CSCI 5454 – Fall 2024
Design and Analysis of Algorithms

Homework 3

Instructions:

- **Topics:** Analysis of algorithms, the Euclidean algorithm and its analysis.
- **Due date:** This homework is due by 11:59PM Mountain Time on **Friday, September 20th**. Late assignments will not be accepted. Your lowest two homework scores for the semester will be dropped.
- You are welcome and encouraged to discuss the problems with classmates, but **you must write up and submit your own solutions and code**. You must also write the names of everyone in your group on the top of your submission.
- The primary resources for this class are the lectures, lecture slides/notes, the CLRS and Erickson algorithms textbooks, the teaching staff, your collaborators, and the class Piazza forum. We strongly encourage you only to use these resources. If you do use another resource, make sure to cite it and explain why you needed it. **Using generative AI tools (e.g., ChatGPT), Q&A forums (e.g., Stack Exchange, Quora), or cheating repositories (e.g., Chegg, CourseHero) is not allowed.** See the syllabus for a more detailed explanation.
- You must justify all of your answers unless specifically stated otherwise.
- We strongly encourage (but do not require) you to write your solutions in LaTeX, and have provided a skeleton file.[1] However, if your homework is illegible then we reserve the right not to grade it.

[1] If you have not used Latex before, it's easy to get started using Overleaf. See their 30-minute tutorial: `https://www.overleaf.com/learn/latex/Learn_LaTeX_in_30_minutes`.

Questions

Question 1. (Calculate the DFT, 10 points.) Compute the DFT of the following sequences. Show your calculations to justify your answer. You should use ideas from the FFT algorithm to make your calculations simpler.

a. (2 points) $0, 1$.

b. (3 points) $0, 0, 1, 1$.

c. (5 points) $0, 0, 0, 0, 1, 1, 1, 1$.

Question 2. (Period finding using FFT, 15 points.) Let n be some number and let p divide n. Consider the sequence a_j for $j = 0, \ldots, n-1$ defined as follows:

$$a_j = \begin{cases} 1 & \text{if } j = lp \text{ for some } l \in \{0, \ldots, \frac{n}{p} - 1\} \\ 0 & \text{otherwise} \end{cases}$$

Calculate the DFT $A_0, \ldots, A_{n-1}$ of $a_0, \ldots, a_{n-1}$.

(**Hint**: Compute FFTs for the sequences: $1, 0, 1, 0, 1, 0, 1, 0$ with $n = 8, p = 2$; and $1, 0, 0, 1, 0, 0, 1, 0, 0$ with $n = 9, p = 3$. Notice that the FFT itself is periodic.)

Question 3. (3-way divide and conquer FFT, 25 points.) In class, we saw a divide-and-conquer scheme for FFT that divides the input into two parts (odd indices and even indices). Now, consider a number $n = 3k$ for some k and the sequence $a_0, \ldots, a_{n-1}$. Let

$$\begin{aligned}
[C_0, \ldots, C_{k-1}] &= \mathrm{FFT}(a_0, a_3, \ldots, a_{3i}, \ldots, a_{n-3})\,, \\
[D_0, \ldots, D_{k-1}] &= \mathrm{FFT}(a_1, a_4, \ldots, a_{3i+1}, \ldots, a_{n-2})\,, \text{ and} \\
[E_0, \ldots, E_{k-1}] &= \mathrm{FFT}(a_2, a_5, \ldots, a_{3i+2}, \ldots, a_{n-1})\,.
\end{aligned}$$

In other words:

$$C_l = \sum_{j=0}^{k-1} a_{3j} (\omega_k^l)^j \qquad \text{for } l \in \{0, \ldots, k-1\}$$

$$D_l = \sum_{j=0}^{k-1} a_{3j+1} (\omega_k^l)^j \qquad \text{for } l \in \{0, \ldots, k-1\}$$

$$E_l = \sum_{j=0}^{k-1} a_{3j+2} (\omega_k^l)^j \qquad \text{for } l \in \{0, \ldots, k-1\}$$

Suppose $A_0, \ldots, A_{n-1}$ is the FFT of the original sequence $a_0, \ldots, a_{n-1}$.

a. (5 points) Write down a formula for each coefficient A_l for $l \in \{0, \ldots, \frac{n}{3} - 1\}$ in terms of the values C_l, D_l and E_l. Note: $\frac{n}{3} = k$.

b. (5 points) Write down a formula for each coefficient $A_{l+\frac{n}{3}}$ for $l \in \{0, \ldots, \frac{n}{3} - 1\}$ in terms of the values C_l, D_l and E_l.

c. (5 points) Write down a formula for each coefficient $A_{l+\frac{2n}{3}}$ for $l \in \{0, \ldots, \frac{n}{3} - 1\}$ in terms of the values C_l, D_l and E_l.

d. (10 points) Write pseudocode for an FFT algorithm using a "3-way divide and conquer" on an input of size n which is a power of 3. Make sure to handle the base case for $n = 1 = 3^0$.