

# Board Work for Lecture 7

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## 1 Relative Performance Evaluation

Let's begin by deriving the worker's certainty equivalent for a wage given by  $w(y_1, y_2) = \alpha + \beta(y_1 - \gamma y_2)$ . We will apply our certainty equivalent formula:

$$d = \mu - r \frac{\sigma^2}{2}$$

where  $\mu = E[w(y_1, y_2)]$ ,  $\sigma^2 = Var[w(y_1, y_2)]$ . The mean of the wage is:

$$E[w(y_1, y_2)] = E[\alpha + \beta(Y_1 - \gamma Y_2)] \quad (1)$$

$$= \alpha + \beta E[Y_1 - \gamma Y_2] \quad (2)$$

$$= \alpha + \beta E[e_1 + v_1 + v_s - \gamma(e_2 + v_2 + v_s)] \quad (3)$$

$$= \alpha + \beta(e_1 - \gamma e_2) \quad (4)$$

$$(5)$$

The variance is:

$$\begin{aligned} Var[w(y_1, y_2)] &= Var[\alpha + \beta(Y_1 - \gamma Y_2)] \\ &= Var[\alpha] + Var[\beta(Y_1 - \gamma Y_2)] \\ &= 0 + Var[\beta(Y_1 - \gamma Y_2)] \\ &= \beta^2 Var[Y_1 - \gamma Y_2] \\ &= \beta^2 Var[e_1 + v_1 + v_s - \gamma(e_2 + v_2 + v_s)] \\ &= \beta^2 Var[v_1 + v_s - \gamma(v_2 + v_s)] \\ &= \beta^2 Var[v_1 + (1 - \gamma)v_s - \gamma v_2] \\ &= \beta^2 (Var[v_1] + Var[(1 - \gamma)v_s] + Var[\gamma v_2]) \\ &= \beta^2 (\sigma_1^2 + (1 - \gamma)^2 \sigma_s^2 + \gamma^2 \sigma_2^2) \end{aligned}$$

We can now plug this into the certainty equivalent formula to get the worker's incentives:

$$d(w) = \alpha + \beta(e_1 - \gamma e_2) - r \frac{\beta^2(\sigma_1^2 + (1 - \gamma)^2 \sigma_s^2 + \gamma^2 \sigma_2^2)}{2}$$

Consider the worker choosing effort for a fixed wage scheme. We focus on worker 1, but the analysis is the same either way. Worker 1 solves:

$$\max_{e_1} d(w) - c(e_1) = \max_{e_1} \alpha + \beta(e_1 - \gamma e_2) - r \frac{\beta^2(\sigma_1^2 + (1 - \gamma)^2 \sigma_s^2 + \gamma^2 \sigma_2^2)}{2} - c(e)$$

Effort does not impact the middle term, and we have the normal condition:  $\beta = c'(e_1)$ . This proves that  $\gamma$  does not directly impact the worker's choice of effort. It does however impact the risk the worker takes on.

To see this notice the term  $r \frac{\beta^2(\sigma_1^2 + (1-\gamma)^2\sigma_s^2 + \gamma^2\sigma_2^2)}{2}$ . This term enters into the worker's utility negatively. By increasing  $\gamma$ , the firm shifts weight from  $\sigma_s^2$  to  $\sigma_2^2$ . Can someone give an intuition for this? What does this intuitively mean?

The worker accepts the wage scheme if:

$$u(\text{accept}) = \alpha + \beta(e_1 - \gamma e_2) - r \frac{\beta^2(\sigma_1^2 + (1-\gamma)^2\sigma_s^2 + \gamma^2\sigma_2^2)}{2} - c(e_1) \geq \bar{u}$$

The firm sets  $\alpha$  as low as it can subject to the worker accepting. This amounts to making the last line an equality:

$$\alpha + \beta(e_1 - \gamma e_2) - r \frac{\beta^2(\sigma_1^2 + (1-\gamma)^2\sigma_s^2 + \gamma^2\sigma_2^2)}{2} - c(e_1) = \bar{u}$$

Solving for  $\alpha$ :

$$\alpha = \bar{u} - \beta(e_1 - \gamma e_2) + r \frac{\beta^2(\sigma_1^2 + (1-\gamma)^2\sigma_s^2 + \gamma^2\sigma_2^2)}{2} + c(e_1)$$

Now we plug this into the firm's profit, along with the fact that  $\beta = c'(e_1)$ :

$$\begin{aligned} \pi &= E[y_1 - w_1] \\ &= E[e_1 + v_1 - \beta(e_1 + v_1 + v_s - \gamma(e_2 + v_2 + v_s)) - \alpha] \\ &= e_1 - \beta(e_1 - \gamma e_2) - \alpha \\ &= e_1 - \beta(e_1 - \gamma e_2) - \bar{u} + \beta(e_1 - \gamma e_2) - r \frac{\beta^2(\sigma_1^2 + (1-\gamma)^2\sigma_s^2 + \gamma^2\sigma_2^2)}{2} - c(e_1) \\ &= e_1 - \bar{u} - r \frac{\beta^2(\sigma_1^2 + (1-\gamma)^2\sigma_s^2 + \gamma^2\sigma_2^2)}{2} - c(e_1) \\ &= e_1 - \bar{u} - r \frac{[c'(e_1)]^2(\sigma_1^2 + (1-\gamma)^2\sigma_s^2 + \gamma^2\sigma_2^2)}{2} - c(e_1) \end{aligned}$$

As before, the firm maximizes this expression.

$$\max_{e_1, \gamma} e_1 - \bar{u} - r \frac{[c'(e_1)]^2(\sigma_1^2 + (1-\gamma)^2\sigma_s^2 + \gamma^2\sigma_2^2)}{2} - c(e_1)$$

Unlike before, notice that the firm has two objects it can control: effort and  $\gamma$ . We need to take two FOCs. let's start with  $e_1$  because it is more familiar:

$$[e_1] : 1 - c'(e_1) - r(\sigma_1^2 + (1-\gamma)^2\sigma_s^2 + \gamma^2\sigma_2^2)c'(e_1)c''(e_1) = 0$$

Simplifying:

$$\begin{aligned} 1 &= c'(e_1) \left( 1 + r(\sigma_1^2 + (1-\gamma)^2\sigma_s^2 + \gamma^2\sigma_2^2)c''(e_1) \right) \\ \frac{1}{1 + r(\sigma_1^2 + (1-\gamma)^2\sigma_s^2 + \gamma^2\sigma_2^2)c''(e_1)} &= c'(e_1) = \beta_{rel} \end{aligned}$$

If we squint at this we will see this looks just like our normal expression, but with more "variance-related" terms. However, we are not done. The firm also gets to choose  $\gamma$ . We need to go all the way back to the profit expression.

$$\max_{e_1, \gamma} e_1 - \bar{u} - r \frac{\beta(e_1)^2(\sigma_1^2 + (1-\gamma)^2\sigma_s^2 + \gamma^2\sigma_2^2)}{2} - c(e_1)$$

Before taking the FOC, notice that only the big variance term depends on  $\gamma$ , and even further, only the inside part of that expression depends on it. Thus we can zoom in on that part of the profit expression. The FOC for  $\gamma$  is:

$$[\gamma] : -\frac{r}{2} \left( -2(1-\gamma)\sigma_s^2 + 2\gamma\sigma_2^2 \right) = 0$$

Simplifying:

$$\begin{aligned}
(1 - \gamma)\sigma_s^2 - \gamma\sigma_2^2 &= 0 \\
-\sigma_s^2\gamma - \sigma_2^2\gamma &= -\sigma_s^2 \leftrightarrow \gamma(-\sigma_s^2 - \sigma_2^2) = -\sigma_s^2 \\
\gamma_{rel} &= \frac{-\sigma_s^2}{-\sigma_s^2 - \sigma_2^2} = \frac{\sigma_s^2}{\sigma_s^2 + \sigma_2^2}
\end{aligned}$$

## 2 Informativeness Principle

We can actually use the ideas from relative performance evaluation to ask a more general question. Suppose that  $y_2$  is not necessarily the performance of another worker, but rather some piece of data. We wish to understand whether it should be incorporated in the wage. Formally, if the wage is given by:

$$w(y_1, y_2) = \alpha + \beta y_1 + b y_2$$

then using the information is setting  $b \neq 0$ . We can easily just do a change of variables, and define  $b = -\beta\gamma$ , and we are back to our old setup:

$$w(y_1, y_2) = \alpha + \beta y_1 - \beta\gamma y_2 = \alpha + \beta(y_1 - \gamma y_2)$$

We now from the work we have done already that the optimal  $\beta$  is positive because:

$$\beta_{rel} = \frac{1}{1 + r\sigma_{TOTAL}^2 c''(e_{rel})} > 0$$

We also know that the optimal  $\gamma$  is given by:

$$\gamma_{rel} = \frac{\sigma_s^2}{\sigma_s^2 + \sigma_2^2} > 0$$

Therefore,  $b \neq 0$  if and only if  $\gamma_{rel} > 0$ , which occurs whenever  $\frac{\sigma_s^2}{\sigma_s^2 + \sigma_2^2} > 0$ . What does this mean?  $\sigma_s^2$  is the variance of the shared component of  $y_1, y_2$ .  $\sigma_s^2 + \sigma_2^2$  is the total variance of  $y_2$ . Thus  $\frac{\sigma_s^2}{\sigma_s^2 + \sigma_2^2}$  is the fraction of total variance of  $y_2$  that is informative about  $y_1$ . It is almost the correlation coefficient. We put more weight on the additional information  $y_2$  whenever it better predicts output  $y_1$ . This is exactly the informativeness principle: we use a piece of information  $y_2$  whenever it helps us better predict output  $y_1$ .