TCSS 343 - Week 1 - Thursday

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Asymptotics

"You're always you, and that don't change, and you're always changing,
and there's nothing you can do about it."
$$\cdots$$
Neil Gaiman

"If one proves the equality of two numbers a and b by showing first that
$a \leqslant b$ and then that $b \geqslant a$, it is unfair; one should instead show that they
are really equal by disclosing the inner ground for their equality".
$$\cdots$$
Emmy Noether

0. Prove the following theorems. Use a **direct proof** to find constants that satisfy the definition of big Θ AND use the **limit test**. Make sure your proof is tidy. That means it is complete, concise, clear and precise.

Theorem 1. $21n - 71 \in \Theta(n)$

Theorem 2. $\log n^n + 21\sqrt{n} \in \Theta(n \log n)$

Theorem 3. $\log (2^n + 2) \in \Theta(n)$

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Algorithm Convert_to_Binary (n) ;
Input: n (a positive integer).
Output: b (an array of bits corresponding to the binary representation of n).

begin
    t := n ; { we use a new variable t to preserve n }
    k := 0 ;
    while t > 0 do
        k := k + 1 ;
        b[k] := t mod 2 ;
        t := t div 2 ;
end
```

Figure 2.6 Algorithm *Convert_to_Binary*.

1. For the following problem we will explore an induction and the notion of an **invariant** of an algorithm, which put simply, is a statement about a variable correct independent of the number of times a loop is executed. For the purposes of this algorithm the expression: $n = t \times 2^k + m$ is a loop invariant. Loop invariants can be hard to find but are often the heart of any algorithm.

   Induction Hypothesis: if m is the integer represented by the binary array $b[1 \ldots k]$, then $n = t \times 2^k + m$. To prove the correctness of this algorithm we must prove three conditions:

   (a) The hypothesis is true at the beginning of the loop.

   (b) The truth of the hypothesis at steps k implies the step $k + 1$.

   (c) When the loop terminates, the hypothesis implies the correctness of the algorithm.

   Attempt to complete the proof given the information I've given you to start.

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2. Let $f(n) \in g(n) + h(n)$. Given $g(n) \in \Theta(F(n))$ and $h(n) \in O(F(n))$, prove or disprove $f(n) \in \Theta(F(n))$. (Use the definition of O and Θ in your notes).

3. Using what you know prove or disprove that if $f(n) \in O(g(n))$ and $g(n) \in O(f(n))$ then $f(n) = g(n) \forall n$.

4. Using what you know prove or disprove that $f(n) \in O(f(n)^2)$.

5. Using what you know prove or disprove that $f(n) + O(f(n)) \in \Theta(f(n))$.