TCSS 343 - Week 5

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August 22, 2018

Dynamic Programming

"Perhaps thinking should be measured not by what you do but by how you do it."

...

Richard Hamming

"For the last sixty five years(speaking in 2018), due to Moore's law, with a clockwork precision, computer capability has been doubling every year and a half. Without fast algorithms you cannot bring to bare Moore's Law. A dramatic increase in computer speed needs to be coupled with efficient algorithms."

...

Christos Papadimitriou

"Nobody expects plumbers to have a physics degree but they do have to know some things about water physics, and that can be learned in a way that doesnt necessarily involve getting a physics degree. And that is super cool, totally valid, and not a problem. But that doesnt mean that physics degrees are bullshit or not useful to plumbers."

...

Steve Klabnik

(analogy on studying CS theory as a programmer)

i	w_i	h_i	n_i
1	1	1	3
2	2	4	2
3	3	6	2
4	4	5	1
5	5	7	1
6	6	8	1

1. Suppose Santa has 6 kinds of toys, each kind of toy has its own weight w_i in tons, happiness rating h_i in ... joy, and quantity n_i. Santa would like to maximize the total hapiness of the children but the total weight of his bag cannot exceed 17 tons. Their weight, hapiness rating and quantity are defined above. Please help Santa by filling in the DP table below, where dp[i][j] indicates the maximum value you can get with weight less or equal to j using toys 1 to i. What is the final solution to this problem and briefly explain how you came to this solution. To help you get started, 23 was generated by solving the equation $i_1 + 2i_2 + 3i_3 \leqslant 15$ which gives you the most value. That value was found by $3 \cdot 1 + 2 \cdot 4 + 2 \cdot 6 = 23$. 12 was found by solving the equation $i_1 + 2i_2 + 3i_3 + 4i_4 + 5i_5 + 6i_6 \leqslant 6$ which gives you the most value. That value was found by $2 \cdot 6 = 12$

i\w	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17
{1}	0																	
{1,2}	0																	
{1,2,3}	0															23		
{1,2,3,4}	0																	
{1,2,3,4,5}	0																	
{1,2,3,4,5,6}	0						12											

2. In mathematics, a sequence of positive real numbers s_1, s_2,... is called *superincreasing* if each element in the sequence is greater than the sum of all previous elements in the sequence:

$$s_{n+1} > \sum_{i=1}^{n} s_i$$

For example: $\{2, 3, 7, 16, 65, 321, 4546\}$ is a superincreasing sequence, but $\{1, 1, 2, 5, 15, 52, 203, 877\}$ us not a superincreasing sequence.

Describe an algorithm that takes as input superincreasing sequence $s_1, \ldots, s_n$ and a positive integer k, please find a sequnce of $s_1, \ldots, s_n$ with the sum equal to k. It is possible and desirable to find an algorithm that can accomplish this task in $O(n)$ time using dynamic progrmaming. If you think you've come up with an algorithm that can accomplish this task attempt to prove that it is correct.

3. For the next few problems we will explore the "placing parenthesis" problem. First I want you to compute the maximum value of the expression given that you can placeins a parenthesis between any two pair of numbers:

$$1 + 2 - 3 \times 4 - 5$$

(example of one possible ordering of parenthesis:$((((1+2)-3)\times 4)-5) = -5$)

4. How many different ordering of arithmetic operations are there in this problem? Why do we care about solving this problem with dynamic programming(find runtime of brute force)?

5. How many possible orderings are there for the expression:

$$5 - 8 + 7 \times 4 - 8 + 9$$

(do not attempt to solve for the maximum possible ordering just yet!)

Placing Parentheses:
Input: A sequence of digits $d_1, \ldots, d_n$ and a sequence of operations $op_1, \ldots, op_{n-1} \in \{+, -, \times\}$.
Output: An order of applying these operations that maximizes the value of the expression.

INTUITION: Assume that the last operation in an optimal parenthesizing is when multiplication is done last, which with our previous example limits our possible perumtations to the including this parenthesization:

$$(5 - 8 + 7) \times (4 - 8 + 9)$$

6. Now above, we've gone from having 5! possible permutations of orderings to now having 2! + 2! possible orderings because we've reduced our global problem into two more managable subproblems by taking advantage of the shape of the problem. You may not understand why yet, but find each parenthesization that maximizes and minimizes the following subproblems:

$min\{5 - 8 + 7\} =$
$max\{5 - 8 + 7\} =$
$min\{4 - 8 + 9\} =$
$max\{4 - 8 + 9\} =$

7. Using what you've found find the maximum value of the given expression below and it's parenthesization:

$max\{(5 - 8 + 7) \times (4 - 8 + 9)\} =$

8. Why can't we be greedy and choose the maximum value at each stage?

Let E_{ij} be the subexpression

$$d_i op_i \ldots op_{j-1} d_j$$

Where are subproblems are defined as:

$$M(i,j) = \text{maximum value of } E_{ij}$$

$$m(i,j) = \text{minimum value of } E_{ij}$$

9. Given the information above and what you did on the prior page, can you write down a recurrence relation that solves the placing parenthesis problem? (HINT: remember that there are $2! + 2!$ possible orderings now that we usered our intuition! Both M and m will have at least 4 possible orderings each)

A. What do we mean when we say: " Our current relation expresses the solution for an expression (i,j) for a solution for smaller sub subexpressions"?

B. When computing $M(i,j)$ the values of ______________ should already be computed.

a) $M(i,k)$ and $M(k+1,j)$

b) $M(i-1,k)$ and $M(i,k-1)$

c) $M(\frac{i}{2},k)$ and $M(i,\frac{k}{2})$

C. We need to solve all subproblems in order of ______________.

a) decreasing $(j-i)$

b) increasing $(j-i)$

D. For this algorithm we have roughly ____________ number of subproblems.

a) Linear

b) Quadratic

c) Exponential

d) Tetratic

Below I will give you the algorithm to compute the parenthesis problem. I didn't ask you to construct it because it is quite complicated, I am much more interested in you seeing what it look like having explored the problem suffiecently up to this point. Before you move on: **Make sure you understand what each and every line is doing.** Please ask questions if you have them. This is as important as answering the questions.

MinAndMax(i,j):

$min \leftarrow +\infty$

$max \leftarrow -\infty$

for k from i to $j-1$:

    $a \leftarrow M(i,k)\ op_k\ M(k+1,j)$

    $b \leftarrow m(i,k)\ op_k\ M(k+1,j)$

    $c \leftarrow M(i,k)\ op_k\ M(k+1,j)$

    $d \leftarrow m(i,k)\ op_k\ M(k+1,j)$

    $min \leftarrow min\{min, a, b, c, d\}$

    $max \leftarrow min\{max, a, b, c, d\}$

return (min, max)

Parentheses($d_1 op_1$**,** $d_2 op_2$**,** $\ldots$**,** $d_n op_n$**):**

for i from 1 to n:

    $m(i,i) \leftarrow d_i$, $M(i,i) \leftarrow d_i$ for s from 1 to $n-1$:

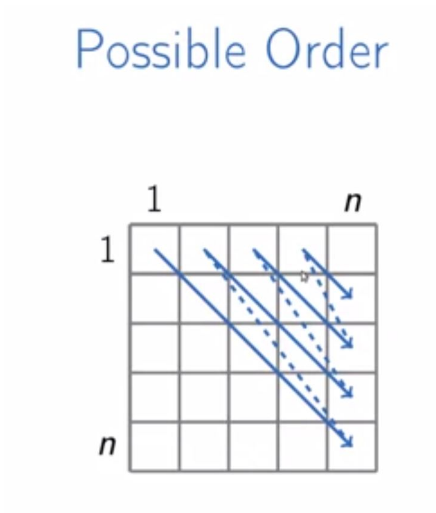
    for s from 1 to $n-s$:

        $j \leftarrow i+s$

        $m(i,j), M(i,j) \leftarrow MinAndMax(i,j)$

return $M(1,n)$

E. What is the worstcase runtime of the **Parentheses** algorithm above? (HINT: What is $j-i$ at most?)



Above is the possible ordering of how the algorithm stores the results of m and M where i is row-wise and j is column-wise.

Down each diagonal will be filled with each value of the arithmetic expression.

F. I will now ask you, using the algorithm stated previously, to constrcut the table produced by the algorithm. I was kind and gave you two of the values in each of the tables to help you along. (HINT to get started: can you break down the subexpression $5 - 8$ any further?)

Reminder of the expression we're maximizing: $5 - 8 + 7 \times 4 - 8 + 9$

m

i\j	1					n
1	5		-10			
		8				
			7			
				4		
					8	
n						9

M

i\j	1					n
1	5					
		8				
			7			
				4		5
					8	
n						9

10. How would you use the table to reconstruct the solution? I know that at this point you probably know the optimal subexpression is but we must think algorithmically so that we can tell the dumbest thing on the planet, namely a computer, how to do it. Sorry I ran out of space on the last page. Please try to trace the steps in the matrix and think about how you would reconstruct the solution by stepping forward and backward that got you to $M(6,6)$. Can you write that expression as an ordering of m and M ordered pairs?