Skycrane System

States = $\eta, \dot{\eta}, z, \dot{z}, \phi, \dot{\phi}$

Equations of motion:

$$\ddot{\eta} = \frac{\left[T_1(\cos\beta \sin\phi + \sin\beta \cos\phi) + T_2(\cos\beta \sin\phi - \sin\beta\cos\phi) - F_{D,\eta}\right]}{m_b + m_F} + \tilde{\omega}_1$$

$$\ddot{z} = \frac{\left[T_1(\cos\beta\cos\phi - \sin\beta\sin\phi) + T_2(\cos\beta\cos\phi + \sin\beta\sin\phi) - F_{D,z}\right]}{m_b + m_F} - g + \tilde{\omega}_2$$

$$\ddot{\phi} = \frac{1}{I_\eta}\left[(T_1 - T_2)\cos\beta \cdot \frac{w_b}{2} + (T_2 - T_1)\sin\beta\, h_{cm}\right] + \tilde{\omega}_3$$

$$I_\eta = \frac{1}{12}\left[m_b(w_b^2 + h_b^2) + m_f(w_f^2 + h_f^2)\right]$$

$$F_{D,\eta} = \frac{1}{2} C_D \rho \left[A_{side} \cdot \cos(\phi - \alpha) + A_{bot}\sin(\phi - \alpha)\right]\dot{\eta} \cdot \sqrt{\dot{\eta}^2 + \dot{z}^2}$$

$$F_{D,z} = \frac{1}{2} C_D \rho \left[A_{side} \cdot \cos(\phi - \alpha) + A_{bot}\sin(\phi - \alpha)\right] \cdot \dot{z}\sqrt{\dot{\eta}^2 + \dot{z}^2}$$

$$\alpha = \tan^{-1}(\dot{z}/\dot{\eta})$$

state Equations

$$\begin{bmatrix} \dot{\eta} \\ \ddot{\eta} \\ \dot{z} \\ \ddot{z} \\ \dot{\phi} \\ \ddot{\phi} \end{bmatrix} = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \\ \dot{x}_5 \\ \dot{x}_6 \end{bmatrix} = \begin{bmatrix} x_2 \\ f(x_2, x_4, x_5, u_1, u_2, \tilde{\omega}_1) \\ x_4 \\ f(x_2, x_4, x_5, u_1, u_2, \tilde{\omega}_2) \\ x_6 \\ f(u_1, u_2, \tilde{\omega}_3) \end{bmatrix}$$

Simplify $F_{D,y}$:

$$F_{D,y} = \frac{1}{2} C_D \rho \left[A_{side} \cos(\phi - \alpha) + A_{bot} \sin(\phi - \alpha) \right] \dot{y} \sqrt{\dot{y}^2 + \dot{z}^2}$$

Trig identities:

$$\cos(\phi - \alpha) = \cos(\phi)\cos(\alpha) + \sin(\phi)\sin(\alpha)$$

$$\cos\left(\tan^{-1}(\dot{z}/\dot{y})\right) = \frac{1}{\sqrt{1 + \dot{z}^2/\dot{y}^2}} = \frac{1}{\frac{1}{\dot{y}} \cdot \sqrt{\dot{y}^2 + \dot{z}^2}} = \frac{\dot{y}}{\sqrt{\dot{y}^2 + \dot{z}^2}}$$

$$\sin\left(\tan^{-1}(\dot{z}/\dot{y})\right) = \frac{\dot{z}/\dot{y}}{\sqrt{1 + \dot{z}^2/\dot{y}^2}} = \frac{\dot{z}/\dot{y}}{\frac{1}{\dot{y}} \cdot \sqrt{\dot{y}^2 + \dot{z}^2}} = \frac{\dot{z}}{\sqrt{\dot{y}^2 + \dot{z}^2}}$$

$$\sin(\phi - \alpha) = \sin(\phi)\cos(\alpha) - \cos(\phi)\sin(\alpha)$$

Plugging back in:

$$F_{D,y} = \frac{1}{2} C_D \rho \left[A_{side}\left(\cos(\phi) \cdot \frac{\dot{y}}{\sqrt{\dot{y}^2 + \dot{z}^2}} + \sin(\phi) \frac{\dot{z}}{\sqrt{\dot{y}^2 + \dot{z}^2}} \right) + A_{bot}\left(\sin(\phi) \cdot \frac{\dot{y}}{\sqrt{\dot{y}^2 + \dot{z}^2}} - \cos(\phi) \cdot \frac{\dot{z}}{\sqrt{\dot{y}^2 + \dot{z}^2}} \right) \right] \cdot \dot{y} \cdot \sqrt{\dot{y}^2 + \dot{z}^2}$$

$$F_{D,y} = \frac{1}{2} C_D \rho \left[A_{side}\left(\cos(\phi)\dot{y} + \sin(\phi)\dot{z} \right) + A_{bot}\left(\sin(\phi)\dot{y} - \cos(\phi)\dot{z} \right) \right] \dot{y}$$

$$\boxed{F_{D,y} = \frac{1}{2} C_D \rho \left[\left(A_{side} \sin\phi - A_{bot} \cos\phi \right) \dot{z}\dot{y} + \left(A_{side} \cos\phi + A_{bot} \sin\phi \right) \dot{y}^2 \right]}$$

Simplify $F_{D,z}$

$$F_{D,z} = \frac{1}{2} C_D \rho \left[A_{side} \cos(\phi - \alpha) + A_{bot} \sin(\phi - \alpha) \right] \dot{z} \sqrt{\dot{y}^2 + \dot{z}^2}$$

Same trig identity substitution:

$$F_{D,z} = \frac{1}{2} C_D \rho \left[A_{side} \left(\cos\phi \frac{\dot{y}}{\sqrt{\dot{y}^2 + \dot{z}^2}} + \sin\phi \frac{\dot{z}}{\sqrt{\dot{y}^2 + \dot{z}^2}} \right) \right.$$

$$\left. + A_{bot} \left(\sin\phi \frac{\dot{y}}{\sqrt{\dot{y}^2 + \dot{z}^2}} - \cos\phi \frac{\dot{z}}{\sqrt{\dot{z}^2 + \dot{y}^2}} \right) \right] \cdot \dot{z} \sqrt{\dot{y}^2 + \dot{z}^2}$$

$$\boxed{F_{D,z} = \frac{1}{2} C_D \rho \left[\left(A_{side} \sin\phi - A_{bot} \cos\phi \right) \dot{z}^2 + \left(A_{side} \cos\phi + A_{bot} \sin\phi \right) \dot{y} \dot{z} \right]}$$

Partial derivatives of $\dot{y}$

$$\boxed{\frac{\partial \dot{y}}{\partial y} = 0, \quad \frac{\partial \dot{y}}{\partial \dot{y}} = 1, \quad \frac{\partial \dot{y}}{\partial z} = 0, \quad \frac{\partial \dot{y}}{\partial \dot{z}} = 0, \quad \frac{\partial \dot{y}}{\partial \phi} = 0, \quad \frac{\partial \dot{y}}{\partial \dot{\phi}} = 0}$$

Partial derivatives of $\ddot{y}$

$$\boxed{\frac{\partial \ddot{y}}{\partial y} = 0, \quad \frac{\partial \ddot{y}}{\partial z} = 0, \quad \frac{\partial \ddot{y}}{\partial \dot{\phi}} = 0}$$

$$\boxed{\frac{\partial \ddot{y}}{\partial \dot{y}} = \frac{-C_D \rho}{2(m_b + m_f)} \cdot \left[(A_{side} \sin\phi - A_{bot} \cos\phi)\, \dot{z} + 2 \cdot (A_{side} \cos\phi + A_{bot} \sin\phi)\, \dot{y} \right]}$$

$$\boxed{\frac{\partial \ddot{y}}{\partial \dot{z}} = \frac{-C_D \rho}{2(m_b + m_f)} \left[(A_{side} \sin\phi - A_{bot} \cos\phi)\, \dot{y} \right]}$$

$$\boxed{\begin{aligned} \frac{\partial \ddot{y}}{\partial \phi} = {} & \frac{T_1 (\cos\beta \cos\phi - \sin\beta \sin\phi) + T_2 (\cos\beta \cos\phi + \sin\beta \sin\phi)}{m_b + m_f} \\ & - \frac{C_D \rho}{2(m_b + m_f)} \left[(A_{side} \cos\phi + A_{bot} \sin\phi)\, \dot{y} \dot{z} + (-A_{side} \sin\phi + A_{bot} \cos\phi)\, \dot{y}^2 \right] \end{aligned}}$$

Partial derivatives of $\dot{z}$

$$\boxed{\frac{\partial \dot{z}}{\partial y} = 0, \quad \frac{\partial \dot{z}}{\partial \dot{y}} = 0, \quad \frac{\partial \dot{z}}{\partial z} = 0, \quad \frac{\partial \dot{z}}{\partial \dot{z}} = 1, \quad \frac{\partial \dot{z}}{\partial \phi} = 0, \quad \frac{\partial \dot{z}}{\partial \dot{\phi}} = 0}$$

Partial derivatives of $\ddot{z}$

$$\frac{\partial \ddot{z}}{\partial z} = 0 \quad , \quad \frac{\partial \ddot{z}}{\partial y} = 0 \quad , \quad \frac{\partial \ddot{z}}{\partial \phi} = 0$$

$$\frac{\partial \ddot{z}}{\partial \dot{y}} = \frac{-C_D \rho}{2(m_b + m_f)} \left[\left(A_{side} \cos\phi + A_{bot} \sin\phi \right) \dot{z} \right]$$

$$\frac{\partial \ddot{z}}{\partial \dot{z}} = \frac{-C_D \rho}{2(m_b + m_f)} \left[\left(A_{side} \sin\phi - A_{bot} \cos\phi \right) \cdot 2 \cdot \dot{z} + \left(A_{side} \cos\phi + A_{bot} \sin\phi \right) \dot{y} \right]$$

$$\frac{\partial \ddot{z}}{\partial \phi} = \frac{\left[T_1 \left(-\cos\beta \sin\phi - \sin\beta \cos\phi \right) + T_2 \left(-\cos\beta \sin\phi + \sin\beta \cos\phi \right) \right]}{m_b + m_f} - \frac{C_D \rho}{2(m_b + m_f)} \left[\left(A_{side} \cos\phi + A_{bot} \sin\phi \right) \dot{z}^2 + \left(-A_{side} \sin\phi + A_{bot} \cos\phi \right) \dot{y} \dot{z} \right]$$

Partial derivatives of $\dot{\phi}$

$$\frac{\partial \dot{\phi}}{\partial y} = 0 \, , \quad \frac{\partial \dot{\phi}}{\partial \dot{y}} = 0 \, , \quad \frac{\partial \dot{\phi}}{\partial z} = 0 \, , \quad \frac{\partial \dot{\phi}}{\partial \dot{z}} = 0 \, , \quad \frac{\partial \dot{\phi}}{\partial \phi} = 0 \, , \quad \frac{\partial \dot{\phi}}{\partial \dot{\phi}} = 1$$

Partial derivatives of $\ddot{\phi}$

$$\frac{\partial \ddot{\phi}}{\partial y} = 0 \, , \quad \frac{\partial \ddot{\phi}}{\partial \dot{y}} = 0 \, , \quad \frac{\partial \ddot{\phi}}{\partial z} = 0 \, , \quad \frac{\partial \ddot{\phi}}{\partial \dot{z}} = 0 \, , \quad \frac{\partial \ddot{\phi}}{\partial \phi} = 0 \, , \quad \frac{\partial \ddot{\phi}}{\partial \dot{\phi}} = 0$$

Partial derivatives with respect to u (or T)

$$\frac{\partial \dot{y}}{\partial u_1} = 0 \;,\; \frac{\partial \dot{y}}{\partial u_2} = 0 \;,\; \frac{\partial \dot{z}}{\partial u_1} = 0 \;,\; \frac{\partial \dot{z}}{\partial u_2} = 0 \;,\; \frac{\partial \dot{\phi}}{\partial u_1} = 0 \;,\; \frac{\partial \dot{\phi}}{\partial u_2} = 0$$

$$\frac{\partial \ddot{y}}{\partial u_1} = \frac{\cos\beta \sin\phi + \sin\beta\cos\phi}{m_b + m_f}$$

$$\frac{\partial \ddot{y}}{\partial u_2} = \frac{\cos\beta \sin\phi - \sin\beta\cos\phi}{m_b + m_f}$$

$$\frac{\partial \ddot{z}}{\partial u_1} = \frac{\cos\beta \cos\phi - \sin\beta\sin\phi}{m_b + m_f}$$

$$\frac{\partial \ddot{z}}{\partial u_2} = \frac{\cos\beta \cos\phi + \sin\beta\sin\phi}{m_b + m_f}$$

$$\frac{\partial \ddot{\phi}}{\partial u_1} = \frac{1}{I_\eta}\left[\cos\beta \frac{w_b}{2} - \sin\beta\, h_{cm}\right]$$

$$\frac{\partial \ddot{\phi}}{\partial u_2} = \frac{1}{I_\eta}\left[-\cos\beta \frac{w_b}{2} + \sin\beta\, h_{cm}\right]$$

Linearization Point

$\eta_{nom} = 0$ $z_{nom} = 20$ $\phi_{nom} = 0$ $u_{1\,nom} = \dfrac{g\cdot(m_b + m_f)}{2\cos\beta}$

$\dot{\eta}_{nom} = 0$ $\dot{z}_{nom} = 0$ $\dot{\phi}_{nom} = 0$ $u_{2\,nom} = \dfrac{g\cdot(m_b + m_f)}{2\cos\beta}$

Substituting into Jacobian equations:

$$\frac{\partial \dot{x}_i}{\partial x_j} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & g & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} = A \text{ for CT Linearized system}$$

$$\frac{\partial \dot{x}_i}{\partial u_j} = \begin{bmatrix} 0 & 0 \\ \frac{\sin\beta}{m_b + m_f} & \frac{-\sin\beta}{m_b + m_f} \\ 0 & 0 \\ \frac{\cos\beta}{m_b + m_f} & \frac{-\cos\beta}{m_b + m_f} \\ 0 & 0 \\ \frac{1}{I_\eta}\left[\cos\beta\, w_{cm} - \sin\beta\, h_{cm}\right] & \frac{1}{I_\eta}\left[\sin\beta\, h_{cm} - \cos\beta\, w_{cm}\right] \end{bmatrix} = B \text{ for CT Linearized system}$$

Measurements

states = $y, z, \dot{\phi}, \ddot{y}$

$$\frac{\partial y_i}{\partial x_j} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & g & 0 \end{bmatrix} = C \text{ for CT Linearized system}$$

$$\frac{\partial y_i}{\partial u_j} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ \frac{\sin\beta}{m_b + m_f} & \frac{-\sin\beta}{m_b + m_f} \end{bmatrix} = D \text{ for CT Linearized system}$$