1 Hadronic parameters

Interference effects can occur within the same final state, that is, the interference is always between $D^0 \to F$ and $\overline{D}^0 \to F$, not between $D^0 \to F$ and $D^0 \to \overline{F}$ for some generic final state F. The hadronic parameters in the ith bin of some final state F are:

$$\begin{aligned} A_{F_i} &= \int_i \left|\mathcal{A}_{D^0\to F}(\mathbf{x})\right|^2 \mathrm{d}\mathbf{x} \\ \overline{A}_{F_i} &= \int_i \left|\mathcal{A}_{\overline{D}^0\to F}(\mathbf{x})\right|^2 \mathrm{d}\mathbf{x}, \end{aligned} \tag{1}$$

where $\mathcal{A}$ is the given amplitude at some position in the phase-space $\mathbf{x}$. That is, these quantities are proportional to some rate into the ith bin for a D of well-defined initial flavour. It is useful to define the fractional quantities:

$$\begin{aligned} K_{F_i} &= \frac{\int_i \left|\mathcal{A}_{D^0\to F}(\mathbf{x})\right|^2 \mathrm{d}\mathbf{x}}{\int \left|\mathcal{A}_{D^0\to F}(\mathbf{x})\right|^2 \mathrm{d}\mathbf{x}} \\ \overline{K}_{F_i} &= \frac{\int_i \left|\mathcal{A}_{\overline{D}^0\to F}(\mathbf{x})\right|^2 \mathrm{d}\mathbf{x}}{\int \left|\mathcal{A}_{\overline{D}^0\to F}(\mathbf{x})\right|^2 \mathrm{d}\mathbf{x}}, \end{aligned} \tag{2}$$

The interference between the two decay paths can be parameterised by the complex number $\mathcal{Z}_{F_i}$, which is defined as

$$\mathcal{Z}_{F_i} = \left(A_{F_i}\overline{A}_{F_i}\right)^{-\frac{1}{2}} \int_i \mathcal{A}_{\overline{D}^0\to F}(\mathbf{x})\mathcal{A}^*_{D^0\to F}(\mathbf{x})\mathrm{d}\mathbf{x}, \tag{3}$$

i.e. the coherence factor / average strong phase is this quantity in polar coordinates, while the (c_i, s_i) of, for example, $D^0 \to K^0_S\pi^+\pi^-$, are this quantity in cartesian coordinates:

$$R_{F_i}e^{i\delta_{F_i}} = c_i - is_i = \mathcal{Z}_{F_i}. \tag{4}$$

For the *CP* tags, $\mathcal{Z}_F = \pm 1$. The hadronic parameters can be summed to calculate, for example, the global coherence factor etc. using the relation

$$\begin{aligned} \mathcal{Z}_F &= \left(A_F\overline{A}_F\right)^{-\frac{1}{2}} \int \mathcal{A}_{\overline{D}^0\to F}(\mathbf{x})\mathcal{A}^*_{D^0\to F}(\mathbf{x})\mathrm{d}\mathbf{x} \\ &= \left(A_F\overline{A}_F\right)^{-\frac{1}{2}} \sum_i \left(A_{F_i}\overline{A}_{F_i}\right)^{\frac{1}{2}} \mathcal{Z}_{F_i} \\ &= \sum_i \sqrt{K_{F_i}\overline{K}_{F_i}}\mathcal{Z}_{F_i} \end{aligned} \tag{5}$$

Another useful parameter is the phase-space averaged relative rate, r_F, which is defined as

$$r_F^2 = \frac{\int \left|\mathcal{A}_{D^0\to F}(\mathbf{x})\right|^2 \mathrm{d}\mathbf{x}}{\int \left|\mathcal{A}_{\overline{D}^0\to F}(\mathbf{x})\right|^2 \mathrm{d}\mathbf{x}}, \tag{6}$$

and the corresponding parameter in one of the phase-space bins by

$$r_{F_i}^2 = \frac{\int_i \left|\mathcal{A}_{D^0\to F}(\mathbf{x})\right|^2 \mathrm{d}\mathbf{x}}{\int_i \left|\mathcal{A}_{\overline{D}^0\to F}(\mathbf{x})\right|^2 \mathrm{d}\mathbf{x}} = r_F^2\frac{K_{F_i}}{\overline{K}_{F_i}} \tag{7}$$

Double tag observables

All of the observables for all tags can be related using these parameters. Beginning with the wavefunction of the $\psi(3770)$:

$$|\psi(3770)\rangle = \frac{1}{\sqrt{2}}\left(|D^0\rangle\,|\overline{D}^0\rangle - |\overline{D}^0\rangle\,|D^0\rangle\right), \tag{8}$$

Consider the generic final state(s) F, G, where these could be the $K^+\pi^-\pi^-\pi^+$, a CP-tag such as K^+K^- etc, at positions in the phase-spaces $\mathbf{x}_F$ and $\mathbf{x}_G$ The amplitude is given by

$$\begin{aligned}\langle F(\mathbf{x}_F)G(\mathbf{x}_G)|\psi(3770)\rangle &= \frac{1}{\sqrt{2}}\left(\langle F(\mathbf{x}_F)|D^0\rangle\langle G(\mathbf{x}_G)|\overline{D}^0\rangle - \langle F(\mathbf{x}_F)|\overline{D}^0\rangle\langle G(\mathbf{x}_G)|D^0\rangle\right)\\ &= \frac{1}{\sqrt{2}}\left(\mathcal{A}_{D^0\to F}(\mathbf{x}_F)\mathcal{A}_{\overline{D}^0\to G}(\mathbf{x}_G) - \mathcal{A}_{\overline{D}^0\to F}(\mathbf{x}_F)\mathcal{A}_{D^0\to G}(\mathbf{x}_G)\right)\end{aligned} \tag{9}$$

Taking the modulus-squared of this equation to get a rate gives:

$$\begin{aligned}\Gamma(\mathbf{x}_F,\mathbf{x}_G) \propto \frac{1}{2}&\left(|\mathcal{A}_{D^0\to F}(\mathbf{x}_F)|^2|\mathcal{A}_{\overline{D}^0\to G}(\mathbf{x}_G)|^2 + |\mathcal{A}_{\overline{D}^0\to F}(\mathbf{x}_F)|^2|\mathcal{A}_{D^0\to G}(\mathbf{x}_G)|^2\right)\\ &- \mathrm{Re}\left(\mathcal{A}_{\overline{D}^0\to F}(\mathbf{x}_F)\mathcal{A}^*_{D^0\to F}(\mathbf{x}_F)\mathcal{A}^*_{\overline{D}^0\to G}(\mathbf{x}_G)\mathcal{A}_{D^0\to G}(\mathbf{x}_G)\right)\end{aligned} \tag{10}$$

So to get the rate integrated over some bins (i,j) in F, G is given by

$$\int_i\int_j \Gamma(\mathbf{x}_F,\mathbf{x}_G)\mathrm{d}\mathbf{x}_F\mathrm{d}\mathbf{x}_G = \frac{1}{2}\left(A_{F_i}\overline{A}_{G_j} + \overline{A}_{F_i}A_{G_j}\right) - \sqrt{A_{F_i}\overline{A}_{F_i}A_{G_j}\overline{A}_{G_j}}\,\mathrm{Re}\left(\mathcal{Z}_{F_i}\mathcal{Z}^*_{G_j}\right) \tag{11}$$

Therefore, neglecting contributions from mixing, which to an excellent approximation only effect the no-coherence rate, the ρ observable in this region of the (double) phase-space is

$$\begin{aligned}\rho_{F_iG_j} &= 1 - 2\frac{\sqrt{A_{F_i}\overline{A}_{F_i}A_{G_j}\overline{A}_{G_j}}}{A_{F_i}\overline{A}_{G_j} + \overline{A}_{F_i}A_{G_j}}\mathrm{Re}\left(\mathcal{Z}_{F_i}\mathcal{Z}^*_{G_j}\right)\\ &= 1 - 2\frac{r_{F_i}r_{G_j}}{r^2_{F_i} + r^2_{G_j}}\mathrm{Re}\left(\mathcal{Z}_{F_i}\mathcal{Z}^*_{G_j}\right)\end{aligned} \tag{12}$$

this can be written in terms of the coherence factors and strong phases as:

$$\rho_{F_iG_j} = 1 - 2\frac{r_{F_i}r_{G_j}}{r^2_{F_i} + r^2_{G_j}}R_{F_i}R_{G_j}\cos\left(\delta_{F_i} - \delta_{G_j}\right) \tag{13}$$

Another observable is the fraction of candidates in bin (i,j). These may become useful observables with very large datasets for due to the lack of reliance on an absolute normalisation, and with a larger number of bins, for example in $D^0 \to K^0_{\mathrm{S}}\pi^+\pi^-$ where the phase-space integrated interference effects are small and the number of bins is large. These observables may also be useful for some debugging purposes.

$$\begin{aligned}\mathcal{F}_{F_iG_j} &= \frac{A_{F_i}\overline{A}_{G_j} + \overline{A}_{F_i}A_{G_j} - 2\sqrt{A_{F_i}\overline{A}_{F_i}A_{G_j}\overline{A}_{G_j}}\,\mathrm{Re}\left(\mathcal{Z}_{F_i}\mathcal{Z}^*_{G_j}\right)}{A_F\overline{A}_G + \overline{A}_FA_G - 2\sqrt{A_F\overline{A}_FA_G\overline{A}_G}\,\mathrm{Re}\left(\mathcal{Z}_F\mathcal{Z}^*_G\right)}\\ &= \frac{r^2_{F_i}\overline{K}_{G_j} + r^2_{G_j}\overline{K}_{F_i} - 2r_{F_i}r_{G_j}\mathrm{Re}\left(\mathcal{Z}_{F_i}\mathcal{Z}^*_{G_j}\right)}{r^2_F + r^2_G - 2r_Fr_G\mathrm{Re}\left(\mathcal{Z}_F\mathcal{Z}^*_G\right)}\end{aligned} \tag{14}$$

Some examples

Taking the signal side of the decay in all examples to be $K^+\pi^-\pi^-\pi^+$ (i.e. so $r_F \sim \tan^4\theta_c$). The first example is the phase-space integrated *CP* tags. In this case, $R_G = K_G = r_G = 1, \delta_G = 0, 180^{\mathrm{o}}$, the ρ observable simplifies to

$$\rho^{CP}_{F_i} = 1 \mp 2\frac{r_{F_i}}{r^2_{F_i} + 1}R_{F_i}\cos\left(\delta_{F_i}\right). \tag{15}$$

For the same-sign self-tag $F = G$, the ρ observables simplify to

$$\rho_{F_iF_j} = 1 - 2\frac{r_{F_i}r_{F_j}}{r^2_{F_i} + r^2_{F_j}}R_{F_i}R_{F_j}\cos\left(\delta_{F_i} - \delta_{F_j}\right), \tag{16}$$

which for the diagonal bins simplifies further to

$$\rho_{F_iF_i} = 1 - R_i^2 \tag{17}$$

For the phase-space integrated flavour tags, i.e. $K^+\pi^-$ and $K^+\pi^-\pi^0$, the rate simplifies to

$$\rho_{F_iG} = 1 - 2\frac{r_{F_i}r_G}{r_{F_i}^2 + r_G^2} R_{F_i} R_G \cos\left(\delta_{F_i} - \delta_G\right) \tag{18}$$

Mixing observables

The binned observables for charm mixing are the relative rates of the WS to RS transitions as a function of proper decay time, i.e.

$$R_{F_i}(t) = \frac{\Gamma(D^0 \to F)(t)}{\Gamma(\overline{D}^0 \to F)(t)} \tag{19}$$

For a D meson tagged at production as a D^0, the wavefunction as a function of decay time to lowest order in the mixing parameters (x, y) is given by

$$|D(t)\rangle = e^{-i\omega t}\left(|D^0\rangle + (y + ix)\frac{t}{2\tau}\frac{p}{q}|\overline{D}^0\rangle\right) \tag{20}$$

So the amplitude for transitions into the final state F is given by

$$\langle F|D(t)\rangle = e^{-i\omega t}\left(\langle F|D^0\rangle + (y + ix)\frac{t}{2\tau}\frac{p}{q}\langle F|\overline{D}^0\rangle\right) = e^{-i\omega t}\left(\mathcal{A}_{D^0\to F}(\mathbf{x}) + (y + ix)\frac{t}{2\tau}\frac{p}{q}\mathcal{A}_{\overline{D}^0\to F}(\mathbf{x})\right), \tag{21}$$

neglecting CP violation (so $p/q = 1$) and mod-squaring to get the rate gives

$$\Gamma(D^0(t) \to F(\mathbf{x})) \propto |\mathcal{A}_{D^0\to F}(\mathbf{x})|^2 + (x^2 + y^2)\frac{t^2}{4\tau^2}|\mathcal{A}_{\overline{D}^0\to F}(\mathbf{x})|^2 - \frac{t}{\tau}\mathrm{Re}\left((y + ix)\mathcal{A}_{\overline{D}^0\to F}(\mathbf{x})\mathcal{A}^*_{D^0\to F}(\mathbf{x})\right) \tag{22}$$

So integrating over some region of the phase space i and writing in terms of the previously defined parameters gives the rate:

$$\begin{aligned}\Gamma(D^0(t) \to F_i) &\propto e^{-t/\tau}\left(N_{F_i} + (x^2 + y^2)\frac{t^2}{4\tau^2}\overline{N}_{F_i} - \frac{t}{\tau}\mathrm{Re}\left((y + ix)\mathcal{Z}_{F_i}\right)\right)\\ &= \overline{N}_{F_i}e^{-t/\tau}\left(r_{F_i}^2 + (x^2 + y^2)\frac{t^2}{4\tau^2} - \frac{t}{\tau}r_{F_i}R_F(y\cos\delta_F - x\sin\delta_F)\right)\end{aligned} \tag{23}$$

So neglecting mixing in denominator of Eq.17, the ratio as a function of decay time is given by

$$R_{F_i}(t) = r_{F_i}^2 - \frac{t}{\tau}r_{F_i}R_F(y\cos\delta_F - x\sin\delta_F) + (x^2 + y^2)\frac{t^2}{4\tau^2} \tag{24}$$

The time-integrated rate is therefore

$$\Gamma(D^0 \to F_i) = \overline{A}_{F_i}\left(r_{F_i}^2 + \frac{x^2 + y^2}{2} - r_{F_i}R_{F_i}\left(y\cos\delta_{F_i} - x\sin\delta_{F_i}\right)\right), \tag{25}$$

On the other hand, for the conjugate process, which in this example is probably Cabibbo favoured, the rate is given by

$$\Gamma(\overline{D}^0 \to F_i) = \overline{A}_{F_i}\left(1 + \frac{x^2 + y^2}{2}r_{F_i}^2 - R_{F_i}r_{F_i}\left(y\cos\delta_{F_i} + x\sin\delta_{F_i}\right)\right) \sim \overline{A}_{F_i} \tag{26}$$

on our third hands, for *CP* tags $r_F = 1$, and hence the rate is approximately given by

$$\Gamma(D^0 \to F_i) = \Gamma(\overline{D}^0 \to F_i) = \overline{A}_{F_i}\left(1 \mp y\right) \tag{27}$$

Mixing corrections to ρ observables

Although the quantum-correlated rates are largely free from mixing corrections, the ρ observables do depend on the mixing parameters via the normalisation, i.e. as the no-coherence hypothesis will be affected by mixing. The correct definition of Eq.11 is really:

$$\rho_{F_iG_j} = \frac{\overline{A}_{F_i}\overline{A}_{G_j}\left(r_{F_i}^2 + r_{G_j}^2 - 2r_{F_i}r_{G_j}R_{F_i}R_{G_j}\cos(\delta_{F_i} - \delta_{G_i})\right)}{\Gamma(D^0 \to F_i)\Gamma(\overline{D}^0 \to G_j) + \Gamma(\overline{D}^0 \to F_i)\Gamma(D^0 \to G_j)} \tag{28}$$

Hence, for the CP tags, the mixing corrections take the form

B-observables

$$\begin{aligned}\left(e^{-t/\tau}\overline{A}_{F_i}\right)^{-1}\Gamma(B \to \left[D \to [F_i(t)]\,K^{\pm}\right]) = {} & r_{F_i}^2 + r_B^2 + 2r_{F_i}r_BR_{F_i}\cos(\delta_B + \delta_{F_i} \mp \gamma) \\ & + \frac{yt}{\tau}\left(r_B\cos(\delta_B - \gamma) + r_{F_i}R_{F_i}\cos\delta_D\right) \\ & + \frac{xt}{\tau}\left(r_B\sin(\delta_B - \gamma) - r_{F_i}R_{F_i}\sin\delta_D\right) \\ & + \frac{x^2+y^2}{4\tau^2}\end{aligned} \tag{29}$$

2 $K_S^0\pi^+\pi^-$ binning convention

- Define everything in the coordinates of $D^0 \to K_S^0\pi^+\pi^-$, with the lower half of the binning histogram in Fig. 4 having positive bin numbers, which is the convention used in Ref. [1].

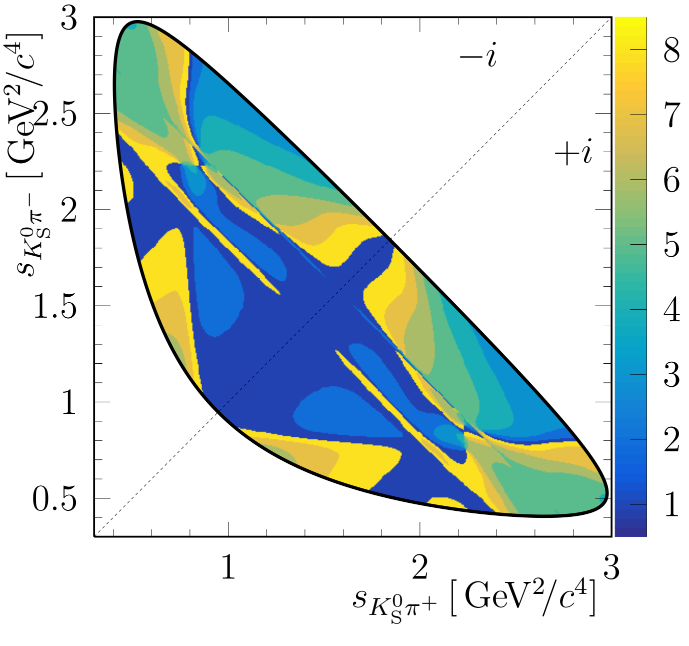
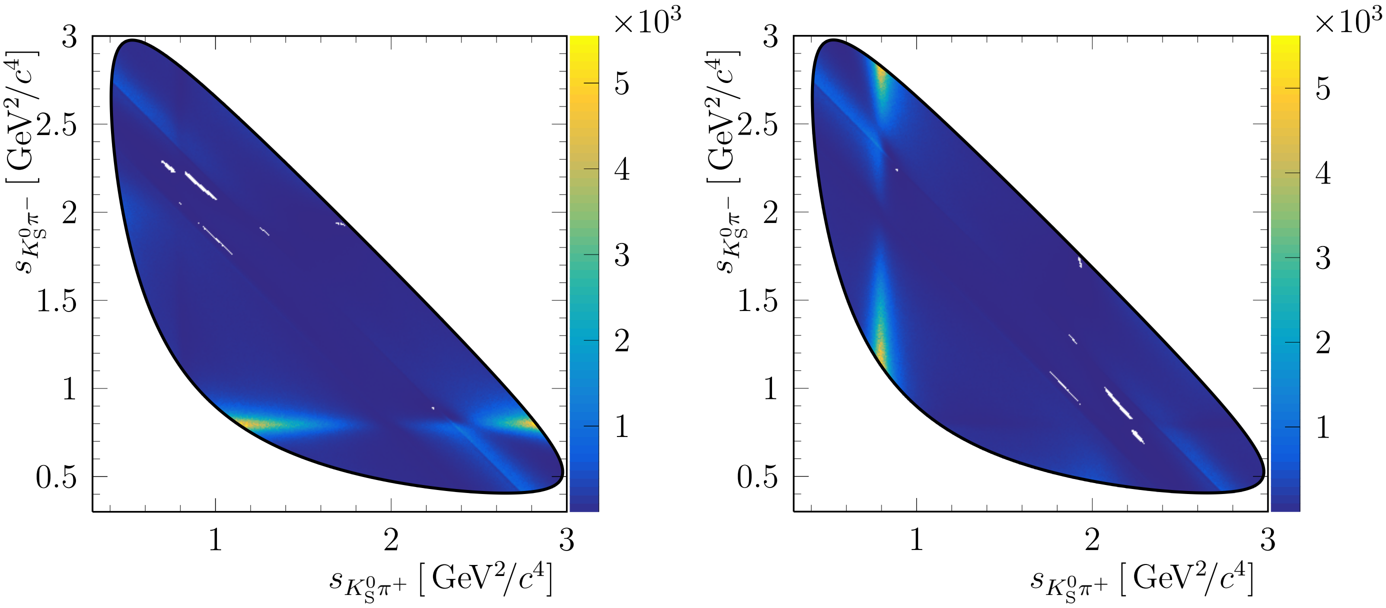


Figure 1: Binning Histogram

- The models for each amplitude are shown in Fig. 2



(a) $D^0 \to K_S^0\pi^+\pi^-$

(b) $\overline{D}^0 \to K_S^0\pi^+\pi^-$

Figure 2: Dalitz plots for $D^0(\overline{D}^0) \to K_S^0\pi^+\pi^-$

- Define the following quantities:

$$
\begin{aligned}
K_i &= \frac{\int_i \left|\mathcal{A}_{D^0\to K_S^0\pi^+\pi^-}(\mathbf{x})\right|^2 \mathrm{d}\mathbf{x}}{\int \left|\mathcal{A}_{D^0\to K_S^0\pi^+\pi^-}(\mathbf{x})\right|^2 \mathrm{d}\mathbf{x}} \\
\overline{K}_i &= \frac{\int_i \left|\mathcal{A}_{\overline{D}^0\to K_S^0\pi^+\pi^-}(\mathbf{x})\right|^2 \mathrm{d}\mathbf{x}}{\int \left|\mathcal{A}_{\overline{D}^0\to K_S^0\pi^+\pi^-}(\mathbf{x})\right|^2 \mathrm{d}\mathbf{x}} \\
\mathcal{Z}_i &= \frac{\int_i \mathcal{A}_{\overline{D}^0\to K_S^0\pi^+\pi^-}(\mathbf{x})\mathcal{A}^*_{D^0\to K_S^0\pi^+\pi^-}(\mathbf{x})\mathrm{d}\mathbf{x}}{\sqrt{\int_i \left|\mathcal{A}_{\overline{D}^0\to K_S^0\pi^+\pi^-}(\mathbf{x})\right|^2 \mathrm{d}\mathbf{x} \int_i \left|\mathcal{A}_{D^0\to K_S^0\pi^+\pi^-}(\mathbf{x})\right|^2 \mathrm{d}\mathbf{x}}},
\end{aligned}
\tag{30}
$$

where the integrals are over the ith bin. There are the following relations between the these quantites in the positive to negative bins.

$$
\begin{aligned}
K_i &= \overline{K}_{-i} \\
\mathcal{Z}_i &= \mathcal{Z}^*_{-i}
\end{aligned}
\tag{31}
$$

- Calculating these quanties from models gives:

i	$\mathcal{Z}_i$		K_i	$\overline{K}_i$
1	0.656	0.039	0.174	0.079
2	0.628	-0.397	0.087	0.017
3	0.095	-0.831	0.069	0.020
4	-0.485	-0.764	0.026	0.016
5	-0.944	0.027	0.086	0.051
6	-0.586	0.550	0.059	0.014
7	0.025	0.798	0.127	0.013
8	0.436	0.421	0.135	0.027
-1	0.656	-0.039	0.079	0.173
-2	0.627	0.396	0.018	0.087
-3	0.094	0.831	0.020	0.069
-4	-0.485	0.764	0.016	0.026
-5	-0.944	-0.027	0.051	0.086
-6	-0.586	-0.551	0.014	0.059
-7	0.025	-0.799	0.013	0.127
-8	0.435	-0.421	0.027	0.136

- So, with respect to what is written in Ref. [2], $\mathcal{Z}_i = c_i - is_i$
- In this binning convention, the number of candidates in the bin (j,i) where j is the bin for the signal-side, i.e. $D^0 \to K^+\pi^-\pi^-\pi^+$ is proportional to, and define the equivalent set of sumbols to Eq. 1 for this final state as $K', \overline{K}', \mathcal{Z}'$

$$N_{ji} = r_{K3\pi}^2 K_j' \overline{K}_i + \overline{K}_j' K_i - 2r_{K3\pi}\sqrt{K_i\overline{K}_i K_j' \overline{K}_j'}\mathrm{Re}\left(\mathcal{Z}_j'\mathcal{Z}_i^*\right) \tag{32}$$

- In terms of the coherence factor/strong phase $\mathcal{Z}_j' = R_i e^{j\delta_j}$, this is

$$Y_{ji} \propto r_{K3\pi}^2 K_j' K_{-i} + \overline{K}_j' K_i - 2r_{K3\pi}\sqrt{K_i K_{-i} K_j' \overline{K}_j'} R_j \left(c_i \cos\delta_j - s_i \sin\delta_j\right) \tag{33}$$

- Which, if we integrate over the bins of $D^0 \to K^+\pi^-\pi^-\pi^+$, gets us to the equation used in Ref. [3].

$$Y_i \propto K_i + r_{K3\pi}^2 K_{-i} - 2r_{K3\pi}\sqrt{K_i K_{-i}} R_{K3\pi}\left(c_i \cos\delta_{K3\pi} - s_i \sin\delta_{K3\pi}\right) \tag{34}$$

- Verify all of this on a large Toy MC sample ($\sim 200,000$ or so events), show consistency by plotting the pull for the 80 observables:

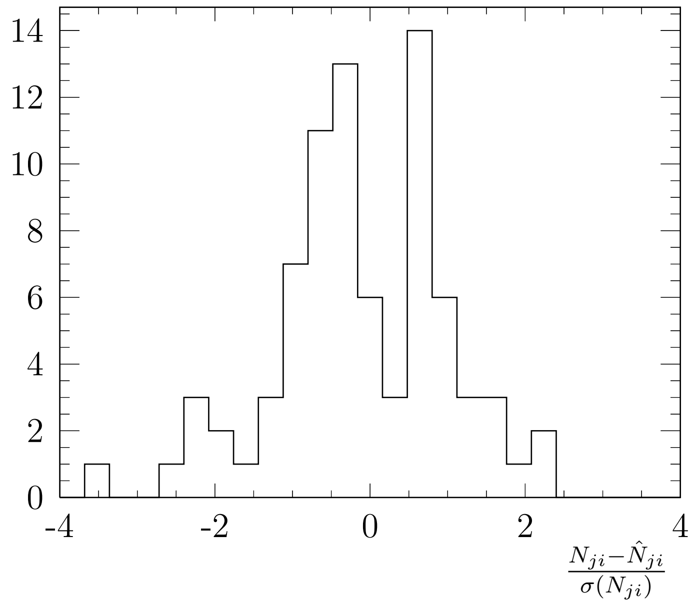


Figure 3: Pulls for observables

An example of the Dalitz plot for this decay (with the 'tag') being $D \to K^+\pi^-\pi^-\pi^+$ is shown below:

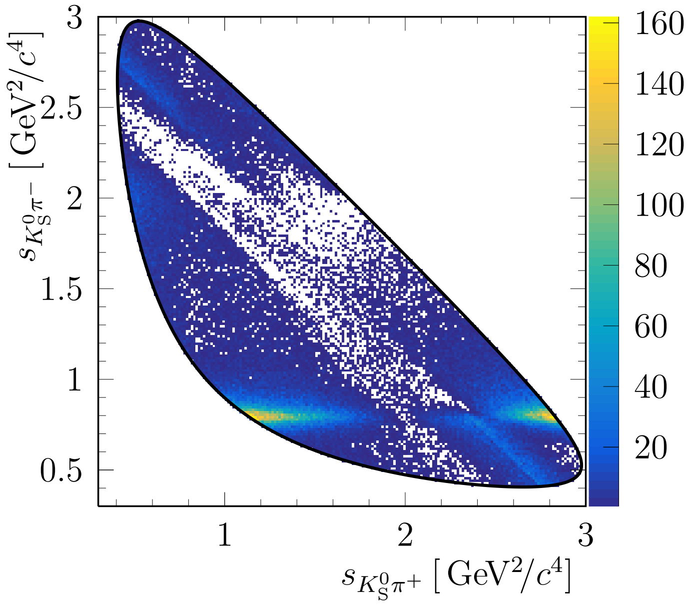


Figure 4: Quantum-correlated Dalitz plot for $D \to K^0_S\pi^-\pi^+$ vs $D \to K^+\pi^-\pi^-\pi^+$

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[2] J. Libby *et al.* (CLEO collaboration), *Model-independent determination of the strong-phase difference between the decays D^0 and $\bar{D}^0 \to K^0_{S,L}h^+h^-$ $(h = \pi, K)$ and its impact on the measurement of the CKM angle γ*, Phys. Rev. **D 82** (2010) 112006, arXiv:1010.2817 [hep-ex].

[3] T. Evans *et al.*, *Improved determination of the $D \to K^-\pi^+\pi^+\pi^-$ coherence factor and associated hadronic parameters from a combination of $e^+e^- \to \psi(3770) \to c\bar{c}$ and $pp \to c\bar{c}X$ data*, Phys. Lett. **B 757** (2016) 520, arXiv:1602.07430 [hep-ex], Erratum ibid. **765** (2017) 402.