TCSS 343 - Week 2 - Tuesday

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Asymptotics with Divide and Conquer

"I am incapable of conceiving infinity, and yet I do not accept finity. I want this adventure that is the context of my life to go on without end".

. . .

Simone de Beauvoir

"Man has not given himself the taste for the infinite and the love of what is immortal. These sublime instincts do not arise from a caprice of the will; they have their unchanging foundation in his nature; they exist despite his efforts. He can hinder and deform them, but not destroy them."

. . .

Alexis de Tocqueville

0. Prove the theorem below using the techniques of binding the term and splitting the sum to find a tight bound for the sum. Make sure your proof is complete, concise, clear and precise.

Theorem 0:

$$\sum_{i=1}^{n} i^4 \in \Theta(n^5)$$

1. Prove the theorem below using the techniques of binding the term and splitting the sum to find a tight bound for the sum. Make sure your proof is complete, concise, clear and precise.
   Theorem 1:
   $$\sum_{i=1}^{n}(\log_2 i)^2 \in \Theta(n(\log_2 n)^2)$$

2. Prove the theorem below using the techniques of binding the term and splitting the sum to find a tight bound for the sum. Make sure your proof is complete, concise, clear and precise.
   Theorem 2:

$$\sum_{i=1}^{\sqrt{n}} i^2 \in \Theta(n^{\frac{3}{2}})$$

Consider the task of detecting whether a given array has an element that is repeated in more than half of the positions of the array. For example. the value 2 is the majority element in the array $\{3, 2, 5, 2, 3, 2, 7, 2, 2\}$, while the array $\{5, 8, 8, 3, 10, 8, 5, 8\}$ has no majority element. In the present task you will develop a divide and conquer algorithm for solving this problem, and you will analyze its running time.

3. Write a formal statement of this problem. That is, state the input and output criteria as precisely as possible.

4. Suppose that an array $a[1 \ldots n]$ has an element v_L that occurs in strictly more than half of the first floor($\frac{n}{2}$) positions of a, and an element v_H that occurs in strictly more than half of the remaining ceiling($\frac{n}{2}$) positions. Argue carefully why if there is an element v of a that occurs in over half of all n positions of a, then v must be either v_L or v_H. Note that it would not be enough for v to occur more times in a than either v_L or v_H. We explicitly require that v occur in strictly more than half of all positions of the entire array.

5. Using the information you came to in the prior pages, design a divide and conquer algorithm that returns the majority element of an array or the value -1 if no such element exists. Give detailed tidy psuedocode.

6. Prove the theorem below using the techniques of binding the term and splitting the sum to find a tight bound for the sum. Make sure your proof is complete, concise, clear and precise.
   Theorem 3:

$$\sum_{i=1}^{n} \log_2 (\frac{n}{i}) \in \Theta(n)$$