Board Work for Lecture 7

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1 Relative Performance Evaluation

Let's begin by deriving the worker's certainty equivalent for a wage given by $w(y_1, y_2) = \alpha + \beta(y_1 - \gamma y_2)$. We will apply our certainty equivalent formula:

$$d = \mu - r\frac{\sigma^2}{2}$$

where $\mu = E[w(y_1, y_2)]$, $\sigma^2 = Var[w(y_1, y_2)]$. The mean of the wage is:

$$\begin{aligned}
E[w(y_1, y_2)] &= E[\alpha + \beta(Y_1 - \gamma Y_2)] && (1)\\
&= \alpha + \beta E[Y_1 - \gamma Y_2] && (2)\\
&= \alpha + \beta E[e_1 + v_1 + v_s - \gamma(e_2 + v_2 + v_s)] && (3)\\
&= \alpha + \beta(e_1 - \gamma e_2) && (4)\\
& && (5)
\end{aligned}$$

The variance is:

$$\begin{aligned}
Var[w(y_1, y_2)] &= Var[\alpha + \beta(Y_1 - \gamma Y_2)]\\
&= Var[\alpha] + Var[\beta(Y_1 - \gamma Y_2)]\\
&= 0 + Var[\beta(Y_1 - \gamma Y_2)]\\
&= \beta^2 Var[Y_1 - \gamma Y_2]\\
&= \beta^2 Var[e_1 + v_1 + v_s - \gamma(e_2 + v_2 + v_s)]\\
&= \beta^2 Var[v_1 + v_s - \gamma(v_2 + v_s)]\\
&= \beta^2 Var[v_1 + (1-\gamma)v_s - \gamma v_2]\\
&= \beta^2 (Var[v_1] + Var[(1-\gamma)v_s] + Var[\gamma v_2])\\
&= \beta^2(\sigma_1^2 + (1-\gamma)^2\sigma_s^2 + \gamma^2\sigma_2^2)
\end{aligned}$$

We can now plug this into the certainty equivalent formula to get the worker's incentives:

$$d(w) = \alpha + \beta(e_1 - \gamma e_2) - r\frac{\beta^2(\sigma_1^2 + (1-\gamma)^2\sigma_s^2 + \gamma^2\sigma_2^2)}{2}$$

Consider the worker choosing effort for a fixed wage scheme. We focus on worker 1, but the analysis is the same either way. Worker 1 solves:

$$\max_{e_1} d(w) - c(e_1) = \max_{e_1} \alpha + \beta(e_1 - \gamma e_2) - r\frac{\beta^2(\sigma_1^2 + (1-\gamma)^2\sigma_s^2 + \gamma^2\sigma_2^2)}{2} - c(e)$$

Effort does not impact the middle term, and we have the normal condition: $\beta = c'(e_1)$. This proves that γ does not directly impact the worker's choice of effort. It does however impact the risk the worker takes on. To see this notice the term $r\frac{\beta^2(\sigma_1^2+(1-\gamma)^2\sigma_s^2+\gamma^2\sigma_2^2)}{2}$. This term enters into the worker's utility negatively. By increasing γ, the firm shifts weight from σ_s^2 to σ_2^2. Can someone give an intuition for this? What does this intuitively mean?

Let's then proceed as we normally do and define $\beta(e_1) = c'(e_1)$. The worker accepts the wage scheme if:

$$u(accept) = \alpha + \beta(e_1)(e_1 - \gamma e_2) - r\frac{\beta(e_1)^2(\sigma_1^2 + (1-\gamma)^2\sigma_s^2 + \gamma^2\sigma_2^2)}{2} - c(e_1) \geq \bar{u}$$

The firm sets α as low as it can subject to the worker accepting. This amounts to making the last line an equality:

$$\alpha + \beta(e_1)(e_1 - \gamma e_2) - r\frac{\beta(e_1)^2(\sigma_1^2 + (1-\gamma)^2\sigma_s^2 + \gamma^2\sigma_2^2)}{2} - c(e_1) = \bar{u}$$

Solving for α:

$$\alpha = \bar{u} - \beta(e_1)(e_1 - \gamma e_2) + r\frac{\beta(e_1)^2(\sigma_1^2 + (1-\gamma)^2\sigma_s^2 + \gamma^2\sigma_2^2)}{2} + c(e_1)$$

Now we plug this into the firm's profit:

$$\begin{aligned}
\pi &= E[y_1 - w_1] \\
&= E[e_1 + v_1 - \beta(e_1)(e_1 + v_1 + v_s - \gamma(e_2 + v_2 + v_s)) - \alpha] \\
&= e_1 - \beta(e_1)(e_1 - \gamma e_2) - \alpha \\
&= e_1 - \beta(e_1)(e_1 - \gamma e_2) - \bar{u} + \beta(e_1)(e_1 - \gamma e_2) - r\frac{\beta(e_1)^2(\sigma_1^2 + (1-\gamma)^2\sigma_s^2 + \gamma^2\sigma_2^2)}{2} - c(e_1) \\
&= e_1 - \bar{u} - r\frac{\beta(e_1)^2(\sigma_1^2 + (1-\gamma)^2\sigma_s^2 + \gamma^2\sigma_2^2)}{2} - c(e_1)
\end{aligned}$$

As before, the firm maximizes this expression.

$$\max_{e_1,\gamma} e_1 - \bar{u} - r\frac{\beta(e_1)^2(\sigma_1^2 + (1-\gamma)^2\sigma_s^2 + \gamma^2\sigma_2^2)}{2} - c(e_1)$$

Unlike before, notice that the firm has two objects it can control: effort and γ. (Remember β is determined by e_1, so we have to choose to maximize with respect to β or e_1). We need to take two FOCs. let's start with e_1 because it is more familiar:

$$[e_1] : 1 - c'(e_1) - r\beta(e_1)(\sigma_1^2 + (1-\gamma)^2\sigma_s^2 + \gamma^2\sigma_2^2)\beta'(e_1) = 0$$

Now, like before, remember that $\beta(e_1) = c'(e_1)$ therefore $\beta'(e_1) = c''(e_1)$. So our expression becomes:

$$1 - c'(e_1) - r\beta(e_1)(\sigma_1^2 + (1-\gamma)^2\sigma_s^2 + \gamma^2\sigma_2^2)c''(e_1) = 0$$

Now we just call $\beta(e_1)$, β_{rel} and plug in that $c'(e_1) = \beta_{rel}$:

$$1 - \beta_{rel} - r\beta_{rel}(\sigma_1^2 + (1-\gamma)^2\sigma_s^2 + \gamma^2\sigma_2^2)c''(e_1) = 0$$

Simplifying:

$$1 = \beta_{rel}\bigg(1 + r(\sigma_1^2 + (1-\gamma)^2\sigma_s^2 + \gamma^2\sigma_2^2)c''(e_1)\bigg)$$

$$\frac{1}{1 + r(\sigma_1^2 + (1-\gamma)^2\sigma_s^2 + \gamma^2\sigma_2^2)c''(e_1)} = \beta_{rel}$$

if we squint at this we will see this looks just like our normal expression, but with more "variance-related" terms. However, we are not done. The firm also gets to choose γ. We need to go all the way back to the profit expression.

$$\max_{e_1,\gamma} e_1 - \bar{u} - r\frac{\beta(e_1)^2(\sigma_1^2 + (1-\gamma)^2\sigma_s^2 + \gamma^2\sigma_2^2)}{2} - c(e_1)$$

before taking the FOC, notice that only the big variance term depends on γ, and even further, only the inside part of that expression depends on it. Thus we can zoom in on that part of the profit expression. The FOC for γ is:

$$[\gamma] : -\frac{r}{2}\bigg(-2(1-\gamma)\sigma_s^2 + 2\gamma\sigma_2^2\bigg) = 0$$

Simplifying:

$$(1-\gamma)\sigma_s^2 - \gamma\sigma_2^2 = 0$$
$$-\sigma_s^2\gamma - \sigma_2^2\gamma = -\sigma_s^2 \leftrightarrow \gamma(-\sigma_s^2 - \sigma_2^2) = -\sigma_s^2$$
$$\gamma_{rel} = \frac{-\sigma_s^2}{-\sigma_s^2 - \sigma_2^2} = \frac{\sigma_s^2}{\sigma_s^2 + \sigma_2^2}$$

This was a lot of work. I want to mention a trick that could have helped us obtain the answer faster. Recall that γ does not impact effort. Also, let's look at the worker's wage again:

$$
\begin{aligned}
w(y_1, y_2) &= \alpha + \beta(Y_1 - \gamma Y_2) \\
&= \alpha + \beta[e_1 + v_s + v_1 - \gamma(e_2 + v_s + v_2)] \\
&= \alpha + \beta[e_1 - \gamma e_2 + (1-\gamma)v_s + v_1 - \gamma v_2)] \\
&= \alpha + \beta e_1 - \beta\gamma e_2 + \beta(1-\gamma)v_s + \beta v_1 - \beta\gamma v_2 \\
&= \underbrace{\alpha - \beta\gamma e_2}_{\text{constant stuff}} + \underbrace{\beta e_1}_{\text{effort stuff}} + \beta \underbrace{(1-\gamma)v_s + v_1 - \gamma v_2}_{\text{random stuff that is mean 0!}}
\end{aligned}
$$

So relative performance pay (γ) only shifts the constant stuff and then impacts the variance. In your homework, you show that shifting constant stuff ($\bar{y}$) just changes the base pay. So the only useful thing γ does is change the variance. We now the risk-incentive trade-off hurts profit. This means that we could have just tried to minimize the variance with respect to γ directly:

$$\min_{\gamma} Var(w_1(y_1, y_2)) = \min_{\gamma} \beta^2(\sigma_1^2 + (1-\gamma)^2\sigma_s^2 + \gamma^2\sigma_2^2)$$

The FOC is:

$$[\gamma] : \beta^2(-2(1-\gamma)\sigma_s^2 + 2\gamma\sigma_2^2) = 0$$

Solving:

$$
\begin{aligned}
(-(1-\gamma)\sigma_s^2 + \gamma\sigma_2^2) &= 0 \\
\gamma\sigma_s^2 + \gamma\sigma_2^2 &= \sigma_s^2 \\
\gamma_{rel} &= \frac{\sigma_s^2}{\sigma_s^2 + \sigma_2^2}
\end{aligned}
$$

To get the optimal β we can just call all the random stuff $\epsilon_{TOTAL} = (1-\gamma)v_s + v_1 - \gamma v_2$ where $\sigma_{TOTAL}^2 = \sigma_1^2 + (1-\gamma)^2\sigma_s^2 + \gamma^2\sigma_2^2$ (we got this a few steos back) and use our formula from the theorem with σ^2 replaced with σ_{TOTAL}^2

$$\beta_{rel} = c'(e_{rel}) = \frac{1}{1 + r\sigma_{TOTAL}^2 c''(e_{rel})} = \frac{1}{1 + r[\sigma_1^2 + (1-\gamma_{rel})^2\sigma_s^2 + \gamma_{rel}^2\sigma_2^2]c''(e_{rel})}$$

2 Informativeness Principle

We wish to understand: when should we use the additional information y_2. Let's start by re-writing:

$$
\begin{aligned}
w(y_1, y_2) &= \alpha + \beta(y_1 - \gamma Y_2) \\
&= \alpha + \beta y_1 - \beta\gamma y_2 \\
&= \alpha + \beta y_1 + b y_2
\end{aligned}
$$

where $b = -\beta\gamma$. Using the additional information means that $b \neq 0$. We know that:

$$\gamma_{rel} = \frac{\sigma_s^2}{\sigma_s^2 + \sigma_2^2}$$

We also know that:

$$\beta^* = \frac{1}{1 + r\sigma_{TOTAL}^2 c''(e^*)} > 0$$

Since this is always positive, whether we use the additional information depends only on γ_{rel}. Therefore we use the additional information whenever:

$$\gamma_{rel} = \frac{\sigma_s^2}{\sigma_s^2 + \sigma_2^2} > 0$$

Let's think through what this final expression means. σ_s^2 is the variance of the shared component of y_1, y_2. $\sigma_s^2 + \sigma_2^2$ is the total variance of y_2. Thus $\frac{\sigma_s^2}{\sigma_s^2+\sigma_2^2}$ is the fraction of total variance of y_2 that is informative about y_1. It is almost the correlation coefficient. We put more weight on the additional information y_2 whenever it better predicts output y_1. How does this impact effort and output?

$$e^* = \beta^* = \frac{1}{1 + r\sigma_{TOTAL}^2}$$

where:$\sigma_{TOTAL}^2 = \sigma_1^2 + \gamma^2\sigma_2^2 + (1-\gamma)^2\sigma_s^2$ Plug in γ_{rel}:

$$\sigma_{TOTAL}^2 = \sigma_1^2 + \left(\frac{\sigma_s^2}{\sigma_s^2 + \sigma_2^2}\right)^2 \sigma_2^2 + \left(1 - \frac{\sigma_s^2}{\sigma_s^2 + \sigma_2^2}\right)^2 \sigma_s^2$$

$$\sigma_{TOTAL}^2 = \sigma_1^2 + 2\left(\frac{\sigma_s \sigma_2}{\sigma_s^2 + \sigma_2^2}\right)^2$$