy apples or oranges.
- But in reality many objects are not like this.
  - **Example:** Used cars, insurance (I don't know if I will get into a car accident), lottery tickets, investments.
- Our goal is to extend our toolbox so we can work with objects that are uncertain.
- Practical Applications: These tools are used broadly in finance, actuarial science and business.

Notation, Terms, Definitions

In this class:

- I denote lottery/gamble a as X_a, where X_a is a random variable with a pmf or pdf giving the probability of receiving different amounts of money
- $E[X_a]$ always means expectation of X_a.
- $Var(X_a)$ always means variance of X_a.
  - It is often useful to recall this alternative formula for variance: $Var(X_a) = E[X_a^2] - E[X_a]^2$.
- *log* always means natural log (base e).

A Survey

Suppose you could enter one of three lotteries for free:

a. X_a: \$100 with probability 50%, \$0 with probability 50%.

b. X_b: \$49 with probability 100%

c. X_c: \$200 with probability 24%, \$0 with probability 76%.

Which would you choose? (Answer in the survey)

Risk Attitudes

- Before looking at the results, notice some facts:
  - b has lower expectation than a: $E[X_a] = 50 > 49 = E[X_b]$.
  - But a has higher variance than b: $Var(X_a) = 2500 > 0 = Var(X_b)$.
  - So if you dislike variance or uncertainty, you will prefer a.
  - $E[X_c] = 48 < 49 < 50$. So in terms of expected money, c is worse than both a and b.
  - However, the variance of c is higher: $Var(X_c) \approx 7296 > 2500 > 0$.
  - You will only choose c if you like uncertainty/risk.
- A rough interpretation of the results:
  - If you chose b you are *risk averse*: you dislike uncertainty/risk, and are willing to pay to reduce it.
  - If you chose a you are either *risk neutral* or slightly *risk loving*.
  - If you chose c you are *risk loving.*
- We can take these concepts and use a new tool to analyze them.

Expected Utility Theory

- We analyze uncertainty using a **Expected utility Theory.**
- Basically two things matter for decisions: **preferences** and **risk attitudes.**
- Main theorem:

Theorem 1

Under a set of axioms (which you do not need to know), we can represent an individual's risk attitudes and preferences using an expected utility function *u such that:*

$$E[u(X_a)] \geq E[u(X_b)] \textit{if and only if } X_a \textit{ is preferred to } X_b$$

- Main idea: We can represent risk attitudes using a utility function over money. This utility function is actually uniquely defined up to a constant.

Example

Consider a person with expected utility function $u(x) = log(x+1)$.

- Given utility function we can compare any two lotteries over money by comparing $E[u(X_a)]$ and $E[u(X_b)]$.
- If a discrete random variable, $E[X_A] = \sum_i f(x_i)x_i$ for pmf f.
- If continuous, $E[X_A] = \int xf(x)dx$ for pdf f.
- **Exercise:** Let's rank the lotteries from the example.

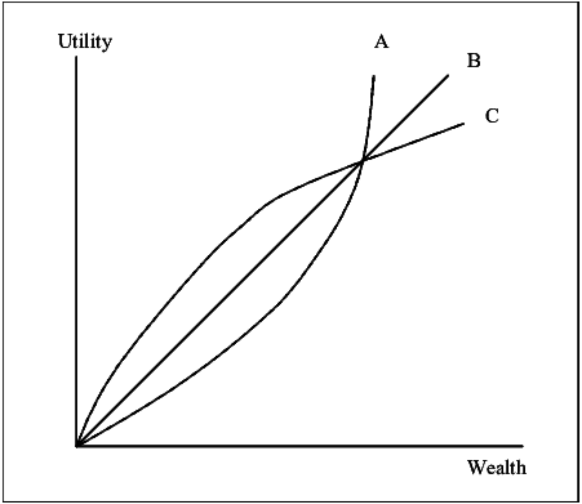
Risk Attitudes As Functions

A person with expected utility function u is...

- **risk averse** if u is concave.
- **risk neutral** if u is linear.
- **risk loving** if u is convex.

Risk Attitudes As Functions

Which function is risk averse? Answer in the survey.



Source: Fleming et. al. (2003)

Certainty Equivalent

- We often want to rank lotteries/gambles. The following concept is useful for ranking lotteries:

Definition 2

The amount of money for sure a decision maker is willing to pay for lottery *a* is the **certainty equivalent** (d_a). Mathematically:

$$u(d_a) = E[u(X_a)]$$

- Given a lottery that gives me d dollars for sure and X_a, it is the value of d where I am indifferent.

- We can use this to rank lotteries. if $d_a > d_b$ then the decision maker prefers lottery a.

- We can also use this to calculate the willingness to pay for insurance.

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- We can use this to rank lotteries. if $d_a > d_b$ then the decision maker prefers lottery a.
- We can also use this to calculate the willingness to pay for insurance.
- Gut check: What is the certainty equivalent of a lottery a with $E[X_a] = 10$ when the decision maker is risk neutral?

Certainty Equivalents and Risk Attitudes

Question: Given a lottery X, what can we say about the certainty equivalent of X relative to $E[X]$ for a risk neutral person? A risk averse person? A risk loving person?

Application: Insurance

Consider a driver who faces monetary damages from driving given by a random variable X. The driver has wealth given by w and has expected utility function $u(x)$.

- We can think of these damages as a lottery X. Then the driver faces uncertain future wealth of $w - X$ (wealth less damages). This is the lottery they care about.

Suppose there is an insurance company which is risk neutral. This is reasonable because:

- Insurance company cares about profits.
- Company will have many customers, so it cares about the average rather than the risk.
- If you recall from Econ 41, by law of Large Numbers sample average converges to expectation: $n^{-1} \sum_i X_i \to E[X]$

Application: Insurance

Question: Suppose $u(x) = log(x)$, $w = \$10,000$, and the probability of an accident is 10%. if an accident happens, the driver incurs \$5,000 in damages. What is the certainty equivalent of the lottery the driver faces without insurance?

1. **Define the problem.** The lottery $w - X$ is $\$10,000$ with probability 90% and $10,000 - 5,000 = \$5,000$ with probability 10%.
2. The certainty equivalent is the amount the driver would take for sure that makes them indifferent between $w - X$. Call it d. d solves:

$$E[u(w - X)] = u(d)$$

3. Solving:

$$E[log(w - X)] = 0.1 log(5000) + 0.9 log(10000) = log(d)$$
$$e^{9.1410} = 9,330 = d$$

Application: Insurance

Question: Keep the same utility function and damage lottery. How much of a premium is the driver willing to pay for full insurance, where full insurance is defined as paying all the damages if there is an accident and 0 if there is no accident.

1. **Define the problem.** Full insurance means that there is no uncertainty: wealth is constant in all states of the world, accident or not.
2. **Write it in math.** Call the maximum premium $\tilde{p}$. The maximum premium the driver will pay is:

$$u(1000 - \tilde{p}) = E[u(w - X)]$$

3. **Notice something clever.** Note that $1000 - \tilde{p}$ is equal to the certainty equivalent. So we can either redo our work, or just solve:

$$1000 - \tilde{p} = 9330 \implies \tilde{p} = 670$$

4. Try the long way for yourself to verify you get the same answer! Also feel free to do this again by switching up X or the utility function.

Application: Insurance

Question: Keep the same utility function and damage lottery. Think about the insurance company now. Define the actuarially fair premium to be the premium p^* which makes expected profit 0 from full insurance. How much is the actuarially fair premium? What are the range of premiums under which both parties are willing to participate?

1. **Define the problem.** Profit is $p - X$. We want to find p^* that solves:

$$E[\pi] = E[p - X] = 0$$

2. **Working it out:**

$$E[p - X] = 0.1(p^* - 5000) + 0.9p^* = p^* - 500 = 0 \implies p^* = 500$$

3. The driver says yes when the premium is less than the $\tilde{p}$ we derived before. The insurance company says yes when the premium is above p^*. This gives a range of premiums which generate gains from trade:

$$500 = p^* < p < 670 = \tilde{p}$$

Exponential Utility

In many settings, it is convenient to use exponential utility because it makes analysis easier. Exponential utility takes the form:

$$u(x) = \frac{1 - e^{-\theta x}}{\theta}$$

where θ is a parameter capturing risk aversion. When a lottery X is normally distributed with mean μ and variance σ^2 it can be shown that:

$$E[u(X)] = \theta^{-1}\left(1 - e^{-\theta\mu + \theta^2\sigma^2/2}\right)$$

you can verify this but it will not ever be tested. The proof is mainly just calculus and algebra.

Exponential Utility - Questions

1. What is the certainty equivalent of X?

$$u(d) = E[u(X)] \implies \frac{1 - e^{-\theta d}}{\theta} = \theta^{-1}\left(1 - e^{-\theta\mu + \theta^2\sigma^2/2}\right)$$

$$\implies e^{-\theta d} = e^{-\theta\mu + \theta^2\sigma^2/2} \implies d = \mu - \theta\sigma^2/2$$

   So utility depends simply on mean and variance. Nice for finance, empirical economic research.

2. What happens when $\theta \to 0$? Hint: use L'Hopsital's rule. Interpret.

$$\lim_{\theta \to 0} \frac{1 - e^{-\theta x}}{\theta} = x$$

   This is risk neutrality.

Normal Random Variables

Suppose we have two normal independent lotteries, X_a, X_b with means μ_a, μ_b and variances σ_a^2, σ_b^2. Suppose we have \$1 to invest total. Let a be the amount we put in a and $1 - a$ be the amount we put in b. It is always true that:

$$\mu_{portfolio} \sim N(a\mu_a + (1-a)\mu_b, a^2\sigma_a^2 + (1-a)^2\sigma_b^2)$$

that is the resulting portfolio is normal.

Application: Diversification

We will now use exponential utility with normal lotteries to see why you should "diversify your portfolio." Suppose you have two independent assets a, b where $\mu_a = \mu_b$,$\sigma_a^2 = \sigma_b^2$. What is the optimal amount you should invest in each asset?

Application: Diversification

- **Define the question.** We want to maximize the certainty equivalent with respect to a.

- **Write in math.**

$$\max_a \mu_{portfolio} - \theta\sigma^2_{portfolio}/2$$

- Simplify and plug in $\mu_{portfolio} = \mu_a$ and $\sigma^2_{portfolio} = (1 - 2a + 2a^2)\sigma_1^2$:

$$\max_a \mu_a - \theta(1 - 2a + 2a^2)\sigma_1^2$$

- Take FOC:

$$-\theta\sigma_1^2(-2 + 4a) = 0 \implies a = 1/2$$

  Note that this is ONLY valid if $\theta > 0$, that is the person is risk averse.

- Challenge: What is the optimal weighting if a person is risk neutral?