TRADING POST PROBLEM

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Jake McKenzie, Joseph Musngi, and Matthew Skipworth

Design And Analysis Of Algorithms

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PROBLEM STATEMENT

There are n trading posts numbered 1 to n, as you travel downstream. At any trading post i, you can rent a canoe to be returned at any of the downstream posts$i > i$. You are given a cost array $R(i, j)$ giving the cost of these rentals for $1 \le i \le j \le n$. We will have to assume that $R(i, i) = 0$ and $R(i, j) = \infty$ if $i > j$. For example, with $n = 4$, the cost array may look as follows: The rows are the sources $(i - s)$ and the columns are the destinations $(j's)$.

	1	2	3	4
1	0	2	3	7
2		0	2	4
3			0	2
4				0

The problem is to find a solution that computes the cheapest sequence of rentals taking you from post 1 all the way to post n. In this example, the cheapest sequence is to rent form post 1 to post 3 (cost 3), then from post 3 to post 4 (cost 2), with a total cost of 5 (less than the direct rental form post 1 to post 7, which would cost 7).

For this problem we used a graph approach, solving each of the desired problems presented in the problem statement including a naive brute force approach (which wasn't so naive to implement), a recursive breadth first search approach and a Dijkstra's algorithm approach for the dynamic programming portion. Most problems that require dynamic programming, especially problems on sets, can be converted to Single Source Shortest Path (SSSA) with a directed acyclic graph (MIT OCW, Eric Demaine).

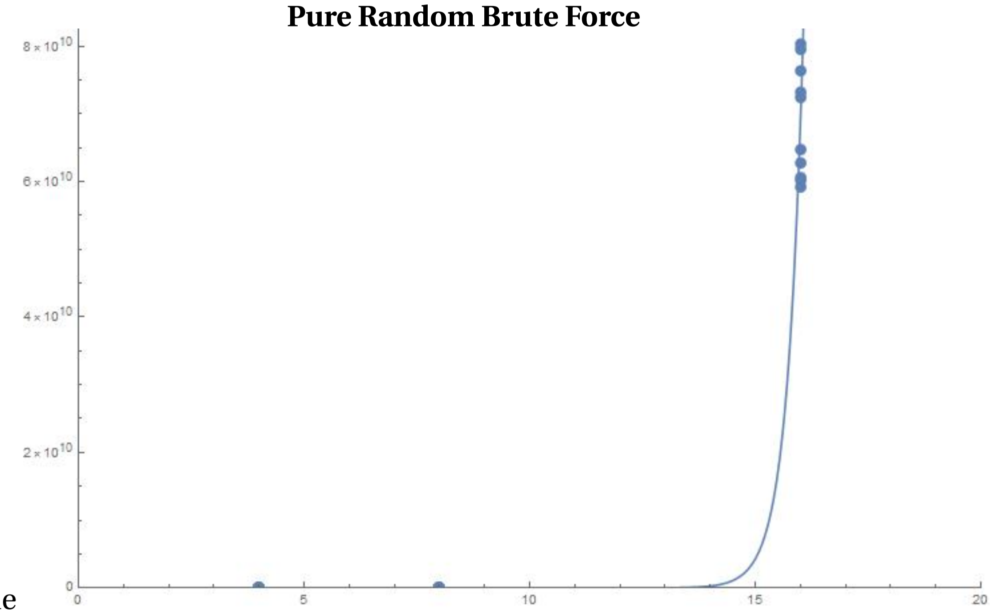
A really important tradeoff in constructing efficient computer programs is between data structures and algorithms. The more fancy stuff you keep in your data structures, usually the less work your algorithms have to do. The less fancy stuff you keep in your data structure the more work you have to do in your algorithms. By assuming nothing fancy in our data structure for our naive brute force approach, we find that first algorithm that comes to mind not only works slower than it need to but also that the work in constructing such an algorithm is far greater. With each successive algorithm the work our algorithm does becomes a great deal less than the prior, with far better overall runtime. This tradeoff between preprocessing and making our data structure more fundamentally functional makes the algorithms faster.

That tradeoff is always there, independent of graphs. One should never think of an algorithm without it's corresponding data structure. Shai Simonson's lecture at Ars Digita was used heavily in constructing the brute force approach and the preprocessing step.

BRUTE FORCE

The brute force approach that we constructucted worked by checking all adjacent paths for each vertice. Most students didn't solve this problem using graphs, for their brute force approach the runtime will be $O(2^n)$ where n is the number of inputs. That brute force approach breaks the problem into partitions. Graphs work differently by breaking the problem into permutations. In our graph the number of paths is $V-1$ where V is the number of vertices in the graph. The total number of paths is then $(V-1)!$ which means the worst case runtime is $O(V-1)!$ if check all adjacent paths. To understand just how worse this is, assume the following. $(V-1)!$ $V^V = 2^{V \log V}$. This may not seem that worse but $n = V-1$. As the number of inputs to the problem grows, Our brute force becomes quite inefficient.
The brute force approach, while running with an exponetial time complexity, can be verified to be correct in polynomial time. Our naive brute force approach can be said to be NP Complete for this reason.
Below we can see the runtimes for the two testcases, with purely random numbers and pure random pluse a purely random constant. The fitted constant out front of the purely random test case was $0.00329433V!$ and for increasing order $0.00240501v!$. The experimental results match the theoretical results I expected to get.

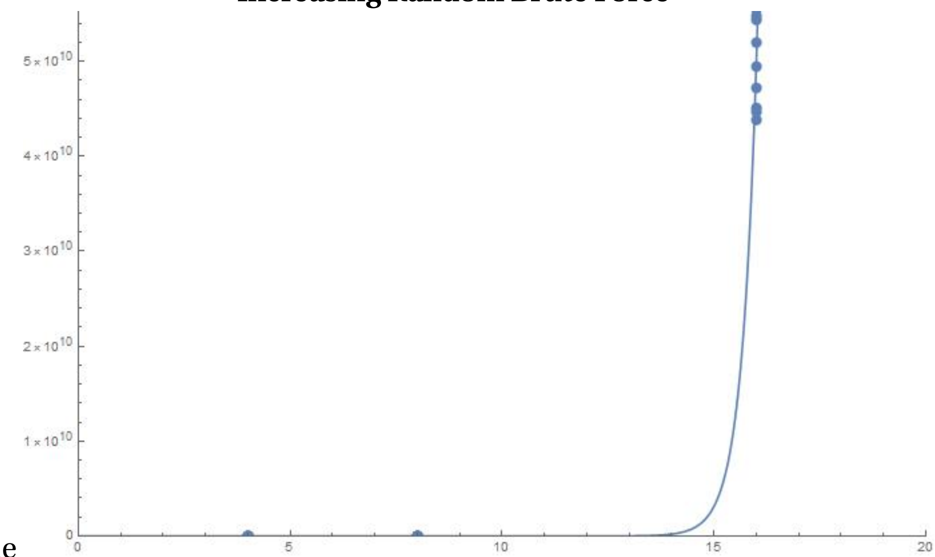
Pure Random Brute Force



runtime

number of inputs

Increasing Random Brute Force



runttime

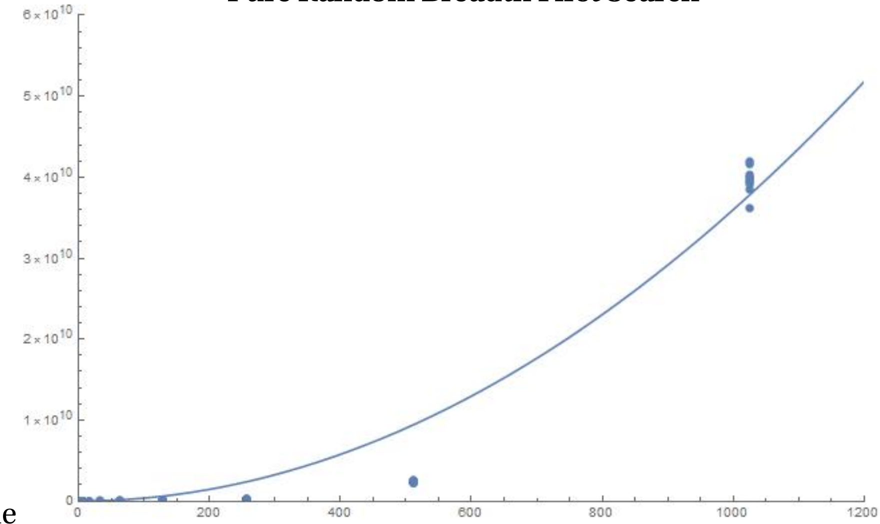
number of inputs

DIVIDE AND CONQUER

To solve this problem from a divide and conquer approach, we bent the rules a little bit and utilized a *decrease and conquer* technique we learned from Data Structures: the breadth

first search technique. Rather than traversing every single permutation of a path like in the brute force solution, we recursively scan the graph breadth first to find the optimal solution and return. Now rather than an worst case runtime of $O(V-1)!$ we now obtain a worst case runtime of $O(|V|+|E|)$. This is misleading. Depending on the number of nodes you have this number can change. We can either have $O(1)$ number of edges for a sparse graph or $O(V^2)$ for a dense graph. Our real number of edges is somewhere greater than $O(V)$ and less than $O(V^2)$. The actual runtimes when we ran the simulations were clearly parabolic so the graph was assumed to be dense. This gives us a worst case runtime of $O(V^2)$.
Below we can see the experimental results for breadth first search. The runtime was fitted to a model of $36002.6V^2$ while the increasing order was fitted to $12902.7V^2$. From the graph a clear parabolic relationship can be seen. In reality there are some $\log V$ and other V terms giving it the exact shape that it has but the parabolic term is clearly the dominating one.

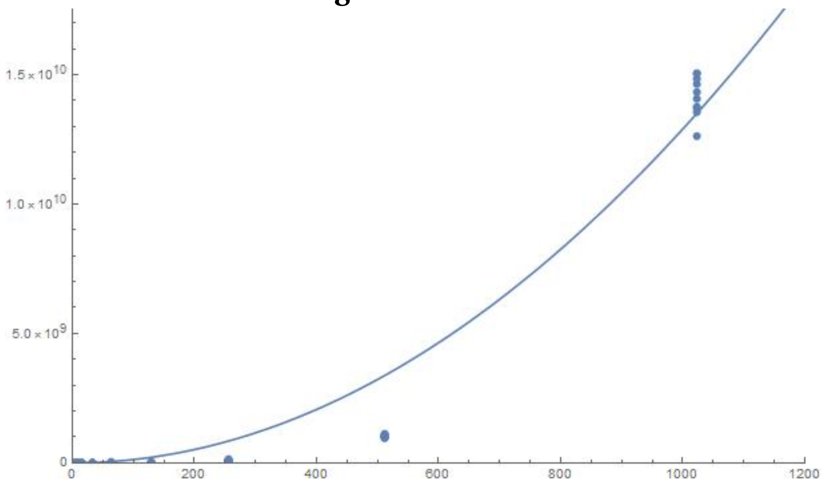
Pure Random Breadth First Search



runtime

number of inputs

Increasing Random Breadth First Search



runttime

number of inputs

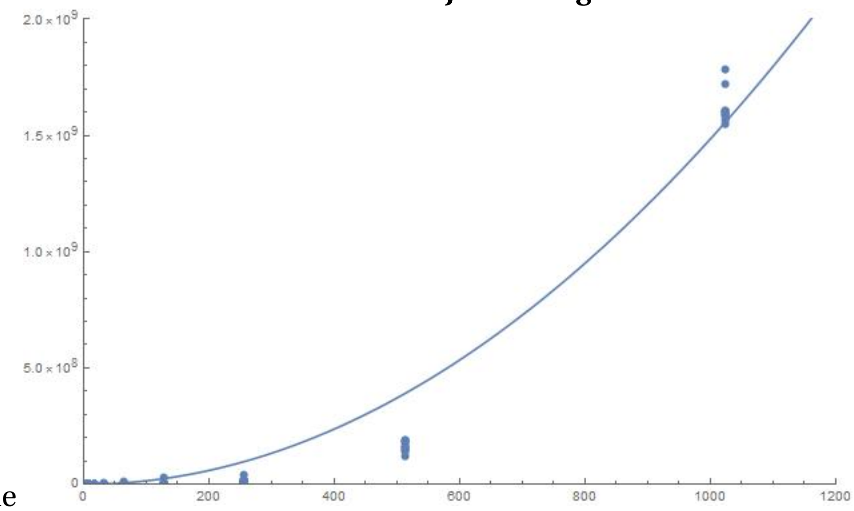
DIJKSTRA'S ALGORITHM

For the dynamic approach to this problem we will utilize Dijkstra's shortest path algorithm. The dynamic programming table is initialized first by taking care of the base cases (setting $R(i,i) = 0$ and $R(i,j) = \infty$ if $i > j$) and setting the estimated "distance" to the rest of the vertices to infinity. Next using Dijkstra's algorithm we traverse the graph from vertex to vertex the estimated infinity distances with the smallest edge weight sum that it takes to traverse to that vertex. By filling in the smallest edge weights in for vertex distances, we're also storing a partial solution to the answer that we're looking for.

The asymptotic complexity of this algorithm is the same as breadth first search : $O(|V|+|E|)$ and like breadth first search the complexity reduces down to: $O(|E|)$. For the same reasons discussed in the breadth first search section the runtime is $O(V^2)$. Given infinite time and more sleep a priority queue or fibonacci que would have been attempted but in the interest of my exam grades, the quickest solution that worked was implemented. The constants are clearly smaller for Dijkstra's algorithm as opposed to breadth first search. Below we can see the experimental results for Dijkstra's algorithm. The pure random runtime was fitted to a model of $1486.33V^2$ while the increasing order was fitted to $1482.97V^2$. From the graph a clear parabolic relationship can be seen. In reality there are some $\log V$ and other V terms giving it the exact shape that it has but the parabolic term is clearly the dominating one.

Pure Random Dijkstra's Algorithmh

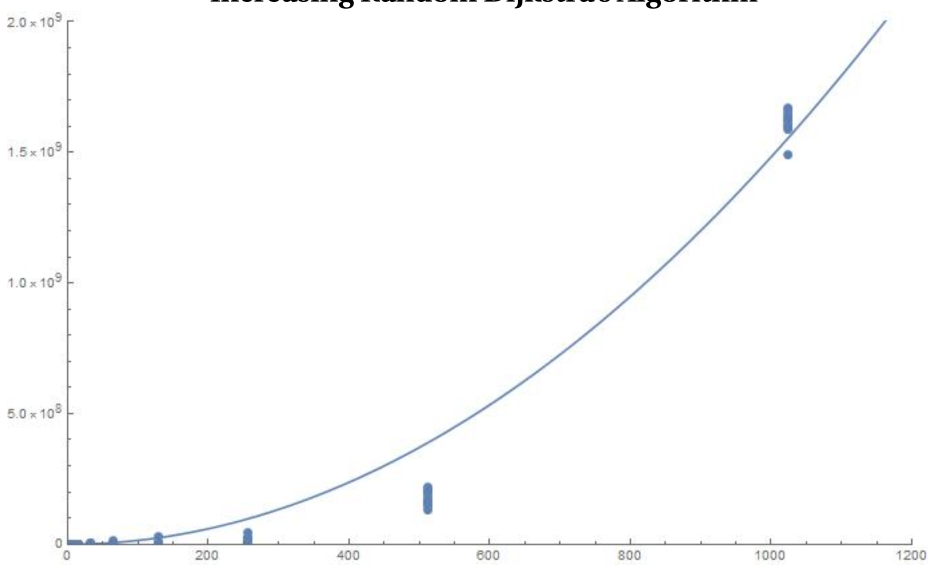
runtime



number of inputs

Increasing Random Dijkstra's Algorithm

runttime



number of inputs

CONCLUSION

The runtime between BFS and Dijkstra's algorithm is a full factor of ten less. While they are asymptotic to each other, this is worth knowing in practice. It is also worth noting that the constants essentially stay the same for Dijkstra depending on the different inputs. This would to me make me think that Dijkstra's algorithm is better than BFS for solving this problem. Not only is it faster by a factor of ten but is is more reliable. As Knuth says, an algorithm must be seen to be believed. All group members worked on this project but a majority of the graph construction and algorithm implementation was completed by Jake. Dijkstra's A Discipline of Programming changed Jake's life. He read it four years ago while he was a political science student and decided that he wanted to move the entire distance across America and study Computer Engineering at the University of Washington. It was a pleasure to finally implement Dijkstra's algorithm. It's been a long road to get to this point, and he didn't take the shortest path, but the destination was finally found. Thank you for this quarter and I'll see you in the fall.

References

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