Board Sheet Lsn24 Soln

Monday, March 23, 2020 4:22 PM

1. An airport limousine service can accommodate 4 passengers on any one trip. The company will accept a maximum of six reservations for a trip, and each passenger must have a reservation. From data, we know that 20% of all those making reservations do not appear for the trip. Answer the following questions, assuming independence wherever appropriate:

a) If six reservations are made, what is the probability that at least one individual with a reservation cannot be accommodated?

Y ≡ the number out of 6 customers w/ reservation who show for the trip

$Y \sim bin(6, 0.8)$

$P(Y=5) + P(Y=6) = (\text{go to R}) =$

```
dbinom(5,6,.8)
dbinom(6,6,.8)
```

$0.393 + 0.262 = \underline{0.6553}$

b) If six reservations are made, what is the expected number of available spaces when the limousine departs?

y	0	1	2	3	4	5	6
Spaces	4	3	2	1	0	0	0
p(y)	.000064	.001536	.01536	.08192	.24576	.393216	.262144

← use R

```
y = seq(0,6)
py = dbinom(y,6,0.8)
res = data.frame(y=y,prob=py)
print(res)
```

```
  y     prob
1 0 0.000064
2 1 0.001536
3 2 0.015360
4 3 0.081920
5 4 0.245760
6 5 0.393216
7 6 0.262144
```

use R to multiply (dot product) spaces * p(y)

$= \underline{0.117504}$

c) Suppose the probability distribution of the number of reservations made is given in the accompanying table and suppose that X is the number of passengers on a randomly selected trip. Obtain the probability mass function.

Number of reservations	3	4	5	6
probability	0.1	0.2	0.3	0.4

X ≡ the # of passengers on a trip.

```
spaces = c(4,3,2,1,0,0,0)
sum(py*spaces)
```

Find pmf for X - support for $X = \{0,1,2,3,4\}$

$P(X=0)$

$= P(X=0 \mid \#\,res=3)\cdot P(\#\,res=3) + P(X=0 \mid \#\text{ of } res=4)\cdot P(\#\,res=4) + P(X=0 \mid res=5)\cdot P(res=5) + P(X=0 \mid res=6)\cdot P(res=6)$

$= binom(0;3,0.8)(0.1) + binom(0;4,0.8)(0.2) + binom(0;5,0.8)(0.3) + binom(0;6,0.8)(0.4)$

$=$ use R → 0.0012 $\xrightarrow{X=0}$

$P(X=1) = (\text{use R}) = 0.0172$ $\xrightarrow{X=1}$

```
pr = c(0.1,0.2,0.3,0.4)
r = seq(3,6)
py = dbinom(X,r,0.8) #change this: X=0,1,2,3,4,5,6
sum(pr*py)
```

$P(X=2) = 0.0906$

$P(X=3) = 0.2273$

$P(X=4) = \underbrace{0.303104}_{\text{4 of 6 show}} + \underbrace{\underbrace{0.2556}_{\text{5 of 6 show}} + \underbrace{0.1049}_{\text{6 of 6 show}}}_{\text{Can still only accommodate only 4!}} = \underline{0.6636}$

#7 in book) The Michigan state lottery runs a game in which you pay $1 to buy a ticket containing a three–digit number of your choice. If your number is drawn at the end of the day, you win $500.

(a) Suppose you were to buy one ticket per week for a year. What are your chances of coming out a winner for the year? [Hint: It is easy to compute the probability of coming out a loser!]

X ≡ the number of winning tickets purchased over the course of a year.

$X \sim bin(52 * W, .001)$

per week

$P(\text{winning ticket}) = p = \frac{1}{1000} = .001$ n = # of tickets purchased over the course of a year $= 52 * W$

$P(\text{win at least once per year}) = 1 - P(\text{did not win}) = 1 - binom(0; 52*1, .001)$

$= 1 - \binom{52}{0} .001^{0}\, .999^{52}$

$$P(\text{win at least once per year}) = 1 - P(\text{did not win}) = 1 - \text{binom}(0; 52 * 1, .001)$$
$$= 1 - \binom{52}{0} .001^0 \cdot .999^{52}$$
$$= \underline{0.051}$$

(b) Can you improve your chances of coming out a winner this year by purchasing more than one ticket per week? Calculate the prob- ability of coming out a winner if you buy n tickets a week, for n = 1,2,3,...,9. – called these w instead of n.

yes

w = weekly # of tickets

Prob of win | weekly tickets = w

w	P(win)
1	.051
2	.099
3	.145
4	.187
5	.229
6	.268
7	.305
8	.340
9	.373

```
  n       pWin
1 1 0.05069583
2 2 0.09882160
3 3 0.14450759
4 4 0.18787748
5 5 0.22904871
6 6 0.26813273
7 7 0.30523535
8 8 0.34045702
9 9 0.37389310
```

```
n = seq(1,9)
f = function(x){1-dbinom(0,52*x,0.001)}
print(f(n))
result = data.frame(n=n,pWin=f(n))
print(result)
```

(c) Suppose that the state lottery sells 1,000,000 tickets this week. What is the range of likely variation in the amount of money the state will make this week? How likely is it that the state will lose money this week? Use the central limit theorem.

$X_1 + X_2 + \dots + X_{1000000} = T_0 \equiv$ # of winners out for 1,000,000

where $X_i \sim \text{Bernoulli}(p = 0.001)$

$E(X) = \mu_x = p = 0.001$

$V(X) = \sigma_x^2 = p(1-p) = .000999$

$T_0 \underset{CLT}{\sim} N(n\mu_x, n\sigma^2)$

$T_0 \sim N(1000, 999)$

$Y = 1{,}000{,}000 - 500\, T_0$

$E(Y) = 1{,}000{,}000 - 500\, E(T_0) = 500{,}000 = \mu_y$

$V(Y) = 500^2\, V(T_0) = 500^2 \cdot 999 = \sigma_y^2 \Rightarrow \sigma_y = 500\sqrt{999}$

so, 2 std deviations from the mean $\Rightarrow 500{,}000 \pm 2(500)(\sqrt{999})$

$= [468393, 531607]$ //

(d) What is wrong with using the central limit theorem to answer the question in part (a) or (b)?

The distn must be reasonably symmetric (n must be sufficiently large)

the sample size n, depends on p. If $p = 0.5$ (or close, then it is always reasonably symmetric $n \geq 30$)

otherwise, if p is very small (like this), then $np \geq 10$ and $n(1-p) \geq 10$ ensures that n is large enough to overcome the skewness in the underlying Bernoulli Distn. (P236-Devore)

3. Automobiles arrive at a vehicle equipment inspection station according to a Poisson Process (α=10 per hour). Suppose that with a probability 0.5 an arriving vehicle will have no equipment violations.

a) What is the probability that exactly 10 arrive during the hour and have no equipment violations?

Conditional Prob

$$P(A|B) = \frac{P(A \cap B)}{P(B)} \Rightarrow \boxed{P(A \cap B) = P(A|B) \cdot P(B)}$$

X = the # of cars that arrive in the next hour

$X \sim \text{Poisson}(10 \cdot 1)$

$$P(X = 10 \cap \text{no violations}) = P(\text{no violations} \mid X = 10) \cdot P(X = 10)$$

Assume each car is indep. $\rightarrow \underbrace{\left(\tfrac{1}{2}\right)^{10}}_{\substack{.0009766 \\ R}} \cdot \underbrace{\text{dpois}(10; 10)}_{\substack{0.12511 \\ R}} = \underline{0.0001222}$

b) For any fixed y ≥ 10, what is the probability that y arrive during the hour, of which 10 have no violations?

$y=10$

$$\underbrace{P(\text{10 have no violation} \mid X = y)} \cdot \underbrace{P(X = y)}$$

```
   y      py
1  10 0.00012
2  11 0.00061
3  12 0.00153
4  13 0.00255
```

```
y = seq(10,40)
```

$y=10$

$$\underbrace{P(\text{10 have no violation} \mid X = y)}_{\text{binom}\left(y,\frac{1}{2}\right)} \cdot \underbrace{P(X=y)}_{\text{Poisson}(y;10)}$$

$y = 10, 11, 12, \ldots \to$ use R

	y	py
1	10	0.00012
2	11	0.00061
3	12	0.00153
4	13	0.00255
5	14	0.00318
6	15	0.00318
7	16	0.00265
8	17	0.00189
9	18	0.00118
10	19	0.00066
11	20	0.00033
12	21	0.00015
13	22	0.00006
14	23	0.00002
15	24	0.00001

```
y = seq(10,40)
py = dbinom(10,y,.5)*dpois(y,10)
result=data.frame(y=y,py=round(py,5))
print(result)
```

c) What is the probability that 10 "no-violation" cars arrive during the next hour? (Hint: Sum the prob in part (b) from y = 10 to ∞.

$$P(\text{exactly 10 without a violation}) = \sum_{y=10}^{\infty} \underbrace{\binom{y}{10}\left(\frac{1}{2}\right)^{y}}_{\text{binomial}} \cdot \frac{e^{-10}\,10^{y}}{y!} = \left[\frac{\cancel{y!}}{10!\,(y-10)!}\right]\frac{5^{y}e^{-10}}{\cancel{y!}}$$

$$\binom{y}{10}\underbrace{\left(\frac{1}{2}\right)^{10}\left(\frac{1}{2}\right)^{y-10}}_{\left(\frac{1}{2}\right)^{y}} \qquad \left(\frac{1}{2}\right)^{y} 10^{y} = (5)^{y}$$

extra

$$= \sum_{y=10}^{\infty} \frac{5^{y}e^{-10}}{10!\,(y-10)!} = \frac{e^{-10}\cdot 5^{10}}{10!}\sum_{y=10}^{\infty}\frac{(5)^{y-10}}{(y-10)!} = \frac{e^{-10}\,5^{10}}{10!}\boxed{\sum_{u=0}^{\infty}\frac{5^{u}}{u!}} = e^{5}$$

$$= \frac{e^{-10}\,5^{10}}{10!}\,e^{5} = \frac{e^{-5}\,5^{10}}{10!} = 0.0181328 \Rightarrow \text{go to R.}$$