csci-5454-hw2

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1 Q1

1.1 Q1a

The maximum prefix sum (**pref**) of the original array can either be the max prefix from the left array or the sum of the left array ($\mathbf{s}_l$) plus the prefix of the right array. Likewise, max suffix sum (**suff**) can either be the max suffix from the right array or the sum of the right ($\mathbf{s}_r$) array plus the suffix of the left array.

$$\mathbf{pref} = max(\mathbf{pref}_l, \mathbf{s}_l + \mathbf{pref}_r) \tag{1}$$

$$\mathbf{suff} = max(\mathbf{suff}_r, \mathbf{s}_r + \mathbf{suff}_l) \tag{2}$$

1.2 Q1b

Algorithm 1 max-subarray(A, low, high)

if high == low **then**
  return (low, low, low, low) ▷ base case
else
  $mid \leftarrow \lfloor \frac{\text{low+high}}{2} \rfloor$
  $(s_l, pref_l, suff_l, m_l) \leftarrow$ max-subarray(A, low, mid)
  $(s_r, pref_r, suff_r, m_r) \leftarrow$ max-subarray$(A, mid+1, high)$
  $pref \leftarrow max(pref_l, s_l + pref_r)$
  $suff \leftarrow max(suff_r, s_r + suff_l)$
  $s_A \leftarrow s_l + s_r$
  $m_A \leftarrow max(m_l, m_r, suff_l + pref_r)$
  return $(s_A, pref, suff, m_A)$
end if

Recursion splits arrays in half (by self-calling max-subarray) until the base case of size 1 is reached. As values start to return from max-subarray calls in the **else** block, the combination step computes max-prefix (pref), max-suffix (suff), total sum (s_A), and max subarray at the current step (m_A). This continues until the 2nd level of the recursion tree (now traveling upward) resolves and is combined into final results of pref, suff, s_A, and m_A .

2 Q2

2.1 Q2a

If squish, s, is already computed from $r1$ to $r2$, then the squish from $r1$ to $r2+1$ is just the element wise sum of the previous squish and the column $r2+1$.

$$s = \mathsf{squish}(A, r_1, r_2) \tag{3}$$

$$\mathsf{squish}(A, r_1, r_2 + 1) = s + A[0 \dots n-1, r_2 + 1] \tag{4}$$

Where $A[0 \dots n-1, r_2+1]$ is the column vector of the matrix at $r_2 + 1$.

2.2 Q2b

Algorithm 2 max-subarray(A)

$s_{current} \leftarrow 0$
$s_{max} \leftarrow -\infty$
$n \leftarrow LENGTH(A)$
for $i = 0$ **upto** $n-1$ **do**
  $s_{current} \leftarrow max(0, s_{current} + A[i])$
  $s_{max} \leftarrow max(s_{max}, s_{current})$
end for
return s_{max}

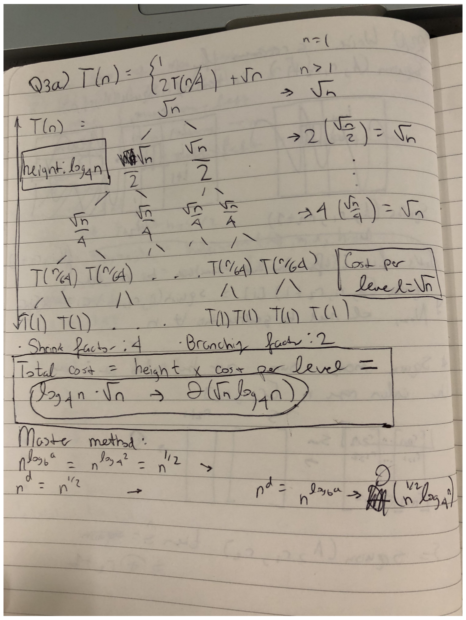
Algorithm 3 max-submatrix(A)

$n \leftarrow LENGTH(A)$
$m_A \leftarrow -\infty$
for $r_1 \leftarrow 0$ **upto** $n-1$ **do** ▷ left column idx
  $sq \leftarrow ARRAY(n)$ ▷ init array of length n
  for $r_2 \leftarrow r1$ **upto** $n-1$ **do** ▷ $r_2 \geq r_1$ ▷ right column idx
    for $i \leftarrow 0$ **upto** $n-1$ **do** ▷ row idx
      $sq[i] \leftarrow sq[i] + A[i, r2]$ ▷ squish columns for Kadane
    end for
    $m_A \leftarrow$ max-subarray(sq)
  end for
end for
return m_A

Max-subarray uses Kadane's algorithm to iteratively track and test sums in $O(n)$. For a **max-submatrix**, we iterate through pairs of column start and stop indices, r_1, r_2. For each r_1,r_2 pair the subarray is squished by column vector additions $O(n^2)$, and the squished column vector is fed into max-subarray to compute the max-submatrix sum since each index in the submatrix is a possible sum with r_1, r_2 column start,stop indices over the rows represented as elements, i, of the squished array. The total cost is $O(n^3)$.

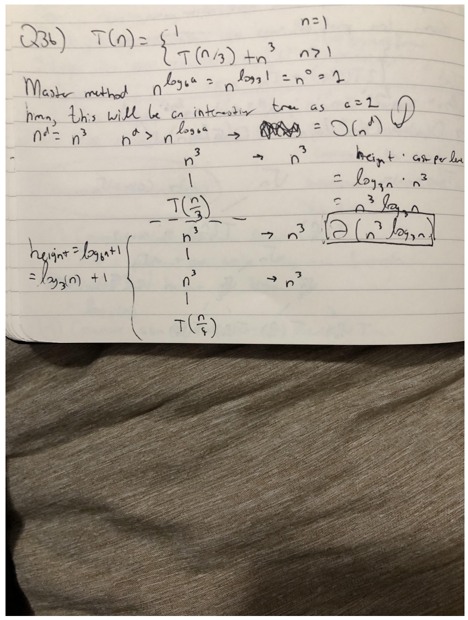
3 Q3

3.1 Q3a



Recurrence $T(n) = 2T(\frac{n}{4}) + \sqrt{n}$ implies that problem size recursively shrinks by a factor of 4, and the cost of combining is $\sqrt{n}$. The recursion tree has a height of $\log_4(n) + 1$ and the cost of each level in the tree is $\sqrt{n}$. Multiplying the number of levels (height) by the cost at each level gives a total cost asymptotic to $\Theta(\sqrt{n}\log_4 n)$.

3.2 Q3b



Q3b) $T(n) = \begin{cases} 1 & n=1 \\ T(n/3) + n^3 & n>1 \end{cases}$

Master method $n^{\log_b a} = n^{\log_3 1} = n^0 = 1$

hmm, this will be an interesting tree as $a = 2$

$n^d = n^3$ $\quad n^d > n^{\log_b a} \rightarrow = O(n^d)$

$n^3 \rightarrow n^3$

$T(\frac{n}{3})$

$n^3 \rightarrow n^3$

$n^3 \rightarrow n^3$

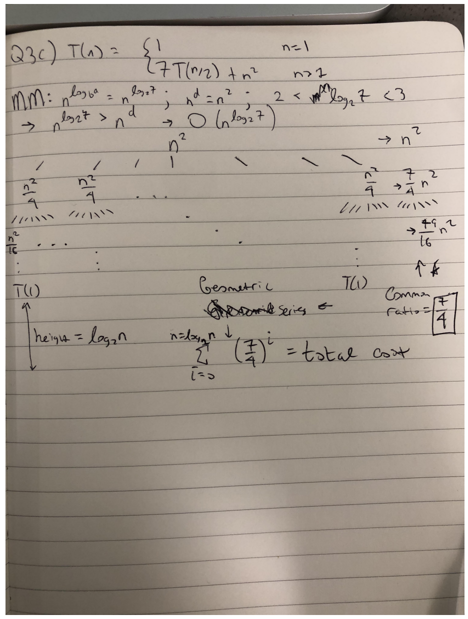
$T(\frac{n}{9})$

height $= \log_b n + 1$
$= \log_3(n) + 1$

height · cost per level
$= \log_3 n \cdot n^3$
$= n^3 \log_3 n$
$\boxed{\Theta(n^3 \log_3 n)}$

The recursion tree for $T(n) = 2T(\frac{n}{3}) + n^3$ has a height of $\log_3(n) + 1$. The cost at each level is n^3. The total cost is $\Theta(n^3 \log_3 n)$.

3.3 Q3c



The recursion tree for $T(n) = 7T(n/2) + n^2$ has a shrink factor of 2, and a combination cost of n^2. The cost of the first level is n^2 while the cost at the ith level is $(\frac{7}{4})^i n^2$. The total cost is a function of the level i and can be written as a geometric sum:

$$\sum_{i=0}^{n=\log_2(n)} (\frac{7}{4})^i n^2 = \tag{5}$$

$$n^2 \sum_{i=0}^{n=\log_2(n)} (\frac{7}{4})^i = \tag{6}$$

$$\frac{7}{4}(\frac{1-(\frac{7}{4})^{\log_2(n)+1}}{1-\frac{7}{4}}) = \tag{7}$$

$$7(\frac{1-(\frac{7}{4})^{\log_2(n)+1}}{-3}) = \tag{8}$$

$$-\frac{7}{3}(1-(\frac{7}{4})^{\log_2(n)+1}) = \tag{9}$$

$$-\frac{7}{3}(1-(\frac{7}{4})^{\log_2 n} * (\frac{7}{4})) = \tag{10}$$

$$-\frac{7}{3}(1-\frac{7}{4}n^{\log_2 \frac{7}{4}}) \tag{11}$$

The total cost is $\Theta(n^{\log_2(\frac{7}{4})})$.