7

Generating Functions

THE MOST POWERFUL WAY to deal with sequences of numbers, as far as anybody knows, is to manipulate infinite series that "generate" those sequences. We've learned a lot of sequences and we've seen a few generating functions; now we're ready to explore generating functions in depth, and to see how remarkably useful they are.

7.1 DOMINO THEORY AND CHANGE

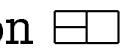
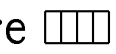
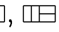
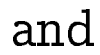
Generating functions are important enough, and for many of us new enough, to justify a relaxed approach as we begin to look at them more closely. So let's start this chapter with some fun and games as we try to develop our intuitions about generating functions. We will study two applications of the ideas, one involving dominoes and the other involving coins.

How many ways T_n are there to completely cover a $2 \times n$ rectangle with 2×1 dominoes? We assume that the dominoes are identical (either because they're face down, or because someone has rendered them indistinguishable, say by painting them all red); thus only their orientations — vertical or horizontal — matter, and we can imagine that we're working with domino-shaped tiles. For example, there are three tilings of a 2×3 rectangle, namely , , and ; so $T_3 = 3$.

"Let me count the ways."
— E. B. Browning

To find a closed form for general T_n we do our usual first thing, look at small cases. When $n = 1$ there's obviously just one tiling, ; and when $n = 2$ there are two, and .

How about when $n = 0$; how many tilings of a 2×0 rectangle are there? It's not immediately clear what this question means, but we've seen similar situations before: There is one permutation of zero objects (namely the empty permutation), so $0! = 1$. There is one way to choose zero things from n things (namely to choose nothing), so $\binom{n}{0} = 1$. There is one way to partition the empty set into zero nonempty subsets, but there are no such ways to partition a nonempty set; so $\left\{ {n \atop 0} \right\} = [n = 0]$. By such reasoning we can conclude that there's just one way to tile a 2×0 rectangle with dominoes, namely to use no dominoes; therefore $T_0 = 1$. (This spoils the simple pattern $T_n = n$ that holds when $n = 1$, 2, and 3; but that pattern was probably doomed anyway, since T_0 wants to be 1 according to the logic of the situation.) A proper understanding of the null case turns out to be useful whenever we want to solve an enumeration problem.

Let's look at one more small case, $n = 4$. There are two possibilities for tiling the left edge of the rectangle — we put either a vertical domino or two horizontal dominoes there. If we choose a vertical one, the partial solution is  and the remaining 2×3 rectangle can be covered in T_3 ways. If we choose two horizontals, the partial solution  can be completed in T_2 ways. Thus $T_4 = T_3 + T_2 = 5$. (The five tilings are , , , , and .)

We now know the first five values of T_n:

n	0	1	2	3	4
T_n	1	1	2	3	5

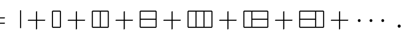
These look suspiciously like the Fibonacci numbers, and it's not hard to see why: The reasoning we used to establish $T_4 = T_3 + T_2$ easily generalizes to $T_n = T_{n-1} + T_{n-2}$, for $n \geqslant 2$. Thus we have the same recurrence here as for the Fibonacci numbers, except that the initial values $T_0 = 1$ and $T_1 = 1$ are a little different. But these initial values are the consecutive Fibonacci numbers F_1 and F_2, so the T's are just Fibonacci numbers shifted up one place:

$$T_n = F_{n+1}, \qquad \text{for } n \geqslant 0.$$

(We consider this to be a closed form for T_n, because the Fibonacci numbers are important enough to be considered "known." Also, F_n itself has a closed form (6.123) in terms of algebraic operations.) Notice that this equation confirms the wisdom of setting $T_0 = 1$.

But what does all this have to do with generating functions? Well, we're about to get to that — there's another way to figure out what T_n is. This new way is based on a bold idea. Let's consider the "sum" of all possible $2 \times n$ tilings, for all $n \geqslant 0$, and call it T:

To boldly go where no tiling has gone before.

$T =$  (7.1)

(The first term '|' on the right stands for the null tiling of a 2×0 rectangle.) This sum T represents lots of information. It's useful because it lets us prove things about T as a whole rather than forcing us to prove them (by induction) about its individual terms.

The terms of this sum stand for tilings, which are combinatorial objects. We won't be fussy about what's considered legal when infinitely many tilings are added together; everything can be made rigorous, but our goal right now is to expand our consciousness beyond conventional algebraic formulas.

We've added the patterns together, and we can also multiply them — by juxtaposition. For example, we can multiply the tilings ▯ and ⊟ to get the new tiling ▯⊟. But notice that multiplication is not commutative; that is, the order of multiplication counts: ▯⊟ is different from ⊟▯.

Using this notion of multiplication it's not hard to see that the null tiling plays a special role — it is the multiplicative identity. For instance, $\text{|} \times \text{⊟} = \text{⊟} \times \text{|} = \text{⊟}$.

Now we can use domino arithmetic to manipulate the infinite sum T:

$$\begin{aligned} \mathsf{T} &= \text{|} + \text{▯} + \text{▯▯} + \text{⊟} + \text{▯▯▯} + \text{▯⊟} + \text{⊟▯} + \cdots \\ &= \text{|} + \text{▯}(\text{|} + \text{▯} + \text{▯▯} + \text{⊟} + \cdots) + \text{⊟}(\text{|} + \text{▯} + \text{▯▯} + \text{⊟} + \cdots) \\ &= \text{|} + \text{▯}\mathsf{T} + \text{⊟}\mathsf{T}. \end{aligned} \tag{7.2}$$

I have a gut feeling that these sums must converge, as long as the dominoes are small enough.

Every valid tiling occurs exactly once in each right side, so what we've done is reasonable even though we're ignoring the cautions in Chapter 2 about "absolute convergence." The bottom line of this equation tells us that everything in T is either the null tiling, or is a vertical tile followed by something else in T, or is two horizontal tiles followed by something else in T.

So now let's try to solve the equation for T. Replacing the T on the left by |T and subtracting the last two terms on the right from both sides of the equation, we get

$$(\text{|} - \text{▯} - \text{⊟})\mathsf{T} = \text{|}. \tag{7.3}$$

For a consistency check, here's an expanded version:

$$\begin{array}{l} \ \text{|} + \text{▯} + \text{▯▯} + \text{⊟} + \text{▯▯▯} + \text{▯⊟} + \text{⊟▯} + \cdots \\ - \text{▯} - \text{▯▯} - \text{▯▯▯} - \text{▯⊟} - \text{▯▯▯▯} - \text{▯▯⊟} - \text{▯⊟▯} - \cdots \\ - \text{⊟} - \text{⊟▯} - \text{⊟▯▯} - \text{⊟⊟} - \text{⊟▯▯▯} - \text{⊟▯⊟} - \text{⊟⊟▯} - \cdots \\ \hline \ \text{|} \end{array}$$

Every term in the top row, except the first, is cancelled by a term in either the second or third row, so our equation is correct.

So far it's been fairly easy to make combinatorial sense of the equations we've been working with. Now, however, to get a compact expression for T we cross a combinatorial divide. With a leap of algebraic faith we divide both sides of equation (7.3) by $\text{|} - \text{▯} - \text{⊟}$ to get

$$\mathsf{T} = \frac{\text{|}}{\text{|} - \text{▯} - \text{⊟}}. \tag{7.4}$$

(Multiplication isn't commutative, so we're on the verge of cheating, by not distinguishing between left and right division. In our application it doesn't matter, because $\vert$ commutes with everything. But let's not be picky, unless our wild ideas lead to paradoxes.)

The next step is to expand this fraction as a power series, using the rule

$$\frac{1}{1-z} \;=\; 1+z+z^2+z^3+\cdots\,.$$

The null tiling $\vert$, which is the multiplicative identity for our combinatorial arithmetic, plays the part of 1, the usual multiplicative identity; and $\text{▯}+\boxminus$ plays z. So we get the expansion

$$\begin{aligned}\frac{\vert}{\vert-\text{▯}-\boxminus} &= \vert+(\text{▯}+\boxminus)+(\text{▯}+\boxminus)^2+(\text{▯}+\boxminus)^3+\cdots\\ &= \vert+(\text{▯}+\boxminus)+(\text{▯▯}+\text{▯}\boxminus+\boxminus\text{▯}+\boxminus\boxminus)\\ &\qquad+(\text{▯▯▯}+\text{▯▯}\boxminus+\text{▯}\boxminus\text{▯}+\text{▯}\boxminus\boxminus+\boxminus\text{▯▯}+\boxminus\text{▯}\boxminus+\boxminus\boxminus\text{▯}+\boxminus\boxminus\boxminus)+\cdots\,.\end{aligned}$$

This is T, but the tilings are arranged in a different order than we had before. Every tiling appears exactly once in this sum; for example, $\text{▯}\boxminus\boxminus\text{▯▯}\boxminus\text{▯}$ appears in the expansion of $(\text{▯}+\boxminus)^7$.

We can get useful information from this infinite sum by compressing it down, ignoring details that are not of interest. For example, we can imagine that the patterns become unglued and that the individual dominoes commute with each other; then a term like $\text{▯}\boxminus\boxminus\text{▯▯}\boxminus\text{▯}$ becomes $\text{▯}^4\text{▭}^6$, because it contains four verticals and six horizontals. Collecting like terms gives us the series

$$T \;=\; \vert+\text{▯}+\text{▯}^2+\text{▭}^2+\text{▯}^3+2\text{▯}\text{▭}^2+\text{▯}^4+3\text{▯}^2\text{▭}^2+\text{▭}^4+\cdots\,.$$

The $2\text{▯}\text{▭}^2$ here represents the two terms of the old expansion, $\text{▯}\boxminus$ and $\boxminus\text{▯}$, that have one vertical and two horizontal dominoes; similarly $3\text{▯}^2\text{▭}^2$ represents the three terms $\text{▯▯}\boxminus$, $\text{▯}\boxminus\text{▯}$, and $\boxminus\text{▯▯}$. We're essentially treating ▯ and ▭ as ordinary (commutative) variables.

We can find a closed form for the coefficients in the commutative version of T by using the binomial theorem:

$$\begin{aligned}\frac{\vert}{\vert-(\text{▯}+\text{▭}^2)} &= \vert+(\text{▯}+\text{▭}^2)+(\text{▯}+\text{▭}^2)^2+(\text{▯}+\text{▭}^2)^3+\cdots\\ &= \sum_{k\geqslant 0}(\text{▯}+\text{▭}^2)^k\\ &= \sum_{j,k\geqslant 0}\binom{k}{j}\text{▯}^j\,\text{▭}^{2k-2j}\\ &= \sum_{j,m\geqslant 0}\binom{j+m}{j}\text{▯}^j\,\text{▭}^{2m}\,. \qquad (7.5)\end{aligned}$$

(The last step replaces $k-j$ by m; this is legal because we have $\binom{k}{j}=0$ when $0 \leqslant k < j$.) We conclude that $\binom{j+m}{j}$ is the number of ways to tile a $2\times(j+2m)$ rectangle with j vertical dominoes and $2m$ horizontal dominoes. For example, we recently looked at the 2×10 tiling ▯⊟⊟▯▯⊟▯, which involves four verticals and six horizontals; there are $\binom{4+3}{4} = 35$ such tilings in all, so one of the terms in the commutative version of T is 35 ▯^{4} ▭^{6}.

We can suppress even more detail by ignoring the orientation of the dominoes. Suppose we don't care about the horizontal/vertical breakdown; we only want to know about the total number of $2 \times n$ tilings. (This, in fact, is the number T_n we started out trying to discover.) We can collect the necessary information by simply substituting a single quantity, z, for ▯ and ▭. And we might as well also replace | by 1, getting

Now I'm disoriented.

$$T \;=\; \frac{1}{1-z-z^2}\,. \tag{7.6}$$

This is the generating function (6.117) for Fibonacci numbers, except for a missing factor of z in the numerator; so we conclude that the coefficient of z^n in T is F_{n+1}.

The compact representations |/(|−▯−⊟), |/(|−▯−▭^{2}), and $1/(1-z-z^2)$ that we have deduced for T are called *generating functions*, because they generate the coefficients of interest.

Incidentally, our derivation implies that the number of $2 \times n$ domino tilings with exactly m pairs of horizontal dominoes is $\binom{n-m}{m}$. (This follows because there are $j = n-2m$ vertical dominoes, hence there are

$$\binom{j+m}{j} \;=\; \binom{j+m}{m} \;=\; \binom{n-m}{m}$$

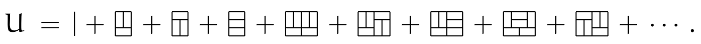
ways to do the tiling according to our formula.) We observed in Chapter 6 that $\binom{n-m}{m}$ is the number of Morse code sequences of length n that contain m dashes; in fact, it's easy to see that $2\times n$ domino tilings correspond directly to Morse code sequences. (The tiling ▯⊟⊟▯▯⊟▯ corresponds to '•−−••−•'.) Thus domino tilings are closely related to the continuant polynomials we studied in Chapter 6. It's a small world.

We have solved the T_n problem in two ways. The first way, guessing the answer and proving it by induction, was easier; the second way, using infinite sums of domino patterns and distilling out the coefficients of interest, was fancier. But did we use the second method only because it was amusing to play with dominoes as if they were algebraic variables? No; the real reason for introducing the second way was that the infinite-sum approach is a lot more powerful. The second method applies to many more problems, because it doesn't require us to make magic guesses.

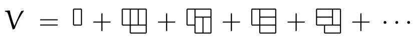
Let's generalize up a notch, to a problem where guesswork will be beyond us. How many ways U_n are there to tile a $3 \times n$ rectangle with dominoes?

The first few cases of this problem tell us a little: The null tiling gives $U_0 = 1$. There is no valid tiling when $n = 1$, since a 2×1 domino doesn't fill a 3×1 rectangle, and since there isn't room for two. The next case, $n = 2$, can easily be done by hand; there are three tilings, , , and , so $U_2 = 3$. (Come to think of it we already knew this, because the previous problem told us that $T_3 = 3$; the number of ways to tile a 3×2 rectangle is the same as the number to tile a 2×3.) When $n = 3$, as when $n = 1$, there are no tilings. We can convince ourselves of this either by making a quick exhaustive search or by looking at the problem from a higher level: The area of a 3×3 rectangle is odd, so we can't possibly tile it with dominoes whose area is even. (The same argument obviously applies to any odd n.) Finally, when $n = 4$ there seem to be about a dozen tilings; it's difficult to be sure about the exact number without spending a lot of time to guarantee that the list is complete.

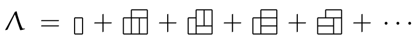
So let's try the infinite-sum approach that worked last time:

 (7.7)

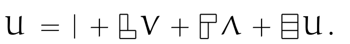
Every non-null tiling begins with either  or  or ; but unfortunately the first two of these three possibilities don't simply factor out and leave us with U again. The sum of all terms in U that begin with  can, however, be written as V, where



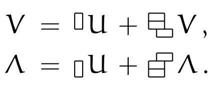
is the sum of all domino tilings of a mutilated $3 \times n$ rectangle that has its lower left corner missing. Similarly, the terms of U that begin with  can be written Λ, where



consists of all rectangular tilings lacking their upper left corner. The series Λ is a mirror image of V. These factorizations allow us to write



And we can factor V and Λ as well, because such tilings can begin in only two ways:



Now we have three equations in three unknowns (U, V, and Λ). We can solve them by first solving for V and Λ in terms of U, then plugging the results into the equation for U:

$$
\begin{aligned}
V &= (\vert - \text{╤})^{-1}\,\text{▯}\,U\,, \qquad \Lambda = (\vert - \text{╧})^{-1}\,\text{▯}\,U\,;\\
U &= \vert + \text{╘}(\vert - \text{╤})^{-1}\,\text{▯}\,U \;+\; \text{╒}(\vert - \text{╧})^{-1}\,\text{▯}\,U \;+\; \text{☰}\,U\,.
\end{aligned}
$$

And the final equation can be solved for U, giving the compact formula

$$
U = \frac{\vert}{\vert - \text{╘}(\vert - \text{╤})^{-1}\text{▯} \;-\; \text{╒}(\vert - \text{╧})^{-1}\text{▯} \;-\; \text{☰}}\,. \tag{7.8}
$$

This expression defines the infinite sum U, just as (7.4) defines T.

The next step is to go commutative. Everything simplifies beautifully when we detach all the dominoes and use only powers of ▯ and ▭:

I learned in another class about "regular expressions." If I'm not mistaken, we can write
$$U = (\text{╘}\,\text{╤}^{*}\,\text{▯} + \text{╒}\,\text{╧}^{*}\,\text{▯} + \text{☰})^{*}$$
in the language of regular expressions; so there must be some connection between regular expressions and generating functions.

$$
\begin{aligned}
U &= \frac{1}{1 - \text{▯}^2\text{▭}(1-\text{▭}^3)^{-1} - \text{▯}^2\text{▭}(1-\text{▭}^3)^{-1} - \text{▭}^3}\\
&= \frac{1-\text{▭}^3}{(1-\text{▭}^3)^2 - 2\text{▯}^2\text{▭}}\\
&= \frac{(1-\text{▭}^3)^{-1}}{1 - 2\text{▯}^2\text{▭}(1-\text{▭}^3)^{-2}}\\
&= \frac{1}{1-\text{▭}^3} + \frac{2\text{▯}^2\text{▭}}{(1-\text{▭}^3)^3} + \frac{4\text{▯}^4\text{▭}^2}{(1-\text{▭}^3)^5} + \frac{8\text{▯}^6\text{▭}^3}{(1-\text{▭}^3)^7} + \cdots\\
&= \sum_{k\geqslant 0} \frac{2^k\text{▯}^{2k}\text{▭}^k}{(1-\text{▭}^3)^{2k+1}}\\
&= \sum_{k,m\geqslant 0} \binom{m+2k}{m} 2^k\,\text{▯}^{2k}\,\text{▭}^{k+3m}\,.
\end{aligned}
$$

(This derivation deserves careful scrutiny. The last step uses the formula $(1-w)^{-2k-1} = \sum_m \binom{m+2k}{m} w^m$, identity (5.56).) Let's take a good look at the bottom line to see what it tells us. First, it says that every $3 \times n$ tiling uses an even number of vertical dominoes. Moreover, if there are $2k$ verticals, there must be at least k horizontals, and the total number of horizontals must be $k+3m$ for some $m \geqslant 0$. Finally, the number of possible tilings with $2k$ verticals and $k+3m$ horizontals is exactly $\binom{m+2k}{m}2^k$.

We now are able to analyze the 3×4 tilings that left us doubtful when we began looking at the $3 \times n$ problem. When $n = 4$ the total area is 12, so we need six dominoes altogether. There are $2k$ verticals and $k+3m$ horizontals, for some k and m; hence $2k + k + 3m = 6$. In other words, $k + m = 2$. If we use no verticals, then $k = 0$ and $m = 2$; the number of possibilities is $\binom{2+0}{2}2^0 = 1$. (This accounts for the tiling .) If we use two verticals, then $k = 1$ and $m = 1$; there are $\binom{1+2}{1}2^1 = 6$ such tilings. And if we use four verticals, then $k = 2$ and $m = 0$; there are $\binom{0+4}{0}2^2 = 4$ such tilings, making a total of $U_4 = 11$. In general if n is even, this reasoning shows that $k + m = \frac{1}{2}n$, hence $\binom{m+2k}{m} = \binom{n/2+k}{n/2-k}$ and the total number of $3 \times n$ tilings is

$$U_n \;=\; \sum_k \binom{n/2+k}{n/2-k} 2^k \;=\; \sum_m \binom{n-m}{m} 2^{n/2-m} \,. \tag{7.9}$$

As before, we can also substitute z for both ▯ and ▭, getting a generating function that doesn't discriminate between dominoes of particular persuasions. The result is

$$U \;=\; \frac{1}{1 - z^3(1-z^3)^{-1} - z^3(1-z^3)^{-1} - z^3} \;=\; \frac{1-z^3}{1-4z^3+z^6} \,. \tag{7.10}$$

If we expand this quotient into a power series, we get

$$U \;=\; 1 + U_2\, z^3 + U_4\, z^6 + U_6\, z^9 + U_8\, z^{12} + \cdots \,,$$

a generating function for the numbers U_n. (There's a curious mismatch between subscripts and exponents in this formula, but it is easily explained. The coefficient of z^9, for example, is U_6, which counts the tilings of a 3×6 rectangle. This is what we want, because every such tiling contains nine dominoes.)

We could proceed to analyze (7.10) and get a closed form for the coefficients, but it's better to save that for later in the chapter after we've gotten more experience. So let's divest ourselves of dominoes for the moment and proceed to the next advertised problem, "change."

How many ways are there to pay 50 cents? We assume that the payment must be made with pennies ①, nickels ⑤, dimes ⑩, quarters ㉕, and half-dollars ㊿. George Pólya [298] popularized this problem by showing that it can be solved with generating functions in an instructive way.

Ah yes, I remember when we had half-dollars.

Let's set up infinite sums that represent all possible ways to give change, just as we tackled the domino problems by working with infinite sums that represent all possible domino patterns. It's simplest to start by working with fewer varieties of coins, so let's suppose first that we have nothing but pennies. The sum of all ways to leave some number of pennies (but just pennies) in change can be written

P = ⁄ + ① + ①① + ①①① + ①①①① + ⋯
  = ⁄ + ① + ①^2 + ①^3 + ①^4 + ⋯ .

The first term stands for the way to leave no pennies, the second term stands for one penny, then two pennies, three pennies, and so on. Now if we're allowed to use both pennies and nickels, the sum of all possible ways is

$$\begin{aligned} N &= P + ⑤P + ⑤⑤P + ⑤⑤⑤P + ⑤⑤⑤⑤P + \cdots \\ &= (\cancel{5} + ⑤ + ⑤^2 + ⑤^3 + ⑤^4 + \cdots)\,P\,, \end{aligned}$$

since each payment has a certain number of nickels chosen from the first factor and a certain number of pennies chosen from P. (Notice that N is *not* the sum $\cancel{1} + ① + ⑤ + (① + ⑤)^2 + (① + ⑤)^3 + \cdots$, because such a sum includes many types of payment more than once. For example, the term $(① + ⑤)^2 = ①① + ①⑤ + ⑤① + ⑤⑤$ treats ①⑤ and ⑤① as if they were different, but we want to list each set of coins only once without respect to order.)

Similarly, if dimes are permitted as well, we get the infinite sum

$$D = (\cancel{10} + ⑩ + ⑩^2 + ⑩^3 + ⑩^4 + \cdots)\,N\,,$$

which includes terms like $⑩^3⑤^3①^5 = ⑩⑩⑩⑤⑤⑤①①①①①$ when it is expanded in full. Each of these terms is a different way to make change. Adding quarters and then half-dollars to the realm of possibilities gives

Coins of the realm.

$$\begin{aligned} Q &= (\cancel{25} + ㉕ + ㉕^2 + ㉕^3 + ㉕^4 + \cdots)\,D\,; \\ C &= (\cancel{50} + ㊿ + ㊿^2 + ㊿^3 + ㊿^4 + \cdots)\,Q\,. \end{aligned}$$

Our problem is to find the number of terms in C worth exactly 50¢.

A simple trick solves this problem nicely: We can replace ① by z, ⑤ by z^5, ⑩ by z^{10}, ㉕ by z^{25}, and ㊿ by z^{50}. Then each term is replaced by z^n, where n is the monetary value of the original term. For example, the term ㊿⑩⑤⑤① becomes $z^{50+10+5+5+1} = z^{71}$. The four ways of paying 13 cents, namely $⑩①^3$, $⑤①^8$, $⑤^2①^3$, and $①^{13}$, each reduce to z^{13}; hence the coefficient of z^{13} will be 4 after the z-substitutions are made.

Let P_n, N_n, D_n, Q_n, and C_n be the numbers of ways to pay n cents when we're allowed to use coins that are worth at most 1, 5, 10, 25, and 50 cents, respectively. Our analysis tells us that these are the coefficients of z^n in the respective power series

$$\begin{aligned} P &= 1 + z + z^2 + z^3 + z^4 + \cdots\,, \\ N &= (1 + z^5 + z^{10} + z^{15} + z^{20} + \cdots)P\,, \\ D &= (1 + z^{10} + z^{20} + z^{30} + z^{40} + \cdots)N\,, \\ Q &= (1 + z^{25} + z^{50} + z^{75} + z^{100} + \cdots)D\,, \\ C &= (1 + z^{50} + z^{100} + z^{150} + z^{200} + \cdots)Q\,. \end{aligned}$$

How many pennies are there, really? If n is greater than, say, 10^{10}, I bet that $P_n = 0$ in the "real world."

Obviously $P_n = 1$ for all $n \geqslant 0$. And a little thought proves that we have $N_n = \lfloor n/5 \rfloor + 1$: To make n cents out of pennies and nickels, we must choose either 0 or 1 or ... or $\lfloor n/5 \rfloor$ nickels, after which there's only one way to supply the requisite number of pennies. Thus P_n and N_n are simple; but the values of D_n, Q_n, and C_n are increasingly more complicated.

One way to deal with these formulas is to realize that $1 + z^m + z^{2m} + \cdots$ is just $1/(1-z^m)$. Thus we can write

$$\begin{aligned} P &= 1/(1-z)\,, \\ N &= P/(1-z^5)\,, \\ D &= N/(1-z^{10})\,, \\ Q &= D/(1-z^{25})\,, \\ C &= Q/(1-z^{50})\,. \end{aligned}$$

Multiplying by the denominators, we have

$$\begin{aligned} (1-z)\,P &= 1\,, \\ (1-z^5)\,N &= P\,, \\ (1-z^{10})\,D &= N\,, \\ (1-z^{25})\,Q &= D\,, \\ (1-z^{50})\,C &= Q\,. \end{aligned}$$

Now we can equate coefficients of z^n in these equations, getting recurrence relations from which the desired coefficients can quickly be computed:

$$\begin{aligned} P_n &= P_{n-1} + [n=0]\,, \\ N_n &= N_{n-5} + P_n\,, \\ D_n &= D_{n-10} + N_n\,, \\ Q_n &= Q_{n-25} + D_n\,, \\ C_n &= C_{n-50} + Q_n\,. \end{aligned}$$

For example, the coefficient of z^n in $D = (1-z^{25})Q$ is equal to $Q_n - Q_{n-25}$; so we must have $Q_n - Q_{n-25} = D_n$, as claimed.

We could unfold these recurrences and find, for example, that $Q_n = D_n + D_{n-25} + D_{n-50} + D_{n-75} + \cdots$, stopping when the subscripts get negative. But the non-iterated form is convenient because each coefficient is computed with just one addition, as in Pascal's triangle.

Let's use the recurrences to find C_{50}. First, $C_{50} = C_0 + Q_{50}$; so we want to know Q_{50}. Then $Q_{50} = Q_{25} + D_{50}$, and $Q_{25} = Q_0 + D_{25}$; so we also want to know D_{50} and D_{25}. These D_n depend in turn on D_{40}, D_{30}, D_{20}, D_{15}, D_{10}, D_5, and on N_{50}, N_{45}, ..., N_5. A simple calculation therefore suffices to determine all the necessary coefficients:

n	0	5	10	15	20	25	30	35	40	45	50
P_n	1	1	1	1	1	1	1	1	1	1	1
N_n	1	2	3	4	5	6	7	8	9	10	11
D_n	1	2	4	6	9	12	16		25		36
Q_n	1					13					49
C_n	1										50

The final value in the table gives us our answer, C_{50}: There are exactly 50 ways to leave a 50-cent tip.

(Not counting the option of charging the tip to a credit card.)

How about a closed form for C_n? Multiplying the equations together gives us the compact expression

$$C = \frac{1}{1-z}\,\frac{1}{1-z^5}\,\frac{1}{1-z^{10}}\,\frac{1}{1-z^{25}}\,\frac{1}{1-z^{50}}\,, \tag{7.11}$$

but it's not obvious how to get from here to the coefficient of z^n. Fortunately there is a way; we'll return to this problem later in the chapter.

More elegant formulas arise if we consider the problem of giving change when we live in a land that mints coins of every positive integer denomination (①, ②, ③, ...) instead of just the five we allowed before. The corresponding generating function is an infinite product of fractions,

$$\frac{1}{(1-z)(1-z^2)(1-z^3)\ldots}\,,$$

and the coefficient of z^n when these factors are fully multiplied out is called $p(n)$, the number of *partitions* of n. A partition of n is a representation of n as a sum of positive integers, disregarding order. For example, there are seven different partitions of 5, namely

$$5 = 4+1 = 3+2 = 3+1+1 = 2+2+1 = 2+1+1+1 = 1+1+1+1+1\,;$$

hence $p(5) = 7$. (Also $p(2) = 2$, $p(3) = 3$, $p(4) = 5$, and $p(6) = 11$; it begins to look as if $p(n)$ is always a prime number. But $p(7) = 15$, spoiling the pattern.) There is no closed form for $p(n)$, but the theory of partitions is a fascinating branch of mathematics in which many remarkable discoveries have been made. For example, Ramanujan proved that $p(5n+4) \equiv 0 \pmod 5$, $p(7n+5) \equiv 0 \pmod 7$, and $p(11n+6) \equiv 0 \pmod{11}$, by making ingenious transformations of generating functions (see Andrews [11, Chapter 10]).