TCSS 343 - Week 0 - Tuesday

Jake McKenzie

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Recursion and some Mathematical Review

"The problem with believing is you have to believe that believing will work, you have to believe in belief".

...

Tommy Orange

"So plant your own gardens and decorate your own soul, instead of waiting for someone to bring you flowers."

...

Jorge Luis Borges

Listen to the **MUSTN'TS**, child,
Listen to the **DON'TS**
Listen to the **SHOULDN'TS**
The **IMPOSSIBLES**, the **WONT'S**
Listen to the **NEVER HAVES**
Then listen close to me-
Anything can happen, child,
ANYTHING can be

...

Shel Silverstein

0. For the following problem I want you to trace the code. What is the final value that is printed? To accomplish this, ***correctly***, you will need to keep track of each the stack trace for each recursive call. That means keeping track of the values of x, y, u, v for 6 recursive calls.

```
def debug_week0_thursday(x,y,u,v):
    if x > y:
        x = x - y;
        u = u + v;
        debug_week0_thursday(x,y,u,v);
    elif y > x:
        y = y - x;
        v = v + u;
        debug_week0_thursday(x,y,u,v);
    else:
        print((x + y) / 2 * (u + v) / 2);
debug_week0_thursday(36,10,36,10)
```

Calculates GCD(X,Y)

Calculates LCM(X,Y)

Calls	X	Y	U	V
0	36	10	36	10
1	26	10	46	10
2	16	10	56	10
3	6	10	66	10
4	6	4	66	76
5	2	4	142	76
6	2	2	142	218

$$\frac{2+2}{2} \times \frac{142+218}{2} = 360 = 10 \times 36$$

$$GCD(X,Y) \times LCM(X,Y) = X \times Y$$

1. For this problem we will prove the proposition $P(n) : \mathscr{F}(n) < 2^n$ where $\mathscr{F}(n) = \mathscr{F}(n-1) + \mathscr{F}(n-2)$ and $\mathscr{F}(1) = 1$ and $\mathscr{F}(2) = 1$ $\forall n > 2$ using strong induction.

Step 1: Show the "Basis", which means simply, show for some finite values that the proposition holds for those values. For instance, let $a = 2$, show that say $P(a), P(a+1), \ldots, P(5)$, that the proposition holds. These needn't be exhaustive but it is a good step to do to get a feel for the problem itself. What would be a good ~~a~~ b to pick for this problem?

Step 2: Show to inductive hypothesis, which means simply, assume $P(i)$ for $a \leqslant i \leqslant k$ where i is some index variable that runs between a (some arbitrary starting point) and k (the substitution for n, we must replace n with k to for tidiness). We do this to establish our finite collection of claims in a tidy way. As good computer scientists we want to keep our proofs and our code we write as tidy as possible. This may seem like a redundant step at times but it is arguably the most important. What is the inductive hypothesis for this problem?

Step 3: Show the inductive step, which means simply, prove $P(k+1)$. This involves introducing our inductive hypothesis into the $P(k+1)$. We assume that the inductive hypothesis is true. You must always introduce the inductive hypothesis into the inductive step. Finish the proof.

Base Case:

$1 + 1 < 2^2$

$2 < 4$ ✓

Inductive Hypothesis:

$P(K): \underbrace{\mathscr{F}(K-1) + \mathscr{F}(K-2)}_{\mathscr{F}(K)} < 2^K$

Inductive Step:

$P(K+1): \mathscr{F}(K) + \mathscr{F}(K-1) < 2^{K+1}$

since $\mathscr{F}(K) > \mathscr{F}(K-1) \wedge \mathscr{F}(K) + \mathscr{F}(K) = 2\mathscr{F}(K)$

$\therefore$

$2\mathscr{F}(K) > \mathscr{F}(K) + \mathscr{F}(K-1)$

$\mathscr{F}(K+1) < 2\mathscr{F}(K) < 2^{K+1}$

$\therefore$

$\mathscr{F}(K+1) \leq 2^{K+1}$

$\therefore$

$\mathscr{F}(K) \leq 2^K$

∎

2. Calculate the following anti-derivatives from 1 to n with respect to x. (Because I am a nice person. Remember that $\log_2 x = \frac{\ln x}{\ln 2}$). I have run into all of these integrals when computing runtimes of algorithms, I know you didn't expect this to be so much math but please, please don't hate me. I added these because I care.

   (a) $6 + 7x + 2x^2$
   (b) $\frac{1}{x^2}$
   (c) $\frac{1}{x}$
   (d) 2^x (HINT: Use the fact that $\frac{d}{dx}A^x = \ln A * A^x$ where A is some finite real number)
   (e) $\log_2 x$ (HINT: set $dv = 1 * dx$)

$$\int_1^n 6 + 7x + 2x^2\,dx = 6x + \frac{7}{2}x^2 + \frac{2}{3}x^3 \Big|_1^n = \boxed{6n + \frac{7}{2}n^2 + \frac{2}{3}n^3 - \frac{61}{6}}$$

$$\int_1^n x^{-2}\,dx = -\frac{1}{n} - \left(-\frac{1}{1}\right) = \boxed{1 - \frac{1}{n}}$$

$$\int_1^n x^{-1}\,dx = \ln n - \ln 1 \boxed{= \ln n}$$

$$\frac{d}{dx}2^x = \ln 2 * 2^x \therefore \frac{1}{\ln 2}\frac{d}{dx}2^x = 2^x \therefore \frac{1}{\ln 2}\int_1^n \frac{d}{dx}2^x\,dx = 2^x \therefore$$

$$\int_1^n 2^x\,dx = \frac{1}{\ln 2}2^x \Big|_1^n = \boxed{\frac{1}{\ln 2}2^n - \frac{1}{\ln 2}2^1}$$

$$\int_1^n 1 \cdot \log_2 x\,dx = \frac{1}{\ln 2}\int_1^n 1 \cdot \ln x\,dx = \frac{1}{\ln 2}\left(x\ln x - \int_1^n \frac{x}{x}\,dx\right)$$

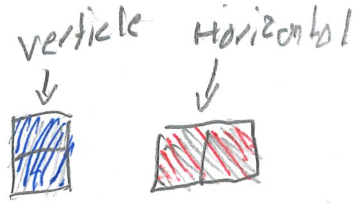
$$= \frac{1}{\ln 2}(x\ln x - x)\Big|_1^n$$

$v = x \quad u = \ln x$

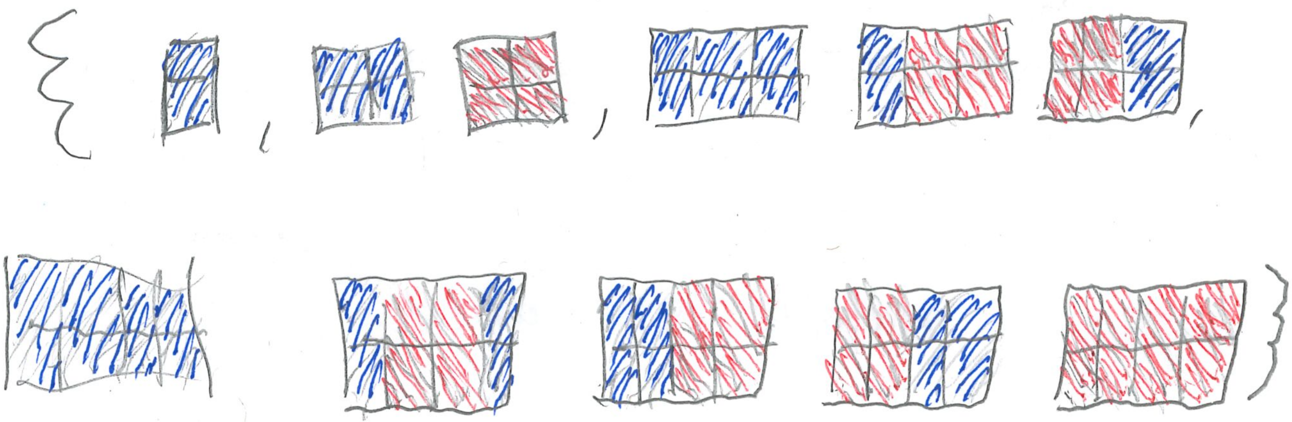
$dv = 1 \cdot dx \quad du = \frac{1}{x}dx$

$$\boxed{n\log_2 n - n + 1}$$

4



3. Emily loves figuring out all the ways to arrange dominos. Help her find all the ways to arrange dominos in that are 2×1 in a 2×1,2×2,2×3 and 2×4 grid! Avoid repeats and double counting by labeling each square of the domino the number of that domino. A domino can either be arranged vertically or horizontally in this paradigm.



4. We've had a lot of fun arranging dominos but now Emily wants a recursive formula for the ways to arrange 2×1 dominos. The key to finding recursive definitions is to find the answer to larger problems by finding the answer to smaller problems. Assume for the sake of argument that 2×0 grid has 1 way to arrange it. This is one of those annoying philosophical things. If I want to tile something does it exist? Is there zero ways or one way? We will avoid the philosophical questions and just follow the mathematics. In this case one is the convenient number so we will choose it arbitrarily by saying there is of course one way to time empty division which is ... to not do it. Is that not convincing? Well you will see, hopefully, that if the recurrence below holds then D_0 must be 1.

$D_n =$# of tilings of a $2 \times n =$ $D_{n-1} + D_{n-2}$

#of tiling of a $2 \times n-1$ + #of tilings of $2 \times n-2$

5. Give a direct proof for the following theorem:
Theorem 0. If x and y are positive reals and $x < y$ then, $x^2 < y^2$.

let $x, y \in \mathbb{R} \wedge x < y \Rightarrow x^2 < y^2$

$x < y$

maltiply by x on both sides

$\therefore$

$x^2 < xy$

maltiply by y on both sides

$y^2 > xy$

$\therefore$

$x^2 < xy < y^2$

$\therefore$

$x^2 < y^2$ ■